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Rational General Solutions of High Order Non-autonomous ODEs*

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Abstract

In this paper, we generalize the results of Ngô and Winkler [18, 20, 21] to the case of high order non-autonomous algebraic ODE with a birational parametrization of the corresponding algebraic hypersurface. First, we reduce the problem for finding rational general solutions of non-autonomous $n-1$ ($n>2$) order ODE to finding rational general solutions of an associated first order rational system of autonomous ODEs in n indeterminates based on the parametrization of hypersurface. Next, the correspondence of the rational general solutions between the original non-autonomous algebraic ODE and the associated system of autonomous ODEs is proved. Finally, a criterion is presented for existence of rational general solutions of the associated system of autonomous ODEs if the degree bound of its rational general solutions is given. Moreover, we give some nice properties of polynomial system of autonomous ODEs.

Keywords: Rational general solutions, non-autonomous ODE, associated system of autonomous ODEs, hypersurface, parametrization.

1 Introduction

The conversion between implicit and parametric representations of (differential) varieties is one of the classic and basic topics in (differential) algebraic geometry [8, 16, 22, 29]. Much of differential algebra or differential algebraic geometry can be regarded as a generalization of the algebraic geometry theory to the analogous theory for the differential equations. In recent years, a few relevant methods have been proposed for implicitization and parametrization problems in the differential case [9–12, 18, 20, 25, 26]. In this paper, we are interested in finding the rational general solutions for algebraic ODEs, which is motivated by developing efficient algorithms to the rational parametrization problem for differential varieties.

From an algorithmic point of view, many approaches have been proposed for finding the solutions of differential equations [3, 5–7, 28, 30, 31]. In the linear case, this problem can be traced back to the work of Liouvillian. Risch [23, 24] presented an algorithm to find elementary function solutions for the simplest differential equation $y' = F(x)$, and Kovacic [15] proposed an effective method for finding Liouvillian solutions of second order linear homogeneous differential equations. The general framework for the Liouvillian solutions for the general linear homogeneous ODEs was established by Singer [27]. There are also a few studies in the direction of algebraic

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(nonlinear) differential equations or some special type nonlinear equations. Bronstein [4] gave an effective method to compute rational solutions of Ricatti equations. Hubert [14] proposed an approach for computing a basis of the general solutions of first order ODEs and applied it to study the local behavior of the solutions. Li and Schwarz [17] presented the first method to find the rational solutions for a class of partial differential equations. In addition, the method based on rational parametrization of plane curves for computing the rational general solutions of first order autonomous ODEs was given by Feng and Gao [9, 11]. Subsequently, Ngô and Winkler [18, 20, 21] presented an approach to compute the rational general solutions of first order non-autonomous ODEs by using the birational parametrization of the correponding algebraic surface in 2009.

In this paper, the results of Ngô and Winkler (in [18, 20, 21]) are generalized to the case of non-autonomous ODEs with order $n-1$ ($n>2$). Based on the birational parametrization of algebraic hypersurface, we obtain an associated first order rational system of autonomous ODEs in n new indeterminates, which has a special structure and some good properties. In addition, we prove the correspondence of a rational general solution of original higer order non-autonomous ODE and a rational general solution of the associated first order system of autonomous ODEs. Furthermore, we present a criterion for existence of rational general solutions of the associated system of autonomous ODEs provided a degree bound of its rational general solutions, and give some nice properties of the first order polynomial system of autonomous ODEs.

The rest of this paper is organized as follows. In the next section, some known concepts and results about differential polynomials and rational general solutions are introduced. In section 3, it is explained how to derive the associated first order rational system of autonomous ODEs. Section 4 is devoted to proving the correspondence of rational general solutions between the original ODE and the associated system of ODEs. In Section 5, we present a criterion for existence of rational general solutions to the associated system of autonomous ODEs. Section 6 gives some properties of polynomial system of ODEs. This paper is concluded with a brief summary and some open problems in Section 7.

2 Preliminaries

In the following, let $\mathcal{K}=\mathbb{Q}(x)$ be the differential field of rational functions in x with differential operator $\frac{d}{dx}$ and we also use $'$ notation for an abbreviation of this derivation. Let $s_1,\ldots,s_n$ be indeterminates over $\mathcal{K}$. The *j-th* derivative of s_i is denoted by s_{ij}. The *differential polynomial ring* $\mathcal{K}\{s_1,\ldots,s_n\}$ is the ring consisting of all polynomials in s_i ($1\le i\le n$) and all their derivatives up to any order. Let $\mathcal{U}$ be a universal extension of the differential field $\mathcal{K}$ and Σ a set of differential polynomials in $\mathcal{K}\{s_1,\ldots,s_n\}$. A set of n elements $\{\eta_1,\ldots,\eta_n\}\in\mathcal{U}^n$ is a *zero of* Σ if all differential polynomials in Σ reduce to zero when each s_i is replaced by η_i.

Definition 1. Let Σ be a nontrivial prime ideal in $\mathcal{K}\{s_1,\ldots,s_n\}$. A zero $\{\eta_1,\ldots,\eta_n\}$ of Σ is called a *generic zero of* Σ if for any differential polynomial $F\in\mathcal{K}\{s_1,\ldots,s_n\}$, $F(\eta_1,\ldots,\eta_n)=0$ implies that $F\in\Sigma$.

Let $F\in\mathcal{K}\{s_1,\ldots,s_n\}$ be a differential polynomial. The *i-th* derivative of F is denoted by $F^{(i)}$. We simply write s_i instead of s_{i0}, or simply write F' instead of $F^{(1)}$. The *order* of F with respect to s_i is the greatest j such that s_{ij} occurring in F, denoted by $\mathrm{ord}_{s_i}(F)$. For convention we define $\mathrm{ord}_{s_i}(F)=-1$ if F does not involve any derivative of s_i.

Definition 2. Let $F,G\in\mathcal{K}\{s_1,\ldots,s_n\}$. Suppose that the indeterminate s_p appears effectively in both of them, where $1\le p\le n$. F is said to be *of higher rank than* G(or G *of lower rank than* F) in s_p if one of the following conditions holds:

(a) $\mathrm{ord}_{s_p}(F)>\mathrm{ord}_{s_p}(G)$;

(b) $\mathrm{ord}_{s_p}(F) = \mathrm{ord}_{s_p}(G) = q$ and $\deg_{s_{pq}}(F) > \deg_{s_{pq}}(G)$.

Definition 3. Let $A = \{s_{ik} : i = 1, \dots, n, k \in \mathbb{N}\}$. The *ord-lex ranking* on A is the total order defined as follows:

(a) $s_i < s_j$ if $i < j$;

(b) $s_{ik} < s_{jl}$ if $k < l$ or $k = l$ and $i < j$.

For any differential polynomial $F \in \mathcal{K}\{s_1, \dots, s_n\} \backslash \mathcal{K}$, the greatest derivative occurring in F with respect to ord-lex ranking is called the *leader* of F. The leading coefficient with respect to the leader of F is called the *initial* of F, the partial derivative with respect to the leader of F is called the *separant* of F. The initial of any $F \in \mathcal{K}$ is defined to be itself.

Definition 4. Let F and G be two differential polynomials in $\mathcal{K}\{s_1, \dots, s_n\}$ with the ord-lex ranking. G is said to be *reduced with respect to* F if G is lower rank than F in the indeterminate defining the leader of F.

Let $\mathbb{A} \subset \mathcal{K}\{s_1, \dots, s_n\}$, the differential polynomial set $\mathbb{A}$ is called *autoreduced* if no elements of $\mathbb{A}$ belongs to $\mathcal{K}$ and each elements of $\mathbb{A}$ is reduced with respect to all the others.

Definition 5. Let $F \in \mathcal{K}\{s_1, \dots, s_n\}$. For any $G \in \mathcal{K}\{s_1, \dots, s_n\}$, there exists a unique representation

$$S^k I^l G = \sum_i Q_i F^{(i)} + R,$$

where S is the separant of F, I is the initial of F, $Q_i \in \mathcal{K}\{s_1, \dots, s_n\}$, $F^{(i)}$ is the i-th derivatives of F, $k, l \in \mathbb{N}$ and $R \in \mathcal{K}\{s_1, \dots, s_n\}$ is reduced with respect to F. Here, R is called the *differential pseudo remainder* of G with respect to F, denoted by $\mathrm{prem}(G, F)$.

Let $F \in \mathcal{K}\{s_1, \dots, s_n\}/\mathcal{K}$ be an irreducible polynomial and

$$\Sigma_F = \{G \in \mathcal{K}\{s_1, \dots, s_n\} : SG \in \{F\}\},$$

where S is the separant of F, $\{F\}$ is the perfect differential ideal generated by F. It is well known by [22, chap II, sect. 13] that

Lemma 6. *Σ_F is a prime differential ideal. Furthermore, $G \in \Sigma_F$ if and only if* $\mathrm{prem}(G, F) = 0$.

Definition 7. Let $F \in \mathcal{K}\{s_1, \dots, s_n\}$ be an irreducible differential polynomial. A generic zero of the prime differential ideal Σ_F is called a *general solution* of $F = 0$. A *rational general solution* $(s_1, \dots, s_n)$ of $F = 0$ is defined as a general solution with every s_i has the following form

$$\frac{a_k\, x^k + a_{k-1}\, x^{k-1} + \cdots + a_0}{x^l + b_{l-1}\, x^{l-1} + \cdots + b_0},$$

where a_i, b_j are constants in the universal extension of $\mathbb{Q}$.

As a consequence of Lemma 6, we have

Corollary 8. *Let $F \in \mathcal{K}\{s_1, \dots, s_n\}/\mathcal{K}$ be an irreducible differential polynomial. If $(\eta_1, \dots, \eta_n)$ is a general solution of $F = 0$, then for any differential polynomial $G \in \mathcal{K}\{s_1, \dots, s_n\}$, we have*

$$G(\eta_1, \dots, \eta_n) = 0 \iff \mathrm{prem}(G, F) = 0.$$

3 Associated first order system of autonomous ODEs

Consider a non-autonomous algebraic ODE

$$F(x, y, y', \dots, y^{(n-1)}) = 0, \tag{1}$$

where $F \in \mathbb{Q}[x, y, y_1, \dots, y_{n-1}]$ is an irreducible polynomial over $\bar{\mathbb{Q}}$. A rational solution $y = f(x) \in \bar{\mathbb{Q}}(x)$ of (1) should satisfy the following equation

$$F(x, f(x), f'(x), \dots, f^{(n-1)}(x)) = 0. \tag{2}$$

By regarding $x, y, y', \dots, y^{(n-1)}$ as independent variables, whose values are in the field $\bar{\mathbb{Q}}$, the equation $F(x, y, y_1, \dots, y_{n-1}) = 0$ defines an algebraic hypersurface $\mathcal{S}$ in the space $\mathbb{A}^{n+1}(\bar{\mathbb{Q}})$, here $F(x, y, y_1, \dots, y_{n-1})$ denotes the algebraic polynomial F in $n+1$ variables $x, y,\ y_1, \dots\ y_{n-1}$. It follows from the condition (2) that the parametric space curve $\mathcal{C} = (x, f(x), f'(x), \dots, f^{(n-1)}(x))$ lies on the hypersurface $\mathcal{S}$, where $\mathcal{C}$ is called the *solution curve* of $y = f(x)$.

Assume that the hypersurface $\mathcal{S}$ can be parametrized properly by rational functions in $\bar{\mathbb{Q}}(s_1, \dots, s_n)$:

$$\mathcal{P}(s_1, \dots, s_n) = (\mathcal{X}_1(s_1, \dots, s_n), \dots, \mathcal{X}_{n+1}(s_1, \dots, s_n)).$$

Since $\mathcal{P}$ is a birational map $\mathbb{A}^n(\bar{\mathbb{Q}}) \to \mathcal{S} \subset \mathbb{A}^{n+1}(\bar{\mathbb{Q}})$, there exists a birational inverse map $\mathcal{P}^{-1}$ defining on the hypersurface $\mathcal{S}$ except finitely many algebraic sets with dimension less than n (e.g. space curves, points and so on).

Definition 9. A solution $y = f(x)$ of the equation $F(x, y, y', \dots, y^{(n-1)}) = 0$ is *parametrizable by* $\mathcal{P}$ if the solution curve $\mathcal{C}$ is almost contained in $\mathrm{im}(\mathcal{P}) \cap \mathrm{dom}(\mathcal{P}^{-1})$, where $\mathrm{im}(\mathcal{P})$ is the image of $\mathcal{P}$, $\mathrm{dom}(\mathcal{P}^{-1})$ is the domain of $\mathcal{P}^{-1}$. Here "almost" means except for finitely many points.

Proposition 10. *Let $F(x, y, y_1, \dots, y_{n-1}) = 0$ be a rational hypersurface with a proper parametrization*

$$\mathcal{P}(s_1, \dots, s_n) = (\mathcal{X}_1(s_1, \dots, s_n), \dots, \mathcal{X}_{n+1}(s_1, \dots, s_n)).$$

Then $F(x, y, y', \dots, y^{(n-1)}) = 0$ has a rational solution, which is parametrizable by $\mathcal{P}$, if and only if there exist rational functions $s_1(x)$,…,$s_n(x)$ such that

$$\begin{cases} \mathcal{X}_1(s_1(x), \dots, s_n(x)) = x \\ \frac{d\mathcal{X}_2(s_1(x), \dots, s_n(x))}{dx} = \mathcal{X}_3(s_1(x), \dots, s_n(x)) \\ \quad\vdots \\ \frac{d\mathcal{X}_n(s_1(x), \dots, s_n(x))}{dx} = \mathcal{X}_{n+1}(s_1(x), \dots, s_n(x)). \end{cases} \tag{3}$$

Proof. ($\Longrightarrow$) Assume that $y = f(x)$ is a rational solution of the differential equation

$$F(x, y, y', \dots, y^{(n-1)}) = 0,$$

which is parametrizable by $\mathcal{P}$. Then let

$$(s_1(x), \dots, s_n(x)) = \mathcal{P}^{-1}(x, f(x), f'(x), \dots, f^{(n-1)}(x)).$$

It follows that

$$\begin{aligned} \mathcal{P}(s_1(x), \dots, s_n(x)) &= \mathcal{P}(\mathcal{P}^{-1}(x, f(x), f'(x), \dots, f^{(n-1)}(x))) \\ &= (x, f(x), f'(x), \dots, f^{(n-1)}(x)), \end{aligned}$$

which means

$$\begin{cases} \mathcal{X}_1(s_1(x), \dots, s_n(x)) = x \\ \mathcal{X}_2(s_1(x), \dots, s_n(x)) = f(x) \\ \quad\vdots \\ \mathcal{X}_{n+1}(s_1(x), \dots, s_n(x)) = f^{(n-1)}(x). \end{cases}$$

Moreover, $(s_1(x), \dots, s_n(x))$ is a rational curve because $\mathcal{P}^{-1}$ is a birational map and the coordinate functions of $(x, f(x), f'(x), \dots, f^{(n-1)}(x))$ are also rational functions in x.

($\Longleftarrow$) If rational functions $s_1 = s_1(x), \dots, s_n = s_n(x)$ satisfy the system (3), then it is obvious that $y = \mathcal{X}_2(s_1(x), \dots, s_n(x))$ is a rational solution of the differential equation $F(x, y, y', \dots, y^{(n-1)}) = 0$. $\square$

Suppose that $s_1 = s_1(x), \dots, s_n = s_n(x)$ are n rational functions satisfying the system (3). We can get the following system by differentiating the first equation of (3) and expanding the remaining equations of (3)

$$\begin{cases} \frac{\partial \mathcal{X}_1(s_1(x),\dots,s_n(x))}{\partial s_1} \cdot s_1'(x) + \dots + \frac{\partial \mathcal{X}_1(s_1(x),\dots,s_n(x))}{\partial s_n} \cdot s_n'(x) = 1 \\ \frac{\partial \mathcal{X}_2(s_1(x),\dots,s_n(x))}{\partial s_1} \cdot s_1'(x) + \dots + \frac{\partial \mathcal{X}_2(s_1(x),\dots,s_n(x))}{\partial s_n} \cdot s_n'(x) = \mathcal{X}_3(s_1(x), \dots, s_n(x)) \\ \vdots \\ \frac{\partial \mathcal{X}_n(s_1(x),\dots,s_n(x))}{\partial s_1} \cdot s_1'(x) + \dots + \frac{\partial \mathcal{X}_n(s_1(x),\dots,s_n(x))}{\partial s_n} \cdot s_n'(x) = \mathcal{X}_{n+1}(s_1(x), \dots, s_n(x)). \end{cases}$$

If

$$\det \begin{pmatrix} \frac{\partial \mathcal{X}_1(s_1(x),\dots,s_n(x))}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_1(s_1(x),\dots,s_n(x))}{\partial s_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{X}_n(s_1(x),\dots,s_n(x))}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_n(s_1(x),\dots,s_n(x))}{\partial s_n} \end{pmatrix} \neq 0, \tag{4}$$

then $(s_1(x), \dots, s_n(x))$ is a solution of the following system of differential equations

$$\begin{cases} s_1'(x) = \frac{M_1(s_1,\dots,s_n)}{N(s_1,\dots,s_n)} \\ \vdots \\ s_n'(x) = \frac{M_n(s_1,\dots,s_n)}{N(s_1,\dots,s_n)}, \end{cases} \tag{5}$$

where

$$M_i = \det \begin{pmatrix} \frac{\partial \mathcal{X}_1}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_1}{\partial s_{i-1}} & 1 & \frac{\partial \mathcal{X}_1}{\partial s_{i+1}} & \dots & \frac{\partial \mathcal{X}_1}{\partial s_n} \\ \frac{\partial \mathcal{X}_2}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_2}{\partial s_{i-1}} & \mathcal{X}_3 & \frac{\partial \mathcal{X}_2}{\partial s_{i+1}} & \dots & \frac{\partial \mathcal{X}_2}{\partial s_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{X}_n}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_n}{\partial s_{i-1}} & \mathcal{X}_{n+1} & \frac{\partial \mathcal{X}_n}{\partial s_{i+1}} & \dots & \frac{\partial \mathcal{X}_n}{\partial s_n} \end{pmatrix},$$

$$N = \det \begin{pmatrix} \frac{\partial \mathcal{X}_1}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_1}{\partial s_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{X}_n}{\partial s_1} & \dots & \frac{\partial \mathcal{X}_n}{\partial s_n} \end{pmatrix}.$$

If the determinant (4) is equal to 0, then $(s_1(x), \dots, s_n(x))$ is a solution of the system

$$\begin{cases} \bar{M}_1(s_1, \dots, s_n) = 0 \\ \vdots \\ \bar{M}_{n-1}(s_1, \dots, s_n) = 0 \\ \bar{N}(s_1, \dots, s_n) = 0, \end{cases}$$

where $\bar{M}_i(s_1, \dots, s_n)$ and $\bar{N}(s_1, \dots, s_n)$ are numerators of $M_i(s_1, \dots, s_n)$ $(1 \leq i \leq n-1)$ and $N(s_1, \dots, s_n)$ respectively.

Definition 11. The system (5) is called *associated system of autonomous ODEs* of the non-autonomous ODE (1) with respect to $\mathcal{P}(s_1, \dots, s_n)$.

Remark 12. The associated system of autonomous ODEs in new indeterminates $s_1, \dots, s_n$ is of order 1 in $s_1, \dots, s_n$ and degree 1 with respect to $s_1', \dots, s_n'$.

4 Correspondence of rational general solutions between original ODE and associated system of ODEs

Let $U_i, V_i \in \bar{\mathbb{Q}}[s_1, \ldots, s_n]$ be the numerator and denominator of $\frac{M_i(s_1,\ldots,s_n)}{N(s_1,\ldots,s_n)}$ in the associated system (5) of autonomous ODEs. The polynomials U_i and V_i $(1 \le i \le n)$ introduced here will be used throughout the paper. From now on, we consider the differential polynomial set

$$\mathbb{A} = \{A_1, \ldots, A_n\} \subset \mathcal{K}\{s_1, \ldots, s_n\}, \quad A_i = s_i' V_i - U_i \text{ for any } 1 \le i \le n.$$

According to the definition of autoreduced set and Proposition 1 in [16, chap I, sect. 9], we have

Proposition 13. *Let $\mathbb{A} = \{A_1, \ldots, A_n\}$, where $A_i = s_i' V_i - U_i$ for $1 \le i \le n$. Then $\mathbb{A}$ is an autoreduced set relative to the ord-lex ranking. Furthermore, for any differential polynomial $G \in \mathcal{K}\{s_1, \ldots, s_n\}$, there exists the following unique representation by consecutive reductions with respect to $\mathbb{A}$*

$$S_1^{k_1} \cdots S_n^{k_n} I_1^{l_1} \cdots I_n^{l_n} G = \sum_j Q_{1j} A_1^{(j)} + \sum_j Q_{2j} A_2^{(j)} + \cdots + \sum_j Q_{nj} A_n^{(j)} + R,$$

where S_i and I_i are the separant and initial of A_i respectively, $k_i, l_i \in \mathbb{N}$, $A_i^{(j)}$ is the j-th derivatives of A_i, $Q_{ij} \in \mathcal{K}\{s_1, \ldots, s_n\}$, $i = 1, \ldots, n$, and R is reduced with respect to $\mathbb{A}$. Here, R is called the differential pseudo remainder of G with respect to $\mathbb{A}$, denoted by $\mathrm{prem}(G, \mathbb{A})$.

Definition 14. A rational solution $(\bar{s}_1(x), \ldots, \bar{s}_n(x))$ of the associated system (5) of autonomous ODEs is called a *rational general solution*, if for any differential polynomial $G \in \mathcal{K}\{s_1, \ldots, s_n\}$,

$$G(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = 0 \iff \mathrm{prem}(G, \mathbb{A}) = 0,$$

where $\mathbb{A} = \{A_1, \ldots, A_n\}$ and $A_i = s_i' V_i - U_i$ for $1 \le i \le n$.

Remark 15. As the degree of s_i' $(1 \le i \le n)$ is 1, it follows that $\mathrm{prem}(G, \mathbb{A}) \in \mathcal{K}[s_1, \ldots, s_n]$ for any differential polynomial $G \in \mathcal{K}\{s_1, \ldots, s_n\}$.

Proposition 16. *Let $(\bar{s}_1(x), \ldots, \bar{s}_n(x))$ be a rational general solution of the associated system (5) of autonomous ODEs and $G \in \mathcal{K}[s_1, \ldots, s_n]$. If $G(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = 0$, then $G = 0$ in $\mathcal{K}[s_1, \ldots, s_n]$.*

Proof. Since $G \in \mathcal{K}[s_1, \ldots, s_n]$, we have $\mathrm{prem}(G, \mathbb{A}) = G$. It follows from Definition 14 that $G(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = 0$ implies $G = 0$. □

Theorem 17. *Let $\bar{y} = f(x)$ be a rational general solution of non-autonomous differential equation $F(x, y, y', \ldots, y^{(n-1)}) = 0$, if $\bar{y} = f(x)$ is parametrizable by $\mathcal{P}$, then*

$$(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = \mathcal{P}^{-1}(x, f(x), f'(x), \ldots, f^{(n-1)}(x))$$

is a rational general solution of the associated system (5) of autonomous ODEs when

$$\det \begin{pmatrix} \frac{\partial \mathcal{X}_1(\bar{s}_1(x),\ldots,\bar{s}_n(x))}{\partial \bar{s}_1} & \cdots & \frac{\partial \mathcal{X}_1(\bar{s}_1(x),\ldots,\bar{s}_n(x))}{\partial \bar{s}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{X}_n(\bar{s}_1(x),\ldots,\bar{s}_n(x))}{\partial \bar{s}_1} & \cdots & \frac{\partial \mathcal{X}_n(\bar{s}_1(x),\ldots,\bar{s}_n(x))}{\partial \bar{s}_n} \end{pmatrix} \neq 0.$$

Conversely, let $(\hat{s}_1(x), \ldots, \hat{s}_n(x))$ is a rational general solution of the associated system (5) of autonomous ODEs, then

$$\hat{y} = \mathcal{X}_2(\hat{s}_1(x - c), \ldots, \hat{s}_n(x - c))$$

is a rational general solution of $F(x, y, y', \ldots, y^{(n-1)}) = 0$, where c is constant.

Proof. Obviously, $(\bar{s}_1(x), \ldots, \bar{s}_n(x))$ is a solution of (5). Suppose that $G \in \mathcal{K}\{s_1, \ldots, s_n\}$ is a differential polynomial such that $G(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = 0$. Let $R = \text{prem}(G, \mathbb{A})$, then $R \in \mathcal{K}[s_1, \ldots, s_n]$. Moreover, we have

$$R(\bar{s}_1(x), \ldots, \bar{s}_n(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x), \ldots, f^{(n-1)}(x))) = 0.$$

Let

$$R(\mathcal{P}^{-1}(x, y, y_1, \ldots, y_{n-1})) = \frac{M(x, y, y_1, \ldots, y_{n-1})}{N(x, y, y_1, \ldots, y_{n-1})},$$

then $M(x, y, y', \ldots, y^{(n-1)})$ is a differential polynomial satisfying the condition

$$M(x, f(x), f'(x), \ldots, f^{(n-1)}(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F(x, y, y', \ldots, y^{(n-1)}) = 0$ and both F and M are the $n-1$ order differential polynomials, we have

$$IM(x, y, y', \ldots, y^{(n-1)}) = M_0 F,$$

where I is the initial of F and M_0 is a differential polynomial of order $n-1$ in $\mathcal{K}\{y\}$. Therefore,

$$\begin{aligned} R(s_1, \ldots, s_n) &= R(\mathcal{P}^{-1}(\mathcal{P}(s_1, \ldots, s_n))) \\ &= \frac{I(\mathcal{P}(s_1, \ldots, s_n))M(\mathcal{P}(s_1, \ldots, s_n))}{I(\mathcal{P}(s_1, \ldots, s_n))N(\mathcal{P}(s_1, \ldots, s_n))} \\ &= \frac{M_0(\mathcal{P}(s_1, \ldots, s_n))F(\mathcal{P}(s_1, \ldots, s_n))}{I(\mathcal{P}(s_1, \ldots, s_n))N(\mathcal{P}(s_1, \ldots, s_n))} \\ &= 0. \end{aligned}$$

According to Definition 14, we know that $(\bar{s}_1(x), \ldots, \bar{s}_n(x))$ is a rational general solution of (5).

Next, we need to construct a rational general solution of $F(x, y, y', \ldots, y^{(n-1)}) = 0$ from a rational general solution of the associated system (5) of autonomous ODEs. Assume that $(\hat{s}_1(x), \ldots, \hat{s}_n(x))$ is a rational general solution of (5). We have $\mathcal{X}_1(\hat{s}_1(x), \ldots, \hat{s}_n(x)) = x + c$ by substituting $\hat{s}_1(x), \ldots, \hat{s}_n(x)$ into $\mathcal{X}_1(s_1, \ldots, s_n)$, where c is constant. It follows that $\mathcal{X}_1(\hat{s}_1(x - c), \ldots, \hat{s}_n(x - c)) = x$. Therefore, $\hat{y} = \mathcal{X}_2(\hat{s}_1(x - c), \ldots, \hat{s}_n(x - c))$ is a rational solution of $F(x, y, y', \ldots, y^{(n-1)}) = 0$. Moreover, it is necessary to prove that $\hat{y}$ is a rational general solution. Let $G \in \mathcal{K}\{s_1, \ldots, s_n\}$ such that $G(\hat{y}) = 0$ and $R = \text{prem}(G, F)$ the differential pseudo remainder of G with respect to F. Obviously, $R(\hat{y}) = 0$. We only need to prove that $R = 0$. If $R \neq 0$, then

$$R(\mathcal{X}_1(s_1, \ldots, s_n), \ldots, \mathcal{X}_n(s_1, \ldots, s_n)) = \frac{U(s_1, \ldots, s_n)}{V(s_1, \ldots, s_n)} \in \bar{\mathbb{Q}}(s_1, \ldots, s_n).$$

As $R(\mathcal{X}_1(\hat{s}_1, \ldots, \hat{s}_n), \ldots, \mathcal{X}_n(\hat{s}_1, \ldots, \hat{s}_n)) = 0$, it follows that $U(\hat{s}_1, \ldots, \hat{s}_n) = 0$. By Proposition 16, we have $U(s_1, \ldots, s_n) = 0$. Hence

$$R(\mathcal{X}_1(s_1, \ldots, s_n), \ldots, \mathcal{X}_n(s_1, \ldots, s_n)) = 0.$$

Since F is irreducible and $\deg_{y^{(n-1)}}(R) < \deg_{y^{(n-1)}}(F)$, it follows that $R = 0$ in $\mathbb{Q}[x, y, y_1, \ldots, y_{n-1}]$. Therefore, $\hat{y}$ is a rational general solution of differential equation $F(x, y, y', \ldots, y^{(n-1)}) = 0$. □

5 Criterion for existence of rational general solutions of associated system of ODEs

It can be seen from Definition 14 that a rational general solution of the associated system (5) of autonomous ODEs is a generic zero of the following ideal

$$\mathcal{I} = \{G \in \mathcal{K}\{s_1, \ldots, s_n\} : \text{prem}(G, \mathbb{A}) = 0\},$$

where $\mathbb{A} = \{A_1, \ldots, A_n\}$ and $A_i = s_i' V_i - U_i$ for $1 \leq i \leq n$.

Proposition 18. *Let* $\mathcal{I} = \{G \in \mathcal{K}\{s_1, \dots, s_n\} : \operatorname{prem}(G, \mathbb{A}) = 0\}$*, then* $\mathcal{I}$ *is differential prime ideal in* $\mathcal{K}\{s_1, \dots, s_n\}$*.*

Proof. It is easy to prove that $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ is a differential prime ideal by an argument similar to that in [22, chap V, sect. 3], where

$$S_{\mathbb{A}}^{\infty} = \langle \prod_i S_i^{k_i} I_i^{l_i} : S_i \text{ and } I_i \text{ are the separant and initial of } A_i \in \mathbb{A}, k_i, l_i \in \mathbb{N} \rangle.$$

In what follows, we claim that $\mathcal{I} = [\mathbb{A}] : S_{\mathbb{A}}^{\infty}$. In fact, the inclusion relation "$\subseteq$" is obvious. We only need to prove $\mathcal{I} \supseteq [\mathbb{A}] : S_{\mathbb{A}}^{\infty}$. For any $G \in [\mathbb{A}] : S_{\mathbb{A}}^{\infty}$, let $R = \operatorname{prem}(G, \mathbb{A}) \in \mathcal{K}[s_1, \dots, s_n]$. Then there exists $G_1 = \prod_i S_i^{k_i} I_i^{l_i} \in S_{\mathbb{A}}^{\infty}$, such that $G_1 G \equiv 0 \mod [\mathbb{A}]$ and $0 \neq G_1 \in \mathcal{K}[s_1, \dots, s_n]$. On the other hand, we have $G_2 G \equiv R \mod [\mathbb{A}]$, where $G_2 = \prod_i S_i^{k_i'} I_i^{l_i'}$. Therefore, $G_1 R \equiv G_1 G_2 G \equiv 0 \mod [\mathbb{A}]$, i.e. $G_1 R \in [\mathbb{A}]$. It follows from $G_1 R \in \mathcal{K}[s_1, \dots, s_n]$ that $G_1 R = 0$. As $G_1 \neq 0$ and $\mathcal{K}[s_1, \dots, s_n]$ is integral domain, we have $R = 0$, which means $G \in \mathcal{I}$. The proposition is proved. □

Let

$$D_{n,m}(y) = \begin{vmatrix} C_{n+1}^0 \, y^{(n+1)} & C_{n+1}^1 \, y^{(n)} & \cdots & C_{n+1}^m \, y^{(n+1-m)} \\ C_{n+2}^0 \, y^{(n+2)} & C_{n+2}^1 \, y^{(n+1)} & \cdots & C_{n+2}^m \, y^{(n+2-m)} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n+1+m}^0 y^{(n+1+m)} & C_{n+1+m}^1 y^{(n+m)} & \cdots & C_{n+1+m}^m y^{(n+1)} \end{vmatrix}.$$

In [9, Lemma 2.6], it has been proved that any solution $\hat{y}$ of the differential equation $D_{n,m}(y) = 0$ has the following form

$$\hat{y} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdot + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0},$$

where a_i, b_j are constants. In fact, if $D_{n,m}(\hat{y}) = 0$, then $D_{\max\{n,m\},\max\{n,m\}}(\hat{y}) = 0$. Furthermore, we have $D_{l,k}(\hat{y}) = 0$ for any $l \geq n, k \geq m$. Therefore, we have the following criterion.

Theorem 19. *The associated system* (5) *of autonomous ODEs has a rational general solution* $(s_1(x), \dots, s_n(x))$ *with* $\deg(s_i(x)) \leq m_i$ *if and only if for all* i $(1 \leq i \leq n)$, $\operatorname{prem}(D_{m_i,m_i}(s_i), \mathbb{A}) = 0$.

Proof. Suppose that $(s_1(x), \dots, s_n(x))$ with $\deg(s_i) \leq m_i$ is a rational general solution of the associated system (5) of autonomous ODEs. According to Definition 14 and the above discussion, there exist n differential polynomials $D_{m_1,m_1}(s_1), \dots, D_{m_n,m_n}(s_n)$ such that $(s_1(x), \dots, s_n(x))$ is a solution of them. It follows that $\operatorname{prem}(D_{m_i,m_i}(s_i), \mathbb{A}) = 0$ for all i $(1 \leq i \leq n)$.

If $\operatorname{prem}(D_{m_i,m_i}(s_i), \mathbb{A}) = 0$ holds, then $D_{m_i,m_i}(s_i) \in \mathcal{I}$ for all i $(1 \leq i \leq n)$, where $\mathcal{I}$ is the differential prime ideal as defined in Proposition 18. As every prime ideal has a generic zero, it follows that $\mathcal{I}$ has a generic zero $(s_1(x), \dots, s_n(x))$. According to Definition 1, $(s_1(x), \dots, s_n(x))$ is a zero of differential polynomials $D_{m_1,m_1}(s_1), \dots, D_{m_n,m_n}(s_n)$. By the results in [9], these differential polynomials have only rational solutions and $\deg(s_i(x)) \leq m_i$ for all i $(1 \leq i \leq n)$. Therefore, the generic zero $(s_1(x), \dots, s_n(x))$ of $\mathcal{I}$ must be rational, i.e. the associated system (5) of ODEs has a rational general solution $(s_1(x), \dots, s_n(x))$ with $\deg(s_i(x)) \leq m_i$. □

Theorem 19 gives a criterion for existence of rational general solutions of the associated system (5) of autonomous ODEs if the degree bound of rational solutions of this system is given.

6 Polynomial system of autonomous ODEs

In the section, some good properties of the first order polynomial system of autonomous ODEs is studied. First, we consider the first order linear system of ODEs:

$$\begin{cases} s_1' = c_{1,1}s_1 + \cdots + c_{1,n}s_n + c_{1,n+1} \\ \quad\vdots \\ s_n' = c_{n,1}s_1 + \cdots + c_{n,n}s_n + c_{n,n+1}, \end{cases} \tag{6}$$

where $c_{i,j}$ $(1 \leq i \leq n, 1 \leq j \leq n+1)$ are constants. It is a special case of polynomial systems of ODEs.

Proposition 20. *Every rational solution of the first order linear system* (6) *of ODEs is a polynomial solution.*

Proof. Assume that $s_1(x) = \frac{K_1}{p^{m_1}L_1}, \ldots, s_n(x) = \frac{K_n}{p^{m_n}L_n}$, where p is an irreducible polynomial with respect to x, K_i, L_i have no factor of p for all i $(1 \leq i \leq n)$. If there exists $m_i > 0$, then

$$\mathrm{ord}_p(s_i'(x)) = m_i + 1.$$

In this case, let k be an index of $\max\{m_i : i = 1, \ldots, n\}$. In particular,

$$\mathrm{ord}_p(s_k'(x)) = m_k + 1.$$

On the other hand, we have

$$c_{k,1}s_1 + \cdots + c_{k,n}s_n + c_{k,n+1} = c_{k,1}\frac{K_1}{p^{m_1}L_1} + \cdots + c_{k,n}\frac{K_n}{p^{m_n}L_n} + c_{k,n+1},$$

Hence,

$$\mathrm{ord}_p(c_{k,1}s_1 + \cdots + c_{k,n}s_n + c_{k,n+1}) \leq \max\{m_1, \ldots, m_n\} = m_k.$$

If follows that

$$\mathrm{ord}_p(c_{k,1}s_1 + \cdots + c_{k,n}s_n + c_{k,n+1}) \leq m_k < \mathrm{ord}_p(s_k'(x)),$$

which is impossible. Therefore, $m_i = 0$ for all i $(1 \leq i \leq n)$. From the conclusion we have proved, we know that for any rational solution of linear system (6) of ODEs

$$\hat{s}_1(x) = \frac{\hat{K}_1}{\hat{L}_1}, \ldots, \hat{s}_n(x) = \frac{\hat{K}_n}{\hat{L}_n},$$

$\hat{L}_i$ has no irreducible factors with respect to x for any $i \in \{1, \ldots, n\}$, i.e. all $\hat{s}_i$ are polynomials. □

Proposition 21. *Every rational general solution of the first order linear system* (6) *is n polynomials of degree at most n.*

Proof. The linear system (6) can be written in the following form

$$\begin{pmatrix} s_1' \\ \vdots \\ s_n' \end{pmatrix} = M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} + \begin{pmatrix} c_{1,n+1} \\ \vdots \\ c_{n,n+1} \end{pmatrix},$$

where

$$M = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix}.$$

It follows that
$$\begin{pmatrix} s_1^{(m+1)} \\ \vdots \\ s_n^{(m+1)} \end{pmatrix} = M^{m+1} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} + M^m \begin{pmatrix} c_{1,n+1} \\ \vdots \\ c_{n,n+1} \end{pmatrix}.$$
By Theorem 19 and Proposition 20, we know that the first order linear system (6) of ODEs has a polynomial general solution of degree m ($m > n$) if and only if
$$\begin{cases} \operatorname{prem}(s_1^{(m+1)}, \mathbb{B}) = 0 \\ \quad\vdots \\ \operatorname{prem}(s_n^{(m+1)}, \mathbb{B}) = 0, \end{cases}$$
where $\mathbb{B} = \{B_1, \dots, B_n\}$ and $B_i = s_i' - c_{i,1}s_1 - \cdots - c_{i,n}s_n - c_{i,n+1}$ for all i ($1 \leq i \leq n$), i.e.
$$M^{m+1} = 0 \text{ and } M^m \begin{pmatrix} c_{1,n+1} \\ \vdots \\ c_{n,n+1} \end{pmatrix} = 0. \tag{7}$$
We will prove that (7) holds if and only if
$$M^n = 0.$$
In fact, the "if" is obvious. Conversely, since $M^{m+1} = 0$ ($m > n$), there exists an invertible matrix P, such that
$$M = P^{-1}NP,$$
where
$$N = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}$$
is the Jordan form of M and $N^n = 0$. It follows that
$$M^n = (P^{-1}NP)^n = P^{-1}N^nP = 0.$$
The proposition is proved. □

In the following, we consider the first order polynomial system of ODEs:
$$\begin{cases} s_1' = P_1(s_1, \dots, s_n) \\ \quad\vdots \\ s_n' = P_n(s_1, \dots, s_n), \end{cases} \tag{8}$$
where $P_i \in \bar{\mathbb{Q}}[s_1, \dots, s_n]$ and $\gcd(P_1, \dots, P_n) = 1$.

Let $(s_1(x), \dots, s_n(x))$ be a rational solution of the first order polynomial system (8) of ODEs, then it defines a parametric space curve $\mathcal{C}$. Let
$$\mathcal{ID} = \{K \in \mathcal{K}[s_1, \dots, s_n] : K(s_1(x), \dots, s_n(x)) = 0\}$$
be the implicit ideal determined by $\mathcal{C}$. It is clear that $\mathcal{ID}$ is a prime ideal. According to the results in [13, sects. 3 and 4], we can compute a basis $\mathbb{H}$ of $\mathcal{ID}$ under the lexicographical order $s_1 < \cdots < s_n$ by the theory of Gröbner bases, where
$$\mathbb{H} = \{H_1(s_1, s_2), H_2(s_1, s_2, s_3), \dots, H_{n-1}(s_1, \dots, s_n)\}, \tag{9}$$
i.e. $\mathcal{ID} = \langle \mathbb{H} \rangle = \langle H_1, \dots, H_{n-1} \rangle$.

Proposition 22. *Let* $(s_1(x),\dots,s_n(x))$ *be a rational solution of the first order polynomial system* (8) *of ODEs and* $\mathbb{H}=\{H_1(s_1,s_2),\dots,H_{n-1}(s_1,\dots,s_n)\}$ *the basis of the implicit ideal* $\mathcal{ID}$ *under the lexicographical order* $s_1<\cdots<s_n$ *determined by the parametric space curve* $(s_1(x),\dots,s_n(x))$*. Then there exist* $(n-1)^2$ *polynomials* $W_{i,1}(s_1,\dots,s_n),\dots,W_{i,n-1}(s_1,\dots,s_n)$*, where* $1\le i\le n-1$*, such that*

$$H_{is_1}P_1+\cdots+H_{is_{i+1}}P_{i+1}=H_1W_{i,1}+\cdots+H_{n-1}W_{i,n-1},\quad i=1,\dots,n-1,$$

where H_{is_j} *is the partial derivative of* H_i *with respect to* s_j *for* $1\le j\le n$*.*

Proof. Suppose that the first order polynomial system (8) of ODEs has a rational solution $(s_1(x),\dots,s_n(x))$. Let $\mathbb{H}=\{H_1(s_1,s_2),\dots,H_{n-1}(s_1,\dots,s_n)\}$ be the basis of the implicit prime ideal $\mathcal{ID}$ determined by parametric space curve $(s_1(x),\dots,s_n(x))$. Since $H_i\in\mathcal{ID}$, we have

$$H_i(s_1(x),\dots,s_{i+1}(x))=0,$$

for all i $(1\le i\le n-1)$. By differentiating the above equation with respect to x, we have

$$H_{is_1}(s_1(x),\dots,s_{i+1}(x))\cdot s_1'(x)+\cdots+H_{is_{i+1}}(s_1(x),\dots,s_{i+1}(x))\cdot s_{i+1}'(x)=0.$$

It follows that

$$\begin{aligned}H_{is_1}(s_1(x),\dots,s_{i+1}(x))\cdot P_1(s_1(x),\dots,s_n(x))+\cdots\\ +H_{is_{i+1}}(s_1(x),\dots,s_{i+1}(x))\cdot P_{i+1}(s_1(x),\dots,s_n(x))=0,\end{aligned}$$

i.e. $H_{is_1}P_1+\cdots+H_{is_{i+1}}P_{i+1}\in\mathcal{ID}$. Therefore, there exist polynomials $W_{i,1},\dots,W_{i,n-1}\in\mathcal{K}[s_1,\dots,s_n]$ such that

$$H_{is_1}P_1+\cdots+H_{is_{i+1}}P_{i+1}=H_1W_{i,1}+\cdots+H_{n-1}W_{i,n-1}$$

holds for all i $(1\le i\le n-1)$. The proposition is proved. □

Proposition 23. *Let* $H_i(s_1,\dots,s_{i+1})$ *be the polynomial such that*

$$H_{is_1}P_1+\cdots+H_{is_{i+1}}P_n=H_1W_{i,1}+\ldots+H_{n-1}W_{i,n-1}$$

for some $W_{i,j}(s_1,\dots,s_n)$*, where* $i=1,\dots,n-1,j=1,\dots,n-1$*. If*

$$H_i(0,\dots,0)=H_{is_1}(0,\dots,0)=\cdots=H_{is_{i+1}}(0,\dots,0)=0$$

and

$$\det\begin{pmatrix}H_{is_1s_1}(0,\dots,0) & \cdots & H_{is_1s_{i+1}}(0,\dots,0)\\ \vdots & \ddots & \vdots\\ H_{is_1s_{i+1}}(0,\dots,0) & \cdots & H_{is_{i+1}s_{i+1}}(0,\dots,0)\end{pmatrix}\neq 0, \tag{10}$$

then $P_k(0,\dots,0)=0$ *for all* k $(1\le k\le i+1)$*.*

Proof. From the Taylor expansion of $H_i(s_1,\dots,s_{i+1})$ at $(0,\dots,0)$ and the assumption $H_i(0,\dots,0)=H_{is_1}(0,\dots,0)=\cdots=H_{is_{i+1}}(0,\dots,0)=0$, we have

$$\begin{aligned}H_i(s_1,\dots,s_{i+1})=&\frac{1}{2!}(H_{is_1s_1}(0,\dots,0)s_1^2+\cdots+H_{is_{i+1}s_{i+1}}(0,\dots,0)s_{i+1}^2)\\ &+H_{is_1s_2}(0,\dots,0)s_1s_2+\cdots+H_{is_1s_{i+1}}(0,\dots,0)s_1s_{i+1}\\ &+H_{is_2s_3}(0,\dots,0)s_2s_3+\cdots+H_{is_2s_{i+1}}(0,\dots,0)s_2s_{i+1}\\ &+\cdots+H_{is_is_{i+1}}(0,\dots,0)s_is_{i+1}+\text{higher order}.\end{aligned}$$

It follows that

$$H_{is_j}(s_1,\dots,s_{i+1}) = \sum_{m=1}^{i+1} H_{is_js_m}(0,\dots,0)s_m + \text{higher order}, \quad j=1,\dots,i+1.$$

Let

$$P_k(s_1,\dots,s_n) = p_{k,0} + p_{k,1}s_1 + \cdots + p_{k,n}s_n + \text{higher order}, \quad k=1,\dots,n,$$

then

$$\begin{aligned} H_{is_1}P_1 + \cdots + H_{is_i+1}P_{i+1} &= \sum_{l=1}^{i+1}(p_{l,0}\sum_{m=1}^{i+1} H_{is_ls_m}(0,\dots,0)s_m) + \text{higher order} \\ &= \sum_{m=1}^{i+1}(\sum_{l=1}^{i+1} p_{l,0}H_{is_ls_m}(0,\dots,0))s_m + \text{higher order.} \end{aligned}$$

According to the Taylor expansion of $H_{is_1,\dots,s_{i+1}}$, we have $H_1W_{i,1} + \ldots + H_{n-1}W_{i,n-1}$ is of order at least 2. Therefore,

$$\begin{cases} p_{1,0}H_{is_1s_1}(0,\dots,0) + \cdots + p_{i+1,0}H_{is_1s_{i+1}}(0,\dots,0) = 0 \\ \vdots \\ p_{1,0}H_{is_1s_{i+1}}(0,\dots,0) + \cdots + p_{i+1,0}H_{is_{i+1}s_{i+1}}(0,\dots,0) = 0. \end{cases}$$

It follows from the assumption (10) that

$$p_{k,0} = 0, \quad k = 1,\dots,i+1,$$

i.e. $P_k(0,\dots,0) = 0$ for all k $(1 \le k \le i+1)$, where $i = 1,\dots,n-1$. □

7 Conclusion and future work

In this paper, we studied the rational general solutions of $n-1$ ($n > 2$) order non-autonomous ODEs. The main strategy is to reduce the problem for finding the rational general solutions of high order non-autonomous ODEs to that of finding the rational general solutions of the first order rational system of autonomous ODEs. Moreover, it is proved that the rational general solutions between the original non-autonomous ODE with order $n-1$ and the associated first order system of autonomous ODEs are corresponding. Furthermore, a criterion for existence of rational general solutions of the associated system of autonomous ODEs is presented, and some nice properties of the first order polynomial system of autonomous ODEs are introduced.

The following are several open problems on determining the rational general solutions of high order non-autonomous ODEs.

(a) When $n = 2$, each solution curve is corresponding to a plane curve, which is an invariant algebraic curve of the polynomial system (8) of ODEs in the sense that its defining equation $H(s_1, s_2) = 0$ satisfies

$$H_{s_1}P_1 + H_{s_2}P_2 = HW,$$

for some polynomial $W \in \bar{\mathbb{Q}}[s_1, s_2]$ and H_{s_i} is the partial derivative of H with respect to s_i for $i = 1, 2$. When $n > 2$, e.g $n = 3$, the solution curves are now space curves. Let

$$\{H_1(s_1, s_2), H_2(s_1, s_2, s_3)\}$$

be a basis of implicit ideal $\{K \in \mathcal{K}[s_1, s_2, s_3] : K(s_1(x), s_2(x), s_3(x)) = 0\}$ determined by the rational solution $(s_1(x), s_2(x), s_3(x))$ of

$$\begin{cases} s_1' = P_1(s_1, s_2, s_3) \\ s_2' = P_2(s_1, s_2, s_3) \\ s_3' = P_3(s_1, s_2, s_3). \end{cases}$$

We have the following relationship between H_1, H_2 and P_1, P_2, P_3:

$$\begin{cases} H_{1s_1}P_1 + H_{1s_2}P_2 = H_1W_{1,1} + H_2W_{1,2} \\ H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3 = H_1W_{2,1} + H_2W_{2,2}, \end{cases}$$

where H_{is_j} is the partial derivative of H_i with respect to s_j for $j = 1, 2, 3$, and $W_{i,j} \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$. The problem is how to compute H_i? Can we do this by the method of undetermine coefficients with some upper bound on the degree of H_1 and H_2 which is similar to that in [19]?

(b) Suppose that

$$\mathbb{H} = \{H_1(s_1, s_2), H_2(s_1, s_2, s_3), \ldots, H_{n-1}(s_1, \ldots, s_n)\} \subseteq \mathcal{K}[s_1, \ldots, s_n]$$

is a basis of implicit ideal

$$\mathcal{ID} = \{K \in \mathcal{K}[s_1, \ldots, s_n] : K(s_1(x), \ldots, s_n(x)) = 0\}$$

determined by the parametric space curve $(s_1(x), \ldots, s_n(x))$ under the lexicographical order $s_1 < \cdots < s_n$, and $(s_1(x), \ldots, s_n(x))$ is a rational solution of the first order polynomial system (8) of ODEs. How do we parametrize the implicit variety (i.e. space curve $\boldsymbol{Z}(\mathbb{H})$) rationally? As we all know, this problem can be done (e.g. [1, 2]). Moreover, the space curve also can be parametrized by proper parametrization.

(c) Assume that $\mathbb{H}$ is the basis of implicit ideal $\mathcal{ID}$ and $(s_1(x), \ldots, s_n(x))$ is a rational parametrization of $\boldsymbol{Z}(\mathbb{H})$. When is $(s_1(x), \ldots, s_n(x))$ a solution of the polynomial system (8) of ODEs? Are all rational solutions of (8) proper?

(d) Can we find the rational general solutions of (8) from a family of $\mathbb{H}_i$?

It would be very interesting to develop some methods for solving the above problems in the near future.

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