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Shape-explicit constants for some boundary integral operators

Clemens Pechstein

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SHAPE-EXPLICIT CONSTANTS FOR SOME BOUNDARY INTEGRAL OPERATORS

CLEMENS PECHSTEIN*

Abstract. Among the well-known constants in the theory of boundary integral equations are the coercivity constants of the single layer potential and the hypersingular boundary integral operator, and the contraction constant of the double layer potential. Whereas there have been rigorous studies how these constants depend on the size and aspect ratio of the underlying domain, only little is known on their dependency on the shape of the boundary.

In this article, we consider the homogeneous Laplace equation and derive explicit estimates for the above mentioned constants in three dimensions. Using an alternative trace norm we make the dependency explicit in two geometric parameters, the so-called Jones parameter and the constant in Poincaré's inequality. The latter one can be tracked back to the constant in an isoperimetric inequality. There are many domains with quite irregular boundaries, where these parameters stay bounded. Our results provide a new tool in the analysis of numerical methods for boundary integral equations.

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1. Introduction

Boundary integral equations for strongly elliptic partial differential equations and their variational framework provide a profound mathematical basis for the Galerkin boundary element method (BEM), see [13] for an early work. The BEM has become a rather popular method for the numerical solution of certain partial differential equations which occur in many problems from physics and engineering. For a comprehensive introduction to boundary integral equations we refer the reader e. g. to [15, 23] and to [30, 33]. The latter references also cover the BEM and related computational aspects.

This article discusses a special issue of boundary integral equations for the Laplace problem. In case of Laplace's equation, there are two equations relating the Dirichlet trace of the solution to its Neumann trace. These equations are usually posed in Sobolev spaces on the boundary of the computational domain, see e. g. [1, 22, 23], and they involve several boundary integral operators, among them the single and double layer potential operator, as well as the hypersingular boundary integral operator. The properties of these operators are well-studied, see e. g. [4, 7] and the references given above. The probably most important properties are the (semi-)coercivity and boundedness of the single layer potential operator and the hypersingular operator, which hold true under suitable conditions. However, the constants of coercivity and boundedness depend on the choice of the Sobolev norms and are in general not accessible.

* Institute of Computational Mathematics, Johannes Kepler University, Altenberger Str. 69, 4040 Linz, Austria, phone: (+43) 732 / 2468-9169, e-mail: clemens.pechstein@jku.at
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In the course of this paper, we elaborate explicit estimates for coercivity and boundedness constants in three dimensions with respect to particularly chosen Sobolev norms. Moreover, we can explicitly bound the contraction constant of the double layer potential operator, cf. [34]. The dependence on the computational domain is made explicit in two geometric parameters. The first one is due to Jones, cf. [16, 29], the other one the constant in Poincaré's inequality, [2, 24], which can be tracked back to the constant in an isoperimetric inequality due to Maz'ja, cf. [11, 21, 22]. There are many domains with quite irregular boundaries, where all these parameters stay bounded. This fact was recently used by Dohrmann, Klawonn, and Widlund [8, 9] (see also Klawonn, Rheinbach, and Widlund [18]) to prove shape-robust estimates for domain decomposition methods for finite element equations. Actually, our research was inspired by their work.

The results from our work can be used to obtain more explicit estimates in BEM-based domain decomposition methods, cf. e. g. [14, 25, 26], or to show rigorous error estimates for so-called BEM-based finite element methods, cf. [3].

The remainder of this article is organized as follows. In Section 2 we describe the problem in more detail and present the main statements. In Section 3 we give precise definitions for our geometric parameters. We recall an extension theorem by Jones and a Poincaré type inequality by Maz'ja. Combining these results we state and prove an auxiliary extension result (Lemma 3.9) that will be used a couple of times in the sequel. Section 4 summarizes some properties of boundary integral operators. In Section 5 we define our alternative norms and prove related inequalities for the Dirichlet and Neumann traces. Finally, Section 6 contains the main results: explicit coercivity and boundedness estimates together with proofs.

2. Problem description and main statement

Let $\Omega^{\mathrm{int}} \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded Lipschitz domain with boundary Γ, unit outward normal n, and its exterior $\Omega^{\mathrm{ext}} := \mathbb{R}^d \setminus \overline{\Omega}^{\mathrm{int}}$. We consider Laplace's problem

$$-\Delta u \;=\; 0$$

both in Ω^{int} and Ω^{ext}. The fundamental solution of the Laplace operator is given by

$$U^*(x,\,y) \;=\; \begin{cases} \frac{1}{2\pi}\log\frac{1}{|x-y|} & \text{for } d=2\,, \\ \frac{1}{4\pi}\,\frac{1}{|x-y|} & \text{for } d=3\,. \end{cases} \tag{2.1}$$

Let $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ denote the single layer potential operator, $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the double layer potential operator, and $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ the hypersingular boundary integral operator. For more precise definitions see Section 4 and [15, 23, 30, 33]. For smooth functions w and v we have the integral representations

$$\begin{aligned}
(Vw)(x) \;&=\; \int_\Gamma U^*(x,\,y)\,w(y)\,ds_y\,,\\
(Kv)(x) \;&=\; \int_\Gamma \frac{\partial U^*}{\partial n_y}(x,\,y)\,v(y)\,ds_y\,,\\
(D\,v)(x) \;&=\; -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial U^*}{\partial n_y}(x,\,y)\,\big(v(y)-v(x)\big)\,ds_y\,,
\end{aligned}$$

where $x \in \Gamma$. Let γ_0^{int} and γ_1^{int} denote the interior Dirichlet, respectively the Neumann trace on Γ, i. e., for smooth functions $u : \overline{\Omega}^{\text{int}} \to \mathbb{R}$,

$$\gamma_0^{\text{int}} u = u|_\Gamma \,, \qquad \gamma_1^{\text{int}} u = \frac{\partial u}{\partial n} \,.$$

Then the solution u to the interior Laplace problem fulfills the two integral equations

$$\begin{aligned} V\,\gamma_1^{\text{int}} u &= (\tfrac{1}{2} I + K)\gamma_0^{\text{int}} u \,, \\ D\,\gamma_0^{\text{int}} u &= (\tfrac{1}{2} I - K')\gamma_1^{\text{int}} u \,, \end{aligned}$$

where K' is the operator adjoint to K. The operators V, K, K', and D are linear and bounded. One can show that in three dimensions, there exist positive constants c_V and c_D such that

$$\langle w,\, Vw\rangle \;\geq\; c_V\,\|w\|^2_{H^{-1/2}(\Gamma)} \qquad \forall w \in H^{-1/2}(\Gamma)\,, \tag{2.2}$$

$$\langle D\,v,\, v\rangle \;\geq\; c_D\,|v|^2_{H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}(\Gamma)\,. \tag{2.3}$$

In two dimensions, the second estimate remains true. The first one, however, holds in general only on a subspace (this property is linked to a peculiarity of the two-dimensional exterior problem). Let $\mathbf{1}_\Gamma$ denote the function that takes the value one everywhere on Γ. According to [33, Sect. 6.6.1] (see also [23, Theorem 8.15]), we define the subspace

$$H_*^{-1/2}(\Gamma) := \{w \in H^{-1/2}(\Gamma) : \langle w,\, \mathbf{1}_\Gamma\rangle = 0\}\,. \tag{2.4}$$

Then, in two dimensions, estimate (2.2) holds for all $w \in H_*^{-1/2}(\Gamma)$. If, in addition the domain is small enough (a sufficient condition is $\operatorname{diam}(\Omega^{\text{int}}) < 1$), it also holds on the whole space $H^{-1/2}(\Gamma)$. In the general case, at least the inverse of V restricted to $H_*^{-1/2}(\Gamma)$ must exist. To this end, we define the equilibrium (or natural) density $w_{\text{eq}} \in H^{-1/2}(\Gamma)$ by

$$V w_{\text{eq}} \;=\; \text{const}\,, \qquad \langle w_{\text{eq}},\, \mathbf{1}_\Gamma\rangle \;=\; 1\,. \tag{2.5}$$

Its existence and uniqueness follow basically from the above coercivity result, cf. [23, 33]. Using the equilibrium density, we define the subspace

$$H_*^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) : \langle w_{\text{eq}},\, v\rangle = 0\}\,. \tag{2.6}$$

One can show that $V : H_*^{-1/2}(\Gamma) \to H_*^{1/2}(\Gamma)$ and that V is a isomorphism between the two spaces, coercive and bounded in both directions. Even if V might not be invertible, we denote the inverse of this isomorphism by V^{-1}.

Obviously, $H_*^{1/2}(\Gamma)$ is a subspace of $H^{1/2}(\Gamma)$ with co-dimension 1. By an embedding argument, one can show that a third constant $\widetilde{c}_D > 0$ exists such that

$$\langle D\,v,\, v\rangle \;\geq\; \widetilde{c}_D\,\|v\|^2_{H^{1/2}(\Gamma)} \qquad \forall v \in H_*^{1/2}(\Gamma)\,. \tag{2.7}$$

Then, the constant

$$c_0 := \inf_{v \in H_*^{1/2}(\Gamma)} \frac{\langle D\,v,\, v\rangle}{\langle V^{-1}v,\, v\rangle} \tag{2.8}$$

is well-defined. From the estimates above we can conclude the bound

$$c_0 \;\geq\; c_V\,\widetilde{c}_D\,. \tag{2.9}$$

In [34], Steinbach and Wendland prove that the contraction properties

$$(1 - c_K)\,\|v\|_{V^{-1}} \;\leq\; \|(\tfrac{1}{2} I \pm K)v\|_{V^{-1}} \;\leq\; c_K\,\|v\|_{V^{-1}} \qquad \forall v \in H_*^{1/2}(\Gamma)$$

hold, where

$$c_K := \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_0}\,, \qquad \|v\|_{V^{-1}} := \sqrt{\langle V^{-1}v,\, v\rangle}\,.$$

They show that $c_0 \leq \frac{1}{4}$. Hence, the contraction constant c_K is well-defined and strictly smaller than one. For an alternative proof and historical remarks see [5]. The contraction properties have a series of important consequences, see e. g. [5, 25, 26, 32, 34].

Apparently, the constants c_V and $\widetilde{c}_D$, and thus also the bound (2.9), depend heavily on the choice of the norm in $H^{1/2}(\Gamma)$ that appears in the estimates (2.2) and (2.7). However, the constant c_0 itself is independent of that choice; it depends only on the domain Ω^{int}. To the best of our knowledge, an explicit dependency has not been known, except for special domains, e. g., balls or ellipses, cf. [27, 28].

An inspection of the proof of (2.2) and (2.3) (see e. g. [33, Sect. 6.6.1] or [23, Theorem 7.6, Theorem 8.2]) reveals that the constants c_V, c_D depend on the constants in both the interior and the exterior trace inequality, and on the definition of the (semi-)norms for $H^{\pm 1/2}(\Gamma)$. In order to track the dependency of these constants on the shape of the domain, we introduce an alternative norm $\|\cdot\|_{\star,H^{1/2}(\Gamma)}$ in $H^{1/2}(\Gamma)$. For a function $v \in H^{1/2}(\Gamma)$, the norm $\|v\|_{\star,H^{1/2}(\Gamma)}$ equals (up to some scaling) the H^1-norm of the harmonic extension of v from Γ to Ω^{int}. Furthermore, we will work with the dual norm $\|\cdot\|_{\star,H^{-1/2}(\Gamma)}$ and a semi-norm $|\cdot|_{\star,H^{1/2}(\Gamma)}$.

Also, we use two geometric parameters. On the one hand we have the Jones parameter, cf. [16] and Section 3. On the other hand, we use the constant in Poincaré's inequality, which can be tracked back to the constant in the isoperimetric inequality, see [21, 22]. Typically, the Jones parameter deteriorates when the aspect ratio gets small, and the Poincaré parameter deteriorates when two parts of a domain are linked only via a narrow channel. Nevertheless, both parameters may stay bounded when the domain has quite a rough boundary, cf. [8, 18].

Using these geometric parameters and our alternative norms, we prove the following results. Let B_R a ball of radius $R > 0$ with

$$\overline{\Omega}^{\text{int}} \subset B_R \qquad \text{and} \qquad \operatorname{dist}(\partial B_R,\, \Gamma) \;\geq\; \tfrac{1}{2}\operatorname{diam}(\Omega^{\text{int}}),$$

and set $\Omega_R := B_R \setminus \overline{\Omega}^{\text{int}}$ (cf. Figure 1). Then in three dimensions, there exist constants $c_V^\star$, $c_D^\star$, and C_P^* depending only on the Jones parameters and the Poincaré constants of Ω^{int} and the auxiliary domain Ω_R such that

$$\begin{aligned} c_V^\star\, \|w\|^2_{\star,H^{-1/2}(\Gamma)} \;&\leq\; \langle w,\, Vw\rangle \;\leq\; (1 + 2\, C_P^*)\, \|w\|^2_{\star,H^{-1/2}(\Gamma)} \qquad && \forall w \in H^{-1/2}(\Gamma)\,,\\ c_D^\star\, |v|^2_{\star,H^{1/2}(\Gamma)} \;&\leq\; \langle D\, v,\, v\rangle \;\leq\; |v|^2_{\star,H^{1/2}(\Gamma)} && \forall v \in H^{1/2}(\Gamma)\,. \end{aligned}$$

Since our norm $\|\cdot\|_{\star,H^{1/2}(\Gamma)}$ is defined via an H^1-norm, we get in a certain sence explicit equivalence relations between $\|\cdot\|_{H^1(\Omega^{\text{int}})}$, $|\cdot|_{H^1(\Omega^{\text{int}})}$ and the (semi-)norms induced by V^{-1} and D. Also,

$$\langle D\, v,\, v\rangle \;\geq\; \frac{c_D^\star}{1 + C_P^*}\, \|v\|^2_{\star,H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}_*(\Gamma)\,.$$

Using our results, we obtain the bound

$$c_0 \;\geq\; \frac{c_V^\star\, c_D^\star}{1 + C_P^*}\,,$$

which shows that c_0 is rather insensitive to rough boundaries. In two dimensions, similar results hold. However, not in all estimates explicit constants are obtained, and we will comment on this in Remark 6.10. The focus of the paper is on three-dimensionional case.

3. Extension results and Poincaré's inequality

3.1. **Jones' extension result.** In [16] Jones introduced the notion of $(\varepsilon,\,\delta)$-domains. The following definition is equivalent to that of an $(\varepsilon,\,\infty)$-domain.

Definition 3.1 (Uniform domain)**.** A bounded and connected set $\Omega \subset \mathbb{R}^d$ is called a *uniform domain* if there exists a constant $C_U(\Omega)$ such that any pair of points $x_1 \in \Omega$ and $x_2 \in \Omega$ can be joined by a rectifiable curve $\gamma(t) : [0,\,1] \to \Omega$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, such that the Euclidean arc length of γ is bounded by $C_U(\Omega)\,|x_1 - x_2|$ and

$$\min_{i=1,2} |x_i - \gamma(t)| \;\leq\; C_U(\Omega)\,\operatorname{dist}(\gamma(t),\,\partial\Omega) \qquad \forall t \in [0,\,1]\,.$$

Any Lipschitz domain is also a uniform domain. Due to Jones [16], Sobolev spaces on uniform domains can be extended to the whole of $\mathbb{R}^d$ and the extension operator is bounded. He even proved that uniform domains are the largest class in two dimensions for which such extensions exist. We remark that Rogers [29] recently found a degree-independent extension operator which has both the properties of Jones' operator and of the one by Stein [31, Chap. 6]. In the present paper, however, we do not make use of this fact. The following theorem can be derived from Jones' result [16, Theorem 1].

Theorem 3.2 (Jones)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, uniform domain with* $\operatorname{diam}(\Omega) = 1$. *Then there exists a bounded linear operator*

$$E : H^1(\Omega) \to H^1(\mathbb{R}^d)\,, \qquad (E\,w)_{|\Omega} = w \quad \forall w \in H^1(\Omega)\,,$$

and a positive constant $C_E(\Omega)$ depending only on $C_U(\Omega)$ and the dimension d such that

$$\|E\,w\|_{H^1(\mathbb{R}^d)} \;\leq\; C_E(\Omega)\,\|w\|_{H^1(\Omega)}\,.$$

Theorem 3.2 also holds if $\operatorname{diam}(\Omega) \geq 1$. But if the diameter tends to zero, the Sobolev norms would have to be re-scaled. In the sequel of the current paper we use scaling-invariant norms whenever possible. Hence, the above result is sufficient for our purpose. Note that the Jones parameter $C_U(\Omega)$ itself remains invariant when re-scaling the coordinates. Therefore, in the general case, the statement of Theorem 3.2 holds true with the estimate replaced by

$$\Big(|E\,w|^2_{H^1(\mathbb{R}^d)} + \frac{1}{\operatorname{diam}(\Omega)^2}\,\|E\,w\|^2_{L^2(\mathbb{R}^d)}\Big)^{1/2} \;\leq\; C_E(\Omega)\,\Big(|w|^2_{H^1(\Omega)} + \frac{1}{\operatorname{diam}(\Omega)^2}\,\|w\|^2_{L^2(\Omega)}\Big)^{1/2}\,.$$

3.2. **Poincaré's inequality.** In order to access the constant in Poincaré's inequality, we make use of the following result by Maz'ja [21] and Federer and Fleming [10], which implies a version of Poincaré's inequality.

Lemma 3.3 (Isoperimetric inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a uniform domain and let u be sufficiently smooth. Then, $\gamma(\Omega) > 0$ is the smallest constant such that*

$$\inf_{c\in\mathbb{R}} \Big(\int_\Omega |u - c|^{\frac{d}{d-1}}\,dx\Big)^{\frac{d-1}{d}} \;\leq\; \gamma(\Omega) \int_\Omega |\nabla u|\,dx \qquad \forall u \in C^1(\overline{\Omega})$$

holds if and only if $\gamma(\Omega)$ is the smallest constant such that the isoperimetric inequality

$$\big[\min(|A|,\,|B|)\big]^{\frac{d-1}{d}} \;\leq\; \gamma(\Omega)\,|\partial A \cap \partial B|$$

holds for all measurable sets $A \subset \Omega$ and $B = \Omega \setminus \overline{A}$ with $\partial A \cap \partial B$ being a measurable surface.

Note that the parameter $\gamma(\Omega)$ does not depend on $\operatorname{diam}(\Omega)$, but only on the shape of Ω and on the dimension d. The following result can be found in [8, 9, 18] with an outline of the proof. The rigorous proof for the three-dimensional case is due to Hyea Hyun Kim [17].

Lemma 3.4 (Poincaré's inequality). *Let $\gamma(\Omega)$ denote the best constant in the isoperimetric inequality. Then, for all $u \in H^1(\Omega)$,*

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|u-c\|_{L^2(\Omega)} \;&\leq\; \gamma(\Omega)\, |\Omega|^{\frac{1}{2}}\, |u|_{H^1(\Omega)} && \text{for } d=2\,,\\ \inf_{c\in\mathbb{R}} \|u-c\|_{L^2(\Omega)} \;&\leq\; \tfrac{4}{3}\,\gamma(\Omega)\, |\Omega|^{\frac{1}{3}}\, |u|_{H^1(\Omega)} && \text{for } d=3\,. \end{aligned}$$

The infimum is attained at $c = \overline{u}^\Omega := |\Omega|^{-1}\int_\Omega u\,dx$.

Proof. For $d=2$ (cf. [8, 18]), Lemma 3.3, Cauchy's inequality, and a density argument imply

$$\Big(\inf_{c\in\mathbb{R}}\int_\Omega |u-c|^2\,dx\Big)^{1/2} \;\leq\; \gamma(\Omega)\int_\Omega |\nabla u|\,dx \;\leq\; \gamma(\Omega)\,|\Omega|^{1/2}\,|u|_{H^1(\Omega)} \qquad \forall u\in H^1(\Omega).$$

The fact that the infimum is attained at $c=\overline{u}^\Omega$ is easily seen from a variational argument.

For $d=3$, we first fix $u\in C^1(\overline{\Omega})$ and a constant $\widehat{u}\in\mathbb{R}$ yet to be specified. Introducing

$$f(x) := \big|u(x)-\widehat{u}\big|^{4/3}\,\operatorname{sign}\big(u(x)-\widehat{u}\big)\,,$$

with $\nabla f(x) = \frac{4}{3}|u(x)-\widehat{u}|^{1/3}\operatorname{sign}(u(x)-\widehat{u})\,\nabla u$, we obtain a function $f\in C^1(\overline{\Omega})$. We have

$$\inf_{c\in\mathbb{R}}\|u-c\|_{L^2(\Omega)} \;\leq\; \|u-\widehat{u}\|_{L^2(\Omega)} \;=\; \Big(\int_\Omega |f|^{3/2}\,dx\Big)^{1/2}. \tag{3.1}$$

For a general function $g\in C(\overline{\Omega})$ we define $K_g(t) := \int_\Omega |g(x)-t|^{3/2}\,dx$. It is easy to show that K_g is convex on the whole of $\mathbb{R}$ and that

$$K_g(0) = \inf_{t\in\mathbb{R}} K_g(t) \quad\Longleftrightarrow\quad \int_\Omega \operatorname{sign}\big(g(x)\big)\,|g(x)|^{1/2}\,dx = 0\,. \tag{3.2}$$

Obviously, $\operatorname{sign}(f(x)) = \operatorname{sign}(u(x)-\widehat{u})$. We note that the function

$$F:\mathbb{R}\to\mathbb{R}: s\mapsto \int_\Omega \operatorname{sign}\big(u(x)-s\big)\,\big||u(x)-s|^{4/3}\big|^{1/2}\,dx$$

is continuous, and that $s\to\pm\infty \implies F(s)\to\mp\infty$. Thus, there exists $\widehat{u}$ such that $F(\widehat{u})=0$. Using this particular $\widehat{u}$, we can conclude from (3.2) and Lemma 3.3 that

$$\Big(\int_\Omega |f|^{3/2}\,dx\Big)^{1/2} \;=\; \Big(\inf_{c\in\mathbb{R}}\int_\Omega |f-c|^{3/2}\,dx\Big)^{3/2} \;\leq\; \gamma(\Omega)^{3/4}\Big(\int_\Omega|\nabla f|\,dx\Big)^{3/4}. \tag{3.3}$$

Applying Cauchy's inequality in a first and Hölder's inequality ($p=3$, $q=3/2$) in a second step, we obtain from the above formula for ∇f that

$$\begin{aligned} \Big(\int_\Omega|\nabla f|\,dx\Big)^{3/4} \;&\leq\; \Big(\frac{4}{3}\Big)^{3/4}\Big\{\Big(\int_\Omega \underbrace{|u-\widehat{u}|^{2/3}}_{=|f|^{1/2}}\,dx\Big)^{1/2}\Big(\int_\Omega|\nabla u|^2\,dx\Big)^{1/2}\Big\}^{3/4}\\ &\leq\; \Big(\frac{4}{3}\Big)^{3/4}\Big\{\Big[\Big(\int_\Omega|f|^{3/2}\,dx\Big)^{1/3}|\Omega|^{2/3}\Big]^{1/2}|u|_{H^1(\Omega)}\Big\}^{3/4}. \end{aligned}$$

Combining the last two inequalities we can conclude that

$$\Big(\int_\Omega|f|^{3/2}\,dx\Big)^{1/2} \;\leq\; \Big(\frac{4}{3}\,\gamma(\Omega)\Big)^{3/4}\Big(\int_\Omega|f|^{3/2}\,dx\Big)^{1/8}|\Omega|^{1/4}\,|u|^{3/4}_{H^1(\Omega)}.$$

Therefore,

$$\Big(\int_\Omega |f|^{3/2}\,dx\Big)^{3/8} \;\le\; \Big(\frac{4}{3}\,\gamma(\Omega)\Big)^{3/4}\,|\Omega|^{1/4}\,|u|^{3/4}_{H^1(\Omega)}\,,$$

and so

$$\Big(\int_\Omega |f|^{3/2}\,dx\Big)^{1/2} \;\le\; \frac{4}{3}\,\gamma(\Omega)\,|\Omega|^{1/3}\,|u|_{H^1(\Omega)}\,.$$

Together with (3.1) and a density argument, we finally arrive at the desired estimate. □

Definition 3.5. Let $\gamma(\Omega)$ be the best constant in the isoperimetric inequality. With

$$C_P(\Omega) := \Big(\frac{4}{3}\Big)^{d-2}\gamma(\Omega)\,\frac{|\Omega|^{1/d}}{\operatorname{diam}(\Omega)}\,, \tag{3.4}$$

we have the following version of Poincaré's inequality with $C_P(\Omega)$ independent of $\operatorname{diam}(\Omega)$,

$$\|u-\overline{u}^\Omega\|_{L^2(\Omega)} \;\le\; C_P(\Omega)\operatorname{diam}(\Omega)\,|u|_{H^1(\Omega)} \qquad \forall u\in H^1(\Omega)\,. \tag{3.5}$$

Remark 3.6. In the following, we will use the constant $C_P(\Omega)$ in our estimates to track geometric dependence. Mario Bebendorf [2] showed that for convex domains Ω, Poincaré's inequality above holds with $C_P(\Omega)$ replaced by $1/\pi$. His proof is based on an earlier one by Payne and Weinberger [24], which contains a mistake for the case $d=3$. Explicit estiamtes for star-shaped domains can, e. g., be found in [35],

3.3. **Two new auxiliary extension results.** Since the single and double layer potential in the theory of boundary integral equations are usually not in $H^1(\Omega^{\text{ext}})$, we need to generalize a bit Jones' extension result (Theorem 3.2).

Definition 3.7. Let Ω denote either Ω^{ext} or the whole of $\mathbb{R}^d$. We set

$$\begin{aligned} H^1_{\text{loc}}(\Omega) &:= \big\{v\in L^1_{\text{loc}}(\Omega) : v\in H^1(B_R\cap\Omega)\quad \forall R>0\big\}\,,\\ H^1_{\text{loc},*}(\Omega) &:= \Big\{v\in H^1_{\text{loc}}(\Omega) : \int_{\Omega^{\text{ext}}}|\nabla v|^2\,dx<\infty\Big\}\,, \end{aligned}$$

where $L^1_{\text{loc}}(\Omega)$ is the space of functions that are integrable over every compact subset of Ω, and B_R is the d-dimensional ball with radius $R>0$ and its center in the origin.

Functions from $H^1_{\text{loc},*}(\Omega^{\text{ext}})$ do not necessarily decay to zero at infinity, but they have a finite H^1-semi-norm (finite energy).

The following lemma is not really needed for the results that come afterwards, but is interesting for itself.

Lemma 3.8. *There exists a linear extension operator* $E^{\text{int}} : H^1(\Omega^{\text{int}}) \to H^1_{\text{loc},*}(\mathbb{R}^d)$ *such that*

$$|E^{\text{int}}\,w|_{H^1(\mathbb{R}^d)} \;\le\; C_E(\Omega^{\text{int}})\,\sqrt{1+C_P(\Omega^{\text{int}})^2}\;|w|_{H^1(\Omega^{\text{int}})} \qquad \forall w\in H^1(\Omega^{\text{int}})\,.$$

Proof. Assume first that $\operatorname{diam}(\Omega^{\text{int}})=1$ and let $E: H^1(\Omega^{\text{int}})\to H^1(\mathbb{R}^d)$ denote Jones' extension operator. Secondly, we define

$$E^{\text{int}}\,w := E(w-\overline{w}^{\Omega^{\text{int}}}) + \overline{w}^{\Omega^{\text{int}}}\,,$$

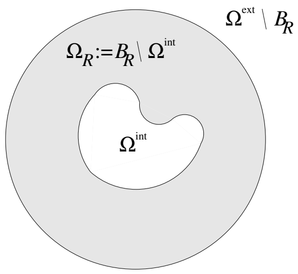

FIGURE 1. Sketch of the domains in Lemma 3.9.

where $\overline{w}^{\Omega^{\rm int}} := |\Omega^{\rm int}|^{-1} \int_{\Omega^{\rm int}} w\,dx$. Obviously, $E^{\rm int}$ is linear and $(E^{\rm int}\,w)_{|\Omega^{\rm int}} = w$. Theorem 3.2 and Poincaré's inequality (3.5) imply

$$\begin{aligned} |E^{\rm int}\,w|_{H^1(\mathbb{R}^d)} \;&=\; |E(w - \overline{w}^{\Omega^{\rm int}})|_{H^1(\mathbb{R}^d)} \;\leq\; C_E(\Omega^{\rm int})\,\|w - \overline{w}^{\Omega^{\rm int}}\|_{H^1(\Omega^{\rm int})} \\ &\leq\; C_E(\Omega^{\rm int})\,\sqrt{1 + C_P(\Omega)^2}\;|w|_{H^1(\Omega^{\rm int})}\,. \end{aligned}$$

The case $\mathrm{diam}(\Omega^{\rm int}) \neq 1$ follows from a simple dilation argument. $\square$

The next lemma is fundamental for our paper.

Lemma 3.9. *We fix a ball B_R with radius $R > 0$ and its center not necessarily in the origin such that*

$$\overline{\Omega}^{\rm int} \subset B_R \qquad \text{and} \qquad \mathrm{dist}(\partial B_R, \Gamma) \;\geq\; \tfrac{1}{2}\,\mathrm{diam}(\Omega^{\rm int})\,,$$

cf. Figure 1. Then there exists a linear extension operator $E^{\rm ext} : H^1_{{\rm loc},*}(\Omega^{\rm ext}) \to H^1_{{\rm loc},*}(\mathbb{R}^d)$ with

$$\begin{aligned} |E^{\rm ext}\,w|_{H^1(\Omega^{\rm int})} \;&\leq\; C_{E^{\rm ext}}\,|w|_{H^1(\Omega^{\rm ext})}\,, \\ |E^{\rm ext}\,w|_{H^1(\mathbb{R}^d)} \;&\leq\; (1 + C_{E^{\rm ext}})\,|w|_{H^1(\Omega^{\rm ext})}\,, \end{aligned}$$

where the constant $C_{E^{\rm ext}}$ is independent of $\mathrm{diam}(\Omega^{\rm int})$ and depends only on the Jones parameter $C_U(B_R \setminus \Omega^{\rm int})$ and on the constant $C_P(B_R \setminus \Omega^{\rm int})$ in Poincaré's inequality (cf. Definition 3.5). Note also that $E^{\rm ext} : H^1(\Omega^{\rm ext}) \to H^1(\mathbb{R}^d)$.

Proof. As a short hand we define $\Omega_R := B_R \setminus \overline{\Omega}^{\rm int}$. Let us assume that $\mathrm{diam}(\Omega_R) = 2R = 1$, the general case is easily obtained by a simple dilation argument. Let $E : H^1(\Omega_R) \to H^1(\mathbb{R}^d)$ denote Jones' extension operator and define

$$(E^{\rm ext}\,w)(x) := \begin{cases} w(x) & \text{for } x \in \Omega^{\rm ext}\,, \\ E\big(w - \overline{w}^{\Omega_R}\big)(x) + \overline{w}^{\Omega_R} & \text{for } x \in \Omega^{\rm int}\,. \end{cases}$$

(Note that if $(E\,\overline{w}^{\Omega_R})_{|\Omega^{\rm int}} = \overline{w}^{\Omega_R}$, then $E^{\rm ext} = E$. However, this is not ensured, at least not in the statement of Jones' theorem.) From Theorem 3.2 and Poincaré's inequality (3.5) we obtain

$$\begin{aligned} |E^{\rm ext} w|_{H^1(\Omega^{\rm int})} \;&=\; |E(w - \overline{w}^{\Omega_R})|_{H^1(\Omega^{\rm int})} \;\leq\; |E(w - \overline{w}^{\Omega_R})|_{H^1(\mathbb{R}^d)} \\ &\leq\; C_E(\Omega_R)\,\|w - \overline{w}^{\Omega_R}\|_{H^1(\Omega_R)} \;\leq\; C_E(\Omega_R)\,\sqrt{1 + C_P(\Omega_R)^2}\,|w|_{H^1(\Omega_R)} \\ &\leq\; C_E(\Omega_R)\,\sqrt{1 + C_P(\Omega_R)^2}\,|w|_{H^1(\Omega^{\rm ext})}\,, \end{aligned}$$

which proves the first estimate in the lemma with $C_{E^{\rm ext}} := C_E(\Omega_R)\,\sqrt{1 + C_P(\Omega_R)^2}$. The second estimate follows from the trivial fact that $|E^{\rm ext} w|_{H^1(\Omega^{\rm ext})} = |w|_{H^1(\Omega^{\rm ext})}$. $\square$

We emphasize once more that the constants in Lemma 3.8 and Lemma 3.9 are scaling-invariant and depend only on Jones parameters and constants in Poincaré's inequality, where the latter can be tracked back to isoperimetric parameters.

4. Potentials and boundary integral operators

Using the fundamental solution (2.1) for the Laplace operator, we can define the single layer potential $\widetilde{V}$ and the double layer potential $\widetilde{W}$, e. g. according to [23]. For smooth functions v and w we have the representations.

$$\begin{aligned} (\widetilde{V}w)(x) &= \int_\Gamma U^*(x,\, y)\, w(y)\, ds_y\,, \\ (\widetilde{W}v)(x) &= \int_\Gamma \frac{\partial U^*}{\partial n_y}(x,\, y)\, v(y)\, ds_y\,. \end{aligned}$$

One can show that $\widetilde{V} : H^{-1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^d)$, and $\widetilde{W} : H^{1/2}(\Gamma) \to H^1(\Omega^{\mathrm{int}}) \cup H^1_{\mathrm{loc}}(\Omega^{\mathrm{ext}})$. Furthermore, we have the jump relations

$$\begin{aligned} [\![\gamma_0 \widetilde{V} w]\!] &= 0\,, \quad [\![\gamma_1 \widetilde{V} w]\!] = -w \qquad \forall w \in H^{-1/2}(\Gamma)\,, \\ [\![\gamma_0 \widetilde{W} v]\!] &= v\,, \quad [\![\gamma_1 \widetilde{W} v]\!] = 0 \qquad \forall v \in H^{1/2}(\Gamma)\,, \end{aligned} \tag{4.1}$$

see e. g. [23]. Here, $[\![\gamma_0 u]\!] := \gamma_0^{\mathrm{ext}} u - \gamma_0^{\mathrm{int}} u$ and $[\![\gamma_1 u]\!] := \gamma_1^{\mathrm{ext}} u - \gamma_0^{\mathrm{ext}} u$. These relations allow to define the single layer potential operator V and the hypersingular operator D,

$$V := \gamma_0\, \widetilde{V} \quad \text{and} \quad D := -\gamma_1\, \widetilde{W}\,.$$

The double layer potential operator $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and its adjoint $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ fulfill the relations

$$\gamma_0^{\mathrm{int}} \widetilde{W} = -\tfrac{1}{2} I + K\,, \qquad \gamma_1^{\mathrm{int}} \widetilde{V} = \tfrac{1}{2} I + K'\,.$$

Recall the definition of the subspaces $H^{-1/2}_*(\Gamma)$ and $H^{1/2}_*(\Gamma)$, as well as the natural density w_{eq}, see (2.4)–(2.6). One easily shows that $\ker(D) = \mathrm{span}\{\mathbf{1}_\Gamma\}$ and $\mathrm{range}(D) = H^{-1/2}_*(\Gamma)$. Also

$$(\tfrac{1}{2} I - K)\mathbf{1}_\Gamma = \mathbf{1}_\Gamma\,, \tag{4.2}$$

$\ker(\frac{1}{2} I + K) = \mathrm{span}\{\mathbf{1}_\Gamma\}$, and the operator $\frac{1}{2} I - K$ is bijective.

Another important issue is the behavior of the two surface potentials at infinity. According to [23, p. 261] (see also [6]), we have

$$(\widetilde{V}w)(x) = \left\{ \begin{array}{ll} \frac{1}{2\pi}\, \langle w,\, \mathbf{1}_\Gamma \rangle \, \log(|x|^{-1}) + \mathcal{O}(|x|^{-1}) & \text{if } d = 2, \\ \mathcal{O}(|x|^{-1}) & \text{if } d = 3, \end{array} \right\} \quad \text{as } |x| \to \infty\,. \tag{4.3}$$

Furthermore, if $d = 3$, or if $d = 2$ and $w \in H^{-1/2}_*(\Gamma)$, then $\widetilde{V} w \in H^1_{\mathrm{loc},*}(\mathbb{R}^d)$. The double layer potential fulfills

$$(\widetilde{W}v)(x) = \mathcal{O}(|x|^{1-d}) \qquad \text{as } |x| \to \infty\,, \tag{4.4}$$

i. e., $\widetilde{W} v \in H^1_{\mathrm{loc},*}(\Omega^{\mathrm{ext}})$, see [23].

5. Alternative trace norms and related inequalities

In the following we define alternative trace norms for $H^{\pm 1/2}(\Gamma)$ using the $H^1(\Omega^{\text{int}})$-norm. This way we can avoid the trace constants on whose values we have no or only little information. Instead, the constant from the extension result in Lemma 3.9 will appear, which depends on Jones and isoperimetric parameters only.

Definition 5.1. For a function $v \in H^{1/2}(\Gamma)$ we define the semi-norm

$$|v|_{\star,H^{1/2}(\Gamma)} := \inf_{\substack{\widetilde{v} \in H^1(\Omega^{\text{int}}) \\ \widetilde{v}_{|\Gamma} = v}} |\widetilde{v}|_{H^1(\Omega^{\text{int}})}\,.$$

In fact, the infimum is attained at the harmonic extension of v, denoted by

$$\mathcal{H}^{\text{int}} v := \operatorname*{argmin}_{\substack{\widetilde{v} \in H^1(\Omega^{\text{int}}) \\ \widetilde{v}_{|\Gamma} = v}} |\widetilde{v}|_{H^1(\Omega^{\text{int}})}\,.$$

We define the full norm

$$\|v\|_{\star,H^{1/2}(\Gamma)} := \Big\{ |v|^2_{\star,H^{1/2}(\Gamma)} + \frac{1}{\operatorname{diam}(\Omega^{\text{int}})^2} \, \|\mathcal{H}^{\text{int}} v\|^2_{L^2(\Omega^{\text{int}})} \Big\}^{1/2},$$

and its associated dual norm

$$\|w\|_{\star,H^{-1/2}(\Gamma)} := \sup_{v \in H^{1/2}(\Gamma)} \frac{|\langle w,\, v\rangle|}{\|v\|_{\star,H^{1/2}(\Gamma)}}\,.$$

From the literature, it is known that the trace operators $\gamma_0^{\text{int}} : H^1(\Omega^{\text{int}}) \to H^{1/2}(\Gamma)$ and $\gamma_0^{\text{ext}} : H^1_{\text{loc},*}(\Omega^{\text{ext}}) \to H^{1/2}(\Gamma)$ are well-defined, linear, and bounded. The following lemma gives precise bounds.

Lemma 5.2. *We have the trace inequalities*

$$\begin{aligned}
&(i) && |\gamma_0^{\text{int}} v|_{\star,H^{1/2}(\Gamma)} \;\le\; |v|_{H^1(\Omega^{\text{int}})} && \forall v \in H^1(\Omega^{\text{int}})\,,\\
&(ii) && |\gamma_0^{\text{ext}} v|_{\star,H^{1/2}(\Gamma)} \;\le\; C_{E^{\text{ext}}}\, |v|_{H^1(\Omega^{\text{ext}})} && \forall v \in H^1_{\text{loc},*}(\Omega^{\text{ext}})\,,
\end{aligned}$$

with $C_{E^{\text{ext}}}$ being the constant in Lemma 3.9.

Proof. The first inequality is easily seen from the fact that our alternative norm is the minimal extension with respect to the $H^1(\Omega^{\text{int}})$-semi-norm and that v extends $\gamma_0^{\text{int}} v$,

$$|\gamma_0^{\text{int}} v|_{\star,H^{1/2}(\Gamma)} \;=\; |\mathcal{H}^{\text{int}} \gamma_0 v|_{H^1(\Omega^{\text{int}})} \;\le\; |v|_{H^1(\Omega^{\text{int}})}\,.$$

For the exterior part, we use that $E^{\text{ext}}\, v$ extends $\gamma_0^{\text{ext}} v$ to Ω^{int} and so Lemma 3.9 yields

$$|\gamma_0^{\text{ext}} v|_{\star,H^{1/2}(\Gamma)} \;=\; |\mathcal{H}^{\text{int}} \gamma_0^{\text{ext}} v|_{H^1(\Omega^{\text{int}})} \;\le\; |E^{\text{ext}}\, v|_{H^1(\Omega^{\text{int}})} \;\le\; C_{E^{\text{ext}}}\, |v|_{H^1(\Omega^{\text{ext}})}\,.$$

□

The following lemma is a slightly modified version of [23, Lemma 4.3] and it provides a generalized definition of the normal derivative for harmonic H^1-functions. For $u \in H^1(\Omega^{\text{int}})$ we say that $\Delta u = 0$ weakly in Ω^{int} if

$$\int_{\Omega^{\text{int}}} \nabla u \cdot \nabla \varphi \, dx \;=\; 0 \qquad \forall \varphi \in \mathcal{D}(\Omega^{\text{int}})\,,$$

where $\mathcal{D}(\Omega^{\text{int}})$ are the $\mathcal{C}^\infty$ functions with compact support in Ω^{int}. The analogous definition can be applied for $u \in H^1_{\text{loc},*}(\Omega^{\text{ext}})$.

Lemma 5.3. (i) *For each function* $u \in H^1(\Omega^{\text{int}})$ *with* $\Delta u = 0$ *weakly in* Ω^{int}*, there exists a unique functional* $\gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$ *such that*

$$\langle \gamma_1^{\text{int}} u,\, \gamma_0^{\text{int}} \widetilde{v} \rangle \;=\; \int_{\Omega^{\text{int}}} \nabla u \cdot \nabla \widetilde{v}\, dx \qquad \textit{for } \widetilde{v} \in H^1(\Omega^{\text{int}})\,.$$

(ii) *For each function* $u \in H^1_{\text{loc},*}(\Omega^{\text{ext}})$ *with* $\Delta u = 0$ *weakly in* Ω^{ext}*, there exists a unique functional* $\gamma_1^{\text{ext}} u \in H^{-1/2}(\Gamma)$ *such that*

$$\langle \gamma_1^{\text{ext}} u,\, \gamma_0^{\text{ext}} \widetilde{v} \rangle \;=\; -\int_{\Omega^{\text{ext}}} \nabla u \cdot \nabla \widetilde{v}\, dx \qquad \forall \widetilde{v} \in H^1(\Omega^{\text{ext}})\,.$$

For smooth functions u*,* $\gamma_1^{\text{int}} u = \frac{\partial u}{\partial n}$ *and* $\gamma_1^{\text{ext}} u = \frac{\partial u}{\partial n}$ *(recall that* n *is* inward *to* Ω^{ext}*).*

Proof. The proof is found in [23] and Case (ii) can easily be generalized to $H^1_{\text{loc},*}(\Omega^{\text{ext}})$. To give a sketch, we note that e. g., the functional $\gamma_1^{\text{ext}} u$ is defined by

$$\langle \gamma_1^{\text{ext}} u,\, v \rangle \;=\; -\int_{\Omega^{\text{int}}} \nabla u \cdot \nabla \mathcal{E}^{\text{ext}} v\, dx \qquad \text{for } v \in H^{1/2}(\Gamma)\,,$$

where $\mathcal{E}^{\text{ext}} : H^{1/2}(\Gamma) \to H^1(\Omega^{\text{ext}})$ is an arbitrary bounded extension. $\square$

From the sketch in the above proof one can see that the operator γ_1^{ext} is linear, and so is γ_1^{int}. Both operators are also bounded, and the following lemma gives precise estimates.

Lemma 5.4. *We have the dual trace inequalities*

$$\begin{array}{llll} (i) & \|\gamma_1^{\text{int}} u\|_{\star,H^{-1/2}(\Gamma)} \;\le\; |u|_{H^1(\Omega^{\text{int}})} & & \forall u \in H^1(\Omega^{\text{int}}),\ \Delta u = 0\,, \\ (ii) & \|\gamma_1^{\text{ext}} u\|_{\star,H^{-1/2}(\Gamma)} \;\le\; C_E(\Omega^{\text{int}})\, |u|_{H^1(\Omega^{\text{ext}})} & & \forall u \in H^1_{\text{loc},*}(\Omega^{\text{ext}}),\ \Delta u = 0\,, \end{array}$$

with the $C_E(\Omega^{\text{int}})$ *depending only on the Jones parameter of* Ω^{int} *(cf. Theorem 3.2).*

Proof. (i) Let $v \in H^{1/2}(\Gamma)$ be arbitrary but fixed. By Cauchy's inequality, using the defining property of γ_1^{int}, and setting $\widetilde{v} := \mathcal{H}^{\text{int}} v$ we obtain that

$$\langle \gamma_1^{\text{int}} u,\, v \rangle \;=\; \int_{\Omega^{\text{int}}} \nabla u \cdot \nabla \mathcal{H}^{\text{int}} v\, dx \;\le\; |u|_{H^1(\Omega^{\text{int}})}\, |\mathcal{H}^{\text{int}} v|_{H^1(\Omega^{\text{int}})} \;=\; |u|_{H^1(\Omega^{\text{int}})}\, |v|_{\star,H^{1/2}(\Gamma)}$$

From the definition of the dual norm we conclude that

$$\|\gamma_1^{\text{int}} u\|_{\star,H^{-1/2}(\Gamma)} \;\le\; \sup_{v \in H^{1/2}(\Gamma)} \frac{|u|_{H^1(\Omega^{\text{int}})}\, |v|_{\star,H^{1/2}(\Gamma)}}{\|v\|_{\star,H^{1/2}(\Gamma)}} \;\le\; |u|_{H^1(\Omega^{\text{int}})}\,.$$

(ii) Assume first that $\operatorname{diam}(\Omega^{\text{int}}) = 1$. Let $E : H^1(\Omega^{\text{int}}) \to H^1(\mathbb{R}^d)$ denote Jones' extension operator. Choosing $\widetilde{v} = E\, \mathcal{H}^{\text{int}} v$ with $v \in H^{1/2}(\Gamma)$ we have

$$|\widetilde{v}|_{H^1(\Omega^{\text{ext}})} \;\le\; C_E(\Omega^{\text{int}})\, \|\mathcal{H}^{\text{int}} v\|_{H^1(\Omega^{\text{int}})} \;=\; C_E(\Omega^{\text{int}})\, \|v\|_{\star,H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}(\Gamma)\,.$$

Using the defining property of $\gamma_1^{\text{ext}} u$, we obtain from Cauchy's inequality and the estimate above that

$$-\langle \gamma_1^{\text{ext}} u,\, v \rangle \;=\; \int_{\Omega^{\text{ext}}} \nabla u \cdot \nabla \widetilde{v}\, dx \;\le\; |u|_{H^1(\Omega^{\text{ext}})}\, |\widetilde{v}|_{H^1(\Omega^{\text{ext}})} \;\le\; |u|_{H^1(\Omega^{\text{ext}})}\, C_E(\Omega^{\text{int}})\, \|v\|_{\star,H^{1/2}(\Gamma)}\,.$$

Estimate (ii) follows then immediately from the definition of the dual norm. The estimate for $\operatorname{diam}(\Omega^{\text{int}}) \ne 1$ can be shown using a simple dilation argument. $\square$

6. Estimates for boundary integral operators

6.1. **Coercivity estimates.** We now present our alternative version of the V-coercivity.

Lemma 6.1 (partial coercivity of V)**.** *We have*

$$\langle w,\, Vw\rangle \;\geq\; \widetilde{c}_V^\star\, \|w\|^2_{\star, H^{-1/2}(\Gamma)}\,, \qquad \forall w \in H_*^{-1/2}(\Gamma)\,,$$

with $\widetilde{c}_V^\star := \frac{1}{2}\, C_E(\Omega^{\text{int}})^{-2}$ *depending only on the Jones parameter* $C_U(\Omega^{\text{int}})$*. In case* $d = 3$*, the same estimate holds for all* $w \in H^{-1/2}(\Gamma)$ *with the same constant.*

Proof. The proof is essentially the one in [33, Sect. 6.6.1] but uses our alternative norms and auxiliary results. We set $u := \widetilde{V} w$, and thus $\Delta u = 0$ in Ω^{int} and Ω^{ext}, separately. Then, with the defining property of the trace operator γ_1^{int} we get

$$\langle \gamma_1^{\text{int}} u,\, \gamma_0^{\text{int}} u\rangle \;=\; \int_{\Omega^{\text{int}}} |\nabla u|^2\, dx\,. \tag{6.1}$$

For the exterior part, even though $u \notin H^1(\Omega^{\text{ext}})$, we obtain

$$\langle \gamma_1^{\text{ext}} u,\, \gamma_0^{\text{ext}} u\rangle \;=\; -\int_{\Omega^{\text{ext}}} |\nabla u|^2\, dx\,, \tag{6.2}$$

due to the decay behavior (4.3) of $u = \widetilde{V} w$ at infinity; see also [23, Theorem 8.12]. Using the jump relations (4.1), we easily obtain that

$$\langle w,\, Vw\rangle \;=\; \langle w,\, \gamma_0\, u\rangle \;=\; -\langle [\![\gamma_1 u]\!],\, \gamma_0 u\rangle \;=\; |u|^2_{H^1(\Omega^{\text{int}})} + |u|^2_{H^1(\Omega^{\text{ext}})}\,. \tag{6.3}$$

By Lemma 5.4 and using the jump relations again, we can conclude from (6.1)–(6.3) that

$$\begin{aligned}\langle w,\, Vw\rangle \;&\geq\; \|\gamma_1^{\text{int}} u\|^2_{\star, H^{-1/2}(\Gamma)} + C_E(\Omega^{\text{int}})^{-2}\, \|\gamma_1^{\text{ext}} u\|^2_{\star, H^{-1/2}(\Gamma)} \\ &\geq\; \tfrac{1}{2}\, \min\big(1,\, C_E(\Omega^{\text{int}})^{-2}\big)\, \|\underbrace{\gamma_1^{\text{int}} u - \gamma_1^{\text{ext}} u}_{=w}\|^2_{\star, H^{-1/2}(\Gamma)}\,.\end{aligned}$$

Obviously, $C_E(\Omega^{\text{int}}) \geq 1$, which finishes the proof. □

Corollary 6.2 (full coercivity of V)**.** *If* $d = 3$ *or if* $d = 2$ *and* $Vw_{\text{eq}} > 0$*,*

$$\langle w,\, Vw\rangle \;\geq\; c_V^\star\, \|w\|^2_{\star, H^{-1/2}(\Gamma)} \qquad \forall w \in H^{-1/2}(\Gamma)\,,$$

with

$$c_V^\star := \begin{cases} \frac{1}{2}\, \min\Big(\widetilde{c}_V^\star,\, \frac{Vw_{\text{eq}}}{\|w_{\text{eq}}\|^2_{\star, H^{-1/2}(\Gamma)}}\Big) & \text{if } d = 2\,, \\ \widetilde{c}_V^\star & \text{if } d = 3\,, \end{cases}$$

and $\widetilde{c}_V^\star$ *according to Lemma 6.1.*

Proof. We only need to show the estimate for $d = 2$, and its proof is analogous to [33, Theorem 6.23]. It is easily seen that there exists the unique decomposition

$$w \;=\; \widetilde{w} + \alpha\, w_{\text{eq}}\,, \qquad \widetilde{w} \in H_*^{-1/2}(\Gamma)\,, \qquad \alpha = \text{const}\,.$$

Using the fact that the decomposition is orthogonal with respect to $\langle\cdot,\, V\cdot\rangle$ and that $\langle w_{\mathrm{eq}},\, V w_{\mathrm{eq}}\rangle = V w_{\mathrm{eq}}$, we can conclude from the previous lemma that

$$\begin{aligned}\langle w,\, Vw\rangle &= \langle \widetilde{w},\, V\widetilde{w}\rangle + \alpha^2\,\langle w_{\mathrm{eq}},\, V w_{\mathrm{eq}}\rangle \\ &\geq \widetilde{c}_V^{\star}\,\|\widetilde{w}\|^2_{\star,H^{-1/2}(\Gamma)} + \frac{V w_{\mathrm{eq}}}{\|w_{\mathrm{eq}}\|^2_{\star,H^{-1/2}(\Gamma)}}\,\|\alpha\, w_{\mathrm{eq}}\|^2_{\star,H^{-1/2}(\Gamma)} \\ &\geq \frac{1}{2}\,\min\Big(\widetilde{c}_V^{\star},\, \frac{V w_{\mathrm{eq}}}{\|w_{\mathrm{eq}}\|^2_{\star,H^{-1/2}(\Gamma)}}\Big)\,\|w\|^2_{\star,H^{-1/2}(\Gamma)}\,.\end{aligned}$$

That concludes our proof. ☐

Remark 6.3. One can show that in two dimensions, the expression $\|w_{\mathrm{eq}}\|_{\star,H^{-1/2}(\Gamma)}$ remains invariant when we re-scale the coordinates. In two as well as in three dimensions, we can choose $v = \mathbf{1}_\Gamma$ in the definition of the dual norm and obtain the lower bound

$$\|w_{\mathrm{eq}}\|^2_{\star,H^{-1/2}(\Gamma)} \;\geq\; \frac{\langle w_{\mathrm{eq}},\, \mathbf{1}_\Gamma\rangle^2}{\|\mathbf{1}_\Gamma\|^2_{\star,H^{1/2}(\Gamma)}} \;=\; \frac{\mathrm{diam}(\Omega^{\mathrm{int}})^2}{|\Omega^{\mathrm{int}}|}\,.$$

An upper bound would be desirable. However, we have only succeeded doing so in three dimensions, see the proof of Lemma 6.9 as well as Remark 6.10.

The following result can be obtained using standard duality techniques in Hilbert spaces.

Corollary 6.4 (Boundedness of V^{-1})**.** *Let $d=3$ then*

$$\langle V^{-1}v,\, v\rangle \;\leq\; (\widetilde{c}_V^{\star})^{-1}\,\|v\|^2_{\star,H^{1/2}(\Gamma)} \qquad \forall v\in H^{1/2}(\Gamma)\,.$$

If $d=2$ let $V^{-1} : H_*^{1/2}(\Gamma)\to H_*^{-1/2}(\Gamma)$ denote the well-defined inverse of the restriction of V to $H_*^{-1/2}(\Gamma)$; then, the same estimate holds for all $v\in H_*^{1/2}(\Gamma)$. If $d=2$ and $V w_{\mathrm{eq}}>0$, the same estimate holds for all $v\in H^{1/2}(\Gamma)$ if $\widetilde{c}_V^{\star}$ above is replaced by $c_V^{\star}$.

By similar techniques as in the proof of Lemma 6.1, we obtain the semi-coercivity of D with respect to our alternative $H^{-1/2}$-semi-norm.

Lemma 6.5 (Semi-coercivity of D)**.** *For all $v\in H^{1/2}(\Gamma)$ we have*

$$\langle D\,v,\, v\rangle \;\geq\; c_D^{\star}\,|v|^2_{\star,H^{1/2}(\Gamma)}\,, \qquad \text{with} \qquad c_D^{\star} = \tfrac{1}{2}(C_{E^{\mathrm{ext}}})^{-2}\,,$$

where $C_{E^{\mathrm{ext}}}$ is the constant from Lemma 3.9.

Proof. We set $u := \widetilde{W}\,v$. Then, $\Delta u = 0$ in Ω^{int} and Ω^{ext} separately. Due to the decay behavior of u at infinity, one can show that the identities (6.1), (6.2) hold, cf. [23, Theorem 8.21]. By the jump relations (4.1) we can conclude that

$$\langle D\,v,\, v\rangle \;=\; \langle -\gamma_1 u,\, [\![\gamma_0 u]\!]\rangle \;=\; -\langle \gamma_1^{\mathrm{ext}}u,\, \gamma_0^{\mathrm{ext}}u\rangle + \langle \gamma_1^{\mathrm{int}}u,\, \gamma_0^{\mathrm{int}}u\rangle \;=\; |u|^2_{H^1(\Omega^{\mathrm{int}})} + |u|^2_{H^1(\Omega^{\mathrm{ext}})}\,.$$

Finally, Lemma 5.2 and the jump relations imply that

$$\begin{aligned}\langle D\,v,\, v\rangle &\geq |\gamma_0^{\mathrm{int}}u|^2_{\star,H^{1/2}(\Gamma)} + C_{E^{\mathrm{ext}}}^{-2}\,|\gamma_0^{\mathrm{ext}}u|^2_{\star,H^{1/2}(\Gamma)} \\ &\geq \tfrac{1}{2}\,\underbrace{\min(1,\, C_{E^{\mathrm{ext}}}^{-2})}_{=C_{E^{\mathrm{ext}}}^{-2}}\,|\underbrace{\gamma_0^{\mathrm{int}}u - \gamma_0^{\mathrm{ext}}u}_{=-v}|^2_{\star,H^{1/2}(\Gamma)}\,,\end{aligned}$$

which concludes the proof. ☐

6.2. **Auxiliary capacity results.** We have learned that the constant Vw_{eq} plays a principal role in our investigation. For $d = 3$, we know that $Vw_{\text{eq}} > 0$ and we can define by

$$\operatorname{Cap}_\Gamma := \frac{1}{Vw_{\text{eq}}} \;=\; \langle V^{-1}\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle \tag{6.4}$$

the *capacity* of Γ, cf. [23, p. 263ff]. In the same reference we find the upper bound

$$\operatorname{Cap}_\Gamma \;\leq\; 4\pi\,\operatorname{diam}(\Gamma),$$

which implies that Vw_{eq} is always strictly positive. Another upper bound for the capacity can be obtained from Corollary 6.4,

$$\operatorname{Cap}_\Gamma \;=\; \langle V^{-1}\mathbf{1}_\Gamma,\, \mathbf{1}_\gamma\rangle \;\leq\; (\widetilde{c}_V^\star)^{-1}\,\|\mathbf{1}_\Gamma\|^2_{\star,H^{1/2}(\Gamma)} \;\leq\; (\widetilde{c}_V^\star)^{-1}\,\frac{|\Omega^{\text{int}}|}{\operatorname{diam}(\Omega^{\text{int}})^2}\,.$$

Remark 6.6. For $d = 2$, the *logarithmic capacity* is defined by

$$\operatorname{Cap}_\Gamma := e^{-2\pi\, Vw_{\text{eq}}}\,,$$

cf. [23, 33]. If Γ is the circle of radius R, then $\operatorname{Cap}_\Gamma = R$. From potential theory (see e. g. [12]) one knows that if $\Omega_1 \subset \Omega_2$ then $\operatorname{Cap}_{\partial\Omega_1} \leq \operatorname{Cap}_{\partial\Omega_2}$. Let $r > 0$ such that $B_r \subset \Omega^{\text{int}}$. Then, $r \leq \operatorname{Cap}_\Gamma \leq \frac{1}{2}\operatorname{diam}(\Omega^{\text{int}})$, and so we obtain the upper and lower bounds

$$-\frac{1}{2\pi}\log\Big(\frac{\operatorname{diam}(\Omega^{\text{int}})}{2}\Big) \;\leq\; Vw_{\text{eq}} \;\leq\; -\frac{1}{2\pi}\,\log(r).$$

We see that if $\operatorname{diam}(\Omega^{\text{int}}) < 2$ then $Vw_{\text{eq}} > 0$.

The next lemma gives a lower bound for the capacity in three dimensions.

Lemma 6.7. *For $d = 3$ we have*

$$\frac{1}{Vw_{\text{eq}}} \;=\; \operatorname{Cap}_\Gamma \;\geq\; \big(48\,\pi^2\,|\Omega^{\text{int}}|\big)^{1/3} \;\geq\; |\Omega^{\text{int}}|^{1/3}\,.$$

Proof. First, we give another characterization of the capacity. The exterior Steklov-Poincaré operator is defined by

$$S^{\text{ext}} := D + (\tfrac{1}{2}I - K')V^{-1}(\tfrac{1}{2}I - K)\,,$$

see e. g. [5, 33], and one can show that for any $v \in H^{1/2}(\Gamma)$

$$S^{\text{ext}}v \;=\; -\gamma_1^{\text{ext}}u$$

whenever $u \in H^1_{\text{loc}}(\Omega^{\text{ext}})$ fulfills

$$\begin{aligned} \Delta u \;&=\; 0 \quad \text{weakly in } \Omega^{\text{ext}}\,,\\ \gamma_0^{\text{ext}}u \;&=\; v\,,\\ u(x) \;&=\; \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty\,. \end{aligned}$$

The last condition (the radiation condition) can be characterized by the norm

$$\|u\|^2_{H^1_{**}(\Omega^{\text{ext}})} := \Big(\int_{\Omega^{\text{ext}}} |\nabla u(x)|^2 + \frac{|u(x)|^2}{1+|x|^2}\,dx\Big)^{1/2}.$$

In can be shown that the space

$$H^1_{**}(\Omega^{\text{ext}}) := \{u \in H^1_{\text{loc}}(\Omega^{\text{ext}}) : \|u\|_{H^1_{**}(\Omega^{\text{ext}})} < \infty\}$$

equipped with the above norm is a Hilbert space, $C_0^\infty(\Omega^{\mathrm{ext}})$ is dense in that space, and the norm $\|\cdot\|_{H^1_{**}(\Omega^{\mathrm{ext}})}$ is equivalent to the H^1-semi-norm. By a variational principle, we have

$$\langle S^{\mathrm{ext}} v,\, v\rangle \;=\; \min_{\substack{u\in H^1_{**}(\Omega^{\mathrm{ext}})\\ \gamma_0^{\mathrm{ext}} u = v}} |u|^2_{H^1(\Omega^{\mathrm{ext}})}\,.$$

We come back to the capacity. Using formula (4.2) and the fact that $D\,\mathbf{1}_\Gamma = 0$, we see that

$$\mathrm{Cap}_\Gamma \;=\; \langle V^{-1}\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle \;=\; \langle S^{\mathrm{ext}}\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle \;=\; \min_{\substack{u\in H^1_{**}(\Omega^{\mathrm{ext}})\\ \gamma_0^{\mathrm{ext}} u = \mathbf{1}_\Gamma}} |u|^2_{H^1(\Omega^{\mathrm{ext}})}\,.$$

Next we make use of a result by Maz'ja [22, Chapter 2]. There, another definition of the capacity of Ω^{int} is used,

$$\widetilde{\mathrm{Cap}}_\Gamma := \inf\Big\{ \int_{\mathbb{R}^d} |\nabla u|^2\, dx : u \in C_0^\infty(\mathbb{R}^d) \text{ and } u(x) \ge 1 \quad \forall x\in\Omega^{\mathrm{int}}\Big\}\,.$$

Exploiting that $C_0^\infty(\mathbb{R}^d)$ is dense in $H^1_{**}(\mathbb{R}^d)$, we easily show that $\mathrm{Cap}_\Gamma \;\ge\; \widetilde{\mathrm{Cap}}_\Gamma$. Maz'ja gives the following lower bound for $d=3$:

$$\widetilde{\mathrm{Cap}}_\Gamma \;\ge\; (4\pi)^{2/3}\, 3^{1/3}\, |\Omega^{\mathrm{int}}|^{1/3}\,.$$

This concludes the proof. □

6.3. **Boundedness.** We start with the boundedness of D with respect to our alternative semi-norm.

Lemma 6.8. *For all $v\in H^{1/2}(\Gamma)$,*

$$\begin{aligned}\langle D\,v,\, v\rangle \;&\le\; |v|^2_{\star,H^{1/2}(\Gamma)}\,,\\ \|D\,v\|_{\star,H^{-1/2}(\Gamma)} \;&\le\; |v|_{\star,H^{1/2}(\Gamma)}\,.\end{aligned}$$

Proof. Let $S^{\mathrm{int}} = V^{-1}(\frac12 I + K)$ denote the Steklov-Poincaré operator, then

$$S^{\mathrm{int}} \;=\; D + (\tfrac12 I + K')V^{-1}(\tfrac12 I + K)\,, \qquad \langle S^{\mathrm{int}} v,\, v\rangle \;=\; |v|^2_{\star,H^{1/2}(\Gamma)} \quad \forall v\in H^{1/2}(\Gamma)\,.$$

Since V^{-1} is $H^{1/2}_*(\Gamma)$-coercive and $\mathrm{range}(\frac12 I + K) = H^{1/2}_*(\Gamma)$,

$$|v|^2_{\star,H^{1/2}(\Gamma)} \;=\; \langle S^{\mathrm{int}} v,\, v\rangle \;=\; \langle D\,v,\, v\rangle + \langle V^{-1}(\tfrac12 I + K)v,\, (\tfrac12 I + K)v\rangle \;\ge\; \langle D\,v,\, v\rangle\,.$$

Let us now fix $v\in H^{1/2}_*(\Gamma)$. On this space, D is coercive and defines an inner product. By duality, Cauchy's inequality, and the estimate from above we obtain

$$\begin{aligned}\|D\,v\|_{\star,H^{-1/2}(\Gamma)} \;&=\; \sup_{y\in H^{1/2}(\Gamma)} \frac{\langle D\,v,\, y\rangle}{\|y\|_{\star,H^{1/2}(\Gamma)}}\\ &=\; \sup_{\substack{\widetilde{y}\in H^{1/2}_*(\Gamma)\\ y_0\in\mathbb{R}}} \frac{\langle D\,v,\, \widetilde{y}\rangle}{\Big(|\widetilde{y}|^2_{\star,H^{1/2}(\Gamma)} + \underbrace{\frac{1}{\mathrm{diam}(\Omega^{\mathrm{int}})^2}\, \|\mathcal{H}^{\mathrm{int}}(\widetilde{y}+y_0)\|^2_{L^2(\Omega^{\mathrm{int}})}}_{\ge 0}\Big)^{1/2}}\\ &\le\; \sup_{\widetilde{y}\in H^{1/2}_*(\Gamma)} \frac{\langle D\,v,\, v\rangle^{1/2}\langle D\,\widetilde{y},\, \widetilde{y}\rangle^{1/2}}{|\widetilde{y}|_{\star,H^{1/2}(\Gamma)}} \;\le\; |v|_{\star,H^{1/2}(\Gamma)}\,.\end{aligned}$$

Since $D\,v$, $\langle D\,v,\, v\rangle$, and $|v|_{\star,H^{1/2}(\Gamma)}$ remain invariant if we add a constant v_0 to v, the same estimate holds for all $v\in H^{1/2}(\Gamma)$. □

Before we can show a bound of Vw in terms of w with respect to our alternative norms, we first need a Poincaré type inequality in $H^{1/2}_*(\Gamma)$.

Lemma 6.9. *In two and three dimensions the Poincaré type inequality*

$$\frac{1}{\operatorname{diam}(\Omega^{\rm int})^2}\,\|\mathcal{H}^{\rm int}v\|^2_{L^2(\Omega^{\rm int})}\;\le\;C_P^*\,|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}\qquad\forall v\in H^{1/2}_*(\Gamma)$$

holds with

$$C_P^*:=\;2\Big\{C_P(\Omega^{\rm int})^2+\big(1+C_P(\Omega^{\rm int})^2\big)\,\frac{|\Omega^{\rm int}|}{\operatorname{diam}(\Omega^{\rm int})^2}\,\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\Big\}.$$

In three dimensions, we have the bound

$$C_P^*\;\le\;2\Big\{C_P(\Omega^{\rm int})^2+(\widetilde{c}_V^{\star})^{-1}\,\big(1+C_P(\Omega^{\rm int})^2\big)\Big\},$$

i. e., C_P^* depends only on the Jones parameter $C_U(\Omega^{\rm int})$ and the constant $C_P(\Omega^{\rm int})$ in the standard Poincaré inequality, but not on $\operatorname{diam}(\Omega^{\rm int})$.

Proof. Our proof uses a Bramble-Hilbert type argumentation. Recall that $Vw_{\rm eq}=\text{const}$ and that $\langle w_{\rm eq},\,\mathbf{1}_\Gamma\rangle=1$. We set $\overline{v}:=\overline{\mathcal{H}^{\rm int}v}^{\Omega^{\rm int}}$ and find that $\langle w_{\rm eq},\,\overline{v}\rangle=\overline{v}$. Using the definition of the dual norm we obtain

$$\big(\overline{v}-\langle w_{\rm eq},\,v\rangle\big)^2\;=\;\langle w_{\rm eq},\,v-\overline{v}\rangle^2\;\le\;\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\,\|v-\overline{v}\|^2_{\star,H^{1/2}(\Gamma)}.$$

Since the operator $\mathcal{H}^{\rm int}$ is linear and $\mathcal{H}^{\rm int}\overline{v}=\overline{v}$, we can conclude from Lemma 3.4 that

$$\begin{aligned}\|v-\overline{v}\|^2_{\star,H^{1/2}(\Gamma)}\;&=\;|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}+\frac{1}{\operatorname{diam}(\Omega^{\rm int})^2}\,\|\mathcal{H}^{\rm int}v-\overline{v}\|^2_{L^2(\Omega^{\rm int})}\\&\le\;\big(1+C_P(\Omega^{\rm int})^2\big)\,|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}.\end{aligned}$$

Combining the last two estimates yields

$$\big(\overline{v}-\langle w_{\rm eq},\,v\rangle\big)^2\;\le\;\big(1+C_P(\Omega^{\rm int})^2\big)\,\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\,|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}.\tag{6.5}$$

Now, we fix $v\in H^{1/2}_*(\Gamma)$ which means that $\langle w_{\rm eq},\,v\rangle=0$. From the standard Poincaré inequality and estimate (6.5), we can conclude that

$$\begin{aligned}\frac{1}{\operatorname{diam}(\Omega^{\rm int})^2}\,\|\mathcal{H}^{\rm int}v\|^2_{L^2(\Omega^{\rm int})}\;&\le\;2\Big\{\frac{1}{\operatorname{diam}(\Omega^{\rm int})^2}\,\|\mathcal{H}^{\rm int}v-\overline{v}\|^2_{L^2(\Omega^{\rm int})}+\frac{|\Omega^{\rm int}|}{\operatorname{diam}(\Omega^{\rm int})^2}\,\overline{v}^2\Big\}\\&\le\;2\Big\{C_P(\Omega^{\rm int})^2\,|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}+\frac{|\Omega^{\rm int}|}{\operatorname{diam}(\Omega^{\rm int})^2}\,\big(\overline{v}-\langle w_{\rm eq},\,v\rangle\big)^2\Big\}\\&\le\;2\Big\{C_P(\Omega^{\rm int})^2+\frac{|\Omega^{\rm int}|}{\operatorname{diam}(\Omega^{\rm int})^2}\,\big(1+C_P(\Omega^{\rm int})^2\big)\,\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\Big\}\,|\mathcal{H}^{\rm int}v|^2_{H^1(\Omega^{\rm int})}.\end{aligned}$$

In three dimensions V^{-1} defines an inner product. Thus, Cauchy's inequality, formula (6.4), and Corollary 6.4 imply that for all $v\in H^{1/2}(\Gamma)$,

$$\begin{aligned}\langle w_{\rm eq},\,v\rangle^2\;&=\;(Vw_{\rm eq})^2\,\langle V^{-1}\mathbf{1}_\Gamma,\,v\rangle^2\le\;(Vw_{\rm eq})^2\,\langle V^{-1}\mathbf{1}_\Gamma,\,\mathbf{1}_\Gamma\rangle\,\langle V^{-1}v,\,v\rangle\\&\le\;Vw_{\rm eq}\,(\widetilde{c}_V^{\star})^{-1}\,\|v\|^2_{\star,H^{1/2}(\Gamma)}.\end{aligned}$$

In other words, $\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\le(\widetilde{c}_V^{\star})^{-1}\,Vw_{\rm eq}$. Finally, we use Lemma 6.7 to see that

$$\frac{Vw_{\rm eq}\,|\Omega^{\rm int}|}{\operatorname{diam}(\Omega^{\rm int})^2}\;\le\;\frac{|\Omega^{\rm int}|}{|\Omega^{\rm int}|^{1/3}\,\operatorname{diam}(\Omega^{\rm int})^2}\;\le\;1.$$

This finishes the proof. $\square$

Remark 6.10. Using the same technique as in the above proof in two dimensions yields only the trivial bound

$$\|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)} \;\le\; \frac{V w_{\rm eq}}{c_V^\star} \;=\; 2\,\max\Big\{\frac{V w_{\rm eq}}{\widetilde{c}_V^\star},\, \|w_{\rm eq}\|^2_{\star,H^{-1/2}(\Gamma)}\Big\}\,.$$

At least, we know that $\|w_{\rm eq}\|_{\star,H^{-1/2}(\Gamma)}$ remains invariant when we re-scale the coordinates, and so C_P^* is independent of $\operatorname{diam}(\Omega^{\rm int})$. In the following we sketch how a direct upper bound could be obtained. The equilibrium density is given by $w_{\rm eq} = \widetilde{t} + |\Gamma|^{-1}\mathbf{1}_\Gamma$, where $\widetilde{t} \in H_*^{-1/2}(\Gamma)$ fulfills the variational equation

$$\langle \tau,\, V\widetilde{t}\rangle \;=\; -|\Gamma|^{-1}\,\langle \tau,\, V\mathbf{1}_\Gamma\rangle \qquad \forall \tau \in H_*^{-1/2}(\Gamma),$$

cf. e. g. [33]. Using the V-coercivity on $H_*^{-1/2}(\Gamma)$, the lemma by Lax and Milgram yields the estimate $\|\widetilde{t}\|_{\star,H^{-1/2}(\Gamma)} \le (\widetilde{c}_V^\star)^{-1}\,|\Gamma|^{-1}\,\|V\mathbf{1}_\Gamma\|_{\star,H^{1/2}(\Gamma)}$, and so we have

$$\|w_{\rm eq}\|_{\star,H^{-1/2}(\Gamma)} \;\le\; |\Gamma|^{-1}\,\big[(\widetilde{c}_V^\star)^{-1}\,\|V\mathbf{1}_\Gamma\|_{\star,H^{1/2}(\Gamma)} + \|\mathbf{1}_\Gamma\|_{\star,H^{-1/2}(\Gamma)}\big].$$

The first term can be bounded by

$$\|V\mathbf{1}_\Gamma\|_{\star,H^{1/2}(\Gamma)} \;\le\; \|\widetilde{V}\mathbf{1}_\Gamma\|_{H^1(\Omega^{\rm int})} \le \frac{|\Gamma|}{2\pi}\int_{\Omega^{\rm int}}\int_\Gamma \big(\log|x-y|\big)^2 + |x-y|^{-2}\,ds_y\,dx\,,$$

assuming that $\operatorname{diam}(\Omega^{\rm int}) = 1$. Using the defining property of Jones' parameter might lead to an explicit bound for this integral. For the second term, we could use that

$$\|\mathbf{1}_\Gamma\|_{\star,H^{-1/2}(\Gamma)} \;\le\; \sup_{v\in H^{1/2}(\Gamma)} \frac{|\Gamma|\,\|v\|_{L^2(\Gamma)}}{\|v\|_{\star,H^{1/2}(\Gamma)}}.$$

This means we would need an explicit Poincaré or trace inequality of the type

$$\|\widetilde{v}\|_{L^2(\Gamma)} \;\le\; C\,\|\widetilde{v}\|_{H^1(\Omega^{\rm int})} \qquad \forall \widetilde{v} \in H^1(\Omega^{\rm int}),$$

for $\operatorname{diam}(\Omega^{\rm int}) = 1$ with C being explicit in $\Omega^{\rm int}$. Maz'ja gives an estimate of this type (cf. [22, Sect. 4.11.4]). However, the dependence of the constant C on $\Omega^{\rm int}$ seems much more complicated than in the standard Poincaré inequality.

Lemma 6.11. *Assume that $V w_{\rm eq} > 0$ (which is always true in three dimensions). Then,*

$$\left.\begin{array}{rcl} \|Vw\|_{\star,H^{1/2}(\Gamma)} & \le & C_V^\star\,\|w\|_{\star,H^{-1/2}(\Gamma)} \\ \langle w,\, Vw\rangle & \le & C_V^\star\,\|w\|^2_{\star,H^{-1/2}(\Gamma)} \end{array}\right\} \qquad \forall w \in H^{-1/2}(\Gamma)\,,$$

where

$$C_V^\star := \begin{cases} \max\Big((1+2\,C_P^*),\, 2\,\frac{|\Omega^{\rm int}|\,V w_{\rm eq}}{\operatorname{diam}(\Omega^{\rm int})^2}\Big) & \text{if } d=2\,,\\ (1+2\,C_P^*) & \text{if } d=3\,,\end{cases}$$

with C_P^* defined according to Lemma 6.9. The same estimate also holds on the subspace $H_*^{-1/2}(\Gamma)$ with $C_V^\star$ replaced by the enhanced constant $\widetilde{C}_V^\star := (1+2\,C_P^*)$.

Proof. We fix $w \in H^{-1/2}(\Gamma)$. By standard arguments one shows that there exists the unique decomposition

$$w \;=\; \widetilde{w} + w_0\,, \quad \text{with} \quad \widetilde{w} \in H_*^{-1/2}(\Gamma) \quad \text{and} \quad w_0 = \langle w,\, \mathbf{1}_\Gamma\rangle\, w_{\rm eq}\,. \tag{6.6}$$

Consequently, $\langle \widetilde{w},\, V w_0\rangle = 0$ and

$$\langle w,\, Vw\rangle \;=\; \langle \widetilde{w},\, V\widetilde{w}\rangle + \langle w_0,\, Vw_0\rangle \;=\; \langle \widetilde{w},\, V\widetilde{w}\rangle + \langle w,\, \mathbf{1}_\Gamma\rangle^2\, V w_{\mathrm{eq}}\,. \tag{6.7}$$

We set $u := \widetilde{V} w$, thus $\gamma_0 u = V w$. Also,

$$\gamma_0 u \;=\; V\widetilde{w} + \langle w,\, \mathbf{1}_\Gamma\rangle\, V w_{\mathrm{eq}}\;, \quad \text{with} \quad V\widetilde{w} \in H^{1/2}_*(\Gamma)\,.$$

From the definition of our alternative norm, the fact that $V w_0 = \mathrm{const}$, and using that $\mathcal{H}^{\mathrm{int}}$ is linear, we can conclude that

$$\begin{aligned} \|Vw\|^2_{\star,H^{1/2}(\Gamma)} &= |V\widetilde{w}|^2_{\star,H^{1/2}(\Gamma)} + \tfrac{1}{\mathrm{diam}(\Omega^{\mathrm{int}})}\, \|\mathcal{H}^{\mathrm{int}} V w\|^2_{L^2(\Omega^{\mathrm{int}})} \\ &\le |V\widetilde{w}|^2_{\star,H^{1/2}(\Gamma)} + \tfrac{2}{\mathrm{diam}(\Omega^{\mathrm{int}})^2} \Big\{ \|\mathcal{H}^{\mathrm{int}} V\widetilde{w}\|^2_{L^2(\Omega^{\mathrm{int}})} + \|\mathcal{H}^{\mathrm{int}} V w_0\|^2_{L^2(\Omega^{\mathrm{int}})} \Big\}\,. \end{aligned}$$

Since $\mathcal{H}^{\mathrm{int}}$ extends a constant function on Γ to the same constant in Ω^{int},

$$\|\mathcal{H}^{\mathrm{int}} V w_0\|^2_{L^2(\Omega^{\mathrm{int}})} \;=\; |\Omega^{\mathrm{int}}|\, \langle w,\, \mathbf{1}_\Gamma\rangle^2\, (V w_{\mathrm{eq}})^2\,.$$

Using this identity, the estimate above, the fact that $V\widetilde{w} \in H^{1/2}_*(\Gamma)$, and Lemma 6.9, we obtain

$$\|Vw\|^2_{\star,H^{1/2}(\Gamma)} \;\le\; (1 + 2\, C_P^*)\, |V\widetilde{w}|^2_{\star,H^{1/2}(\Gamma)} + \tfrac{2\, |\Omega^{\mathrm{int}}|\, V w_{\mathrm{eq}}}{\mathrm{diam}(\Omega^{\mathrm{int}})^2}\, \langle w,\, \mathbf{1}_\Gamma\rangle^2\, V w_{\mathrm{eq}}\,. \tag{6.8}$$

By identity (6.3), we see that

$$\begin{aligned} |V\widetilde{w}|^2_{\star,H^{1/2}(\Gamma)} &= |\mathcal{H}^{\mathrm{int}} V\widetilde{w}|^2_{H^1(\Omega^{\mathrm{int}})} \;\le\; |\widetilde{V}\widetilde{w}|^2_{H^1(\Omega^{\mathrm{int}})} \\ &\le |\widetilde{V}\widetilde{w}|^2_{H^1(\Omega^{\mathrm{int}})} + |\widetilde{V}\widetilde{w}|^2_{H^1(\Omega^{\mathrm{ext}})} \;=\; \langle \widetilde{w},\, V\widetilde{w}\rangle\,. \end{aligned}$$

Combining this estimate with (6.8) and (6.7) yields

$$\|Vw\|^2_{\star,H^{-1/2}(\Gamma)} \;\le\; (1 + 2\, C_P^*)\, \langle \widetilde{w},\, V\widetilde{w}\rangle + \tfrac{2\, |\Omega^{\mathrm{int}}|\, V w_{\mathrm{eq}}}{\mathrm{diam}(\Omega^{\mathrm{int}})^2}\, \langle w_0,\, V w_0\rangle\,.$$

In three dimensions, we see from the proof of Lemma 6.9 that $\frac{V w_{\mathrm{eq}}\, |\Omega^{\mathrm{int}}|}{\mathrm{diam}(\Omega^{\mathrm{int}})^2} \;\le\; C_P^*$. Altogether, this implies the estimates

$$\begin{aligned} \|Vw\|^2_{\star,H^{-1/2}(\Gamma)} &\le \widetilde{C}_V^\star\, \langle w,\, Vw\rangle \qquad && \forall w \in H^{-1/2}_*(\Gamma)\,, \\ \|Vw\|^2_{\star,H^{-1/2}(\Gamma)} &\le C_V^\star\, \langle w,\, Vw\rangle \qquad && \forall w \in H^{-1/2}(\Gamma)\,. \end{aligned}$$

The second estimate in the statement of Lemma 6.11 can be obtained by standard duality arguments. $\square$

By standard duality arguments, the operator V^{-1} exhibits certain coercive properties where the coercivity constant is $(C_V^\star)^{-1}$ or $(\widetilde{C}_V^\star)^{-1}$, similary to Corollary 6.2,

6.4. **Estimates for the constant c_0 and the contraction constant c_K.**

Corollary 6.12 ($H^{1/2}_*$-coercivity of D)**.** *We have that*

$$\langle D\, v,\, v\rangle \;\ge\; \frac{c_D^\star}{1 + C_P^*}\, \|v\|^2_{\star,H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}_*(\Gamma)\,,$$

with the constant C_P^* defined according to Lemma 6.9.

Proof. By our Poincaré type inequality in $H^{1/2}_*(\Gamma)$, we easily obtain that for $v \in H^{1/2}_*(\Gamma)$,

$$\|v\|^2_{\star,H^{1/2}(\Gamma)} \;\le\; (1 + C^*_P)\,|v|^2_{\star,H^{1/2}(\Gamma)}\,.$$

An application of Lemma 6.5 finally proves the statement. □

Corollary 6.13. *In case that V is not fully coercive, let V^{-1} denote the well-defined inverse of V restricted to the space $H^{-1/2}_*(\Gamma)$. Then,*

$$c_0 \;:=\; \inf_{v\in H^{1/2}_*(\Gamma)} \frac{\langle D\,v,\, v\rangle}{\langle V^{-1}\,v,\, v\rangle} \;\ge\; \frac{\widetilde{c}^{\star}_V\, c^{\star}_D}{1 + C^*_P}\,.$$

Proof. The proof follows immediately from Corollary 6.4 and Lemma 6.5. □

Remark 6.14. The above corollary states that in three dimensions, the constant c_0 depends only on the Jones parameters and Poincaré constants of Ω^{int} and $B_R \setminus \overline{\Omega}^{\mathrm{int}}$. Consequently, the contraction constant c_K only depends on these constants. In two dimensions, we have not given an explicit bound for the constant C^*_P ; we have only sketched how such one might be obtained, cf. Remark 6.10.

Remark 6.15. Assume that $V w_{\mathrm{eq}} > 0$ and let $S^{\mathrm{int}} := D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)$ denote the interior Steklov Poincaré operator. For a triangulation $\mathcal{T}_h$ of Γ, denote Z_h the space of piecewise constant function with respect to $\mathcal{T}_h$, and define the (symmetric) approximation $\widetilde{S}^{\mathrm{int}}v := D\,v + (\frac{1}{2}I + K')w_h$ where $\langle \tau_h,\, V w_h\rangle = \langle \tau_h,\, (\frac{1}{2}I + K)v\rangle$ for all $\tau_h \in Z_h$. One can show that

$$\frac{c_0}{c_K}\,\langle S^{\mathrm{int}}v,\, v\rangle \;\le\; \langle \widetilde{S}^{\mathrm{int}}v,\, v\rangle \;\le\; \langle S^{\mathrm{int}}v,\, v\rangle \qquad \forall v \in H^{1/2}(\Gamma),$$

cf. [32]. This means we have a shape-explicit spectral equivalence between the original and the approximated operator. It can be used in the analysis of domain decomposition methods [14, 19, 20, 25, 26] and of BEM-based finite element methods [3].

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Institute of Applied Geometry

Address:

Johannes Kepler University Linz
Doctoral Program "Computational Mathematics"
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-7174

E-Mail:

office@dk-compmath.jku.at

Homepage:

http://www.dk-compmath.jku.at