A Symbolic-Numeric Algorithm for Computing the Alexander Polynomial of a Plane Curve Singularity

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Motivation

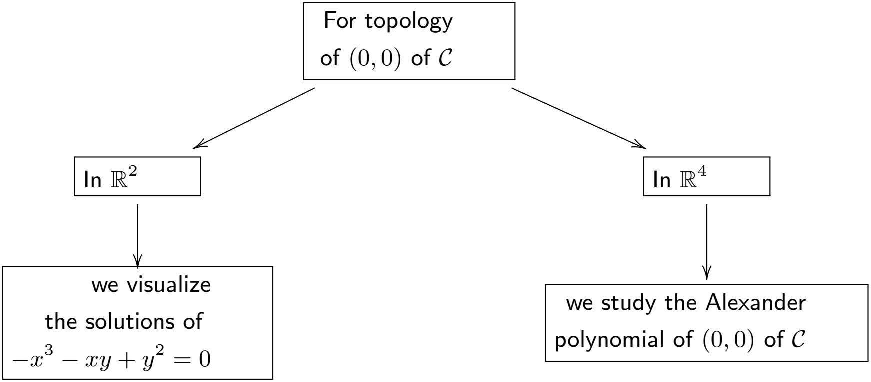
We investigate the topology (i.e. roughly speaking the shape) of plane complex algebraic curves. These curves can be identified with objects in $\mathbb{R}^4$ we cannot visualize! We sketch the equivalent objects in $\mathbb{R}^2$ for a rough "idea"!

Motivation

For instance,
We visualize the topology of the algebraic curve $\mathcal{C} = \{(x, y) | -x^3 - xy + y^2 = 0\}$ in $\mathbb{R}^2$!
We notice an "involved" topology around the point $(0, 0)$, which is called singularity!

Motivation

Roughly, if we consider $\mathcal{C} = \{(x, y) | -x^3 - xy + y^2 = 0\}$ then



Motivation

Our goal is to compute and

Roughly, if we consider $\mathcal{C} = \{(x, y) | -x^3 - xy + y^2 = 0\}$ then

For topology of $(0, 0)$ of $\mathcal{C}$

In $\mathbb{R}^2$

we visualize the solutions of $-x^3 - xy + y^2 = 0$

In $\mathbb{R}^4$

to study the Alexander polynomial of $(0, 0)$ of $\mathcal{C}$

Motivation

Our goal is also to design for:

Rational and floating point numbers → A symbolic-numeric algorithm (in GENOM3CK) (in Axel) → to compute the Alexander polynomial

Because at present (from our knowledge) there exists only for:

Rational numbers → A symbolic algorithm (in Singular) → to compute the Alexander polynomial

Motivation

For instance:

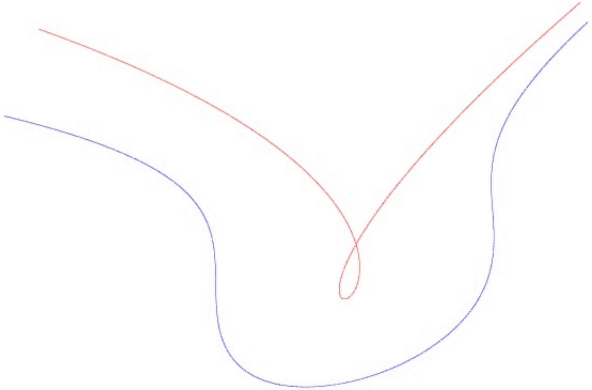
```
                         SINGULAR                             /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-0
                                                           0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann            \   Mar 2009
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "alexpoly.lib";
// ** loaded /sw/share/Singular/LIB/alexpoly.lib (1.18,2009/04/15)
// ** loaded /sw/share/Singular/LIB/hnoether.lib (1.59,2009/04/15)
// ** loaded /sw/share/Singular/LIB/sing.lib (1.34,2009/04/15)
// ** loaded /sw/share/Singular/LIB/primdec.lib (1.147,2009/04/15)
// ** loaded /sw/share/Singular/LIB/absfact.lib (1.7,2008/07/16)
// ** loaded /sw/share/Singular/LIB/triang.lib (1.14,2009/04/14)
// ** loaded /sw/share/Singular/LIB/matrix.lib (1.48,2009/04/10)
// ** loaded /sw/share/Singular/LIB/nctools.lib (1.54,2009/05/08)
// ** loaded /sw/share/Singular/LIB/ring.lib (1.34,2009/04/15)
// ** loaded /sw/share/Singular/LIB/poly.lib (1.53,2009/04/15)
// ** loaded /sw/share/Singular/LIB/elim.lib (1.34,2009/05/05)
// ** loaded /sw/share/Singular/LIB/general.lib (1.62,2009/04/15)
// ** loaded /sw/share/Singular/LIB/random.lib (1.20,2009/04/15)
// ** loaded /sw/share/Singular/LIB/inout.lib (1.34,2009/04/15)
// ** loaded /sw/share/Singular/LIB/primitiv.lib (1.23,2009/04/15)
> ring r=0,(x,y),ls;
> poly f=-x3-x*y+y2;
> list ALEX=alexanderpolynomial(f);
> def ALEXring=ALEX[1];
> setring ALEXring;
> alexpoly;def precision=10;
1
>
. ring r=(real,precision),(x,y),ls;
// ** redefining r **
> poly f=-x3-x*y+y2-0.01;
> list ALEX=alexanderpolynomial(f);
// ** redefining ALEX **
 Singular cannot factorize over 'real' as ground field
```

Problem specifications

- Input:
  - $f(x,y) \in \mathbb{C}[x,y]$ squarefree with symbolic and numeric coefficients;
  - $\mathcal{C} = \{(x,y) \in \mathbb{C}^2 | f(x,y) = 0\} \subseteq \mathbb{C}^2 \simeq \mathbb{R}^4$ of degree m;
  - $\epsilon \in \mathbb{R}_+^*$ input parameter.
- Output:
  - ϵ-Alexander polynomial of each numerical singularity of $\mathcal{C}$;
- We also compute as output:
  - A set of invariants of $\mathcal{C}$ (numerical singularities, algebraic link, Milnor fibration, Milnor number, δ-invariant of each singularity, genus of $\mathcal{C}$, diagram, crossings, arcs of each algebraic link, etc).

Ill-posedness of the problem

The problem is ill-posed! Small changes in input produce huge changes in the output!
Example. Let $s_1 = (0,0)$ of $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 | -x^3 - xy + y^2 = 0\}$ and $s_2 = (0,0)$ of $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 | -x^3 - xy + y^2 - 0.01 = 0\}$!
The topology of $(0,0)$ is not stable under small changes in input!



The same situation happens in $\mathbb{R}^4$, but we cannot visualize it!

Techniques for dealing with the ill-posedness

How to deal with the **ill-posedness** of **a problem**?

- We construct numerical methods that approximate solutions to ill-posed problems, that are stable under small changes of the input! (i.e. regularization method)
- Similar methods are subjects of *approximate algebraic computation* in order to compute: greatest common divisor of polynomials, root of polynomials, etc.

Techniques for dealing with the ill-posedness

How to deal with the **ill-posedness** of **our problem**?

- *Example.* For $s_1 = (0,0)$ of $\mathcal{C} = \{(x,y) \in \mathbb{R}^4 | -x^3 - xy + y^2 = 0\}$ and $s_2 = (0,0)$ of $\mathcal{D} = \{(x,y) \in \mathbb{R}^4 | -x^3 - xy + y^2 - 0.01 = 0\}$, we compute their ϵ-*algebraic links* denoted $L_\epsilon(s_1), L_\epsilon(s_2)$ (**our research**).

For sufficiently small ϵ, L_ϵ are stable under small changes in the input and they characterize the topology of s_1, s_2 (Milnor's research and **our research**).

From L_ϵ we compute the ϵ-Alexander polynomials (**our research**)
This polynomial is a complete invariant for L_ϵ! (Yamamoto's research)

- If the Alexander polynomials of $L_\epsilon(s_1), L_\epsilon(s_2)$ are equal, then s_1, s_2 have the same topology, else they have different topology!

- Next we compute L_ϵ and ϵ-Alexander polynomial.

Mathematical method and algorithm

Plane complex algebraic curve

↓ compute numerically

Singularities moved in origin

↓ compute numerically-symbolically

Algebraic link and operations on it (ϵ needed)

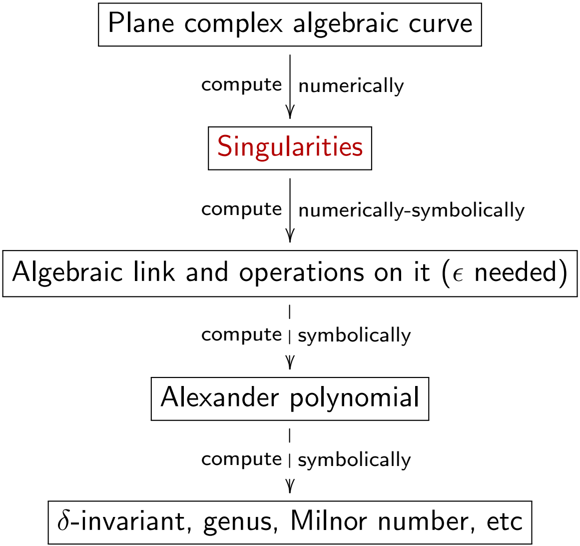
↓ compute symbolically

Alexander polynomial

↓ compute symbolically

δ-invariant, genus, Milnor number, etc

First


Plane complex algebraic curve
compute
numerically
Singularities
compute
numerically-symbolically
Algebraic link and operations on it (ϵ needed)
compute
symbolically
Alexander polynomial
compute
symbolically
δ-invariant, genus, Milnor number, etc

Computing the singularities of the curve

- Input:
  - $f(x,y) \in \mathbb{C}[x,y]$ squarefree with symbolic and numeric coefficients
  - $\mathcal{C} = \{(x,y) \in \mathbb{C}^2 | f(x,y) = 0\}$ complex algebraic curve of degree m.
- Output:
  - $Sing(\mathcal{C}) = \{(x_0,y_0) \in \mathbb{C}^2 | f(x_0,y_0) = 0, \frac{\partial f}{\partial x}(x_0,y_0) = 0, \frac{\partial f}{\partial y}(x_0,y_0) = 0\}$
- Method: We solve the overderminate system in $\mathbb{C}^2$:

$$\begin{cases} f(x_0,y_0) = 0 \\ \frac{\partial f}{\partial x}(x_0,y_0) = 0 \\ \frac{\partial f}{\partial y}(x_0,y_0) = 0 \end{cases} , \qquad (1)$$

using subdivision methods from Axel.

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Next

Plane complex algebraic curve

↓ compute numerically

Singularities moved in the origin

↓ compute numerically-symbolically

Algebraic link and operations on it (ϵ needed)

↓ compute symbolically

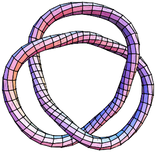
Alexander polynomial

↓ compute symbolically

δ-invariant, genus, Milnor number, etc

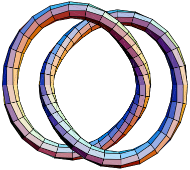
Defining the algebraic link of the singularity

Trefoil Knot



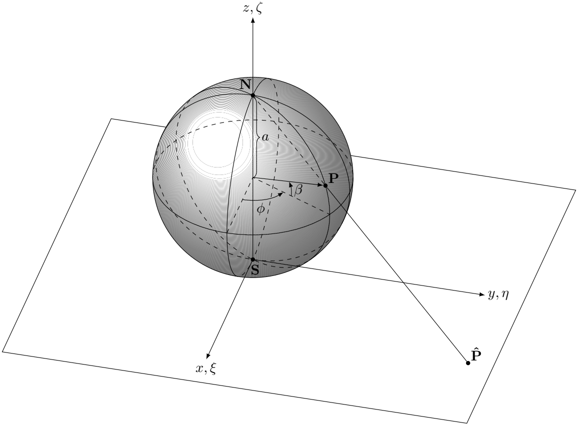
- A **knot** is a piecewise linear or a differentiable simple closed curve in $\mathbb{R}^3$.
- A **link** is a finite union of disjoint knots.
- Links resulted from the intersection of a given curve with the sphere are called **algebraic links**.

Hopf Link



Computing the link of the singularity

- How do we compute the link of a plane curve singularity?
  - use the stereographic projection;



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Computing the link of the singularity

1. Let $\mathcal{C} = \{(a,b,c,d) \in \mathbb{R}^4 | f(a,b,c,d) = 0\}$ s.t. $(0,0,0,0) \in Sing(\mathcal{C})$

2. Consider $S_{(0,\epsilon)} := S = \{(a,b,c,d) \in \mathbb{R}^4 | a^2 + b^2 + c^2 + d^2 = \epsilon^2\}$,
   $X = \mathcal{C} \bigcap S \subset \mathbb{R}^4, f(a,b,c,d) = R(a,b,c,d) + iI(a,b,c,d)$

3. For $N \in S \setminus \mathcal{C}$, $\pi : S \setminus \{N\} \to \mathbb{R}^3, (a,b,c,d) \mapsto (u = \frac{a}{\epsilon - d}, v = \frac{b}{\epsilon - d}, w = \frac{c}{\epsilon - d})$,
   $\pi^{-1} : \mathbb{R}^3 \to S \setminus \{N\}$
   $(u,v,w) \mapsto (a,b,c,d) = (\frac{2u\epsilon}{n}, \frac{2v\epsilon}{n}, \frac{2w\epsilon}{n}, \frac{\epsilon(u^2+v^2+w^2-1)}{n})$, with $n = 1+u^2+v^2+w^2$.

4. Denote $\alpha = (\frac{2u\epsilon}{n}, \frac{2v\epsilon}{n}, \frac{2w\epsilon}{n}, \frac{\epsilon(u^2+v^2+w^2-1)}{n})$. Thus $\pi^{-1}(u,v,w) = \alpha$.

5. Compute $\pi(X) = \{(u,v,w) \in \mathbb{R}^3 | \exists (a,b,c,d) = \pi^{-1}(u,v,w,) \in X = \mathcal{C} \cap S\} \Leftrightarrow$
   $\pi(X) = \{(u,v,w) \in \mathbb{R}^3 | f(\alpha) = 0\} = \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) = I(\alpha) = 0\}$ with
   $R, I \in \mathbb{R}[u,v,w]$. $\pi(X)$ is an implicitly defined algebraic curve!
   For small $\epsilon, \pi(X) := L_\epsilon$ is a link (algebraic link) (based on Milnor's research).

Computing the link of the singularity

We use Axel for implementation.

- For $\mathcal{C} = \{(x, y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4, \epsilon = 1$ we compute with the previous method in Axel:

Computing the link of the singularity

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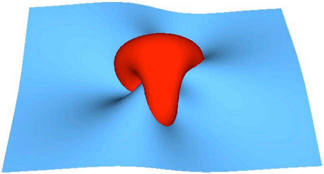
- $\pi(\mathcal{C} \cap S) = \pi(X) := L_\epsilon =$
  $= \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) = 0, I(\alpha) = 0\}$



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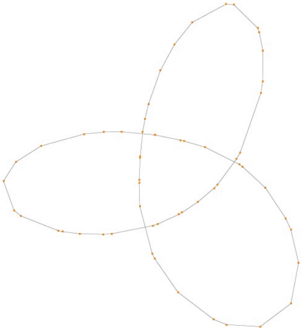
- $\pi(\mathcal{C} \cap S) = \pi(X) := L_\epsilon =$
  $= \{(u, v, w) \in \mathbb{R}^3 | R(\alpha) = 0, I(\alpha) = 0\}$
- $Graph(L_\epsilon) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
  $\mathcal{V} = \{p = (m, n, q) \in \mathbb{R}^3\}$
  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$



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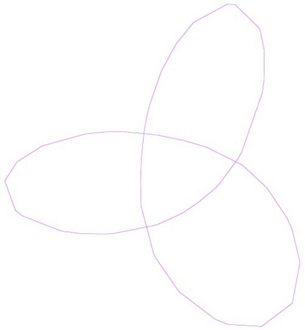
- s.t. $Graph(L_\epsilon) \cong_{isotopic} L_\epsilon$



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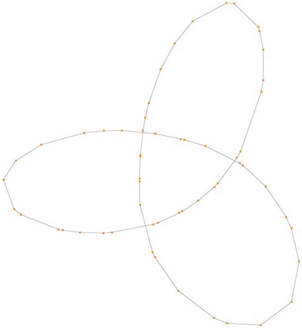
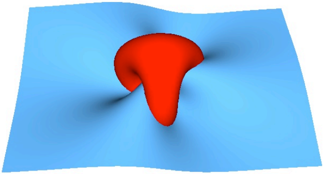
  $= \{(u, v, w) \in \mathbb{R}^3 | R(\alpha) = 0, I(\alpha) = 0\}$
- $Graph(L_\epsilon) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
  $\mathcal{V} = \{p = (m, n, q) \in \mathbb{R}^3\}$
  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- s.t. $Graph(L_\epsilon) \cong_{isotopic} L_\epsilon$
- $Graph(L_\epsilon)$ is a piecewise linear approximation for L_ϵ
- Why Axel? It is the only system to implement a method which returns such an approximation!



Computing the link of the singularity

We use Axel for the implementation. Why Axel?

- For $\mathcal{C} = \{(x,y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4, \epsilon = 1$
- and $L_\epsilon =$
  $= \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) = 0, I(\alpha) = 0\}$



Computing the link of the singularity

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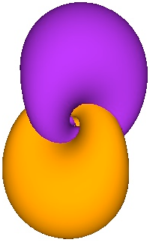
- and $L_\epsilon =$
  $= \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) = 0, I(\alpha) = 0\}$
- we also compute (for visualization reasons)
  $\mathcal{S}' = \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) + I(\alpha) = 0\}$
  $\mathcal{S}'' = \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) - I(\alpha) = 0\}$



Computing the link of the singularity

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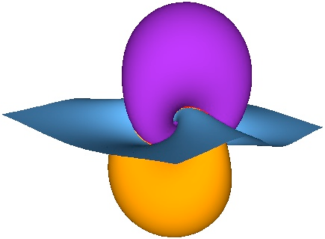
- we also compute (for visualization reasons)
  $\mathcal{S}' = \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) + I(\alpha) = 0\}$
  $\mathcal{S}'' = \{(u,v,w) \in \mathbb{R}^3 | R(\alpha) - I(\alpha) = 0\}$
- L_ϵ is the intersection of any 2 of the surfaces:
  $R(\alpha), I(\alpha)$
  $R(\alpha) + I(\alpha), R(\alpha) - I(\alpha)$



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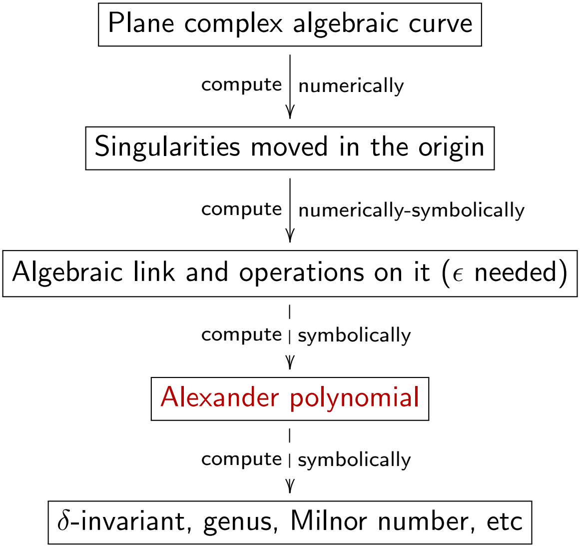

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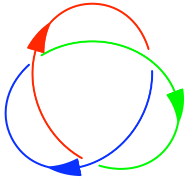
Computing the Alexander polynomial

We give an example to compute the Alexander polynomial Δ_L for a link L with K knots! We need some definitions.

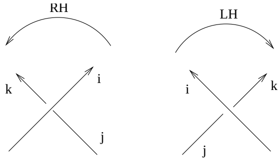
A **diagram** is the image under projection, together with the information on each crossing telling which branch goes over and which under.
An **arc** is the part of a diagram between two undercrossings.

A crossing is:
-**righthanded** if the underpass traffic goes from right to left.
-**lefthanded** if the underpass traffic goes from left to right.

Diagram and arcs

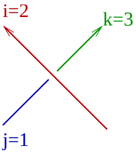


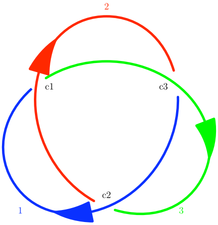
Crossings



Computing the Alexander polynomial of the link

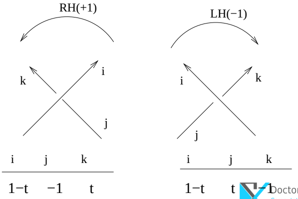
Example: Δ_L for L with $K = 1$ knot (i.e. trefoil knot).





$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & -1 & 2 & 1 & 3 \\ & & & & \\ \end{array} \right)$$

$$P(L) = \left(\quad \right)$$

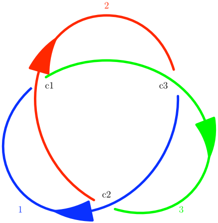


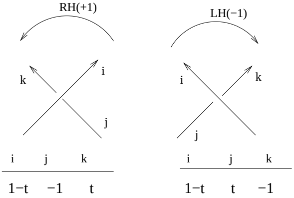
Computing the Alexander polynomial of the link

Example: Δ_L for L with $K = 1$ knot.

$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & -1 & 2 & 1 & 3 \\ & & 1-t & t & -1 \\ & & & & \end{array}\right)$$

$$P(L) = \begin{pmatrix} 2 & 1 & 3 \\ 1-t & t & -1 \\ & & \end{pmatrix}$$





Computing the Alexander polynomial of the link

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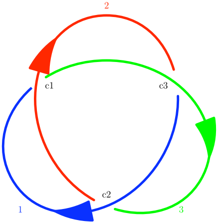
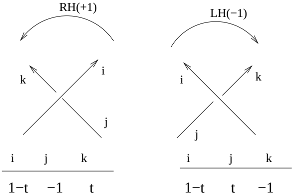
$$P(L) = \begin{pmatrix} 1 & 2 & 3 \\ t & 1-t & -1 \\ & & \end{pmatrix}$$





Computing the Alexander polynomial of the link

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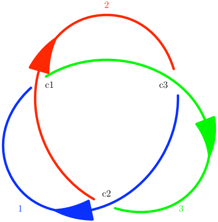
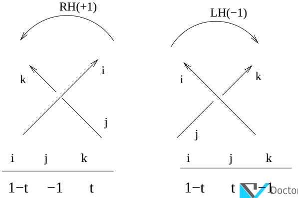


$$P(L) = \begin{pmatrix} t & 1-t & -1 \\ 1-t & -1 & t \\ -1 & t & 1-t \end{pmatrix}$$

$$D := det(minor(P(L))) = -t^2 + t - 1$$

$$\Delta(L) := \Delta(t) = Normalise(D) = t^2 - t + 1$$

For a link L with $K > 1$ knots and n crossings, $\Delta(L)$ is the gcd of all the $(n-1) \times (n-1)$ minor determinants of $P(L)$.



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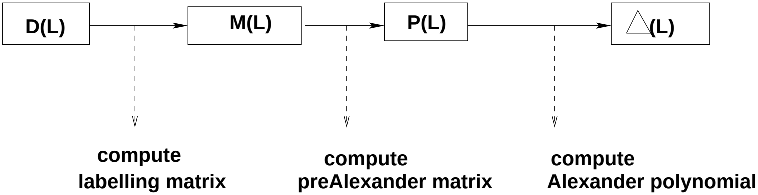
Computing the Alexander polynomial of the link

So, the Alexander polynomial is computed in several steps:



In order to compute it, we need D(L)!

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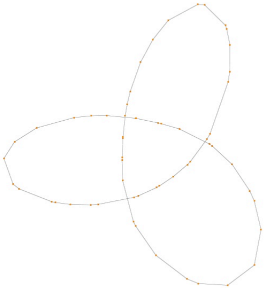
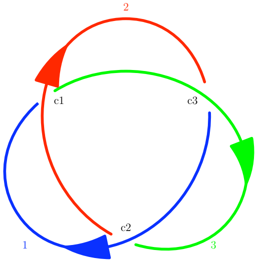
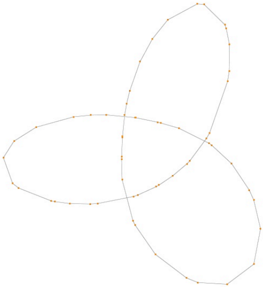
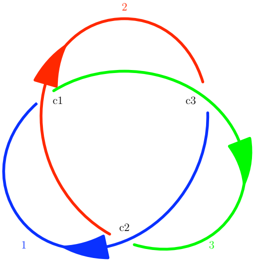
- 

$\overset{?}{\Rightarrow}$



- $G(L_\epsilon) = \langle P, E \rangle$ $D(L_\epsilon)$
- We need to transform the graph data structure $G(L_\epsilon)$ returned by Axel into the diagram of the algebraic link $D(L_\epsilon)$.

Computing the Alexander polynomial of the link

-  $\overset{?}{\Rightarrow}$ 

- We developed several computational geometry and combinatorial algorithms for this! (M. Hodorog, J.Schicho. *Computational geometry and combinatorial algorithms for the genus computation problem.* DK 10-07 Report).

Implementation

- *Axel* free algebraic geometric modeler (INRIA Sophia-Antipolis) [a]

http://axel.inria.fr/

[a] Acknowledgements: Julien Wintz

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  - written in *C*++, Qt Script for Applications (QSA);

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- GENOM3CK-our library in Axel (GEnus cOMputation of plane Complex algebraiC Curves with Knot theory). Support: `http://people.ricam.oeaw.ac.at/m.hodorog/software.html` and `madalina.hodorog@oeaw.ac.at`

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- Version 0.2 of GENOM3CK is released!

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Analysis of the algorithm

With the notations:

- E the symbolic algorithm to compute the Alexander polynomial, which is ill-posed.
- A_ϵ the symbolic-numeric algorithm to compute the ϵ-Alexander polynomial Δ_ϵ from the ϵ-algebraic link L_ϵ.
  - For input polynomial f, $A_\epsilon(f)$ returns as output Δ_ϵ.
  - For perturbed f_δ (for any $\delta : ||f - f_\delta|| < \delta$), $A_\epsilon(f_\delta)$ returns as output Δ_ϵ^δ.

and based on:

- Milnor's theorem, i.e. if $\epsilon \to 0$ then Δ_ϵ tends to the exact solution (**convergence for exact data**)
- and on general results from regularization theory (adapted to our case).

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- Δ_ϵ^δ depends continuously on the perturbed input polynomial f_δ (**continuity**);
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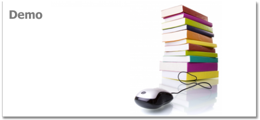
Demo (Numeric and Symbolic Examples)

Equation in $\mathbb{R}^4$	Box
$-x^3 - xy + y^2, \epsilon = 1.00$	$[-4, 4, -6, 6, -6, 6]$
$-x^3 - xy + y^2, \epsilon = 0.25$	$[-4, 4, -6, 6, -6, 6]$
$-x^3 - xy + y^2 - 0.01, \epsilon = 1.00$	$[-4, 4, -6, 6, -6, 6]$
$-x^3 - xy + y^2 - 0.01, \epsilon = 0.25$	$[-4, 4, -6, 6, -6, 6]$



Conclusion and future work

✓ DONE:

- automatization of symbolic-numeric algorithms for plane curves in GENOM3CK (i.e. algorithm to compute the Alexander polynomial);
- describe algorithms with principles from regularization theory;

✗ TO DO's:

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- proofs for continuity and convergence for perturbed data properties are constructed.

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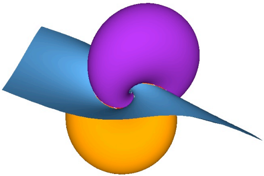
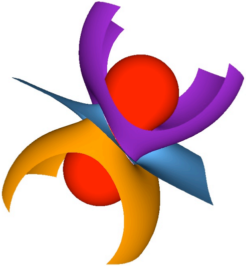
- finalize the proof for convergence for perturbed data property of the algorithm;
- include other operations, i.e. from knot theory, algebraic geometry.





Thank you for your attention.
Questions?