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Fast solvers and a posteriori error estimates in elastoplasticity

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Abstract The paper reports some results on computational plasticity obtained within the Special Research Program "Numerical and Symbolic Scientific Computing" and within the Doctoral Program "Computational Mathematics" both supported by the Austrian Science Fund FWF under the grants SFB F013 and DK W1214, respectively. Adaptivity and fast solvers are the ingredients of efficient numerical methods. The paper presents fast and robust solvers for both 2D and 3D plastic flow theory problems as well as different approaches to the derivations of a posteriori error estimates. In the last part of the paper higher-order finite elements are used within a new plastic-zone concentrated setup according to the regularity of the solution. The theoretical results obtained are well supported by the results of our numerical experiments.

1 Introduction

The theory of plasticity has a long tradition in the engineering literature. These classical results on plasticity together with the introduction of the Finite Element Method (FEM) into engineering computations provides the basis for the modern computational plasticity (see [56] and the references therein). The rigorous mathematical analysis of plastic flow theory problems and of the numerical methods for their solution started in the late 70ies and in the early 80ies by the work of C. Johnson [31, 32], H. Matthies [41, 42], V.G. Korneev and U. Langer [40], and others. Since then many mathematical contributions to Computational Plasticity have been made. We here only refer to the monographs by J.C. Simo and T.J.R. Hughes [51] and W. Han and B.D. Reddy[29], to the habilitation theses by C. Carstensen [12] and C. Wieners [55], to the collection [52], and the references given therein.

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The incremental elastoplasticity problem can be reformulated as a minimization problem for a convex but not-smooth functional, where the unknowns are the displacements u and the plastic strains p. One method to deal with this non-smoothness relies on regularization techniques which were initially studied in [35]. However, eliminating the plastic strains p and using Moreau's theorem, we see that the reduced functional, that is now only a functional in the displacements u, is actually continuously Fréchet differentiable. The elimination of the plastic strains can be done locally and with the help of symbolic techniques. Unfortunately, the second derivative of the reduced functional does not exist. As a remedy, the concept of slanting functions, introduced by X. Chen, Z. Nashed, and L. Qi in [17], allows us to construct and analyze generalized Newton methods which show fast convergence in all our numerical experiments. More precisely, we can proof super-linear convergence of these generalized Newton methods at least in the finite element setting.

The second part of this paper is devoted to the a posteriori error analysis of elastoplastic problems. Two different techniques were developed: the first one is exploring a residual-type estimator respecting certain oscillations, and the second one is based on functional a posteriori estimates introduced by S. Repin [46].

Finally, we consider spatial discretizations of the incremental plasticity problems based on hp finite element techniques. A straightforward application of the classical h-FEM yields algebraic convergence. However, the regularity results presented in [6, 39], namely H^2_{loc} regularity of the displacements in the whole domain, and C^∞ regularity apart from plastic zones and the boundary of the computational domain, justify the application of high order finite element methods in the elastic part, but not necessarily in the plastic part. A few hp-adaptive strategies, as well as a related technique, the so-called Boundary Concentrated Finite Element Method (BC-FEM) introduced by B.N. Khoromskij and J.M. Melenk [33], are discussed in this paper.

The rest of the paper is organized as follows: In Section 2, we describe the initial-boundary value problem of elastoplasticity which is studied in this paper. Section 3 is devoted to the incremental elastoplasticity problems and strategies for their solution. In Section 4 we derive a posteriori error estimates which can be used in the adaptive h-FEM providing an effective spatial discretization in every incremental step. Section 5 deals with the use of the hp-FEM in elastoplasticity. Finally, we draw some conclusions.

2 Modeling of elastoplasticity

There are many mathematical models describing the elastoplastic behavior of materials under loading. In this paper we follow the description given by C. Carstensen in [12, 13, 14, 15]. The classical equations of elastoplasticity can be found in the standard literature on plasticity, see, e.g., [51, 29]. Let us first recall these describing relations. Let $\Theta := [0,T]$ be a (pseudo) time interval, and let Ω be a bounded domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. In the quasi-static case which is considered throughout this paper, the equilibrium of forces reads as follows

$$-\operatorname{div}(\sigma(x,t)) = f(x,t) \qquad \forall\,(x,t) \in \Omega \times \Theta\,, \tag{1}$$

where $\sigma(x,t) \in \mathbb{R}^{3\times 3}$ is called Cauchy's stress tensor and $f(x,t) \in \mathbb{R}^3$ represents the volume force acting at the material point $x \in \Omega$ at the time $t \in \Theta$. Let $u(x,t) \in \mathbb{R}^3$ denote the displacements of the body, and let

$$\varepsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \tag{2}$$

be the linearized Green-St. Venant strain tensor. In elastoplasticity, the total strain ε is additively split into an elastic part e and a plastic part p, that is,

$$\varepsilon = e + p\,. \tag{3}$$

We assume a linear dependence of the stress on the elastic part of the strain, which is defined by Hooke's law

$$\sigma = \mathbb{C}\,e. \tag{4}$$

Since we assume the material to be isotropic, the single components of the elastic stiffness tensor $\mathbb{C} \in \mathbb{R}^{3\times3\times3\times3}$ are defined by $\mathbb{C}_{ijkl} := \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Here, $\lambda > 0$ and $\mu > 0$ denote the Lamé constants, and δ_{ij} the Kronecker symbol.

Let the boundary Γ be split into a Dirichlet part Γ_D and a Neumann part Γ_N such that $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$. We assume the boundary conditions

$$u = u_D \;\text{ on } \Gamma_D \quad \text{and} \quad \sigma \cdot n = g \;\text{ on } \Gamma_N\,, \tag{5}$$

where $n(x,t)$ denotes the exterior unit normal, $u_D(x,t) \in \mathbb{R}^3$ denotes a prescribed displacement and $g(x,t) \in \mathbb{R}^3$ denotes a prescribed traction. If $p = 0$ in (3), the system (1) – (5) describes the linear elastic behavior of the continuum Ω.

Two more properties, incorporating the admissibility of the stress σ with respect to a certain hardening law and the time evolution of the plastic strain p, are required to describe the plastic behavior of some body Ω. Therefore, we introduce the hardening parameter α and define the generalized stress (σ,α), which we call admissible if for a given convex yield functional ϕ the inequality

$$\phi(\sigma,\alpha) \leq 0\,. \tag{6}$$

holds. The explicit form of ϕ depends on the choice of the hardening law, see, e.g., formula (9) for isotropic hardening. The second, specifically elastoplastic, property addresses the time development of the generalized plastic strain $(p,-\alpha)$ that is described by the normality rule

$$\langle(\dot{p},-\dot{\alpha})\,,(\tau,\beta)-(\sigma,\alpha)\rangle_F \leq 0 \qquad \forall\,(\tau,\beta) \text{ which satisfy } \phi(\tau,\beta) \leq 0\,, \tag{7}$$

where $\dot{p}$ and $\dot{\alpha}$ denote the first time derivatives of p and α, respectively. Therefore, we need initial conditions, which read as follows

$$p(x,0) = p_0(x) \quad \text{and} \quad \alpha(x,0) = \alpha_0(x) \qquad \forall x \in \Omega\,, \tag{8}$$

with given initial values $p_0 : \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $\alpha_0 : \Omega \to [0,\infty[$.

Problem 1 (classical formulation). Find (u, p, α), which satisfies (1) – (8).

In this paper we concentrate on the isotropic hardening law, where the hardening parameter α is a scalar function $\alpha : \Omega \to \mathbb{R}$ and the yield functional ϕ is defined by

$$\phi(\sigma,\alpha) := \begin{cases} \|\mathrm{dev}\,\sigma\|_F - \sigma_y(1 + H\alpha) & \text{if } \alpha \geq 0, \\ +\infty & \text{if } \alpha < 0. \end{cases} \tag{9}$$

Here, the Frobenius norm $\|A\|_F := \langle A\,,A\rangle_F^{1/2}$ is defined by the matrix scalar product $\langle A\,,B\rangle_F := \sum_{ij} a_{ij} b_{ij}$ for $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$ and $B = (b_{ij}) \in \mathbb{R}^{3\times 3}$. The deviator is defined for square matrices by $\mathrm{dev} A = A - \frac{\mathrm{tr} A}{\mathrm{tr} I} I$, where the trace of a matrix is defined by $\mathrm{tr} A = \langle A\,,I\rangle_F$ and I denotes the identity matrix. The real material constants $\sigma_y > 0$ and $H > 0$ are called yield stress and modulus of hardening, respectively.

3 The incremental elastoplasticity problems and solvers

We turn to the specification of proper function spaces. For a fixed time $t \in \Theta$, let

$$u \in V := \left[H^1(\Omega)\right]^3, \quad p \in Q := [L_2(\Omega)]^{3\times 3}_{\mathrm{sym}}, \quad \alpha \in L_2(\Omega)\,.$$

We define the hyperplane $V_D := \{v \in V \mid v_{|\Gamma_D} = u_D\}$ and the test space $V_0 := \{v \in V \mid v_{|\Gamma_D} = 0\}$, and the associated scalar products and norms as follows:

$$\langle u\,,v\rangle_V := \int_\Omega \left(u^T v + \langle \nabla u\,, \nabla v\rangle_F\right)\,\mathrm{d}x\,, \qquad \|v\|_V := \langle v\,,v\rangle_V^{1/2}\,,$$
$$\langle p\,,q\rangle_Q := \int_\Omega \langle p\,,q\rangle_F\,\mathrm{d}x\,, \qquad \|q\|_Q := \langle q\,,q\rangle_Q^{1/2}\,.$$

Starting from Problem 1, one can derive a uniquely solvable time dependent variational inequality for unknown displacement $u \in \{v \in H^1(\Theta;V) \mid v_{|\Gamma_D} = u_D\}$ and plastic strain $p \in H^1(\Theta;Q)$ (see [29, Theorem 7.3] for details). However, the numerical treatment requires a time discretization of this variational inequality. Therefore, we pick a fixed number of time ticks $0 = t_0 < t_1 < \ldots < t_{N_\Theta} = T$ out of Θ. We introduce the notation

$$u_k := u(t_k)\,,\;\; p_k := p(t_k)\,,\;\; \alpha_k := \alpha(t_k)\,,\;\; f_k := f(t_k)\,,\;\; g_k := g(t_k)\,,\;\; \ldots\,,$$

and approximate time derivatives by the backward difference quotients

$$\dot{p}_k \approx (p_k - p_{k-1})\,/\,(t_k - t_{k-1}) \quad \text{and} \quad \dot{\alpha}_k \approx (\alpha_k - \alpha_{k-1})\,/\,(t_k - t_{k-1})\,.$$

Consequently, the time dependent problem is approximated by a sequence of time independent variational inequalities of the second kind. Each of these variational inequalities can be equivalently expressed by a minimization problem, which by definition of the set of extended real numbers, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, reads [14, Example 4.5]:

Problem 2. Find $(u_k, p_k) \in V_D \times Q$ such that $J_k(u_k, p_k) = \inf_{(v,q)\in V_D\times Q} J_k(v,q)$, where $J_k : V_D \times Q \to \overline{\mathbb{R}}$ is defined by

$$J_k(v,q) := \frac{1}{2}\|\varepsilon(v) - q\|_{\mathbb{C}}^2 + \psi_k(q) - l_k(v)\,, \tag{10}$$

with

$$\langle q_1\,,\, q_2\rangle_{\mathbb{C}} := \int_\Omega \langle \mathbb{C}\, q_1(x)\,,\, q_2(x)\rangle_F \,\mathrm{d}x\,, \qquad \|q\|_{\mathbb{C}} := \langle q\,,\, q\rangle_{\mathbb{C}}^{\frac{1}{2}}\,, \tag{11}$$

$$\tilde{\alpha}_k(q) := \alpha_{k-1} + \sigma_y H \|q - p_{k-1}\|_F\,, \tag{12}$$

$$\psi_k(q) := \begin{cases} \int_\Omega \left(\frac{1}{2}\tilde{\alpha}_k(q)^2 + \sigma_y\|q - p_{k-1}\|_F\right)\,\mathrm{d}x & \text{if } \operatorname{tr}(q - p_{k-1}) = 0\,, \\ +\infty & \text{else}\,, \end{cases} \tag{13}$$

$$l_k(v) := \int_\Omega f_k \cdot v \,\mathrm{d}x + \int_{\Gamma_N} g_k \cdot v \,\mathrm{d}s\,. \tag{14}$$

The convex functional J_k expresses the mechanical energy of the deformed system at the kth time step. It is smooth with respect to the displacements v, but not with respect to the plastic strains q. Notice, that no minimization with respect to the hardening parameter α_k is necessary. It is computed in the post-processing by $\alpha_k = \tilde{\alpha}_k(p_k)$, with $\tilde{\alpha}_k$ defined as in (12). A short summary on the modeling of Problem 2 starting from the classical formulation can be found in [38]. The problem is uniquely solvable due to [22, Proposition 1.2 in Chapter II].

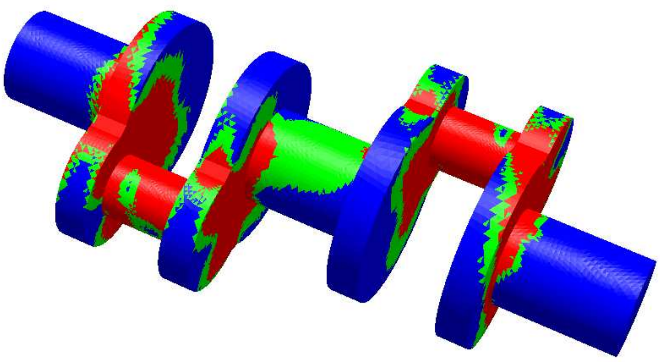

Fig. 1 Example of two-yield plasticity distribution.

J. Valdman together with M. Brokate and C. Carstensen published results on the analysis [10] and numerical treatment [11] of multi-yield elastoplastic models based on the PhD-thesis of J. Valdman [53] and its extension. The main feature of the multi-yield models is a higher number of plastic strains $p_1, \ldots, p_N$ used for more realistic modeling of the elastoplastic-plastic transition. It was possible to prove the existence and uniqueness of the corresponding variational inequalities and design a FEM based solution algorithm. Since the structure of the minimization functional in the multi-yield plasticity model remains the same as for the single-yield model, it was possible to prove the existence and uniqueness of the corresponding variational inequalities and design a FEM based solution algorithm. In terms of a software development, an existing elastoplasticity package [34], written as a part of the NETGEN/NGSolve software of J. Schöberl, was modified to make the computations of a two-yield elastoplastic problem feasible [37]. Figure 1 displays elastic (blue), first (red) and second (green) plastic deformational zones of the shaft model. The numerical treatment of the two-yield problem requires to resolve the plastic-strain increment matrices P_1 and P_2 from a local minimization problem with a convex but non-smooth functional. Since there are typically millions of such minimizations, iterative techniques such as alternating minimizations, Newton based methods or even partially exact analytical solutions were studied in [30].

The first class of algorithms is based on a regularization of the objective, where the modulus is smoothed for making the objective $J_k^{(\delta)}$ twice differentiable. Figure 2 shows the modulus $|p| := \|p_k - p_{k-1}\|_F$ and possible regularizations $|p|^{(\delta)}$ depending on the regularization parameter δ, where δ is here chosen as 10^{-6}. The quadratic regularization has a smooth first derivative within the interval $(-\delta, \delta)$, but the second derivative is piecewise constant and discontinuous. Thus, the local quadratic convergence of Newton type methods cannot be guaranteed. The piecewise cubic spline has a piecewise linear continuous second derivative. Thus, Newton type methods can be applied. As a final choice of regularization, the cubic spline function is shifted to the origin, so that $|p|^{(\delta)} = 0$ holds for $p = 0$.

For instance, in case of a quadratic regularization (green), we have

$$|p|_\delta := \begin{cases} |p| & \text{if } |p| \geq \delta, \\ \frac{1}{2\delta}|p|^2 + \frac{\delta}{2} & \text{if } |p| < \delta, \end{cases}$$

with a small regularization parameter $\delta > 0$.

The algorithm is based on alternating minimization with respect to the two variables, and on the reduction of the objective to a quadratic functional with respect to the plastic strains. This can be interpreted as a linearization of the nonlinear elastoplastic problem.

The minimization problem with respect to the plastic part of the strain is separable and the analytical solution $p^{(\delta)}(u)$ can be calculated in explicit form. Problem 2 formally reduces to

$$J_k^{(\delta)}(u) = \min_v J_k^{(\delta)}(v, p^{(\delta)}(v)). \tag{15}$$

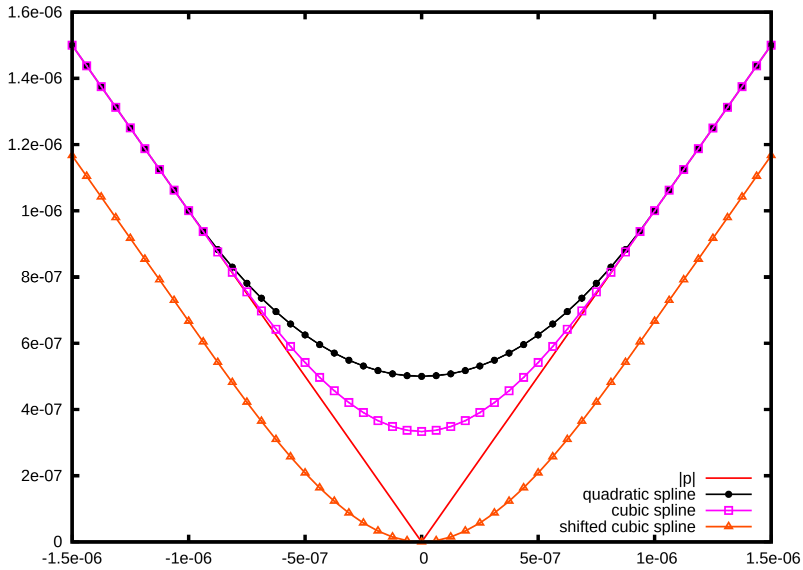

Fig. 2 Plot of $|p|$ and its regularizations.

After the finite element (FE) discretization and the elimination of plastic strains, the FE displacement field results from the solution of a linear Schur complement system. The solution of this linear system can efficiently be computed by a multi-grid preconditioned conjugate gradient solver, see [37, 36].

Using Moreau's theorem, that is well known in the scope of convex analysis [44], we can avoid the regularization of the original functional J_k. The formula for minimizing $J_k(u,p)$ with respect to the plastic strain p for a given displacement u is explicitly known [2], i. e., we know a function $\tilde{p}_k(\varepsilon(u))$, such that there holds

$$J_k(u) := J_k(u, \tilde{p}_k(\varepsilon(u))) = \inf_q J_k(u,q)\,.$$

In detail, the plastic strain minimizer reads as follows

$$\tilde{p}_k(\varepsilon(v)) = \xi \max\{0, \|\operatorname{dev}\sigma_k(\varepsilon(v))\|_F - \sigma_y\} \frac{\operatorname{dev}\sigma_k(\varepsilon(v))}{\|\operatorname{dev}\sigma_k(\varepsilon(v))\|_F} + p_{k-1}\,, \qquad (16)$$

with the constant $\xi := \left(1+\sigma_y^2 H^2\right)^{-1}$, the trial stress $\sigma_k(\varepsilon(v)) := \mathbb{C}\,(\varepsilon(v) - p_{k-1})$, and the deviatoric part $\operatorname{dev}\sigma := \sigma - (\operatorname{tr}\sigma/3)\,I$. Thus, it remains to solve a minimization problem with respect to one variable only, i.e. $J_k(u) \to \min$. The theorem of Moreau says, that, due to the specific structure of $J_k(u,p)$, the functional $J_k(u)$ is continuously Fréchet differentiable and strictly convex. Moreover, the explicit form of the derivative is also provided. The Gâteaux differential is given by the relation

$$\mathrm{D}\,J_k(v;w) = \langle \varepsilon(v) - \tilde{p}_k(\varepsilon(v))\,,\,\varepsilon(w)\rangle_{\mathbb{C}} - l_k(w)\,.$$

Hence, it suffices to find u such that the first derivative of F vanishes. This approach was first discussed in the master thesis [24] by P.G. Gruber. Several numerical examples can also be found in [23, 27, 28].

The second derivative of J_k does not exist. As a remedy, the concept of slanting functions, introduced by X. Chen, Z. Nashed, and L. Qi in [17], allows the application of the following Newton-like method: Let $v^0 \in V_D$ be a given initial guess for the displacement field. Then, for $j = 0,1,2,\ldots$ and given v^j, find $v^{j+1} \in V_D$ such that

$$(\mathrm{D}\,J_k)^o\,(v^j;v^{j+1} - v^j,w) = -\mathrm{D}\,J_k(v^j;w)$$

holds for all $w \in V_0$, where the slanting function of $\mathrm{D}\,J_k$ is defined by the identity

$$(\mathrm{D}\,J_k)^o\,(v\,;w_1,w_2) = \langle \varepsilon(w_1) - \tilde{p}_k{}^o(\varepsilon(v)\,;\,\varepsilon(w_1))\,,\,\varepsilon(w_2)\rangle_{\mathbb{C}} \quad \forall\, w_1,w_2 \in V_0\,.$$

Here, $\tilde{p}_k{}^o$ denotes the slanting function of $\tilde{p}_k$ (16), which, by using the definition $\beta_k(\varepsilon(v)) := 1 - \sigma_y\,\|\mathrm{dev}\,\sigma_k(\varepsilon(v))\|_F^{-1}$, and the abbreviations $\beta_k(\varepsilon(v)) := \beta_k$ and $\sigma_k(\varepsilon(v)) := \sigma_k$, reads

$$\tilde{p}_k{}^o(\varepsilon(v)\,;\,q) = \begin{cases} 0 & \text{if } \beta_k \le 0\,, \\ \xi\left(\beta_k\,\mathrm{dev}\,q + (1-\beta_k)\,\frac{\langle \mathrm{dev}\,\sigma_k\,,\,\mathrm{dev}\,q\rangle_F}{\|\mathrm{dev}\,\sigma_k\|_F^2}\,\mathrm{dev}\,\sigma_k\right) & \text{else}\,. \end{cases}$$

Utilizing this concept, P. G. Gruber and J. Valdman were able to prove the local super-linear convergence of the resulting Newton-like solver in the spatial discretized case (see Table 1), and formulate sufficient regularity conditions which would guarantee super-linear convergence in the non-discretized case [28, 27]. An extension of the numerical solver to other kinds of time-dependent models with internal variables, as discussed in [26], is possible and left for future investigation.

The slant Newton method is tested on a benchmark problem in computational plasticity [52]. The left plot of Figure 3 shows the mesh for the right upper quarter of a plate with geometry $(-10,10)\times(-10,10)\times(0,2)$ and a circular hole of the radius $r = 1$ in the middle. One elastoplastic time step is performed, where a surface load g with the intensity $|g| = 450$ is applied to the plate's upper and lower edge in outer normal direction. Due to the symmetry of the domain, the solution is calculated on one quarter of the domain only. Thus, homogeneous Dirichlet boundary conditions in the normal direction (gliding conditions) are considered for both symmetry axes. The material parameters are set to

$$\lambda = 1.1074 * 10^5\,, \quad \mu = 8.0194 * 10^4\,, \quad \sigma_y = 450\,\sqrt{2/3}\,, \quad H = 0.5\,.$$

Differently to the original problem in [52], the modulus of hardening H is nonzero, i. e. hardening effects are considered. The numerical results for the original problem (H=0) can be found in [23]. The two plots in Figure 3 show the coarsest tetrahedral FE-mesh with the applied traction g (left), and the Frobenius norm of the plastic strain field p (right) on a finer mesh for this three dimensional problem. Table 1 outlines the convergence of the slant Newton method, where the initial values for the displacement are chosen to be zero at each level of refinement. We observe super-linear convergence with respect to a Cauchy test and a constant number of iterations at each refinement level. The implementation was done in C++ using the NETGEN/NGSolve software package developed by J. Schöberl [49].

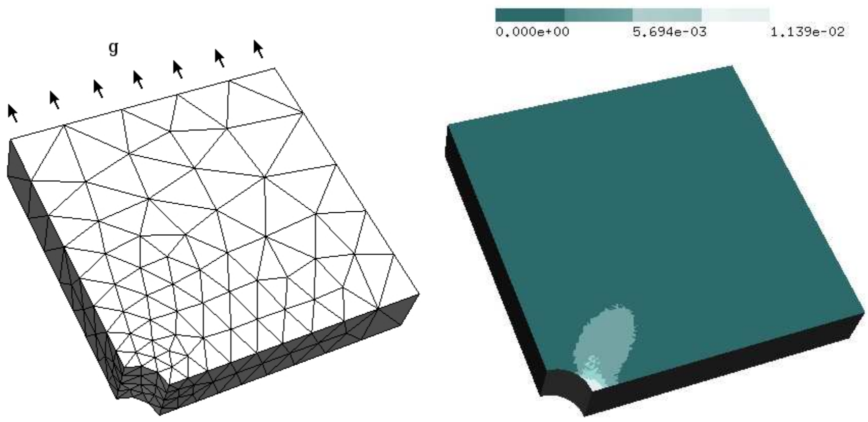

Fig. 3 Coarsest triangulation (left) and the Frobenius norm of the plastic strain field p (right).

dof:	717	5736	45888	367104
0-1	1.000e+00	1.000e+00	1.000e+00	1.000e+00
1-2	1.013e-01	1.254e-01	1.367e-01	1.419e-01
2-3	7.024e-03	6.919e-03	7.159e-03	6.993e-03
3-4	1.076e-04	9.359e-05	1.263e-04	1.176e-04
4-5	2.451e-08	6.768e-07	1.744e-06	1.849e-06
5-6	7.149e-15	6.887e-12	4.874e-09	1.001e-08
6-7			4.298e-13	2.368e-14

Table 1 Convergence behavior of the slant Newton method for different refinement levels.

4 Adaptive h-FEM and a posteriori error estimates for elastoplasticity

The efficient numerical treatment of problems with poor regularity of the solution can be realized with adaptive mesh refinement techniques based on a posteriori error estimators. An $h-$finite element adaptive algorithm consists of successive loops of the form

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE} \tag{17}$$

designed to produce more efficient meshes by targeted local refinements with less computational effort. The a posteriori error analysis of (17) started with the pioneering work of [4] for a two-point elliptic boundary value problem and with the step MARK realized by the max refinement rule. This marking rule currently employed in the engineering literature consists in looking at the elements with the largest error and refining these in order to achieve a better accuracy. Let $\eta^2 := \sum_M \eta_M^2$ denote a typical reliable error estimator with local contributions η_M associated with an edge, face, or element M in the current mesh, the max refinement rule marks a subset $\mathscr{M}$ according to

$$L \in \mathscr{M} \text{ if and only if } \eta_L \geq \Theta \max_M \eta_M \tag{18}$$

with $0 \leq \Theta \leq 1$. The analysis of [4], however, does not provide information on the convergence rate and its extension to higher dimensions still remains unsolved. It is only after the contribution of Dörfler [20] with the introduction of a new marking strategy for error reduction (hereafter referred to as bulk criterion or fixed fraction criterion) that the convergence analysis of AFEMs has experienced significant development. With such criterion, one defines the set $\mathscr{M}$ of the marked objects using the rule

$$\sum_{M \in \mathscr{M}} \eta_M^2 \geq \Theta\, \eta^2 \tag{19}$$

with $0 \leq \Theta \leq 1$. The condition (19) together with local discrete efficiency estimates, and the Galerkin orthogonality yields a linear error reduction rate for the energy norm towards a preassigned tolerance TOL in finite steps for the Poisson problem.

In [16], a proof of convergence of AFEM with indication of the rate of convergence for the primal formulation of plasticity is provided under the application of the bulk criterion (19). Applications include several plasticity models: linear isotropic-kinematic hardening, linear kinematic hardening, multi-surface plasticity as model for nonlinear hardening laws, and perfect plasticity. Exploiting properties of a non-differentiable energy functional J, and the reliability of a new edge-based residual error estimate, we obtain the following results:

(i) *Energy reduction*: for some data oscillations $\mathrm{osc}_\ell^2 \geq 0$ and positive constants ρ_E, C with $\rho_E < 1$ there holds

$$J(w_{\ell+1}) - J(w) \leq \rho_E (J(w_\ell) - J(w)) + C\,\mathrm{osc}_\ell^2\,.$$

Here, $J(w)$ denotes a minimal energy and $J(w_\ell)$ and $J(w_{\ell+1})$ are energies on refined triangulations $\mathscr{T}_\ell$ and $\mathscr{T}_{\ell+1}$.

(ii) *$R-$linear convergence for the stresses*: up to oscillation terms there holds

$$|||\sigma - \sigma_\ell|||_{\mathbb{C}^{-1};\Omega} \leq \alpha_\ell \quad \text{for } \ell = 0,1,2\cdots,$$

with $\alpha_\ell \to 0$ and linear convergent, and $|||\cdot|||_{\mathbb{C}^{-1};\Omega}$ the energy norm induced by the Hook tensor $\mathbb{C}$. Here, σ denotes the stress on an exact solution and σ_ℓ ist approximation on the triangulation $\mathscr{T}_\ell$.

In [48], the framework introduced in the book [45] is applied to elastoplasticity, where the estimates are derived by the analysis of the variational problem and its dual counterpart. A computable upper bound of the error is obtained on a purely functional level without exploitation of specific properties of the approximation or the method used for its computation. Estimates of such a type are often called “functional a posteriori estimates”. Application to linear isotropic hardening allows us to express another reliability estimate

$$\frac{1}{2}|||w-v|||^2 \leq \mathscr{M}(v,\tau,\lambda) \tag{20}$$

which bounds an error of a discrete solution v, i.e. its distance from the exact solution w by an expression on the right-hands side called a functional majorant $\mathscr{M}(v,\tau,\lambda)$. The functional majorant can be generally minimized with respect to free parameters τ,λ to keep the estimate (20) as sharp as possible. Numerical verification of this estimate will be the topic of the forthcoming paper, where it should be profited from the experience in problems with nonlinear boundary conditions [47] and an application of a multigrid preconditioned solver to a majorant computation [54].

5 High order FEM for elastoplasticity: hp-FEM and BC-FEM

In nowadays computer simulations of elastoplasticity, adaptive h-FEM (as presented in Section 4) is probably the most propagated and well known discretization technique. However, as computers become faster, and parallelization is no longer just a scientific topic, the mixture of low and high order finite element methods (hp-FEM) becomes more and more attractive in daily practice. Applying a high order method means to increase the polynomial degree of the shape functions on an element instead of refining it. The major drawback of a high order method is the expensive assembling of the system matrix. As long as this handicap can be settled (e.g., by finding recurrences via symbolic computation [5, 8, 9]), the application of such methods are definitely worth their price. The idea of hp-FEM [3, 50] is to increase the polynomial degree locally on elements, where the solution has high regularity. In such areas of the domain we can expect local exponential convergence of the approximate towards the solution. On other elements, i. e. where the regularity is low, mesh refinement is applied, which locally yields algebraic convergence. Moreover, by choosing proper hp-adaptive refinement strategies, an exponential convergence rate can be achieved globally [3].

In elastoplasticity, the solution in each time step is known to be in $H^2_{\mathrm{loc}}(\Omega)$, and, moreover, analytic in balls where the plastic strain p vanishes [39, 6]. Thus, the application of an hp-FEM is a natural choice. In those parts of the interior domain, where the material reacts purely elastic, the polynomial degree is increased, whereas the mesh is refined in plastic areas and towards rough boundary data or geometry.

The basic hp-adaptive algorithm reads as follows:

Algorithm 1 The hp-adaptive Algorithm:

Require: A mesh $\mathscr{T}$, a polynomial degree vector $(p_K)_{K\in\mathscr{T}}$, a Finite Element Solution u_{FE}.
Ensure: A refined mesh $\mathscr{T}_{\mathrm{ref}}$, a new polynomial degree vector $(p_K)_{K\in\mathscr{T}_{\mathrm{ref}}}$.

1: Determine which elements to refine $\to \mathscr{T}_h$.
2: Determine where the polynomial degree should be increased $\to \mathscr{T}_p$.
3: Obtain a preliminary refined mesh $\to \mathscr{T}'_{\mathrm{ref}}$.
4: Elimination of hanging nodes $\to \mathscr{T}_{\mathrm{ref}}$.
5: Increase the polynomial degree $p_K = p_K + 1$ for all elements $K \in \mathscr{T}_{\mathrm{ref}} \cap \mathscr{T}_p$. In particular: Elements to which an h-refinement is applied inherit the polynomial degree from their father.

Note, that Items 3–5 are straight forward, whereas, one still has to decide on the exact realization of Items 1 and 2. In general, the set of all adaptive strategies divides into two classes: strategies which are problem dependent, and those which are not. In problem dependent strategies, the decision whether to refine in h, or in p, or not at all, relies on the evaluation of problem dependent quantities, typically the error estimator. Algorithms of this class can be found, e.g., in [1, 21]. Problem independent algorithms, such as discussed in [18, 19], estimate the regularity of the solution without using problem dependent quantities.

Due to the lack of a reliable and efficient error estimator for elastoplasticity, the use of problem independent algorithms is a natural choice. The application of an algorithm presented in [21] to elastoplastic problems in two dimensions is discussed in [25]. This adaptive algorithm is based on the following idea:

Expressing the solution u to the (elastoplastic) problem as an expansion with respect to orthogonal Legendre polynomials

$$u = \sum_{p,q\in\mathbb{N}_0} u_{pq}\psi_{pq} \tag{21}$$

results in a sequence of coefficients u_{pq}, which decays exponentially if and only if the solution u is analytic:

Proposition 1. *Define on the reference triangle $\hat{K}$ the $L_2(\hat{K})$-orthogonal basis ψ_{pq}, $p,q \in \mathbb{N}_0$ by*

$$\psi_{pq} = \tilde{\psi}_{pq} \circ D^{-1}\,, \quad \tilde{\psi}_{pq} = P_p^{(0,0)}(\eta_1)\left(\frac{1-\eta_2}{2}\right)^p P_q^{(2p+1,0)}(\eta_2)\,,$$

where $P_p^{(\alpha,\beta)}$ is the (well known) p-th Jacobi polynomial with respect to the weight $\eta \mapsto (1-\eta)^\alpha(1+\eta)^\beta$ and D the Duffy transformation. Let $u \in L_2(\hat{K})$ be written *as in* (21)*. Then u is analytic on* $\overline{\hat{K}}$ *if and only if there exist constants* $C, b > 0$ *such that* $|u_{pq}| \leq Ce^{-b(p+q)}$ *for all* $p, q \in \mathbb{N}_0$.

Proof. See [43].

Since the true solution u is not available, the idea for the hp-adaptive algorithm is to estimate the decay of the coefficients u_{pq} of the expansion of the finite element solution $u_{FE|K} \circ F_K = \sum_{p,q} u_{pq} \psi_{pq}$ instead. If the decay is exponentially, then the polynomial degree p will be increased, otherwise, the mesh will be refined:

Algorithm 2 Realization of Items 1 and 2 in Algorithm 1:

Require: A mesh $\mathscr{T}$, a polynomial degree vector $(p_K)_{K\in\mathscr{T}}$, a parameter $b > 0$, a Finite Element Solution u_{FE}.
Ensure: The marked elements $\mathscr{T}_p$ and $\mathscr{T}_h$.

1: For all elements $K \in \mathscr{T}$ compute the expansion coefficients

$$u_{ij,K} = \|\psi_{ij}\|^{-2}_{L_2(\hat{K})} \langle u_{\mathrm{FE}|K} \circ F_K \, , \, \psi_{ij} \rangle_{L_2(\hat{K})}$$

for $0 \leq i + j \leq p_K$.

2: Estimate the decay coefficient b_K by a least squares fit of

$$\ln|u_{ij,K}| \approx C_K - b_K(i+j)\,.$$

3: Determine $\mathscr{T}_p = \{K \in \mathscr{T} \mid b_K \geq b\}$ and $\mathscr{T}_h = \{K \in \mathscr{T} \mid b_K < b\}$.

Additionally to the presented adaptive strategy in Algorithm 1, a different discretization approach applied to elastoplasticity is investigated in [25]. This approach is still of an hp-adaptive Finite Element type, but with a slightly different aim: Considering a general boundary value problem, where the regularity of the solution is known to be low at the boundary and high in the interior of the domain, the parameters h and p are chosen to be small in a neighborhood of the boundary and to be growing towards the interior of the domain. This growth is done in a manner, such that

- the convergence rate is of the same order as in h-FEM,
- and the number of total unknowns is proportional to the number of unknowns on the boundary (such as in BEM).

Due to the second property, the method is called a Boundary Concentrated Finite Element Method (BC-FEM) [33]. The method exploits the knowledge about the regularity of the solution in a way, that it searches for the smallest (and sparse) system which allows for the same convergence rate as is obtained in a classical h-FEM.

In elastoplasticity, BC-FEM can be applied for the purely elastic region, where the solution is known to be analytic [6], whereas the plastic region, where the solution is known to be in H^2_{loc} [39], is discretized by using h-FEM. However, the interface between plastic ($\|p\| > 0$) and elastic ($\|p\| = 0$) parts of the domain is not known in advance, since the calculation of the plastic strain field p relies on the displacement field, as it is pointed out in equation (16). Thus, one has to estimate, which parts of the domain will be plastic at the next step of refinement. This task can be again handled by Algorithm 2, due to the knowledge about the solutions regularity in the elastic and plastic parts of the domain. The resulting method has the same accuracy as a classical h-FEM, i.e. the error $\|u-u_h\|_{H^1(\Omega)} = O(h)$, but the number of degrees of freedom is significantly smaller: Considering h-FEM in two dimensions ($d=2$), the number degrees of freedom is roughly $O(N^2)$, with $N=h^{-1}$ denoting the number of nodes on the boundary of the domain, whereas in BC-FEM it is $O(N_E)+O(N_P^2)$, where N_E is the number of nodes on the boundary of the purely elastic sub-domain, and N_P the number of nodes on the boundary of the plastic sub-domain (compare Table 2). It is possible to generalize the primal and dual domain decomposition solvers proposed in [7] for solving interface-concentrated finite element equations to the plastic-zone concentrated finite element equations which we have to solve at each incremental step.

Finally, we present the results of following two numerical experiments:

- A **plate with a hole** $\{x \in [-10,10]^2 \ : \ \|x\| \geq 1\}$ is torn on the top and the bottom edge in normal direction with a traction of intensity $|g| = 450$. Due to the symmetry of the problem, only the top right quarter is considered in the numerical simulation. The material parameters are chosen as follows: Young's modulus $E = 20690$, Poisson ratio $\nu = 0.29$, yield stress $\sigma_y = 450\sqrt{2/3}$, and modulus of hardening $H = 0.1$. On the left of Figure 4 one can see the mesh after 5 steps of BC-refinement. The elements are colored from blue to red, indicating its polynomial degree. On the right of Figure 4, the elastic (blue) and plastic (red) zones are plotted. A zoom (Figure 5) shows the adaptive refinement towards the boundary and the elastoplastic interface. Plastic zones are red, elastic zones are blue.

- A **screw wrench** sticks on a screw (homogeneous Dirichlet condition) and is pressed down at its handheld in normal direction with an intensity $|g| = 1e6$. The material parameters are chosen as follows: Young's modulus $E = 2e8$, Poisson ratio $\nu = 0.3$, yield stress $\sigma_y = 1e6$, modulus of hardening $H = 0.01$. On top of Figure 6, one can see the mesh after 5 steps of BC-refinement. The elements are colored from blue to red, indicating its polynomial degree. On bottom of Figure 6, the elastic (blue) and plastic (red) zones are plotted.

Table 2 shows the number of degrees of freedom for both examples in case of an h-FEM and a BC-FEM discretization.

DOFs at Level	1	2	3	4	5	6
Plate with Hole (h-FEM)	2018	7810	30722	121858	485378	1937410
Plate with Hole (BC-FEM)	2018	5010	14658	37874	103050	307330
Screw Wrench (h-FEM)	474	1778	6882	27074	107394	427778
Screw Wrench (BC-FEM)	474	1618	4266	10290	24490	58474

Table 2 Comparison of the degrees of freedom at each numerical example.

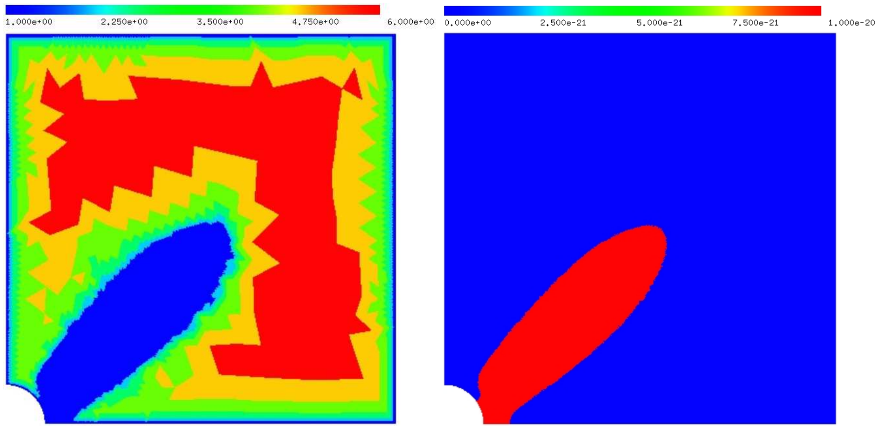

Fig. 4 Plate with a hole: polynomial order (left) and plastic zones (right).

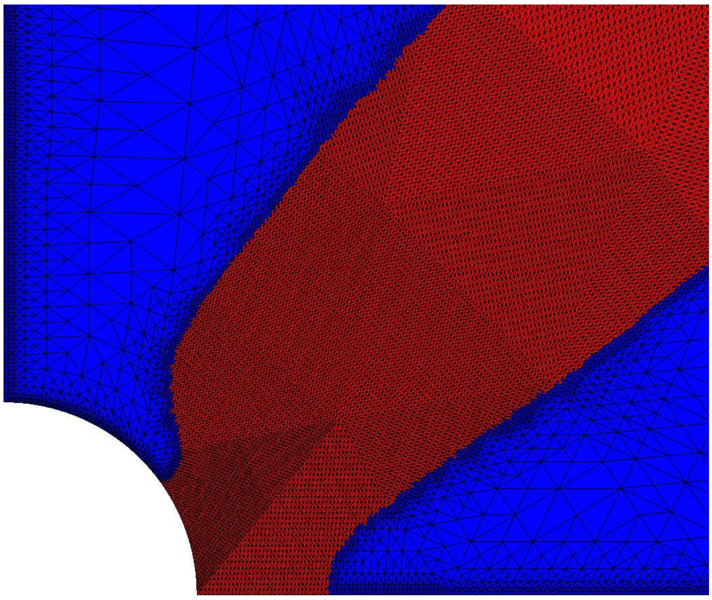

Fig. 5 Plate with a hole: adaptive refinement.

6 Conclusion

We presented two strategies to deal with the non-smoothness of the functional arising at each incremental step in elastoplasticity. The first one uses traditional regularization techniques whereas the second one makes use of Moreau's theorem for the reduced functional. Generalized Newton-methods are derived and analyzed on

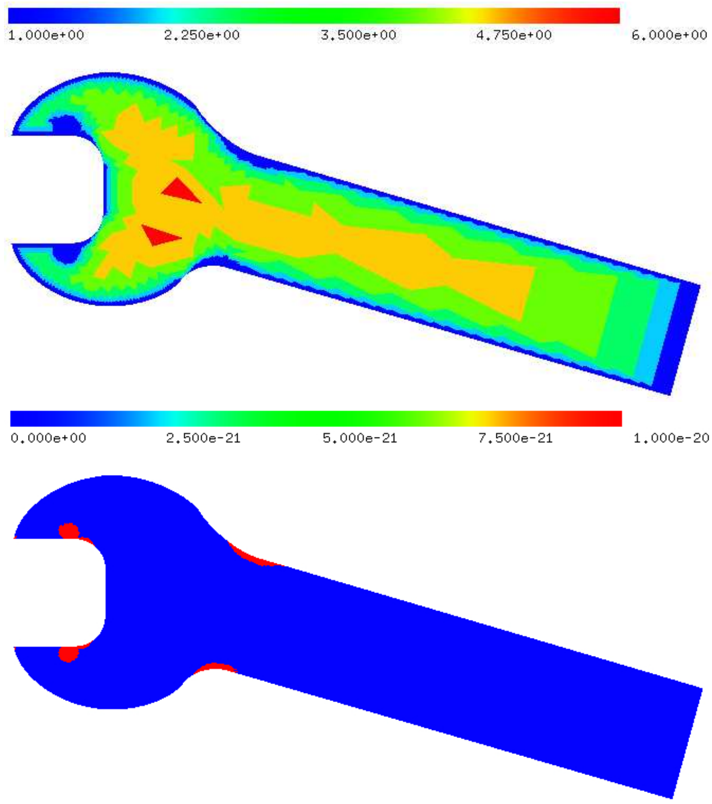

Fig. 6 Screw Wrench: polynomial order (top) and plastic zones (bottom).

the basis of the concept of slanting functions. Furthermore, we proposed residual-based and functional-based a posteriori error estimates for elastoplastic problems which can be used in an AFEM. In some cases the convergence of the AFEM can be shown. Finally, we studied the use of higher-order finite elements in elastoplasticity. The approximation quality of higher-order elements strongly depends on the local regularity of the solution. The new plastic-zone concentrated finite element approximation used low-order elements in the plastic zones and boundary or, more precisely, interface concentrated finite element approximations in the elastic zone where higher and higher order finite elements are used in dependence on the distance to the elastic-plastic interface and the boundary. Regularity detectors can be used to predict the elastic-plastic interface at each incremental step.

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