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An Algorithmic Approach to the Mellin Transform Method

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Abstract. We present proofs for typical entries from the Gradshteyn-Ryzhik Table of Integrals using the Mellin transform method and computer algebra algorithms based on WZ theory. After representing an identity from the Table in terms of multiple contour integrals of Barnes' type and nested sums, we use Wegschaider's summation algorithm to find recurrences satisfied by both sides of this identity and check finitely many initial values.

1. Introduction

Considerable work on proving and verifying the entries in the Gradshteyn-Ryzhik Table of Integrals, Series and Products [**6**] is being done by Victor Moll, et.al., in a series of articles, the latest being [**1**]. Moreover, an introduction to the art of evaluating definite integrals using a variety of techniques can be found in [**5**].

We continue this effort with an approach based on the Mellin transform method for rewriting definite integration problems in terms of nested Mellin-Barnes integrals. This way, we end up with complex contour integrals over hypergeometric terms which are in the input class of summation algorithms based on the WZ-paradigm, as it was shown in [**13**]. Viewing the identities in [**6**] from the perspective of the Mellin transform method seems natural, especially since most entries from the table of Mellin transforms [**9**] are also found there.

In this section we introduce the general Mellin transform method and hint at the role of algorithmic tools like Wegschaider's summation algorithm [**14**]. Let us first recall that the Mellin transform of a locally integrable function $f : (0, \infty) \to \mathbb{C}$ is defined by

$$\tilde{f}(z) = \int_0^\infty x^{z-1} f(x)dx \tag{1}$$

where the integral converges, usually on an infinite strip of the form $\alpha < Re(z) < \beta$. Our main tool will be the inversion formula, given by

$$f(x) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} x^{-z} \tilde{f}(z)dz \tag{2}$$

which uniquely determines $f(x)$ from $\tilde{f}(z)$. The contour of integration is a vertical line in the z-plane and must be placed in the strip of analyticity $\alpha < \delta < \beta$.

To introduce our approach to identities involving definite integrals, we prove the main property of the Mellin convolution:

$$\int_0^\infty g(xy)h(y)dy = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} x^{-z} \tilde{g}(z)\tilde{h}(1-z)dz, \tag{3}$$

where $g, h : (0, \infty) \to \mathbb{C}$ are defined such that the left hand side integral exists and the Mellin transforms $\tilde{g}(z)$ and $\tilde{h}(1-z)$ have a common domain of analyticity with δ lying in this common

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domain. Note that the special case $x = 1$ of (3) is called the Parseval formula for the Mellin transform [**10**].

To prove (3), we start with the Mellin convolution on the left-hand side and use the inversion formula to insert $\tilde{g}(z)$. By Fubini's theorem, we interchange the order of integration

$$\int_0^\infty g(xy)h(y)dy = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}\tilde{g}(z)\left(\int_0^\infty y^{-z}h(y)dy\right)dz$$

and since the inner definite integral is precisely $\tilde{h}(1-z)$ the proof is complete.

From here on we will avoid using properties of the Mellin transform apart from the definition of the reciprocal pair (1) and (2). The proof of property (3) has here merely a didactic purpose since we use a similar technique to prove identities from the table of integrals [**6**].

More precisely, we rewrite the definite integrals appearing in table entries considered here by inserting a Mellin-Barnes integral representation of type (2) for a factor of the integrand. We are doing this in the hope that after interchanging the order of integration, the inner integral becomes an easily computable definite integral and we end up with a contour integral of Barnes' type over a hypergeometric integrand.

For example, when proving the identity ([**6**], **3.383.1**)

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x}dx = B(\mu,\nu)u^{\mu+\nu-1}\,{}_1F_1(\nu;\nu+\mu;\beta u), \quad [Re\,\mu>0, Re\,\nu>0] \tag{4}$$

we rewrite the left-hand side by plugging in the Mellin-Barnes integral representation

$$e^{\beta x} = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{(-\beta)^z}\Gamma(z)x^{-z}dz, \quad \delta>0.$$

This representation of the exponential function is to be found in [**9**] or can be obtained by observing that its Mellin transform is given by ([**2**], 1.1.18)

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad Re\,z>0 \tag{5}$$

and using the inversion formula (2) afterwards.

Hence, the left-hand side of (4) becomes

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x}dx = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{(-\beta)^z}\Gamma(z)\left(\int_0^u x^{\nu-z-1}(u-x)^{\mu-1}dx\right)dz.$$

After several changes of variables, the inner definite integral is given by

$$\int_0^u x^{\nu-z-1}(u-x)^{\mu-1}dx = u^{\nu+\mu-1-z}B(\nu-z,\mu),$$

where B denotes the beta function. The identity (4) is equivalent to

$$\frac{\Gamma(\nu+\mu)}{2\pi i\Gamma(\nu)}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\nu-z)}{\Gamma(\nu+\mu-z)}\Gamma(z)(-u\beta)^{-z}dz = {}_1F_1(\nu;\nu+\mu;\beta u),$$

which is the Barnes' integral representation for the confluent hypergeometric function ${}_1F_1$; see for instance section 4.2 in [**2**]. Note that identity (4) constitutes the base case for a proof by induction in n of the entry **3.478.3** from [**6**].

Proving more involved identities from [**6**] using the Mellin transform method requires inserting the Barnes type integral representations for two or more factors of the integrand. In this case we will end up with multiple nested contour integrals over hypergeometric terms and a sum representation for such integrals is not always easily determined. Examples of such situations are included in section 3.

Section 2 describes how Wegschaider's summation algorithm [**14**] can be used to compute homogeneous and inhomogeneous recurrences not only for nested sums but also for multiple Mellin-Barnes integrals over hypergeometric terms. This algorithmic method of proving and computing recurrences for contour integrals of this type was already used in [**13**] for a class of Ising integrals.

Wegschaider's algorithm [**14**] adds more power to the Mellin transform method. Finding recurrences for both sides of an identity reduces the problem to checking finitely many initial values. Even though several non-algorithmic aspects are involved in the proofs, we are able to tackle more and more involved entries from the table.

2. An algorithmic approach

2.1. Deriving Recurrences Algorithmically. Wegschaider's algorithm [**14**] is an extension of multivariate WZ summation [**16**], and in this context it is used to compute recurrences for sums of the form

$$Sum\,(\mu) = \sum_{\kappa_1 \in \mathcal{R}_1} \cdots \sum_{\kappa_r \in \mathcal{R}_r} \mathcal{F}\,(\mu, \kappa_1, \ldots, \kappa_r)\,. \tag{6}$$

Loosely speaking, this algorithm [**14**] can be applied if the summands $\mathcal{F}\,(\mu, \kappa)$ are hypergeometric in all integer variables μ_i from $\mu = (\mu_1, \ldots, \mu_p)$ and in all summation variables κ_j from $\kappa = (\kappa_1, \ldots, \kappa_r) \in \mathcal{R}$ where $\mathcal{R} := \mathcal{R}_1 \times \cdots \times \mathcal{R}_r \subseteq \mathbb{Z}^r$ is the summation range.

Remark: Recall that an expression $\mathcal{F}\,(\mu, \kappa)$ is called hypergeometric [**17, 16**] if there exists a rational function $r_{m,k}(\mu, \kappa)$ such that $\frac{\mathcal{F}(\mu,\kappa)}{\mathcal{F}(\mu-m,\kappa-k)} = r_{m,k}(\mu, \kappa)$ at the points $m \in \mathbb{Z}^p$ and $k \in \mathbb{Z}^r$ where this ratio is defined.

The algorithm first finds a recurrence for the summand $\mathcal{F}\,(\mu, \kappa)$ called certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} a_m\,(\mu)\,\mathcal{F}(\mu + m, \kappa) = \sum_{j=1}^{r} \Delta_{\kappa_j} \left(\sum_{(m,k) \in \mathbb{S}_j} b_{m,k}\,(\mu, \kappa)\,\mathcal{F}(\mu + m, \kappa + k) \right), \tag{7}$$

where the polynomials $a_m\,(\mu)$, not all zero, $b_{m,k}\,(\mu, \kappa)$ and the sets $\mathbb{S}_j \subset \mathbb{Z}^{p+r}$ are determined algorithmically.

The forward shift operators denoted above with Δ_{κ_j} are defined as

$$\Delta_{\kappa_j} \mathcal{F}\,(\mu, \kappa) := \mathcal{F}\,(\mu, \kappa_1, \ldots, \kappa_j + 1, \ldots, \kappa_r) - \mathcal{F}\,(\mu, \kappa)\,.$$

Moreover, the right hand side of (7) can always be rewritten as

$$\sum_{j=1}^{r} \Delta_{\kappa_j} \left(\sum_{(m,k) \in \mathbb{S}_j} b_{m,k}\,(\mu, \kappa)\,\mathcal{F}(\mu + m, \kappa + k) \right) = \sum_{j=1}^{r} \Delta_{\kappa_j} \left(r_j\,(\mu, \kappa)\,\mathcal{F}(\mu, \kappa) \right), \tag{8}$$

where r_j are rational functions of all variables from $\mu = (\mu_1, \ldots, \mu_p)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$.

Remark: In the certificate recurrence (7), the coefficients $a_m\,(\mu)$ are polynomials free of the summation variables κ_j from κ, while the coefficients $b_{m,k}\,(\mu, \kappa)$ of the delta-parts are polynomials in all the variables from μ and κ.

Finally, the recurrence for the multisum (6) is obtained by summing the certificate recurrence (7) over all variables from κ in the given summation range $\mathcal{R}$. Since it can be easily checked whether the summand $\mathcal{F}(\mu, \kappa)$ indeed satisfies the recurrence (7), the certificate recurrence also provides a proof of the recurrence for the multisum $Sum\,(\mu)$.

Two further remarks are required. First, Wegschaider's algorithm determines certificate recurrences, after making an Ansatz about their structure (i.e., fixing the structure set $\mathbb{S}$), by solving a large system of linear equations over a field of rational functions. If the input of the algorithm is involved, computations will be time consuming. To this purpose, the procedure `FindStructureSet` included in the package `MultiSum` and already used in [**7**], implements an algorithm based on modular computation for finding small structure sets. To use this procedure and the summation algorithm [**14**], one loads the package `MultiSum`[1] within a Mathematica session:

In[1]:= **<< MultiSum.m**

MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

[1]available at `http://www.risc.uni-linz.ac.at/research/combinat/software/`

To be more precise, the algorithm [**14**] terminates successfully, for a large enough structure set, if we restrict our input class to *proper* hypergeometric summands; see [**16**] for the definition of proper hypergeometric terms and also regarding the existence conditions for certificate recurrences.

Secondly, we remark that in many applications the function $\mathcal{F}(\mu, \kappa)$ has finite support. In these cases, if we sum the recurrence (7) over a domain that is larger than the support of the function, the Δ-parts on the right hand side telescope and the values that are not in the support vanish. So, from the summand recurrence one obtains a homogeneous recurrence for the sum

$$\sum_{m \in \mathbb{S}} a_m(\mu)\, Sum(\mu + m) = 0. \tag{9}$$

This is not the case in general; i.e., in specific situations human inspection is still needed to pass from the recurrence (7) to a homogeneous or inhomogeneous recurrence for the sum (6). More information on this subject can be found in [**16**].

2.2. From Summation to Integration. In this section we will show how Wegschaider's algorithm [**14**] can be used to determine recurrences for multiple contour integrals of Barnes' type

$$Int(\mu) = \int_{\mathcal{C}_{\kappa_1}} \dots \int_{\mathcal{C}_{\kappa_r}} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r)\, d\kappa_1 \dots d\kappa_r, \tag{10}$$

where the integrands $\mathcal{F}(\mu, \kappa)$ are hypergeometric in all integer variables μ_i from $\mu = (\mu_1, \dots, \mu_p)$ and in all integration variables κ_j from $\kappa = (\kappa_1, \dots, \kappa_r) \in \mathbb{C}^r$. This idea, going back to W. Zudilin [**3**], was already used in [**13**] to prove recurrences for a class of Ising integrals.

As in the case of the summation problem (6), the fundamental theorem of hypergeometric summation stated by Wilf and Zeilberger in [**16**] proves the existence of non-trivial certificate recurrences of the form (7) for the function $\mathcal{F}(\mu, \kappa)$. Using WZ summation methods, Wegschaider's algorithm [**14**] delivers recurrences of the form (7) for the hypergeometric integrand from (10), and as remarked in section 2.1, the coefficients on the left hand side of this recurrence are free of all integration variables $\kappa = (\kappa_1, \dots, \kappa_r)$.

Remark: Although discrete functions are our main interest, one can evaluate the function $\mathcal{F}(\mu, \kappa)$ also for complex values of the variables μ_i and κ_j for all $1 \le i \le p$ and $1 \le j \le r$ except at certain poles. In our case, the singularities of the numerator gamma functions need to be excluded from the evaluation domain. The function $\mathcal{F}(\mu, \kappa)$ is then continuous on its evaluation domain, and by taking limits it can be shown that the computed recurrences hold in $\mathbb{C}^{p+r}$.

After successively integrating over the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ for $1 \le j \le r$, (7) leads, in some cases, to a homogeneous recurrence for the integration problem (10), i.e.,

$$\sum_{m \in \mathbb{S}} a_m(\mu)\, Int(\mu + m) = 0. \tag{11}$$

However, again in analogy to the summation case, after integrating over the contours of integration $\mathcal{C}_{\kappa_j}$ for $1 \le j \le r$, it is not clear in general that we obtain a homogeneous equation of the type (11). Consequently, one needs to analyze the behavior of the contour integrals over the left hand side of (7).

For this purpose, we study the following integration problems:

$$I_j := \int_{\mathcal{C}_{\kappa_j}} \Delta_{\kappa_j} \mathcal{F}(\mu, \kappa)\, d\kappa_j = \int_{\mathcal{C}'_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j - \int_{\mathcal{C}_{\kappa_j}} \mathcal{F}(\mu, \kappa)\, d\kappa_j, \tag{12}$$

where the Barnes path $\mathcal{C}_{\kappa_j}$ runs vertically over $(c_j - i\infty, c_j + i\infty)$ while $\mathcal{C}'_{\kappa_j}$ denotes the shifted path $(1 + c_j - i\infty, 1 + c_j + i\infty)$ for all $1 \le j \le r$.

For any $1 \leq j \leq r$, consider now the contour integral I_j^N over a rectangle with vertices at the points $c_j - iN$, $c_j + iN$, $1 + c_j + iN$ and $1 + c_j - iN$ with $N \in \mathbb{N}$; i.e.,

$$I_j^N = \int_{c_j-iN}^{c_j+iN} \mathcal{F}(\mu,\kappa)\,d\kappa_j + \int_{c_j+iN}^{1+c_j+iN} \mathcal{F}(\mu,\kappa)\,d\kappa_j - \int_{1+c_j-iN}^{1+c_j+iN} \mathcal{F}(\mu,\kappa)\,d\kappa_j + \int_{1+c_j-iN}^{c_j-iN} \mathcal{F}(\mu,\kappa)\,d\kappa_j. \tag{13}$$

If in any such rectangular region of integration, we have the asymptotic behavior

$$\mathcal{F}(\mu,\kappa) = \mathcal{O}\left(\frac{1}{|\kappa_j|^d} e^{-c|\kappa_j|}\right) \quad \text{as} \quad |\kappa_j| \to \infty \quad \text{with} \quad c \geq 0, \quad d > 0 \tag{14}$$

then $I_j^N \to I_j$ as $N \to \infty$. When the function $\mathcal{F}(\mu,\kappa)$ is dominated by an exponential with negative exponent, it suffices to to analyze the integrals (12) instead of the integrals over the right hand side of (8).

On the other hand, we can calculate the integrals (13) by considering the residues of the function $\mathcal{F}(\mu,\kappa)$ at the poles lying inside the closed rectangular contours. If for all $1 \leq j \leq r$, the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ can be chosen such that the function $\mathcal{F}(\mu,\kappa)$ has no poles inside these rectangular regions, then the integrals (12) will be zero.

Under these restrictions, we obtain from the certificate recurrence (7) a homogeneous recurrence (11) for the multiple Barnes' type integral (10). Note that a different choice of the integration contours will lead to inhomogeneous recurrences for multiple Barnes' integrals which satisfy the asymptotic condition (14).

We consider the entry ([**6**], **6.512.3**) as an explanatory example

$$\int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x)dx = \frac{\beta^{\nu-1}}{\alpha^\nu}, \quad [\beta < \alpha] \tag{15}$$

where J_ν denotes the Bessel function of the first kind of order ν; see for instance ([**2**], 4.5.2). The Mellin-Barnes integral representation of J_ν is given by ([**9**], 10.1)

$$J_\nu(\alpha x) = \frac{1}{4\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{\nu+z}{2}\right)}{\Gamma\left(1+\frac{\nu-z}{2}\right)}\left(\frac{\alpha x}{2}\right)^{-z}. \tag{16}$$

Using the Mellin transform method presented in section 1, we obtain a simple Mellin-Barnes integral representation for the left hand side of (15)

$$\int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x)dx = \frac{1}{2\pi i\beta}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{\beta}{\alpha}\right)^z \frac{dz}{\nu - z} \tag{17}$$

where $-\nu < \delta < \frac{3}{2}$.

We denote the integral on the right hand side of (17) by

$$Int[\nu] = \int F[\nu,z]dz \tag{18}$$

and observe that $F[\nu,z] = \mathcal{O}\left(|z|^{-1} e^{|z|\,log\,\frac{\beta}{\alpha}}\right)$.

Once we input the integrand

In[2]:= $\boldsymbol{F[\nu\_, z\_] := \frac{1}{2\pi i\beta(\nu - z)}\left(\frac{\beta}{\alpha}\right)^z}$

we compute a certificate recurrence in the integer parameter ν using either the Mathematica implementation of Zeilberger's algorithm [**11**] or the command

In[3]:= **FindRecurrence** $[\boldsymbol{F[\nu, z], \nu, z, 1}]$;

from the package `MultiSum` and shift this recurrence accordingly:

In[4]:= **ShiftRecurrence** $[\%[[1]], \{\boldsymbol{\nu, 1}\}, \{\boldsymbol{z, 1}\}]$

Out[4]= $\beta F[\nu, z] - \alpha F[\nu+1, z] = \Delta_z\left[\alpha F[\nu+1, z]\right].$

Here we need to think about the contour of integration. Since δ can be chosen such that the rectangular regions described above do not contain the pole of the function $F[\nu+1, z]$, we find the homogeneous recurrence satisfied by the left hand side of (15) as the output of the following command:

In[5]:= $\boldsymbol{rec1} = \mathbf{SumCertificate}\,[\%]\ /.\mathbf{SUM} \to \mathbf{INT}$

Out[5]= $\beta\mathrm{INT}[\nu] - \alpha\mathrm{INT}[\nu+1] = 0.$

While in this simple case we can read off the solution of the recurrence relation, in general situations, solving might be done using the package `Hyper` [**12**]. In this case since the right side of the identity (15) is given, one can simply check that it also satisfies the recurrence above:

In[6]:= $\mathbf{RHS}[\boldsymbol{\nu}\_] := \dfrac{\boldsymbol{\beta}^{\boldsymbol{\nu}-\mathbf{1}}}{\boldsymbol{\alpha}^{\boldsymbol{\nu}}}$

In[7]:= $\mathbf{CheckRecurrence}\,[\boldsymbol{rec1}, \mathbf{RHS}[\boldsymbol{\nu}]]$

Out[7]= True.

The initial value that needs to be checked is a known property of the Bessel function. A similar approach works for the other two cases given in the table for this identity.

3. Examples

3.1. A simple example. To prove the identity ([**6**],**7.245.1**)

$$\int_0^{2\pi} P_{2m+1}(\cos\theta)\cos\theta d\theta = \frac{\pi}{2^{4m+1}}\binom{2m}{m}\binom{2m+2}{m+1} \tag{19}$$

we use the change of variable $\sin\theta =: x$ and the following representation for the Legendre function of the first kind

$$P_\nu(z) = {}_2F_1\left(\begin{matrix} -\frac{\nu}{2}, \frac{\nu+1}{2} \\ 1 \end{matrix}; 1-z^2\right).$$

Since, in our case, $\nu = 2m+1$ with $m \in \mathbb{N}$ by converting the ${}_2F_1$ to a Barnes integral, reversing the order of integration and evaluating the innermost integral, we rewrite (19) as

$$\begin{aligned} \frac{1}{2\pi i\Gamma(-m-\frac{1}{2})\Gamma(m+1)} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(-m-\frac{1}{2}+s)\Gamma(m+1+s)\Gamma(-s)}{\Gamma(1+s)} \frac{(-1)^s}{(2s+1)} ds \\ = \frac{\pi}{2^{4m+3}}\binom{2m}{m}\binom{2m+2}{m+1}. \end{aligned} \tag{20}$$

This path of integration is curved to put the poles of the gamma functions $\Gamma(-m-\frac{1}{2}+s)$ and $\Gamma(m+1+s)$ to the left of the path and the poles of $\Gamma(-s)$ to the right.

Using Wegschaider's algorithm [**14**], we find a recurrence for the integrand:

In[8]:= $\mathrm{F}\,[\boldsymbol{m}\_, \boldsymbol{s}\_] := \dfrac{\mathbf{\Gamma(-m-1/2+s)\Gamma(m+1+s)\Gamma(-s)(-1)^s}}{\mathbf{2\pi i\Gamma(-m-1/2)\Gamma(m+1)\Gamma(1+s)(2s+1)}}$

In[9]:= $\mathbf{FindRecurrence}\,[\boldsymbol{F}[\boldsymbol{m}, \boldsymbol{s}], \boldsymbol{m}, \{\boldsymbol{s}\}, \mathbf{1}]$

In[10]:= $\boldsymbol{rec1} = \mathbf{ShiftRecurrence}\,[\%[[1]], \{\boldsymbol{m}, \mathbf{1}\}, \{\boldsymbol{s}, \mathbf{1}\}]$

Out[10]= $2(1+m)(1+2m)(3+2m)(9+4m)F[m,s] + 3(7+4m)(11+14m+4m^2)F[1+m,s]$
$- 4(2+m)(3+m)(5+2m)(5+4m)F[2+m,s] = \Delta_s[2(1+2m)(3+2m)(9+4m)sF[m,s]$
$- 2(300+610m+446m^2+140m^3+16m^4+297s+510ms+276m^2s+48m^3s)F[1+m,s]$
$+ 4(2+m)(3+m)(5+2m)(5+4m)F[2+m,s]]$

To check the asymptotic condition (14), we use Stirling's formula ([**2**], 1.4)

$$\log\Gamma(z+a) = (z+a-\frac{1}{2})\log z - z + \mathcal{O}(1). \tag{21}$$

Since $|e^{i\pi y}| = 1$ for any real y, we write PI for the pure imaginary terms and we obtain

$$\log F[m,s] = -\frac{5}{2}\log|s| + (arg(-s) - arg(s) - \pi)\, Im\, s + PI + \mathcal{O}(1).$$

Here we distinguish two cases, either $Im(s) > 0$ or $Im(s) < 0$, and in either of these cases the function $F[m,s]$ is of the form (14).

Integrating over the certificate recurrence with a suitable contour leads to a zero integral over the Δ_s part and we obtain a homogeneous recurrence for the left hand side of (20):

In[11]:= $\boldsymbol{rec2} = \textbf{SumCertificate}[\boldsymbol{rec1}]/.\textbf{SUM} \rightarrow \textbf{INT}$

Out[11]= $2(1+m)(1+2m)(3+2m)(9+4m)\mathrm{INT}[m] + 3(7+4m)(11+14m+4m^2)\mathrm{INT}[1+m] - 4(2+m)(3+m)(5+2m)(5+4m)\mathrm{INT}[2+m] = 0$

Now we check that the right hand side of (20) also satisfies the recurrence:

In[12]:= $\textbf{RHS}[\boldsymbol{m\_}] := \frac{\boldsymbol{\pi}}{\boldsymbol{2^{4m+3}}}\binom{\boldsymbol{2m}}{\boldsymbol{m}}\binom{\boldsymbol{2m+2}}{\boldsymbol{m+1}}$

In[13]:= $\textbf{CheckRecurrence}[\boldsymbol{rec2}, \boldsymbol{RHS}[\boldsymbol{m}]]$

Out[13]= True.

Lastly, we see that we only need to show that identity (19) holds for two initial values $m = 0$ and $m = 1$, and this is done by looking up the appropriate Legendre polynomials.

3.2. Examples involving orthogonal polynomials. For the functions considered so far, the Mellin transform existed as defined in (1) and the contour of integration for its Mellin-Barnes integral representation passing through $\delta \in \mathbb{R}$ lied in the strip of analyticity $\alpha < \delta < \beta$. In the case of a polynomial of order $n \in \mathbb{N}$ we have $\alpha = 0$ and $\beta = -n$. Hence, the Mellin transform does not exist as defined in (1).

A constructive approach to this problem is presented in ([**4**], 4.3). We first decompose the function $f(x)$ into two functions defined on disjoint intervals, for instance,

$$f_1(x) = \begin{cases} f(x), & x \in [0,1) \\ 0, & x \in [1,\infty) \end{cases}, \quad f_2(x) = \begin{cases} 0, & x \in [0,1) \\ f(x), & x \in [1,\infty) \end{cases}.$$

Then, by analytic continuation of their Mellin transforms, we obtain the Mellin transform of the function f as a meromorphic function defined by

$$\tilde{f}(z) = \tilde{f}_1(z) + \tilde{f}_2(z)$$

on the entire z-plane.

Indeed for the function $f(x) = (1-x)^n$ with $Re(n) > 0$, we have

$$\tilde{f}(z) = \Gamma(n+1)\left[\frac{\Gamma(z)}{\Gamma(n+z+1)} + (-1)^n\frac{\Gamma(-n-z)}{\Gamma(1-z)}\right], \tag{22}$$

for all $z \in \mathbb{C}$ except at its simple poles. The asymptotic behavior of these generalized Mellin transforms and the Parseval formula are considered in section 4.5 of [**4**].

Remark: From our algorithmic point of view, the Mellin transform (22) is particularly interesting as it is the sum of two proper hypergeometric terms which are shadows of each other. Therefore, we find the same certificate recurrence for both terms which is also satisfied by their sum. More on this topic can be found in section 4 of [**15**].

In more general situations, in order to compute the recurrence for the sum from those of the terms, we can use the command `REPlus` from the package `GeneratingFunctions` [**8**], since we are working with holonomic recurrences [**18**].

From (22) and Euler's integral representation ([**2**], theorem 2.2.1) we determine the Barnes' type integral form of the terminating ${}_2F_1$

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)\Gamma(n+1)}{2\pi i\Gamma(b)}&\left[\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(z)}{\Gamma(n+z+1)}\frac{\Gamma(b-z)}{\Gamma(c-z)}x^{-z}dz\right. \\ &\left.+(-1)^n\int_{\eta-i\infty}^{\eta+i\infty}\frac{\Gamma(-n-z)}{\Gamma(1-z)}\frac{\Gamma(b-z)}{\Gamma(c-z)}x^{-z}dz\right] \end{aligned} \tag{23}$$

where $Re(c) > Re(b) > 0$, $Re(b) > \delta > 0$ and $\eta < -Re(n)$.

Next we consider two more examples from the table [**6**] involving Gegenbauer polynomials.

7.318 We prove the identity

$$\int_0^1 x^{2\nu}(1-x^2)^{\sigma-1}C_n^{\nu}\left(1-x^2y\right)dx = \frac{\Gamma(2\nu+n)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\sigma)}{2\Gamma(2\nu)\Gamma\left(n+\nu+\sigma+\frac{1}{2}\right)}P_n^{\left(\nu+\sigma-\frac{1}{2},\nu-\sigma-\frac{1}{2}\right)}(1-y) \tag{24}$$

for $Re(\nu) > -\frac{1}{2}$ and $Re(\sigma) > 0$. This identity can be shown by simply applying the Mellin transform method.

Using the definition of the Jacobi polynomials ([**2**], page 99), we have

$$P_n^{\left(\nu+\sigma-\frac{1}{2},\nu-\sigma-\frac{1}{2}\right)}(1-y) = \frac{\left(\nu+\sigma+\frac{1}{2}\right)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\sigma+\frac{1}{2}\end{matrix};\frac{y}{2}\right). \tag{25}$$

On the left hand side of (24), it is convenient to make the change of variable $x^2 = z$. Then use the following representation for the Gegenbauer polynomials ([**2**], 6.4.9 and 6.3.5),

$$C_n^{\nu}(1-zy) = \frac{(2\nu)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\frac{1}{2}\end{matrix};\frac{zy}{2}\right). \tag{26}$$

After this preprocessing step, identity (24) can be rewritten as

$$\int_0^1 z^{\nu-\frac{1}{2}}(1-z)^{\sigma-1}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\frac{1}{2}\end{matrix};\frac{zy}{2}\right)dz = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\sigma)}{\Gamma\left(\nu+\sigma+\frac{1}{2}\right)}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\sigma+\frac{1}{2}\end{matrix};\frac{y}{2}\right). \tag{27}$$

Next, we represent the ${}_2F_1$ on the left hand side as a sum of Barnes' type integrals (23) and identity (27) becomes

(28)

$$\frac{\Gamma(n+1)}{2\pi i\Gamma(n+2\nu)}\left[\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(s)}{\Gamma(n+s+1)}\frac{\Gamma(n+2\nu-s)}{\Gamma\left(\sigma+\nu-s+\frac{1}{2}\right)}\left(\frac{y}{2}\right)^{-s}ds + (-1)^n\right.$$
$$\left.\times\int_{\eta-i\infty}^{\eta+i\infty}\frac{\Gamma(-n-s)}{\Gamma(1-s)}\frac{\Gamma(n+2\nu-s)}{\Gamma\left(\sigma+\nu-s+\frac{1}{2}\right)}\left(\frac{y}{2}\right)^{-s}ds\right] = \frac{1}{\Gamma\left(\nu+\sigma+\frac{1}{2}\right)}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\sigma+\frac{1}{2}\end{matrix};\frac{y}{2}\right),$$

where we also used the property of the Beta integral

$$\int_0^1 z^{\nu-s-\frac{1}{2}}(1-z)^{\sigma-1}dz =: B(\nu-s+\frac{1}{2},\sigma) = \frac{\Gamma(\nu-s+\frac{1}{2})\Gamma(\sigma)}{\Gamma(\nu-s+\sigma+\frac{1}{2})}.$$

At last, identity (28) is equivalent to the Barnes type integral representation of the ${}_2F_1$ appearing on the right hand side.

As a last example, we prove the more involved identity ([**6**], **7.314.1**)

$$\int_{-1}^1 (1-x)^{\nu-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(x)\right]^2dx = \frac{\pi^{1/2}\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!\Gamma(\nu)\Gamma(2\nu)}. \tag{29}$$

We first make a change of variable $\frac{1-x}{2} =: y$ and then use the duplication formula ([**2**], 1.5.1) to write (29) as

$$\int_0^1 y^{\nu-\frac{3}{2}}(1-y)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(1-2y)\right]^2dy = \frac{\Gamma\left(\nu-\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)}{n!\Gamma(2\nu)^2}. \tag{30}$$

For the Gegenbauer polynomials we have the representation (26) with $z = 2$ and the Barnes' type integral representation for the terminating ${}_2F_1$ given by (23). Therefore (30) can be rewritten as

$$\frac{\Gamma\left(\nu+\frac{1}{2}\right)^2}{(2\pi i)^2}\sum_{i,j\in\{1,2\}}\int_{C_i}\int_{C_j}\tilde{f}_i(s)\tilde{f}_j(t)\frac{\Gamma\left(\nu-s-t-\frac{1}{2}\right)}{\Gamma(2\nu-s-t)}ds\ dt=\frac{\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!}, \tag{31}$$

where for simplicity of presentation, we introduced the notations

$$\tilde{f}_1(s)=\frac{\Gamma(s)}{\Gamma(n+s+1)}\frac{\Gamma(n+2\nu-s)}{\Gamma\left(\nu+\frac{1}{2}-s\right)},$$
$$\tilde{f}_2(s)=(-1)^n\frac{\Gamma(-n-s)}{\Gamma(1-s)}\frac{\Gamma(n+2\nu-s)}{\Gamma\left(\nu+\frac{1}{2}-s\right)}$$

and the contours of integrations are of the form $C_1=(\delta-i\infty,\delta+i\infty)$ and $C_2=(\eta-i\infty,\eta+i\infty)$.

Since all the integrands on the left hand side of (31) are shadows of each other and will satisfy the same certificate recurrence, we denote a generic integral of the four by

$$INT[n]=\int\int F[n,s,t]\quad ds\ dt. \tag{32}$$

Wegschaider's algorithm [**14**] delivers a certificate recurrence in the integer parameter n

In[14]:= **FindRecurrence [*F*[*n*, *s*, *t*], *n*, {*s*, *t*} , 1] ;**

In[15]:= **ShiftRecurrence [%[[1]], {*n*, 2} , {*s*, 1} , {*t*, 1}]**

Out[15]= $(n+1)(2n+2\nu+3)(n+2)^2F[n+2,s,t]+(n+1)(n+2\nu)^2(2n+2\nu+1)F[n,s,t]-2(n+1)(n+\nu+1)(2n^2+4\nu n+4n+6\nu+3)F[n+1,s,t]=\Delta_s[2(n+\nu+1)(4\nu n^2-4sn^2-6tn^2-4n^2+4\nu^2n-4\nu n-4\nu sn-4sn-8\nu tn-2stn-8tn-7n-4\nu^2-8\nu-4\nu t-4st-2t-3)F[n+1,s,t]-2(n+1)(n+\nu+1)(4n+6\nu+3)(2\nu-2s-2t-3)F[n+1,s,t+1]+4(n+2)(n+\nu+1)(n+s+2)(t+1)F[n+2,s,t]]+\Delta_t[4(n+\nu+1)(2n^3+6\nu n^2-sn^2+8n^2+4\nu^2n+18\nu n-2\nu sn-3sn+stn-tn+10n+8\nu^2+12\nu-4\nu s-2s-2\nu t+2st-t+4)F[n+1,s,t]-4(n+2)(n+\nu+1)(2n+s+3)(n+t+2)F[n+2,s,t]].$

By integrating over this certificate recurrence, we obtain a recurrence for the sum of integrals from (31). Section 4 of [**13**] describes the conditions that need to be fulfilled by the integrand $F[n,s,t]$ in order to obtain from the certificate recurrence a homogeneous recurrence for our integration problem (32). This homogeneous recurrence is the output of the following command

In[16]:= ***rec*2 = SumCertificate [%] /.SUM → INT**

Out[16]= $(2n+2\nu+3)(n+2)^2\mathrm{INT}[n+2]+(n+2\nu)^2(2n+2\nu+1)\mathrm{INT}[n]-2(n+\nu+1)\left(2n^2+4\nu n+4n+6\nu+3\right)\mathrm{INT}[n+1]=0.$

and it is also satisfied by the right hand side of (31)

In[17]:= **RHS** $[\nu\_,n\_]:=\dfrac{\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!}$

In[18]:= **CheckRecurrence [*rec*2, *RHS*[*n*, *ν*]]**

Out[18]= True.

At last, we only need consider two initial values. In the case $n=0$, we have $C_0^\nu(x)=1$ and (29) is equivalent to the duplication formula. For $n=1$, we have $C_1^\nu(x)=2\nu x$ and the calculations are again trivial.

4. Conclusions

We have introduced an algorithmic approach to the Mellin transform method by applying Wegschaider's algorithm [**14**] to multiple nested Mellin-Barnes integrals. As shown in [**13**], Wegschaider's algorithm computes recurrences for multisums as well as for nested Barnes type integrals over hypergeometric terms.

In analogy with the summation case, we prove entries from [**6**] by first using the Mellin transform method to bring the integrals to a suitable input form and then algorithmically finding a recurrence satisfied by both sides of the identity.

We demonstrate that the idea can be successfully used to enlarge the domain of applicability for this classic integral transform. So far we dealt with table entries containing single definite integrals over functions with known Mellin transforms. This algorithmic twist especially helps in the case of involved examples and its applications deserve further investigation.

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