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A criterion for existence of rational general solutions of planar systems of ODEs *

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Abstract

In the paper [Ngo09] we have studied the algebraic ODE of first order $F(x, y, y') = 0$, where $F \in \mathbb{Q}[x, y, z]$, given its proper rational parametrization of the corresponding surface $F(x, y, z) = 0$. Using this proper parametrization we deduced the problem of finding rational general solutions of the equation $F(x, y, y') = 0$ to finding rational general solutions of its associated system of ODEs in two new indeterminates s, t. This is a planar autonomous system of first order in s, t and of first degree in s', t'.

In this paper we give a criterion for existence of rational general solutions of such an autonomous system provided a degree bound of its rational general solutions. The criterion is based on the vanishing of the differential pseudo remainder of Gao's differential polynomials [FG06] with respect to the chain of the ODE system. As a result, we use this criterion to classify all planar linear systems of ODEs having a rational general solution.

1 Preliminaries

In this section we recall some basic notions in differential algebra such as order, initial, separant, ranking and reduction in a ring of differential polynomials in two indeterminates. The general definitions can be found in [Rit50] and [Kol73].

Let $\overline{\mathbb{Q}}(x)$ be the differential field of rational functions over $\overline{\mathbb{Q}}$ with usual derivation $\frac{d}{dx}$ and we also use $'$ notation for an abbreviation of this derivation. Let s, t be two indeterminates over $\overline{\mathbb{Q}}(x)$. The i-th derivatives of s and t are denoted by s_i and t_i, respectively. The differential polynomial ring $\overline{\mathbb{Q}}(x)\{s, t\}$ is the ring consisting of all polynomials in s, t and all their derivatives up to any order. Let $F \in \overline{\mathbb{Q}}(x)\{s, t\}$ be a differential polynomial. The i-th derivative of F is denoted by $F^{(i)}$. We simply write s and t instead of s_0 and t_0, respectively, or simply write F' instead of $F^{(1)}$. The *order* of F in s (respectively, in t) is the highest n such that s_n (respectively, of t_n) occuring in

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F, denoted by $\text{ord}_s(F)$. For convention we define $\text{ord}_s(F) = -1$ if F does not involve any derivative of s.

Definition 1.1. Let $F, G \in \overline{\mathbb{Q}}(x)\{s,t\}$. F is said to be *of higher rank than* G in s if one of the following conditions holds:

1. $\text{ord}_s(F) > \text{ord}_s(G)$;
2. $\text{ord}_s(F) = \text{ord}_s(G) = n$ and $\deg_{s_n}(F) > \deg_{s_n}(G)$.

If F is of higher rank than G in s, then we also say G is *of lower rank than* F in s. Similarly, we can define the corresponding notion in t.

We order the family $(s_i, t_i)_{i\in\mathbb{N}}$ by a total order as $t < s < t_1 < s_1 < \cdots$. In differential algebra, this total order defines an orderly ranking on the set of derivatives of the differential indeterminates s, t of $\overline{\mathbb{Q}}(x)\{s,t\}$. The *leader* of a differential polynomial F is the greatest derivative occurring in F with respect to this ranking. The *initial* of F is the leading coefficient of F with respect to its leader. The *separant* of F is the partial derivative of F with respect to its leader. It is also the initial of any proper derivative of F.

Definition 1.2. Let F and G be two differential polynomials in $\overline{\mathbb{Q}}(x)\{s,t\}$ with the orderly ranking. G is said to be *reduced* with respect to F if G is lower rank than F in the indeterminate defined by the leader of F.

Definition 1.3. Let $F \in \overline{\mathbb{Q}}(x)\{s,t\}$. By Ritt's reduction, for any $G \in \overline{\mathbb{Q}}(x)\{s,t\}$ there exists a unique representation

$$S^m I^n G = \sum_i Q_i F^{(i)} + R,$$

where S is the separant of F, I is the initial of F, $Q_i \in \overline{\mathbb{Q}}(x)\{s,t\}$, $F^{(i)}$ are the i-th derivatives of F, $m, n \in \mathbb{N}$ and $R \in \overline{\mathbb{Q}}(x)\{s,t\}$ is reduced with respect to F. The R is called the *differential pseudo remainder* of G with respect to F, denoted by

$$R = \text{prem}(G, F).$$

The reduction of G with respect to F is *trivial* if $R = G$.

From now on, we consider $M_1, N_1, M_2, N_2 \in \overline{\mathbb{Q}}[s,t]$ and two special differential polynomials F_1 and F_2 in $\overline{\mathbb{Q}}(x)\{s,t\}$ defined as the following. Let

$$F_1 = M_1 s' - N_1, F_2 = M_2 t' - N_2.$$

Note that the initial and separant of F_1 (respectively, of F_2) are the same. The differential ideal generated by F_1 and F_2 is denoted by $[F_1, F_2]$.

Lemma 1.1. *Let $G \in \overline{\mathbb{Q}}(x)\{s,t\}$, $h = ord_s(G)$ and $F_1 = M_1 s' - N_1$. Let $R = prem(G, F_1)$. Then*

$$ord_s(R) \le 0,\ ord_t(R) \le \max\{ord_t(G), h-1\}.$$

Proof. Since $\mathrm{ord}_s(F_1) = 1$ and $R = \mathrm{prem}(G, F_1)$, by definition of the differential pseudo remainder, we have $\mathrm{ord}_s(R) \leq 0$. If $h < 1$, then $R = G$. Hence

$$\mathrm{ord}_t(R) = \mathrm{ord}_t(G) \leq \max\{\mathrm{ord}_t(G), h - 1\}.$$

Suppose that $h \geq 1$. Then the reduction of G with respect to F_1 is non-trivial. Assume that

$$M_1^{n_1} G = Q_{h-1} F_1^{(h-1)} + R_1,$$

where $\mathrm{ord}_s(R_1) \leq h - 1$ and $Q_{h-1} \in \overline{\mathbb{Q}}(x)\{s, t\}$. We claim that

$$\mathrm{ord}_t(R_1) \leq \max\{\mathrm{ord}_t(G), h - 1\}.$$

By contradiction, if $\mathrm{ord}_t(R_1) > \max\{\mathrm{ord}_t(G), h - 1\}$, then Q_{h-1} would have to involve $t_{\mathrm{ord}_t(R_1)}$ and $Q_{h-1} F_1^{(h-1)}$ would contain a term involving $t_{\mathrm{ord}_t(R_1)}$ and s_h. This term would be balanced neither by R_1 nor by $M_1^{n_1} G$. Therefore the claim is proven. Repeating Ritt's reduction for R_1 with respect to $F_1^{(h-2)}$ we obtain

$$M_1^{n_2} R_1 = Q_{h-2} F_1^{(h-2)} + R_2,$$

where $\mathrm{ord}_s(R_2) \leq h - 2$, $Q_{h-2} \in \overline{\mathbb{Q}}(x)\{s, t\}$ and

$$\mathrm{ord}_t(R_2) \leq \max\{\mathrm{ord}_t(R_1), h - 2\} \leq \max\{\mathrm{ord}_t(G), h - 1\}.$$

Therefore, we eventually reduce G to $R = \mathrm{prem}(G, F_1)$ with property that

$$\mathrm{ord}_t(R) \leq \max\{\mathrm{ord}_t(G), h - 1\}.$$

□

LEMMA 1.2. *Let $F_1 = M_1 s' - N_1, F_2 = M_2 t' - N_2$. Let $G \in \overline{\mathbb{Q}}(x)\{s, t\}$ with $h = ord_s(G)$ and $k = ord_t(G)$. Suppose that $R_1 = prem(G, F_1)$ and $R_2 = prem(R_1, F_2)$. Then*

$$ord_s(R_2) \leq \max\{0, k - 1, h - 2\}, \quad ord_t(R_2) \leq 0.$$

Proof. Let $h_1 = \mathrm{ord}_s(R_1)$ and $k_1 = \mathrm{ord}_t(R_1)$. By Lemma 1.1, we have

$$h_1 \leq 0,\ k_1 \leq \max\{k, h - 1\}.$$

Applying Lemma 1.1 for R_1, we again have

$$\mathrm{ord}_t(R_2) \leq 0, \quad \mathrm{ord}_s(R_2) \leq \max\{h_1, k_1 - 1\} \leq \max\{0, k - 1, h - 2\}.$$

□

For any $G \in \overline{\mathbb{Q}}(x)\{s, t\}$, consider the sequence $R_1 = \mathrm{prem}(G, F_1)$, $R_2 = \mathrm{prem}(R_1, F_2)$, $R_3 = \mathrm{prem}(R_2, F_1)$, $R_4 = \mathrm{prem}(R_3, F_2)$, ... Lemma 1.2 tells us that each two consecutive reduction by F_1 and F_2 returns a differential polynomial having of lower order than the previous one in both s and t. Therefore, we eventually reach a differential polynomial which is reduced with respect to both F_1 and F_2.

Definition 1.4. Let $F_1 = M_1 s' - N_1, F_2 = M_2 t' - N_2$. Let $G \in \overline{\mathbb{Q}}(x)\{s,t\}$. By consecutive reductions with respect to F_1 and F_2 we have a unique representation

$$M_1^m M_2^n G = \sum_i Q_{1i} F_1^{(i)} + \sum_i Q_{2i} F_2^{(i)} + R,$$

where $Q_{1i}, Q_{2i} \in \overline{\mathbb{Q}}(x)\{s,t\}$, $m, n \in \mathbb{N}$, $F_1^{(i)}$ and $F_2^{(i)}$ are the i-th derivatives of F_1 and F_2, respectively, R is reduced with respect to both F_1 and F_2. We denote

$$R = \mathrm{prem}(G, F_1, F_2)$$

and call the *differential pseudo remainder* of G with respect to F_1 and F_2.

Lemma 1.3. *Let*

$$I = \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid prem(G, F_1, F_2) = 0\}.$$

Then I is a differential prime ideal in $\overline{\mathbb{Q}}(x)\{s,t\}$.

Proof. We first prove that I is a differential ideal. It is clear that $I \supset \{F_1, F_2\}$. Let $G_1, G_2 \in I$. Then we have

$$M_1^{n_1} M_2^{m_1} G_1 \in [F_1, F_2], \quad M_1^{n_2} M_2^{m_2} G_2 \in [F_1, F_2],$$

where $[F_1, F_2]$ is the differential ideal generated by F_1 and F_2. Let $n = \max\{n_1, n_2\}$ and $m = \max\{m_1, m_2\}$. Then

$$M_1^n M_2^m (G_1 + G_2) = M_1^n M_2^m G_1 + M_1^n M_2^m G_2 \in [F_1, F_2].$$

Thus

$$\mathrm{prem}(G_1 + G_2, F_1, F_2) = 0.$$

In other words,

$$G_1 + G_2 \in I.$$

Now, for any $G \in \overline{\mathbb{Q}}(x)\{s,t\}$ we will prove that $GG_1 \in I$. Since $M_1^{n_1} M_2^{m_1} G_1 \in [F_1, F_2]$, we have

$$M_1^{n_1} M_2^{m_1} G_1 G \in [F_1, F_2].$$

Thus $\mathrm{prem}(GG_1, F_1, F_2) = 0$, which means $GG_1 \in I$.

We also prove that $G_1' \in I$. We have

$$M_1^{n_1} M_2^{m_1} G_1 \in [F_1, F_2].$$

It follows that $(M_1^{n_1} M_2^{m_1} G_1)' \in [F_1, F_2]$. Thus

$$n_1 M_1' M_1^{n_1 - 1} M_2^{m_1} G_1 + m_1 M_2' M_1^{n_1} M_2^{m_1 - 1} G_1 + M_1^{n_1} M_2^{m_1} G_1' \in [F_1, F_2].$$

Multiplying by $M_1 M_2$ we obtain

$$M_1^{n_1+1} M_2^{m_1+1} G_1' \in [F_1, F_2],$$

which means $\mathrm{prem}(G_1', F_1, F_2) = 0$. Therefore, $G_1' \in I$. Hence I is a differential ideal in $K\{s,t\}$.

Moreover, let $G_1, G_2 \in \overline{\mathbb{Q}}(x)\{s,t\}$ be such that $G_1G_2 \in I$. Suppose that

$$M_1^{n_1}M_2^{m_1}G_1 \equiv R_1 \bmod F_1, F_2 \quad \text{and } M_1^{n_2}M_2^{m_2}G_2 \equiv R_2 \bmod F_1, F_2.$$

Then $R_1, R_2 \in \overline{\mathbb{Q}}(x)[s,t]$ and

$$R_1R_2 = \text{prem}(R_1R_2, F_1, F_2).$$

On the other hand, since $\text{prem}(G_1G_2, F_1, F_2) = 0$, we have

$$M_1^n M_2^m G_1G_2 \equiv 0 \bmod F_1, F_2.$$

Hence

$$\begin{aligned} M_1^n M_2^m R_1R_2 \equiv & M_1^{n_1+n_2}M_2^{m_1+m_2}(M_1^n M_2^m G_1G_2) \bmod F_1, F_2 \\ \equiv & 0 \bmod F_1, F_2 \end{aligned}$$

This means

$$R_1R_2 = \text{prem}(R_1R_2, F_1, F_2) = 0.$$

Since $\overline{\mathbb{Q}}(x)[s,t]$ is an integral domain, we have either $R_1 = 0$ or $R_2 = 0$. Therefore, I is a prime ideal. □

2 The planar autonomous system of ODEs of first order, first degree with rational general solutions

DEFINITION 2.1. Let $M_1, N_1, M_2, N_2 \in \overline{\mathbb{Q}}[s,t]$, $M_1, M_2 \neq 0$. A system of ordinary differential equations of the form

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{1}$$

is called a *planar autonomous system* of ODEs of first order and first degree in s and t.

DEFINITION 2.2. A rational solution $(s(x), t(x))$ of the system (1) is called a *rational general solution* if for any $G \in \overline{\mathbb{Q}}(x)\{s,t\}$ we have

$$G(s(x), t(x)) = 0 \Longleftrightarrow \text{prem}(G, M_1s' - N_1, M_2t' - N_2) = 0.$$

REMARK 2.1.

1. A rational general solution of the system (1) is nothing but a generic zero of the prime differential ideal

$$I = \{G \in \overline{\mathbb{Q}}(x)\{s,t\} \mid \text{prem}(G, M_1s' - N_1, M_2t' - N_2) = 0\}.$$

2. For any $G \in \overline{\mathbb{Q}}(x)\{s,t\}$, $\text{prem}(G, M_1s' - N_1, M_2t' - N_2) \in \overline{\mathbb{Q}}(x)[s,t]$.
3. Observe that if $(s(x), t(x))$ is a rational general solution of the system (1), then

for any polynomial $G \in \overline{\mathbb{Q}}(x)[s,t]$ we have

$$G(s(x), t(x)) = 0 \Longrightarrow G = 0,$$

because $G = \mathrm{prem}(G, M_1 s' - N_1, M_2 t' - N_2)$.

Lemma 2.1. *Let*

$$s(x) = \frac{a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0}{b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0}$$

and

$$t(x) = \frac{c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0}{d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0}$$

be a rational solution of the system (1)*, where a_i, b_i, c_i, d_i are constants in an extension field of $\overline{\mathbb{Q}}(x)$. If $(s(x), t(x))$ is a rational general solution of the system* (1)*, then there exists a constant in the coefficients of $s(x)$ and $t(x)$ such that it is transcendental over $\overline{\mathbb{Q}}$.*

Proof. Let

$$S = (b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0)s - (a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0),$$

and

$$T = (d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0)t - (c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0).$$

Let $G = \mathrm{res}_x(S, T)$. Then G is a polynomial in s and t with the coefficients depending on a_i, b_i, c_i, d_i. If all a_i, b_i, c_i, d_i were in $\overline{\mathbb{Q}}$, then $G \in \overline{\mathbb{Q}}[s,t]$ and $G(s(x), t(x)) = 0$. Since $(s(x), t(x))$ is a rational general solution, it follows that $G = 0$. This is impossible because $G = \mathrm{res}_x(S, T) \neq 0$, which is known for computing the implicit equation of the rational algebraic curve having the rational parametrization $(s(x), t(x))$ [SWPD08]. Therefore, there is a coefficient that does not belong to $\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}}$ is an algebraically closed field, a constant which is not in $\overline{\mathbb{Q}}$ must be a transcendental element over $\overline{\mathbb{Q}}$. $\square$

The following lemma can be found in [FG06].

Lemma 2.2. *There exists a differential polynomial $D_{n,m}(y)$ such that every rational function*

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

is a solution of $D_{n,m}(y)$. Moreover, the differential polynomial $D_{n,m}(y)$ has only rational solutions.

Proof.

$$D_{n,m}(y) = \begin{vmatrix} \binom{n+1}{0} y^{(n+1)} & \binom{n+1}{1} y^{(n)} & \cdots & \binom{n+1}{m} y^{(n+1-m)} \\ \binom{n+2}{0} y^{(n+2)} & \binom{n+2}{1} y^{(n+1)} & \cdots & \binom{n+2}{m} y^{(n+2-m)} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n+1+m}{0} y^{(n+1+m)} & \binom{n+1+m}{1} y^{(n+m)} & \cdots & \binom{n+1+m}{m} y^{(n+1)} \end{vmatrix}.$$

$\square$

Using Gao's differential polynomial we have the following criterion.

THEOREM 2.1. *The system* (1) *has a rational general solution* $(s(x), t(x))$ *with* $\deg s(x) \leq n$ *and* $\deg t(x) \leq m$ *if and only if*

$$\begin{cases} prem(D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2) = 0 \\ prem(D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2) = 0. \end{cases} \tag{2}$$

Proof. Suppose that the system (1) has a rational general solution $(s(x), t(x))$ with $\deg s(x) \leq n$ and $\deg t(x) \leq m$. Then $(s(x), t(x))$ is a solution of $D_{n,n}(s)$ and $D_{m,m}(t)$ as well. By the definition of rational general solutions of the system (1) we have

$$\begin{cases} \text{prem}(D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2) = 0 \\ \text{prem}(D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2) = 0. \end{cases}$$

Conversely, if these two conditions hold, then $D_{n,n}(s)$ and $D_{m,m}(t)$ belong to the prime differential ideal I as defined in Lemma 1.3. Since I is a prime ideal, it has a generic zero. This generic zero is a zero of $D_{n,n}(s)$ and $D_{m,m}(t)$. By Lemma 2.2 these two differential polynomials have only rational solutions. Therefore, the generic zero of I must be rational. □

REMARK 2.2. If we know a degree bound of the rational solutions of the system (1), then Theorem 2.1 gives us a criterion for existence of rational general solutions of the system (1).

In the paper [Ngo09] we have studied the algebraic ODE of first order $F(x, y, y') = 0$ given a proper rational parametrization

$$\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$$

of its corresponding surface $F(x, y, z) = 0$. We know that its associated system is a planar autonomous system. Moreover, every rational solution $(s(x), t(x))$ of this associated system satisfies the condition

$$\chi_1(s(x), t(x)) = x.$$

From this condition we can see that the degree of $t(x)$ is determined in terms of the degree of $s(x)$ and the degree of $\chi_1(s,t)$ with respect to s.

THEOREM 2.2. *Let*

$$\chi_1(s,t) = \frac{a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_0(t)}{b_m(t)s^m + b_{m-1}(t)s^{m-1} + \cdots + b_0(t)} \in \overline{\mathbb{Q}}(x)(s,t).$$

Suppose that $s(x)$ *and* $t(x)$ *are rational functions such that* $\chi_1(s(x), t(x)) = x$. *Let* $\delta = \deg_x s(x)$. *Then*

1. *If* $n \geq m$, *then* $\deg_x t(x) \leq 1 + n\delta$.
2. *If* $n < m$, *then* $\deg_x t(x) \leq 1 + m\delta$.

Proof. We have

$$\chi_1(s(x), t(x)) = x \iff \frac{a_n(t(x))s(x)^n + a_{n-1}(t(x))s(x)^{n-1} \cdots + a_0(t(x))}{b_m(t(x))s(x)^m + b_{m-1}(t)s(x)^{m-1} \cdots + b_0(t(x))} = x.$$

Suppose that $\mathbb{K}$ is the coefficient field of $s(x)$ and $t(x)$. We know that for any rational function $t \in \mathbb{K}(x)$, x is algebraic over $\mathbb{K}(t)$ and

$$\deg_x t(x) = [\mathbb{K}(x) : \mathbb{K}(t)].$$

Therefore, in order to find a degree bound for t, it is enough to find an algebraic equation for x over $\mathbb{K}(t)$. Let $s = \dfrac{P(x)}{Q(x)}$, $\delta = \deg_x s(x) = \max\{\deg_x P(x), \deg_x Q(x)\}$, $l = \deg_x Q(x)$. We have

$$x = \frac{Q(x)^m}{Q(x)^n} \frac{(a_n(t)P(x)^n + \cdots + a_0(t)Q(x)^n)}{(b_m(t)P(x)^m + \cdots + b_0(t)Q(x)^m)}.$$

We need to know the degree of x in the above equation. It follows from $l \leq \delta$ that if $n \geq m$, then

$$\deg_x t(x) \leq \max\{1 + m\delta + l(n - m), n\delta\} \leq 1 + n\delta.$$

If $n < m$, then

$$\deg_x t(x) \leq \max\{1 + m\delta, n\delta + l(m - n)\} \leq 1 + m\delta.$$

□

2.1 The linear system of autonomous ODEs

Consider the linear system

$$\begin{cases} s' = as + bt + e \\ t' = cs + dt + h \end{cases} \tag{3}$$

where a, b, c, d, e, h are constants. We claim that

Lemma 2.3. *Every rational solution of the linear system is a polynomial solution.*

Proof. Assume that $s(x) = \dfrac{A}{p^\alpha B}, t(x) = \dfrac{C}{p^\beta D}$, where p is an irreducible polynomial and $\alpha, \beta > 0$; A, B, C, D have no factor of p. Then

$$\text{ord}_p(s'(x)) = \alpha + 1 \quad \text{ord}_p(t'(x)) = \beta + 1.$$

We have

$$as + bt + e = a\frac{A}{p^\alpha B} + b\frac{C}{p^\beta D} + e.$$

Hence

$$\text{ord}_p(as + bt + e) \leq \max\{\alpha, \beta\}, \quad \text{ord}_p(cs + dt + h) \leq \max\{\alpha, \beta\}.$$

There are two cases

- If $\alpha \geq \beta$, then $\mathrm{ord}_p(as+bt+e) \leq \alpha < \mathrm{ord}_p(s'(x))$. It is impossible.
- If $\alpha < \beta$, then $\mathrm{ord}_p(cs+dt+h) \leq \beta < \mathrm{ord}_p(t'(x))$. It is impossible.

Therefore, $\alpha = \beta = 0$. Thus $s(x), t(x)$ are polynomials. □

THEOREM 2.3. *Every rational general solution of the linear system* (3) *is a couple of polynomials of degree at most 2.*

Proof. We can write the linear system in the form

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} e \\ h \end{pmatrix}$$

Hence

$$\begin{pmatrix} s'' \\ t'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} s^{(n+1)} \\ t^{(n+1)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} e \\ h \end{pmatrix}.$$

The system has a polynomial general solution of degree n iff

$$\begin{cases} \mathrm{prem}(s^{(n+1)}, [s' - as - bt - e, t' - cs - dt - h]) = 0 \\ \mathrm{prem}(t^{(n+1)}, [s' - as - bt - e, t' - cs - dt - h]) = 0. \end{cases}$$

It means

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} e \\ h \end{pmatrix} = 0.$$

We will prove that this holds if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0.$$

In fact, the "if " is clear. Conversely, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = 0.$$

Then $ad - bc = 0$ and the Jordan form of the matrix is

$$\text{either} \begin{pmatrix} 0 & 0 \\ 0 & a+d \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $a + d = 0$ and $n = 2$. □

We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0 \Longleftrightarrow \begin{cases} a^2 + bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 0. \end{cases}$$

Solving this algebraic system we obtain the following cases

- If $b = 0$, then $a = d = 0$.
- If $b \neq 0$, then $a = -d$ and $c = -\frac{d^2}{b}$.

Thus the explicit polynomial solutions of the system given by the following table.

System	Rational general solutions
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = hx + C_2 \end{cases}$
$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = ce\frac{x^2}{2} + (cC_1 + h)x + C_2 \end{cases}$
$\begin{pmatrix} -d & b \\ -\frac{d^2}{b} & d \end{pmatrix}$	$\begin{cases} s(x) = \frac{hb - ed}{2}x^2 + (bC_1 + e)x + C_2 \\ t(x) = \frac{(hb - ed)d}{2b}x^2 + (dC_1 + h)x + \frac{d}{b}C_2 + C_1 \end{cases}$

Note that the last matrix of the table also covers the other symmetric cases, for instance

$$d = 0 \longrightarrow \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad d = -a, b = -\frac{a^2}{c} \longrightarrow \begin{pmatrix} a & -\frac{a^2}{c} \\ c & -a \end{pmatrix}.$$

We can prove that they are rational general solutions of the corresponding system. For instance, consider a simple system

$$\begin{cases} s' = e \\ t' = h \end{cases}$$

where e, h are not all zero. It turns out that the system has a solution given by

$$s(x) = ex + C_1, \ t(x) = hx + C_2$$

where C_1, C_2 are arbitrary constants. The implicit equation of $(s(x), t(x))$ is

$$et - hs + hC_1 - eC_2 = 0.$$

Let $G \in \overline{\mathbb{Q}}[s, t]$ be such that $G(s(x), t(x)) = 0$. Then $et - hs + hC_1 - eC_2$ is a factor of G for arbitrary constant $C = hC_1 - eC_2$. It implies that $G = 0$. Therefore, $(s(x), t(x))$ is a rational general solution of the given system. Using a similar argument we prove for the other systems in the table.

2.2 The polynomial system of autonomous ODEs

In this section we present some simple properties of the polynomial system of autonomous ODEs. Consider the system

$$\begin{cases} s' = P(s,t) \\ t' = Q(s,t) \end{cases} \tag{4}$$

where P and Q are polynomials in $\overline{\mathbb{Q}}[s,t]$ and $\gcd(P,Q) = 1$.

LEMMA 2.4. *Let $(s(x), t(x))$ be a rational solution of the system 4. Let $F(s,t)$ be the implicit irreducible polynomial defining the parametric curve $(s(x), t(x))$. Then there exists a polynomial $G(s,t)$ such that*

$$F_s P + F_t Q = FG \tag{5}$$

where F_s and F_t are partial derivatives of F with respect to s and t. Conversely, suppose that $F(s,t)$ is an irreducible polynomial satisfying the equation (5) for some G and $(s(x), t(x))$ is a rational parametrization of $F(s,t) = 0$. Then

$$s'(x) \cdot Q(s(x), t(x)) - t'(x) \cdot P(s(x), t(x)) = 0.$$

Proof. Suppose that the system (4) has a rational solution $(s(x), t(x))$. Let $F(s,t)$ be the defining polynomial of the rational algebraic curve $(s(x), t(x))$. We have

$$F(s(x), t(x)) = 0.$$

Differentiating this equation with respect to x we get

$$F_s(s(x), t(x)) \cdot s'(x) + F_t(s(x), t(x)) \cdot t'(x) = 0,$$

hence

$$F_s(s(x), t(x)) \cdot P(s(x), t(x)) + F_t(s(x), t(x)) \cdot Q(s(x), t(x)) = 0.$$

Therefore, there exists a polynomial $G(s,t)$ such that

$$F_s P + F_t Q = FG.$$

Conversely, assume that $F(s,t)$ is an irreducible polynomial satisfying the equation

$$F_s P + F_t Q = FG$$

for some $G \in \overline{\mathbb{Q}}[s,t]$. Suppose that $(s(x), t(x))$ is a rational parametrization of $F(s,t) = 0$. We have

$$F_s(s(x), t(x)) \cdot s'(x) + F_t(s(x), t(x)) \cdot t'(x) = 0.$$

On the other hand, because of $F(s(x), t(x)) = 0$,

$$F_s(s(x), t(x)) \cdot P(s(x), t(x)) + F_t(s(x), t(x)) \cdot Q(s(x), t(x)) = F(s(x), t(x)) \cdot G(s(x), t(x)) = 0.$$

It follows that

$$\begin{cases} F_s(s(x),t(x))\cdot s'(x) + F_t(s(x),t(x))\cdot t'(x) = 0 \\ F_s(s(x),t(x))\cdot P(s(x),t(x)) + F_t(s(x),t(x))\cdot Q(s(x),t(x)) = 0. \end{cases}$$

Since $F(s,t)$ is irreducible, $F_s(s(x),t(x))$ and $F_t(s(x),t(x))$ are non-zero rational functions in x. Therefore, the determinant of this homogeneous system is 0, i.e.

$$s'(x)\cdot Q(s(x),t(x)) - t'(x)\cdot P(s(x),t(x)) = 0.$$

□

LEMMA 2.5. *Let $F(s,t)$ be an irreducible polynomial such that*

$$F_sP + F_tQ = FG$$

for some $G(s,t)$. If $F(0,0) = F_s(0,0) = F_t(0,0) = 0$ and $F_{ss}(0,0)F_{tt}(0,0) - F_{st}^2(0,0) \neq 0$, then

$$P(0,0) = Q(0,0) = 0.$$

Proof. Consider the Taylor expansion of $F(s,t)$ at $(0,0)$ we have

$$F(s,t) = \frac{1}{2!}(F_{ss}(0,0)s^2 + 2F_{st}(0,0)st + F_{tt}(0,0)t^2) + \text{higher order}.$$

Then

$$F_s(s,t) = F_{ss}(0,0)s + F_{st}(0,0)t + \text{higher order}$$

and

$$F_t(s,t) = F_{st}(0,0)s + F_{tt}(0,0)t + \text{higher order}.$$

Suppose that

$$P(s,t) = p_0 + p_1 s + p_2 t + \text{higher order},$$

$$Q(s,t) = q_0 + q_1 s + q_2 t + \text{higher order}.$$

Then we have

$$\begin{aligned} F_sP + F_tQ =& p_0(F_{ss}(0,0)s + F_{st}(0,0)t) + q_0(F_{st}(0,0)s + F_{tt}(0,0)t) + \text{higher order} \\ =& (p_0F_{ss}(0,0) + q_0F_{st}(0,0))s + (p_0F_{st}(0,0) + q_0F_{tt}(0,0))t + \text{higher order} \end{aligned}$$

Since FG is of order at least 2, we have

$$\begin{cases} p_0F_{ss}(0,0) + q_0F_{st}(0,0) = 0 \\ p_0F_{st}(0,0) + q_0F_{tt}(0,0) = 0. \end{cases}$$

It follows from the assumption $F_{ss}(0,0)F_{tt}(0,0) - F_{st}^2(0,0) \neq 0$ that

$$p_0 = q_0 = 0.$$

Therefore, $P(0,0) = Q(0,0) = 0$. □

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