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Parity Results for Broken k-diamond Partitions and (2k+1)-cores

Silviu Radu James Sellers

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PARITY RESULTS FOR BROKEN k–DIAMOND PARTITIONS AND $(2k+1)$–CORES

SILVIU RADU AND JAMES A. SELLERS

Abstract. In this paper we prove several new parity results for broken k-diamond partitions introduced in 2007 by Andrews and Paule. In the process, we also prove numerous congruence properties for $(2k+1)$-core partitions. The proof technique involves a general lemma on congruences which is based on modular forms.

1. Introduction

Broken k-diamond partitions were introduced recently by Andrews and Paule [1]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms. Namely,

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_k(n)q^n &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}\\ &= q^{(k+1)/12}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \quad k\geq 1,\end{aligned}$$

where we recall the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \quad (q=e^{2\pi i\tau}).$$

In [1], Andrews and Paule proved that, for all $n\geq 0$, $\Delta_1(2n+1)\equiv 0 \pmod 3$ and conjectured a few other congruences modulo 2 satisfied by certain families of k-broken diamond partitions.

Since then, a number of authors have provided proofs of additional congruences satisfied by broken k-diamond partitions. Hirschhorn and Sellers [8] provided a new proof of the modulo 3 result mentioned above as well as elementary proofs of the following parity results: For all $n\geq 1$,

$$\begin{aligned}\Delta_1(4n+2) &\equiv 0 \pmod 2,\\ \Delta_1(4n+3) &\equiv 0 \pmod 2,\\ \Delta_2(10n+2) &\equiv 0 \pmod 2,\\ \Delta_2(10n+6) &\equiv 0 \pmod 2\end{aligned}$$

The third result in the list above appeared in [1] as a conjecture while the other three did not. Soon after the publication of [8], Chan [3] provided a different proof of the parity results for Δ_2 mentioned above as well as a number of congruences modulo powers of 5.

In this paper, we significantly extend the list of known parity results for broken k-diamonds by proving a large number of congruences which are similar to those mentioned above. Indeed, we will do so by proving a similar set of parity results satisfied by certain t-core partitions.

A partition is called a t-core if none of its hook lengths is divisible by t. These partitions have been studied extensively by many, especially thanks to their strong connection to representation theory. Numerous congruence properties are known for t-cores, although few such results are known modulo

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2. Such parity results can be found in [7], [5], [9], [6], [2], [4]. In all of these papers, the value of t which was considered was even; in this paper, we provide a new set of parity results for t-cores wherein t is odd.

The generating function for t-core partitions (for a fixed $t \geq 1$) is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

Given this fact, we can quickly see a connection between broken k-diamonds and $(2k+1)$-cores which we will utilize below.

Lemma 1.1. *For all $k \geq 1$ we have*

$$\left(\prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}\right)\left(\sum_{n=0}^{\infty}\Delta_k(n)q^n\right) \equiv \sum_{n=0}^{\infty} a_{2k+1}(n)q^n \pmod 2.$$

Proof. Using the relation $(1-q^n)^2 \equiv (1-q^{2n}) \pmod 2$ we find

$$\begin{aligned}
&\left(\prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}\right)\left(\sum_{n=0}^{\infty}\Delta_k(n)q^n\right)\\
&= \prod_{n=1}^{\infty}\frac{(1-q^{(4k+2)n})^k(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3}\\
&\equiv \prod_{n=1}^{\infty}\frac{(1-q^{(2k+1)n})^{2k+1}}{(1-q^n)} \pmod 2\\
&= \sum_{n=0}^{\infty} a_{2k+1}(n)q^n.
\end{aligned}$$

□

We assume throughout that $\Delta_k(v) = a_k(v) = 0$ if $v \leq 0$.

Corollary 1.2. *Let $r \in \mathbb{N}$. Then for all $k \geq 1$ and $n \in \mathbb{Z}$ we have*

$$\Delta_k((4k+2)n+r) \equiv 0 \pmod 2 \Leftrightarrow a_{2k+1}((4k+2)n+r) \equiv 0 \pmod 2.$$

Proof. Let k and r be fixed and assume that $\Delta_k((4k+2)n+r) \equiv 0 \pmod 2$ for all $n \in \mathbb{Z}$. Let

$$\sum_{n\in\mathbb{Z}} b(n)q^{(4k+2)n} = \prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}.$$

Then using Lemma 1.1 we find that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} a_{2k+1}((4k+2)n+r)q^n\\
&\equiv \sum_{\substack{n,m\in\mathbb{Z},\\ (4k+2)n+m\equiv r \pmod{4k+2}}} b(n)\Delta_k(m)q^{(4k+2)n+m}\\
&\equiv \sum_{\substack{n,m\in\mathbb{Z},\\ m\equiv r \pmod{4k+2}}} b(n)\Delta_k(m)q^{(4k+2)n+m}\\
&\equiv \sum_{n,v\in\mathbb{Z}} b(n)\Delta_k((4k+2)v+r)q^{(4k+2)m+(4k+2)v+r}\\
&\equiv 0 \pmod 2.
\end{aligned}$$

The reverse direction is analogous. □

With this motivation, we now state the full list of parity results we will prove in this paper. With the goal of minimizing the notation, we will write

$$f(tn + r_1, r_2, \ldots, r_m) \equiv 0 \pmod 2$$

to mean that, for each $i \in \{1, 2, \ldots, m\}$,

$$f(tn + r_i) \equiv 0 \pmod 2.$$

Theorem 1.3. *For all $n \geq 0$,*

$$\Delta_2(10n + 2, 6) \equiv 0 \pmod 2, \tag{1}$$

$$\Delta_3(14n + 7, 9, 13) \equiv 0 \pmod 2, \tag{2}$$

$$\Delta_5(22n + 2, 8, 12, 14, 16) \equiv 0 \pmod 2, \tag{3}$$

$$\Delta_6(26n + 2, 10, 16, 18, 20, 22) \equiv 0 \pmod 2, \tag{4}$$

$$\Delta_8(34n + 11, 15, 17, 19, 25, 27, 29, 33) \equiv 0 \pmod 2, \tag{5}$$

$$\Delta_9(38n + 2, 8, 10, 20, 24, 28, 30, 32, 34) \equiv 0 \pmod 2, \tag{6}$$

$$\Delta_{11}(46n + 11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45) \equiv 0 \pmod 2. \tag{7}$$

(Note that (1) was proved in [8].) Thanks to Corollary 1.2, we see that Theorem 1.3 is proved once we prove the following corresponding theorem involving t-cores:

Theorem 1.4. *For all $n \geq 0$,*

$$a_5(10n + 2, 6) \equiv 0 \pmod 2, \tag{8}$$

$$a_7(14n + 7, 9, 13) \equiv 0 \pmod 8, \tag{9}$$

$$a_{11}(22n + 2, 8, 12, 14, 16) \equiv 0 \pmod 2, \tag{10}$$

$$a_{13}(26n + 2, 10, 16, 18, 20, 22) \equiv 0 \pmod 2, \tag{11}$$

$$a_{17}(34n + 11, 15, 17, 19, 25, 27, 29, 33) \equiv 0 \pmod 8, \tag{12}$$

$$a_{19}(38n + 2, 8, 10, 20, 24, 28, 30, 32, 34) \equiv 0 \pmod 2, \tag{13}$$

$$a_{23}(46n + 11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45) \equiv 0 \pmod 8. \tag{14}$$

Note that every prime p, $5 \leq p \leq 23$, is represented in Theorem 1.4, which helps to explain why certain families of broken k–diamond partitions appear in Theorem 1.3 (and others do not). Our ultimate goal now is to provide a proof of Theorem 1.4. We close this section by developing the machinery necessary to prove this theorem.

For M a positive integer let $R(M)$ be the set of integer sequences indexed by the positive divisors δ of M. Let $1 = \delta_1, \ldots, \delta_k = M$ be the positive divisors of M and $r \in R(M)$. Then we will write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$.

For s an integer and m a positive integer we denote by $[s]_m$ the set of all elements congruent to s modulo m, in other words $[s]_m \in \mathbb{Z}_m$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $S_m \subset \mathbb{Z}_m^*$ be the set of all squares in $\mathbb{Z}_m^*$.

Definition 1.5. *For $m, M \in \mathbb{N}^*$,$(r_\delta) \in R(M)$ and $t \in \{0, \ldots, m-1\}$ we define the map $\overline{\odot} : \mathbb{S}_{24m} \times \{0, \ldots, m-1\} \to \{0, \ldots, m-1\}$ with $([s]_{24m}, t) \mapsto [s]_{24m}\overline{\odot}t$ and the image is uniquely determined by the relation $[s]_{24m}\overline{\odot}t \equiv ts + \frac{s-1}{24}\sum_{\delta|M} \delta r_\delta \pmod m$. We define the set*

$$P_{m,r}(t) := \{[s]_{24m}\overline{\odot}t | [s]_{24m} \in \mathbb{S}_{24m}\}.$$

Lemma 1.6. *Let $p \geq 5$ be a prime. Let $r^{(p)} := (r_1^{(p)}, r_p^{(p)}) = (-1, p) \in R(p)$. Then*

$$P_{2p,r^{(p)}}(t) = \left\{ t' | \left(\frac{24t-1}{p}\right) = \left(\frac{24t'-1}{p}\right), t \equiv t' \pmod 2, 0 \leq t' \leq 2p-1 \right\}. \tag{15}$$

Proof. First note that

$$\frac{1}{24}\sum_{\delta|p}\delta r_{\delta}^{(p)} = \frac{p^2-1}{24} \in \mathbb{Z}.$$

Let $m = 2p$. If $s_1 \equiv s_2 \pmod m$ then $[s_1]_{24m}\overline{\odot}t = [s_2]_{24m}\overline{\odot}t$ because $\frac{p^2-1}{24}$ is an integer. This implies that

$$P_{2p,r^{(p)}}(t) = \{t' | t' \equiv ts + (s-1)\frac{p^2-1}{24} \pmod p, s \in \mathbb{S}_m, 0 \leq t \leq 2p-1\}. \tag{16}$$

We see that

$$P_{2p,r^{(p)}}(t) \pmod 2 = \{t \pmod 2\}. \tag{17}$$

Next we compute $P_{2p,r^{(p)}}(t) \pmod p$. By (16) we know

$$\begin{aligned} P_{2p,r^{(p)}}(t) \pmod p &= \left\{ t' \pmod p | t' \equiv ts + (s-1)\frac{p^2-1}{24} \pmod p, s \in \mathbb{S}_p \right\} \\ &= \{t' \pmod p | 24t'-1 \equiv s(24t-1) \pmod p, s \in \mathbb{S}_p\} \\ &= \left\{ t' \pmod p | \left(\frac{24t-1}{p}\right) = \left(\frac{24t'-1}{p}\right) \right\}. \end{aligned} \tag{18}$$

By (17) and (18) and the Chinese remainder theorem we obtain $P_{2p,r^{(p)}}(t) \pmod{2p}$ and we obtain the formula (15) by imposing that the elements of $P_{2p,r^{(p)}}(t)$ lie between 0 and $2p-1$. □

We now use Lemma 1.6 to compute $P_{2p,r^{(p)}}(t)$ for $p = 5, 7, 11, 13, 19, 23$ and $t = 2, 7, 2, 2, 11, 2, 11$ below.

$p = 5, t = 2$. We see that $\left(\frac{24t-1}{p}\right) = \left(\frac{2}{5}\right) = -1$. For $t' \in \{1, 2\}$ we have $\left(\frac{24t'-1}{5}\right) = -1$ and for $t' \in \{0, 3, 4\}$ we have $\left(\frac{24t'-1}{5}\right) \in \{0, 1\}$. This implies that $P_{10,r^{(5)}}(2) \equiv \{1, 2\} \pmod 5$. Since $t \equiv 0 \pmod 2$ we have that $P_{10,r^{(5)}}(2) \equiv 0 \pmod 2$. Hence by Lemma 1.6 we have

$$P_{10,r^{(5)}}(2) = \{2, 1+5\} = \{2, 6\}.$$

$p = 7, t = 7$. We see that $\left(\frac{24t-1}{p}\right) = \left(\frac{-1}{7}\right) = -1$. We see that for $t' \in \{0, 2, 6\}$ we have $\left(\frac{24t'-1}{7}\right) = -1$ (and this is all t' with this property) so $P_{14,r^{(7)}} \equiv \{0, 2, 6\} \pmod 7$. Because $t \equiv 1 \pmod 2$ we obtain by Lemma 1.6

$$P_{14,r^{(7)}}(7) = \{0+7, 2+7, 6+7\} = \{7, 9, 13\}.$$

$p = 11, t = 2$. Here $\left(\frac{24t-1}{11}\right) = \left(\frac{5^2}{11}\right) = 1$. We see that for $t' \in \{1, 2, 3, 5, 8\}$ we have $\left(\frac{24t'-1}{11}\right) = 1$ so

$$P_{22,r^{(11)}}(2) = \{1+11, 2, 3+11, 5+11, 8\} = \{2, 8, 12, 14, 16\}.$$

Similarly we get by Lemma 1.6

$$P_{26,r^{(13)}} = \{2, 10, 16, 18, 20, 22\},$$

$$P_{34,r^{(17)}} = \{11, 15, 17, 19, 25, 27, 29, 33\},$$

$$P_{38,r^{(19)}} = \{2, 8, 10, 20, 24, 28, 30, 32, 34\}$$

and

$$P_{46,r^{(23)}} = \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}.$$

We see immediately from the above that Theorem 1.4 is equivalent to the following theorem.

Theorem 1.7. *Let $t : \{5,7,11,13,17,19,23\} \to \{2,7,11\}$ with $p \mapsto t_p$ be defined by*

$$(t_5, t_7, t_{11}, t_{13}, t_{17}, t_{19}, t_{23}) := (2,7,2,2,11,2,11).$$

Then for all $n \geq 0$, p prime with $5 \leq p \leq 23$, and $t' \in P_{2p,r^{(p)}}(t_p)$, we have

$$a_p(2pn + t') \equiv 0 \pmod{2^{i(p)}}, \tag{19}$$

where

$$i(p) = \begin{cases} 1 & \text{if } p = 5,11,13,19, \\ 3 & \text{if } p = 7,17,23. \end{cases}$$

For each $r \in R(M)$ we assign a generating function

$$f_r(q) := \prod_{\delta|M} \prod_{n=1}^{\infty} (1-q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} c_r(n) q^n.$$

Given p a prime, $m \in \mathbb{N}$ and $t \in \{0, \ldots, m-1\}$ we are concerned with proving congruences of the type $c_r(mn+t) \equiv 0 \pmod{p}, n \in \mathbb{N}$. The congruences we are concerned with here have some additional structure; namely $a_r(mn+t') \equiv 0 \pmod{p}, n \geq 0, t' \in P_{m,r}(t)$. In other words a congruence is a tuple (r, M, m, t, p) with $r \in R(M)$, $m \geq 1$, $t \in \{0, \ldots, m-1\}$ and p a prime such that

$$a_r(mn+t') \equiv 0 \pmod{p}, n \geq 0, t' \in P(t).$$

Throughout when we say that $a_r(mn+t) \equiv 0 \pmod{p}$ we mean that $a_r(mn+t') \equiv 0 \pmod{p}$ for all $n \geq 0$ and all $t' \in P(t)$. The purpose of this paper is show the congruences

$$a_p(2pn + t_p) \equiv 0 \pmod{2}$$

when $p = 5,7,11,13,17,19,23$ and $t_p = 2,7,2,2,11,2,11$.

In order to accomplish our goal we need a lemma ([10, Lemma 4.5]). We first state it and then explain the terminology.

Lemma 1.8. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $a = (a_\delta) \in R(N)$, n the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_a^*(\gamma_i) \geq 0$, $i \in \{1, \ldots, n\}$. Let $t_{min} := \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m}.$$

Then if

$$\sum_{n=0}^{\lfloor \nu \rfloor} c_r(mn+t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$ then

$$\sum_{n=0}^{\infty} c_r(mn+t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$.

The lemma reduces the proof of a congruence modulo u to checking that finitely many values are divisible by u. We first define the set Δ^*. Let $\kappa = \kappa(m) = \gcd(m^2-1, 24)$ and $\pi(M, (r_\delta)) := (s, j)$ where s is a non-negative integer and j an odd integer uniquely determined by $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$. Then a tuple $(m, M, N, (r_\delta), t)$ belongs to Δ^* iff

- $m \geq 1, M \geq 1, N \geq 1$, $(r_\delta) \in R(M)$, $t \in \{0, \ldots, m-1\}$;
- $p|m$ implies $p|N$ for every prime p;
- $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$;
- $\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv 0 \pmod{24}$;

- $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$;
- $\dfrac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} \mid N$;
- for $(s,j) = \pi(M,(r_\delta))$ we have $(4|\kappa N$ and $8|Ns)$ or $(2|s$ and $8|N(1-j))$.

Next we need to define the groups $\Gamma, \Gamma_0(N)$ and Γ_∞:

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid N|c \right\}$$

for N a positive integer, and

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\}.$$

For the index we have $[\Gamma : \Gamma_0(N)] := N \prod_{p|N}(1+p^{-1})$ (see, for example, [11]).

Finally for $m \geq 1, M \geq 1$, and $r \in R(M)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m} \tag{20}$$

and

$$p_r^*(\gamma) := \frac{1}{24} \sum_{\delta|M} \frac{r_\delta \gcd^2(\delta, c)}{\delta}.$$

2. The Congruences

Let $r^{(p)} = (-1, p)$ throughout this section where $p \geq 5$ is a prime. Before we prove the congruences we will show that $p_{2p,r^{(p)}}(\gamma) \geq 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we know by (20) that

$$\begin{aligned} p_{2p,r^{(p)}}(\gamma) &= \min_{\lambda \in \{0,\ldots,2p-1\}} \frac{1}{24} \left(-\frac{\gcd^2(a+\kappa\lambda c, 2pc)}{2p} + p\frac{\gcd^2(p(a+\kappa\lambda c), 2pc)}{2p^2} \right) \\ &= \min_{\lambda \in \{0,\ldots,2p-1\}} \frac{1}{24} \left(-\frac{\gcd^2(a+\kappa\lambda c, 2pc)}{2p} + p\frac{\gcd^2(a+\kappa\lambda c, 2c)}{2} \right) \\ &= \min_{\lambda \in \{0,\ldots,2p-1\}} \frac{1}{24} \left(-\frac{\gcd^2(a+\kappa\lambda c, 2p)}{2p} + p\frac{\gcd^2(a+\kappa\lambda c, 2)}{2} \right). \end{aligned}$$

The last rewriting follows from $\gcd(a,c) = 1$ because $ad - bc = 1$. Next we will show that $p_{2p,r^{(p)}}$ is nonnegative by proving that

$$F(a,c,p,\lambda) := -\frac{\gcd^2(a+\kappa\lambda c, 2p)}{2p} + p\frac{\gcd^2(a+\kappa\lambda c, 2)}{2} \geq 0$$

for all integers a, c, p and λ. We split the proof in four cases:

$$\begin{aligned} \gcd(a+\kappa\lambda c, 2p) = 1 &\quad\Rightarrow\quad F(a,c,p,\lambda) = -\frac{1}{2p} + \frac{p}{2} \geq 0 \\ \gcd(a+\kappa\lambda c, 2p) = 2 &\quad\Rightarrow\quad F(a,c,p,\lambda) = -\frac{2}{p} + 2p \geq 0 \\ \gcd(a+\kappa\lambda c, 2p) = p &\quad\Rightarrow\quad F(a,c,p,\lambda) = -\frac{p}{2} + \frac{p}{2} = 0 \\ \gcd(a+\kappa\lambda c, 2p) = 2p &\quad\Rightarrow\quad F(a,c,p,\lambda) = -2p + 2p = 0 \end{aligned}$$

Because $p_{2p,r^{(p)}}(\gamma) = \min_{\lambda \in \{0,\ldots,2p-1\}} \frac{1}{24} F(a,c,p,\lambda)$ we know $p_{2p,r^{(p)}}(\gamma) \geq 0$.

We are now ready to prove the congruences in Theorem 1.4. We start with (8):

$$a_5(10n+2,6) \equiv 0 \pmod 2$$

We apply Lemma 1.8. We see that $(10,5,10,2,r^{(5)}=(-1,5)) \in \Delta^*$. We choose the sequence (a_δ) in Lemma 1.8 to be the zero sequence (this will be so for all the congruences in this paper). Because $(a_\delta) \equiv 0$ and because $p_{10,r^{(5)}} \geq 0$ we see that $p_{10,r^{(5)}}(\gamma) + p_a^*(\gamma) \geq 0$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Finally

$$\nu = \frac{1}{24}(5-1)(5+1)(2+1) - \frac{1}{10} - \frac{1}{5} = 3 - \frac{3}{10}.$$

We choose $u=2$ in the lemma and note that $c_r(n) = a_5(n)$ for all $n \geq 0$. Then (8) is true iff

$$a_5(2) \equiv a_5(12) \equiv a_5(22) \equiv a_5(6) \equiv a_5(16) \equiv a_5(26) \pmod 2.$$

These values of a_5 are all even as can be seen in the Appendix below, so (8) is proven.

A similar approach can be used to prove (9)–(14). In particular let t_p be as in Theorem 1.7 and $r^{(p)} = (-1,p)$. Then

$$(2p, p, 2^{\frac{3-(-1)^{\frac{p-1}{2}}}{2}} p, p, t_p, r^{(p)}) \in \Delta^*.$$

We again set $(a_\delta) \equiv 0$ and see as before that

$$p_{2p,r^{(p)}}(\gamma) + p_a^*(\gamma) \geq 0$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We further obtain

$$\begin{aligned}\nu = \nu_p &= \frac{1}{24}(p-1)2^{\frac{3-(-1)^{\frac{p-1}{2}}}{2}} p(1+\frac{1}{p})(1+\frac{1}{2}) - \frac{p^2-1}{48p} - \frac{t_p}{2p} \\ &= \frac{1}{8}(p^2-1)2^{\frac{1-(-1)^{\frac{p-1}{2}}}{2}} - \frac{p^2-1}{48p} - \frac{t_p}{2p}.\end{aligned}$$

Putting these values in a table we obtain

p	ν_p	$\lfloor \nu_p \rfloor$
5	$3 - \frac{1}{10} - \frac{2}{10}$	2
7	$12 - \frac{1}{7} - \frac{1}{2}$	11
11	$30 - \frac{5}{22} - \frac{2}{22}$	29
13	$21 - \frac{7}{26} - \frac{2}{26}$	20
17	$36 - \frac{6}{17} - \frac{11}{34}$	35
19	$90 - \frac{15}{38} - \frac{2}{38}$	89
23	$132 - \frac{11}{23} - \frac{11}{46}$	131

We conclude by Lemma 1.8 that for all $n \geq 0$ we have

$$a_p(2pn+t') \equiv 0 \pmod u, t' \in P_{2p,r^{(p)}}(t_p),$$

if for $0 \leq n \leq \lfloor \nu_p \rfloor$

$$a_p(2pn+t') \equiv 0 \pmod u, t' \in P_{2p,r^{(p)}}(t_p).$$

In particular we choose $u=2$ in the case $p = 5, 11, 13, 19$ and $u=8$ for $p = 7, 17, 23$.

The values of $a_t(n)$ have been calculated in `MAPLE` for $5 \leq t \leq 23$ and are explicitly given in the Appendix below for $5 \leq p \leq 17$. The values of $a_{19}(n)$ and $a_{23}(n)$ have been suppressed in the Appendix due to the length of those tables of data.

Given that all of these values are congruent to zero modulo 2 (or 8, respectively), it is the case that Theorem 1.4 is proved.

3. Appendix

In this section, we demonstrate the exact values of $a_t(n)$ (where t is prime, $5 \leq t \leq 17$) which are needed to complete the proofs above.

n	$a_5(n)$
2	2
6	6
12	12
16	16
22	22
26	20

n	$a_7(n)$
7	8
9	16
13	24
21	64
23	72
27	112
35	176
37	168
41	224
49	288
51	352
55	360
63	504
65	576
69	616
77	792
79	728
83	864
91	960
93	1080
97	1120
105	1408
107	1472
111	1624
119	1856
121	1680
125	2016
133	1968
135	2368
139	2208
147	2752
149	2816
153	2880
161	3312
163	2928
167	3528

n	$a_{11}(n)$
2	2
8	22
12	66
14	102
16	154
24	552
30	1182
34	1838
36	2214
38	2684
46	5370
52	8382
56	10868
58	12496
60	14010
68	22266
74	30536
78	37224
80	41022
82	45496
90	63978
96	81642
100	96614
102	102798
104	110748
112	148832
118	181742
122	204006
124	219868
126	230934
134	292822
140	347176
144	386562
146	407828
148	435120
156	526794
162	610038
166	679008
168	702540
170	736512
178	890452
184	1012918
188	1088164
190	1149874
192	1181400
200	1387322
206	1554652
210	1678644
212	1738234
214	1826418
222	2082522
228	2311584
232	2505064
234	2558988
236	2645588
244	3052216

n	$a_{11}(n)$
250	3362502
254	3527854
256	3685264
258	3752298
266	4230294
272	4617668
276	4890336
278	5030512
280	5246754
288	5780238
294	6268350
298	6692434
300	6798300
302	6967290
310	7827536
316	8430246
320	8763888
322	9078300
324	9185484
332	10116262
338	10851604
342	11370942
344	11635286
346	12052898
354	13027830
360	13942758
364	14722928
366	14853204
368	15181738
376	16730958
382	17812432
386	18331434
388	18940526
390	19123320
398	20687438
404	21947332
408	22808874
410	23301160
412	24008270
420	25629408
426	27064554
430	28475624
432	28603086
434	29131256
442	31699274
448	33435556
452	34209296
454	35247894
456	35423652
464	37931194
470	39990384
474	41288478
476	41981162
478	43194338
486	45583848
492	47834154

n	$a_{11}(n)$
496	50022974
498	50206764
500	51092422
508	54998418
514	57608716
518	58681458
520	60390624
522	60495450
530	64357218
536	67185690
540	69306996
542	70216028
544	72138928
552	75494322
558	78800040
562	82042510
564	82212408
566	83374324
574	89234708
580	93151696
584	94395102
586	96865032
588	96985812
596	102326006
602	106472872
606	109304196
608	110747244
610	113766418
618	118102494
624	122768718
628	127477340
630	127725018
632	129078022
640	137641988
646	142548818
650	144592250
652	147960912
654	147922632

n	$a_{13}(n)$	n	$a_{13}(n)$
2	2	278	56402964
10	42	280	58183920
16	192	282	60245730
18	294	288	66743616
20	432	296	76634976
22	612	302	84530244
28	1560	304	86942064
36	4392	306	89775762
42	8454	308	93029304
44	10344	314	102269280
46	12504	322	115182990
48	15024	328	126042144
54	25236	330	129892902
62	46956	332	134345760
68	71176	334	137785176
70	80952	340	150342384
72	91968	348	168438696
74	104600	354	183217440
80	149520	356	189126304
88	231120	358	193533876
94	314028	360	198963312
96	346368	366	215763180
98	382782	374	240903748
100	419280	380	260485168
106	550656	382	266187336
114	775080	384	273098736
120	987312	386	281305134
122	1071834	392	303431952
124	1152888	400	333736176
126	1243500	406	359054076
132	1550832	408	367732896
140	2059872	410	378352134
146	2515848	412	385925712
148	2672952	418	414230340
150	2850504	426	454854966
152	3049296	432	487260528
158	3668416	434	500483666
166	4630632	436	509863416
172	5491632	438	521308380
174	5805144	444	557588064
176	6158368	452	611350176
178	6473682	458	652147944
184	7596576	460	663774576
192	9325536	462	678070512
198	10816464	464	695541024
200	11404128	470	740997372
202	11917320	478	801685416
204	12498312	484	852786456
210	14377584	486	870301224
218	17298230	488	891480496
224	19735200	490	906132654
226	20522130	496	962218848
228	21410976	504	1041152112
230	22436860	510	1103787288
236	25424568	512	1129942032
244	29772624	514	1147163922
250	33504906	516	1169346576
252	34827456	522	1237981896
254	36344340	530	1339958130
256	37601184	536	1416507504
262	42091860	538	1436414886
270	48734484	540	1463425008
276	54245328	542	1496544132

n	$a_{17}(n)$	n	$a_{17}(n)$
11	56	271	2426989824
15	176	283	3245729904
17	280	287	3566632112
19	456	289	3737033832
25	1584	291	3912563064
27	2296	297	4488172112
29	3256	299	4697678944
33	6216	301	4913246400
45	32592	305	5369995224
49	52536	317	6965156360
51	65608	321	7576561416
53	81816	323	7904365552
59	151888	325	8240802648
61	184584	331	9324359448
63	222832	333	9707104152
67	320616	335	10112716256
79	862656	339	10952431968
83	1165488	351	13858209488
85	1349280	355	14970028800
87	1555248	357	15543374104
93	2348024	359	16150139104
95	2680064	365	18069637832
97	3051288	367	18751505520
101	3928136	369	19446320864
113	7960592	373	20930459544
117	9918968	385	25947385248
119	11052624	389	27833654208
121	12288864	391	28820095104
127	16734672	393	29822363464
129	18489048	399	33056294656
131	20413208	401	34214397512
135	24749232	403	35390769744
147	42868864	407	37849417312
151	51037968	419	46121197704
153	55557320	423	49179153232
155	60476544	425	50807820064
161	77428368	427	52457960208
163	83910624	433	57686644632
165	90819416	435	59499587168
169	106245144	437	61416536840
181	166516968	441	65320417456
185	192230168	453	78438078008
187	206311800	457	83323981104
189	221170720	459	85804092672
195	271737008	461	88427697592
197	290796720	467	96584805992
199	310846080	469	99443364840
203	354503280	471	102327135376
215	518490320	475	108458323536
219	585660808	487	128606909712
221	622373744	491	135998931184
223	660746160	493	139828062552
229	788287968	495	143686905328
231	834834192	501	156020841952
233	884527736	503	160400341216
237	990281688	505	164813860440
249	1376710768	509	173949954920
253	1532051616	521	204009642112
255	1614146704	525	214872968448
257	1701462600	527	220638376272
263	1985556176	529	226433715240
265	2088907176	535	244608538032
267	2195793456	537	250823331760

n	$a_{17}(n)$
539	257407547216
543	270649209968
555	314370913112
559	330379351536
561	338410621544
563	346919996760
569	373072012776
571	382155478176
573	391243575192
577	410550844032
589	472804320624
593	495270155080
595	506851583400
597	518420413624
603	555241230368
605	568272396240
607	581292530304
611	608101322456
623	694977500640
627	725884834888
629	742277376064
631	758642533248
637	809604714840
639	826852286272
641	845186334064
645	881697155128
657	1000792982152
661	1043915802216
663	1065326040848
665	1088129921376
671	1157457951536
673	1181376923400
675	1205199084424
679	1255759774512
691	1416600616272
695	1473993826176
697	1503432254352
699	1532670755360
705	1625532844920
707	1658296877024
709	1690855114032
713	1757602536520
725	1971662997248
729	2046863934848
731	2086805374264
733	2126416991712
739	2249259722304
741	2290479895504
743	2334443092288
747	2421329693032
759	2702354085200
763	2803280716680
765	2853041651944
767	2906179410752
773	3066493765088
775	3121643396256
777	3176146368064
781	3292096023528
793	3657001864440
797	3786166999920

n	$a_{17}(n)$
799	3852176265936
801	3917376565112
807	4124282712960
809	4197225062160
811	4269323481336
815	4416707600192
827	4885329617184
831	5048409214960
833	5135131873632
835	5220877551720
841	5485338701856
843	5573382742896
845	5667949332416
849	5853060745680
861	6448539632720
865	6661275248376
867	6765247044512
869	6876882060392
875	7211498618488
877	7326086174976
879	7438845040096
883	7679198560944
895	8429681682192
899	8693382000016
901	8827862802264
903	8959909645648
909	9379483781288
911	9527349078512
913	9672678527136
917	9969427797424
929	10906552713056
933	11229960128080
935	11402760570944
937	11572406641296
943	12094229565744
945	12267189623208
947	12453455872088
951	12815797785536
963	13976174115064
967	14389276995504
969	14589644317408
971	14805899114128
977	15450205421352
979	15670243785888
981	15885697223528
985	16346555724408
997	17774538588600
1001	18273693316440
1003	18527480806464
1005	18776225511224
1011	19565501950968
1013	19843654467256
1015	20116540384272
1019	20671443981560
1031	22415569736576
1035	23013799200800
1037	23333967678456
1039	23647168970256
1045	24608464496304
1047	24924891465616

n	$a_{17}(n)$
1049	25267824454744
1053	25930044061624
1065	28046036133912
1069	28796519539752
1071	29158202296704
1073	29550325460792
1079	30713788742384
1081	31110089579736
1083	31495955399328
1087	32325371046432
1099	34878563240664
1103	35766685018640
1105	36218661026976
1107	36658285197352
1113	38056051986944
1115	38550240895168
1117	39031729604736
1121	40010086539664
1133	43071802056728
1137	44116570453312
1139	44677355054440
1141	45223620353352
1147	46896911035152
1149	47444642461712
1151	48041678116608
1155	49187814425240
1167	52839446255568
1171	54131850411216
1173	54750037179320
1175	55426211278560
1181	57417366260416
1183	58093821991728
1185	58751663828288
1189	60167395111200
1201	64503867395472
1205	66007643257976
1207	66770727743184
1209	67510478527224
1215	69867223862720
1217	70700640248904
1219	71509962335016
1223	73151514600576

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Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria, sradu@risc.uni-linz.ac.at

Department of Mathematics, Penn State University, University Park, PA 16802, USA, sellersj@math.psu.edu

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Doctoral Program
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Institute of Applied Geometry

Address:

Johannes Kepler University Linz
Doctoral Program "Computational Mathematics"
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-7174

E-Mail:

office@dk-compmath.jku.at

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