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Rational Solutions of a Rational System of Autonomous ODEs: Generalization to Trivariate Case and Problems *

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Abstract

This paper presents a method for computing the explicitly rational solutions of a trivariate rational system of autonomous ordinary differential equations (ODEs) based on the proper parametrization of its invariant algebraic space curve. It is the generalization of the method in [11]. First, an undetermined coefficients method is given for computing the invariant algebraic space curves of the trivariate polynomial system of autonomous ODEs. Then we extend some nice properties of the proper parametrization for the plane curves to the case of space curves. Moreover, an algorithm is provided for computing the rational solutions of the trivariate polynomial system of autonomous ODEs based on previous preparation. Finally, we generalize this algorithm to compute the rational solutions of the trivariate rational system of autonomous ODEs and give some relations and properties of the rational solutions.

Keywords: Rational solutions, invariant algebraic space curves, trivariate rational system of autonomous ODEs, proper parametrization.

1 Introduction

In [9], we have reduced the problem for computing the rational general solutions of $n-1$ ($n > 2$) order non-autonomous ODEs to finding the rational general solutions of an associated first order rational system of autonomous ODEs in n indeterminates based on the proper parametrization of hypersurface. Based on the previous work, this paper mainly consider the problem for finding the rational solutions of the first order trivariate rational system of autonomous ODEs

$$\begin{cases} s_1' = \dfrac{U_1(s_1, s_2, s_3)}{V_1(s_1, s_2, s_3)} \\ s_2' = \dfrac{U_2(s_1, s_2, s_3)}{V_2(s_1, s_2, s_3)} \\ s_3' = \dfrac{U_3(s_1, s_2, s_3)}{V_3(s_1, s_2, s_3)}, \end{cases} \tag{1}$$

where $U_i, V_i \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$ for $i = 1, 2, 3$. A rational solution of the system (1) is a 3-tuple of rational functions $(s_1(x), s_2(x), s_3(x))$ that satisfies the given system. Each rational solution of

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(1) represents a rational algebraic space curve for which the rational solution itself is a rational parametrization of the space curve. Such a rational algebraic space curve is implicitly defined by the intersection of two algebraic surfaces. Therefore, it is possible to compute a rational solution of (1) by finding the implicit rational algebraic space curve (i.e. the implicit defining equations of two algebraic surfaces such that this space curve is determined by their intersection) of the possible rational solutions first, and then choosing suitable parametrizations of the space curve which satisfy the differential system (1).

2 Invariant algebraic space curves of polynomial system of autonomous ODEs

Consider the trivariate polynomial system of autonomous ODEs

$$\begin{cases} s_1' = P_1(s_1, s_2, s_3) \\ s_2' = P_2(s_1, s_2, s_3) \\ s_3' = P_3(s_1, s_2, s_3), \end{cases} \tag{2}$$

where $P_1, P_2, P_3 \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$ with constant coefficients. It is a special case of trivariate rational system (1) of autonomous ODEs. In [9, sect. 6], we have introduced some nice properties of general polynomial system of autonomous ODEs. In particular, if $(s_1(x), s_2(x), s_3(x))$ is a rational solution of polynomial system (2) of autonomous ODEs , then this solution defines a parametric space curve. Let

$$\mathcal{ID} = \{K \in \bar{\mathbb{Q}}[s_1, s_2, s_3] : K(s_1(x), s_2(x), s_3(x)) = 0\}$$

be the implicit ideal determined by this parametric space curve, then the basis $\mathbb{H}$ of the implicit ideal $\mathcal{ID}$ under the lexicographical order $s_1 < s_2 < s_3$ can be computed by the results in [8, sects. 3 and 4], where $\mathbb{H} = [H_1(s_1, s_2), H_2(s_1, s_2, s_3)]$ and the leading term of H_i is a power of s_{i+1} with coefficient 1 for $i = 1, 2$. Furthermore, according to Proposition 22 in [9], there exists $W_{i,j} \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$ such that

$$\begin{cases} H_{1s_1}P_1 + H_{1s_2}P_2 = H_1W_{1,1} + H_2W_{1,2} \\ H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3 = H_1W_{2,1} + H_2W_{2,2}. \end{cases}$$

In this section, we present a method by using the same technique as in [10,11] for computing the implicit equations of two algebraic surfaces $H_1(s_1, s_2) = 0$ and $H_2(s_1, s_2, s_3) = 0$ such that the possible rational solutions of polynomial system (2) of autonomous ODEs can be computed from the parametrization of the space curve determined by their intersection.

Let $\mathcal{K}$ be an algebraically closed field of characteristic 0, $s_1 < s_2 < s_3$ be 3 ordered variables. For any polynomial $F \in \mathcal{K}[s_1, s_2, s_3] \setminus \mathcal{K}$, the biggest index p such that the degree of F in s_p is greater than 0 is called the *class* of F, denoted by $\mathrm{cls}(F)$. Define $\mathrm{cls}(F) = 0$ if $F \in \mathcal{K}$. Let $p = \mathrm{cls}(F) > 0$, s_p is called the leading variable of F, denoted by $\mathrm{lv}(F)$.

Definition 1. Let $\mathcal{I} = \langle H_1, H_2 \rangle$ be a 1-dimensional ideal, where $\mathrm{lv}(H_1) = s_2$ and $\mathrm{lv}(H_2) = s_3$. An *invariant algebraic space curve* of trivariate polynomial system (2) of autonomous ODEs is an algebraic variety $\boldsymbol{Z}(\mathcal{I})$, such that

$$\begin{cases} H_{1s_1}P_1 + H_{1s_2}P_2 = H_1W_{1,1} + H_2W_{1,2} \\ H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3 = H_1W_{2,1} + H_2W_{2,2}, \end{cases}$$

for some polynomial $W_{i,j} \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$, $i, j = 1, 2$.

Note that $\boldsymbol{Z}(\mathcal{I}) = \boldsymbol{Z}(\langle H_1, H_2\rangle) = \boldsymbol{Z}(H_1, H_2)$. It can be seen from Definition 1 that $H_{1s_1}P_1 + H_{1s_2}P_2, H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3 \in \mathcal{I}$. This refers to the ideal membership problem. We can solve it by the theory of Gröbner bases. Therefore, one way to compute H_1 and H_2 is by the undetermined coefficients method which is similar to the methods used for computing the formal solutions of first order ODEs in [10] and the invariant algebraic curves of planar polynomial differential system in [11]. In fact, if the degrees of H_1 and H_2 are given, the system of equations on the coefficients of H_1 and H_2 can be obtained by equating the normal form of $H_{1s_1}P_1+H_{1s_2}P_2$ and $H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3$ modulo $\mathbb{G}$ to zero, where $\mathbb{G}$ is the Gröbner basis of the ideal $\mathcal{I}$. By solving the obtained system, we can get the defining polynomials of two surfaces $H_1 = 0$ and $H_2 = 0$, their intersection is the invariant algebraic space curve we want to find.

Example 2. Consider the trivariate polynomial system of autonomous ODEs

$$\begin{cases} s_1' = s_1 s_3 - s_2 \\ s_2' = 2s_1^2 - s_1 s_2 \\ s_3' = s_1^2. \end{cases} \tag{3}$$

First, we look for an invariant algebraic space curve $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ satisfying $\deg(H_1) = \deg(H_2) = 1$. Assume

$$H_1 = s_2 + c_1 s_1 + c_2, \quad H_2 = s_3 + c_3 s_2 + c_4 s_1 + c_5.$$

Then the Gröbner basis of $\langle H_1, H_2\rangle$ with respect to the lexicographic order determined by $s_1 < s_2 < s_3$ is

$$\mathbb{G} = [s_2 + c_1 s_1 + c_2, s_3 + (c_4 - c_3 c_1)s_1 + c_5 - c_3 c_2]$$

and

$$\begin{aligned} &H_{1s_1}P_1 + H_{1s_2}P_2 = c_1 s_3 s_1 - s_2 s_1 - c_1 s_2 + 2s_1^2, \\ &H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3 = c_4 s_3 s_1 - c_3 s_2 s_1 - c_4 s_2 + (2c_3 + 1)s_1^2. \end{aligned}$$

The normal form of $H_{1s_1}P_1 + H_{1s_2}P_2$ and $H_{2s_1}P_1 + H_{2s_2}P_2 + H_{2s_3}P_3$ modulo $\mathbb{G}$ are

$$(c_1 - c_1 c_4 + c_1^2 c_3 + 2)s_1^2 + (c_1^2 + c_2 - c_1 c_5 + c_1 c_2 c_3)s_1 + c_1 c_2$$

and

$$(c_1 c_3 - c_4^2 + c_1 c_3 c_4 + 2c_3 + 1)s_1^2 + (c_1 c_4 + c_2 c_3 - c_4 c_5 + c_2 c_3 c_4)s_1 + c_2 c_4$$

respectively. Therefore, the algebraic system of equations on the coefficients of H_1 and H_2 is

$$\begin{cases} c_1 - c_1 c_4 + c_1^2 c_3 + 2 = 0 \\ c_1^2 + c_2 - c_1 c_5 + c_1 c_2 c_3 = 0 \\ c_1 c_2 = 0 \\ c_1 c_3 - c_4^2 + c_1 c_3 c_4 + 2c_3 + 1 = 0 \\ c_1 c_4 + c_2 c_3 - c_4 c_5 + c_2 c_3 c_4 = 0 \\ c_2 c_4 = 0. \end{cases}$$

By solving this system, we obtain the following solution

$$\{c_1 = -1, c_2 = 0, c_3 = -1 - c_4, c_4 = c_4, c_5 = -1\},$$

i.e.

$$H_1 = s_2 - s_1, \quad H_2 = s_3 - (1 + c_4)s_2 + c_4 s_1 - 1. \tag{4}$$

Now we ask for an invariant algebraic space curve $\boldsymbol{Z}(H_1'(s_1,s_2),H_2'(s_1,s_2,s_3))$ such that $\deg(H_1')=2$, $\deg(H_2')=1$. Let

$$H_1' = s_2 + c_1 s_1^2 + c_2 s_1 + c_3, \quad H_2' = s_3 + c_4 s_2 + c_5 s_1 + c_6, \tag{5}$$

then the following solutions are computed by using the same procedure as above

$$\begin{aligned}&\{c_1=0, c_2=-1, c_3=0, c_4=-1-c_5, c_5=c_5, c_6=-1\},\\ &\{c_1=3/2, c_2=-4, c_3=0, c_4=0, c_5=2, c_6=-4\}.\end{aligned}$$

The first solution corresponds to the invariant algebraic space curve determined by (4), where $\deg(H_1)=\deg(H_2)=1$. For the second solution, it determines another invariant algebraic space curve $\boldsymbol{Z}(H_1',H_2')$, where

$$H_1' = s_2 + \frac{3}{2}s_1^2 - 4s_1, \quad H_2' = s_3 + 2s_1 - 4.$$

In Example 2, we computed two invariant algebraic space curves, one of them is a space line determined by the intersection of surfaces $H_1=0$ and $H_2=0$, another is a space conic determined by the intersection of $H_1'=0$ and $H_2'=0$.

Remark 3. In fact, we need to consider the following two cases for computing the invariant algebraic space curve $\boldsymbol{Z}(H_1',H_2')$ such that $\deg(H_1')=2$ and $\deg(H_2')=1$,

$$\begin{cases} H_1' = s_2 s_1 + c_1 s_2 + c_2 s_1^2 + c_3 s_1 + c_4 \\ H_2' = s_3 + c_5 s_2 + c_6 s_1 + c_7, \end{cases} \qquad \begin{cases} H_1' = s_2^2 + c_1 s_2 s_1 + c_2 s_2 + c_3 s_1^2 + c_4 s_1 + c_5 \\ H_2' = s_3 + c_6 s_2 + c_7 s_1 + c_8. \end{cases}$$

But in these two cases, the obtained algebraic systems of equations on the coefficients of H_1' and H_2' have no solution. Therefore, the computed $\boldsymbol{Z}(H_1',H_2')$ in the Example 2 is the only invariant algebraic space curve satisfying $\deg(H_1')=2$ and $\deg(H_2')=1$.

3 Rational solutions of polynomial system of autonomous ODEs

Let $\mathcal{C}$ be a space curve that is implicitly defined as

$$\begin{cases} f(x,y,z)=0 \\ g(x,y,z)=0, \end{cases} \tag{6}$$

i.e. the intersection of two algebraic surfaces $f(x,y,z)=0$ and $g(x,y,z)=0$.

It is known that any space curve can be birationally projected onto a plane curve (see [6], p.155). In addition, the parametrization problem for the algebraic plane curves has been widely studied (e.g. [1, 15]). Therefore, it is important to compute a projected plane curve such that the points on it are birational corresponding to the points of the original space curve $\mathcal{C}$ for computing the parametrization of rational space curve. In [2], Abhyankar and Bajaj presented a method for computing such a projected plane curve. In fact, the problem can be reduced to find an appropriate axis of projection. For example, if we choose z axis as the project direction, it is valid when $\mathrm{res}_z(f,g)$ is not a power of irreducible polynomial. If the z axis is not valid, we need to make a linear transformation of variables. The following general procedure may be adopted. First, compute the transformed equations $f_1(x_1,y_1,z_1)=0$ and $g_1(x_1,y_1,z_1)=0$ by substituting

$$\begin{cases} x = a_1x_1 + b_1y_1 + c_1z_1 \\ y = a_2x_1 + b_2y_1 + c_2z_1 \\ z = a_3x_1 + b_3y_1 + c_3z_1 \end{cases}$$

into the equations of the two surfaces (6) defining the space curve $\mathcal{C}$, where a_i, b_i, c_i are selected such that the determinant of them is nonzero. Then the resultant of f_1 and g_1 with respect to z_1 (denoted by $\mathrm{res}_{z_1}(f_1, g_1)$) is our required projected plane curve when it is not a power of irreducible polynomial. This condition can be achieved by choosing the different linear transformations. In fact, a suitable random choice of coefficients a_i, b_i, c_i can ensure that the projected plane curve $\mathrm{res}_{z_1}(f_1, g_1)$ is in birational correspondence with the original space curve $\mathcal{C}$ and thus of the same genus.

Example 4. Consider the space curve defined by the intersection

$$\begin{cases} H_1' = s_2 + \dfrac{3}{2}s_1^2 - 4s_1 = 0 \\ H_2' = s_3 + 2s_1 - 4 = 0 \end{cases}$$

in Example 2. Obviously, $\mathrm{res}_{s_3}(H_1', H_2') = H_1'$ is not a power of irreducible polynomial. It follows that $H_1' = 0$ is our required projected plane curve without making the linear transformation of variables. Therefore, it is birational corresponding to the original space curve $\boldsymbol{Z}(H_1', H_2')$. Since $(x, -3/2x^2 + 4x)$ is a proper parametrization of $H_1' = 0$, the proper parametrization of the space curve $\boldsymbol{Z}(H_1', H_2')$ is $(x, -3/2x^2 + 4x, -2x + 4)$.

Similarly, consider the invariant algebraic space curve $\boldsymbol{Z}(H_1, H_2)$ in Example 2 determined by

$$\begin{cases} H_1 = s_2 - s_1 = 0 \\ H_2 = s_3 - (1 + c_4)s_2 + c_4 s_1 - 1 = 0, \end{cases}$$

we can get $(x, x, x + 1)$ is a proper parametrization of $\boldsymbol{Z}(H_1, H_2)$.

The parametrization of rational space curve in Example 4 didn't involve the linear transformation of variables. But this strategy is useful and feasible for the general cases. Although $\boldsymbol{Z}(\prod_i H_{1,i}, \prod_j H_{2,j})$ can be decomposed into $\bigcup_{i,j} \boldsymbol{Z}(H_{1,i}, H_{2,j})$, not all their irreducible components $\boldsymbol{Z}(H_{1,i}, H_{2,j})$ can keep the property of invariant algebraic space curve. In other words, there may appear some components in the decomposition which are not the invariant algebraic space curves. Even if some components in the decomposition are the invariant algebraic space curves, we can compute them by setting the degree of H_1 and H_2 lower than before in the undetermined coefficients method. Therefore, from now on we only consider the invariant algebraic space curves determined by the irreducible polynomials.

Based on the above introduction, it is natural to generalize the relevant definitions and results for rational plane curves to the case of rational space curves. Moreover, the definition for the proper rational solutions in [11] can be generalized to the trivariate polynomial differential system as follows.

Definition 5. A rational solution of the trivariate polynomial system (2) of autonomous ODEs is called a *proper rational solution* if it is a proper rational parametrization of its corresponding invariant algebraic space curve. An invariant algebraic space curve of the trivariate polynomial differential system (2) is called a *rational invariant algebraic space curve* if it is a rational space curve, i.e. it has a rational parametrization.

Note that the proper rational solution $(s_1(x), s_2(x), s_3(x))$ means not all rational functions in this solution are constant, i.e. at least one of the components is non-constant.

Lemma 6. *Let $\mathcal{P}_1(t)$ be a proper parametrization of an affine rational space curve $\mathcal{C}$, and let $\mathcal{P}_2(t)$ be any other rational parametrization of $\mathcal{C}$.*

(a) There exists a nonconstant rational function $R(t) \in \mathcal{K}(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(R(t))$.

(b) $\mathcal{P}_2(t)$ is proper if and only if there exists a linear rational function $L(t) \in \mathcal{K}(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(L(t))$.

Proof. Assume that $\varphi : \mathcal{C} \to \widehat{\mathcal{C}}$ is a birational projection between the space curve $\mathcal{C}$ and the plane curve $\widehat{\mathcal{C}}$. As φ is birational, the rationality of $\mathcal{C}$ is consistent with that of $\widehat{\mathcal{C}}$. Let $\mathcal{P}_1(t)$ be a proper parametrization of rational space curve $\mathcal{C}$, then $\widehat{\mathcal{P}}_1(t) = \varphi(\mathcal{P}_1(t))$ is a proper parametrization of rational plane curve $\widehat{\mathcal{C}}$. Similarly, let $\mathcal{P}_2(t)$ be any other rational parametrization of space curve $\mathcal{C}$, then $\widehat{\mathcal{P}}_2(t) = \varphi(\mathcal{P}_2(t))$ is a parametrization of rational plane curve $\widehat{\mathcal{C}}$. According to Lemma 4.17 in [15], we have $\widehat{\mathcal{P}}_2(t) = \widehat{\mathcal{P}}_1(R(t))$, where $R(t) \in \mathcal{K}(t)$ is a nonconstant rational function. It follows that

$$\mathcal{P}_2(t) = \varphi^{-1}(\widehat{\mathcal{P}}_2(t)) = \varphi^{-1}(\widehat{\mathcal{P}}_1(R(t))) = \mathcal{P}_1(R(t)).$$

This proves (a). Since φ is birational, $\mathcal{P}_2(t)$ is proper if and only if $\widehat{\mathcal{P}}_2(t)$ is proper. Therefore, it is easy to prove (b) by a similar argument. $\square$

In this following, we present a method for finding rational solutions of the trivariate polynomial system (2) of autonomous ODEs based on the proper parametrization of the invariant algebraic space curves.

Theorem 7. *Any rational solution of the trivariate polynomial system* (2) *of autonomous ODEs is a proper rational solution.*

Proof. Let $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ be a rational solution of the polynomial system (2) of autonomous ODEs and $\{H_1(s_1, s_2), H_2(s_1, s_2, s_3)\}$ the basis of the implicit ideal $\mathcal{ID}$ under the lexicographical order $s_1 < s_2 < s_3$ determined by the parametric space curve $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$. Then

$$\begin{cases} \widehat{s}_1'(x) = P_1(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) \\ \widehat{s}_2'(x) = P_2(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) \\ \widehat{s}_3'(x) = P_3(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)). \end{cases} \tag{7}$$

Let $(s_1(x), s_2(x), s_3(x))$ be a proper parametrization of rational space curve $\boldsymbol{Z}(H_1, H_2)$. According to Proposition 6(a), there exists a nonconstant rational function $T(x)$, such that the two rational parametrizations of the same algebraic space curve $\boldsymbol{Z}(H_1, H_2)$ have the following relation

$$\begin{cases} \widehat{s}_1(x) = s_1(T(x)) \\ \widehat{s}_2(x) = s_2(T(x)) \\ \widehat{s}_3(x) = s_3(T(x)). \end{cases} \tag{8}$$

By (7) and (8), we have

$$\begin{cases} s_1'(T(x)) \cdot T'(x) = P_1(s_1(T(x)), s_2(T(x)), s_3(T(x))) \\ s_2'(T(x)) \cdot T'(x) = P_2(s_1(T(x)), s_2(T(x)), s_3(T(x))) \\ s_3'(T(x)) \cdot T'(x) = P_3(s_1(T(x)), s_2(T(x)), s_3(T(x))). \end{cases}$$

Note that at least one of rational functions $s_1(x), s_2(x)$ and $s_3(x)$ is non-constant. Therefore,

$$T'(x) = \begin{cases} \dfrac{P_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_1'(T(x))}, & \text{if } s_1'(x) \neq 0 \\ \dfrac{P_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_2'(T(x))}, & \text{if } s_2'(x) \neq 0 \\ \dfrac{P_3(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_3'(T(x))}, & \text{if } s_3'(x) \neq 0. \end{cases}$$

As $T(x)$ is unknown in above three autonomous differential equations which are of degree 1 with respect to $T'(x)$, it follows from Theorem 2.7 and Corollary 3.11 in [7] that the rational solution $T(x)$ is a linear rational function. Therefore, $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ is a proper rational solution of the system (2) by Proposition 6(b). $\square$

From the constructive proof for Theorem 7, we get the following theorem immediately.

Theorem 8. *Suppose that $\boldsymbol{Z}(H_1, H_2)$ is a rational invariant algebraic space curve of the trivariate polynomial system (2) of autonomous ODEs. Let $(s_1(x), s_2(x), s_3(x))$ be an arbitrary proper rational parametrization of the space curve $\boldsymbol{Z}(H_1, H_2)$. Then the polynomial system (2) has a proper rational solution*

$$\widehat{s}_1(x) = s_1(T(x)), \quad \widehat{s}_2(x) = s_2(T(x)), \quad \widehat{s}_3(x) = s_3(T(x))$$

corresponding to $\boldsymbol{Z}(H_1, H_2)$ if and only if there exists a linear rational transformation $T(x) = \frac{ax+b}{cx+d}$ which is a rational solution of the following autonomous differential equation

$$T'(x) = \begin{cases} \dfrac{P_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_1'(T(x))}, & \textit{if } s_1'(x) \neq 0 \\ \dfrac{P_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_2'(T(x))}, & \textit{if } s_2'(x) \neq 0 \\ \dfrac{P_3(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_3'(T(x))}, & \textit{if } s_3'(x) \neq 0. \end{cases}$$

In the following, an algorithm is given for computing the rational solutions of the polynomial system (2) of autonomous ODEs corresponding to the computed invariant algebraic space curves. Algorithm 1 always terminates and its correctness is guaranteed by Theorem 8.

Algorithm 1: Rational solutions of the trivariate polynomial system of autonomous ODEs

Input: Three polynomials $P_1(s_1, s_2, s_3), P_2(s_1, s_2, s_3), P_3(s_1, s_2, s_3) \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$, and the invariant algebraic space curve $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$, where H_1 and H_2 are irreducible.

Output: A rational solution of the polynomial system (2) corresponding to $\boldsymbol{Z}(H_1, H_2)$.

if *$\boldsymbol{Z}(H_1, H_2)$ is not a rational space curve* **then**
  return *No rational solution corresponding to $\boldsymbol{Z}(H_1, H_2)$*;
else
  compute a proper parametrization $(s_1(x), s_2(x), s_3(x))$ of $\boldsymbol{Z}(H_1, H_2)$;
  if $s_1'(x) \neq 0$ **then**
    compute the rational solution $T(x)$ of the autonomous differential equation

$$T'(x) = \frac{P_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_1'(T(x))};$$

  else if $s_2'(x) \neq 0$ **then**
    compute the rational solution $T(x)$ of the autonomous differential equation

$$T'(x) = \frac{P_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_2'(T(x))};$$

  else
    compute the rational solution $T(x)$ of the autonomous differential equation

$$T'(x) = \frac{P_3(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{s_3'(T(x))};$$

  end
  if *$T(x)$ is a linear rational function* **then**
    return $(s_1(T(x)), s_2(T(x)), s_3(T(x)))$;
  else
    return *No rational solution corresponding to $\boldsymbol{Z}(H_1, H_2)$*;
  end
end

Example 9. Continue considering the trivariate polynomial system (3) of autonomous ODEs

$$\begin{cases} s_1' = s_1 s_3 - s_2 \\ s_2' = 2s_1^2 - s_1 s_2 \\ s_3' = s_1^2. \end{cases}$$

From Examples 2 and 4, we have known that $\boldsymbol{Z}(H_1', H_2')$ is an invariant algebraic space curve, where

$$\begin{cases} H_1' = s_2 + \dfrac{3}{2}s_1^2 - 4s_1 \\ H_2' = s_3 + 2s_1 - 4, \end{cases}$$

and

$$(s_1(x), s_2(x), s_3(x)) = (x, -\frac{3}{2}x^2 + 4x, -2x + 4)$$

is a proper parametrization of the space curve $\boldsymbol{Z}(H_1', H_2')$. Since $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = -\frac{1}{2}T^2(x),$$

we have

$$T(x) = \frac{2}{x}.$$

It follows from Algorithm 1 that

$$s_1(T(x)) = \frac{2}{x}, \quad s_2(T(x)) = -\frac{6}{x^2} + \frac{8}{x}, \quad s_3(T(x)) = -\frac{4}{x} + 4$$

is a rational solution of the system (3) corresponding to $\boldsymbol{Z}(H_1', H_2')$.

Similarly, according to the results in Example 4,

$$(s_1(x), s_2(x), s_3(x)) = (x, x, x + 1)$$

is a proper parametrization of invariant algebraic space curve $\boldsymbol{Z}(H_1, H_2)$, where

$$\begin{cases} H_1 = s_2 - s_1 \\ H_2' = s_3 - (1 + c_4)s_2 + c_4 s_1 - 1, \end{cases}$$

Note that $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = T^2(x),$$

we have

$$T(x) = -\frac{1}{x}.$$

According to Algorithm 1,

$$s_1(T(x)) = -\frac{1}{x}, \quad s_2(T(x)) = -\frac{1}{x}, \quad s_3(T(x)) = -\frac{1}{x} + 1$$

is a rational solution of the system (3) corresponding to $\boldsymbol{Z}(H_1, H_2)$.

4 Rational solutions of rational system of autonomous ODEs

In this section, the previous algorithm for computing the rational solutions of polynomial system of autonomous ODEs is generalized to the case of rational system of autonomous ODEs (1)

$$\begin{cases} s_1' = \dfrac{U_1(s_1, s_2, s_3)}{V_1(s_1, s_2, s_3)} \\ s_2' = \dfrac{U_2(s_1, s_2, s_3)}{V_2(s_1, s_2, s_3)} \\ s_3' = \dfrac{U_3(s_1, s_2, s_3)}{V_3(s_1, s_2, s_3)}, \end{cases}$$

where $U_i, V_i \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$ for $i = 1, 2, 3$.

Lemma 10. *Each rational solution of the rational system* (1) *of autonomous ODEs defines a rational invariant algebraic space curve of the polynomial system of autonomous ODEs*

$$\begin{cases} s_1' = U_1(s_1, s_2, s_3)W_1(s_1, s_2, s_3) \\ s_2' = U_2(s_1, s_2, s_3)W_2(s_1, s_2, s_3) \\ s_3' = U_3(s_1, s_2, s_3)W_3(s_1, s_2, s_3), \end{cases} \tag{9}$$

where $W_i = \frac{\mathrm{lcm}(V_1, V_2, V_3)}{V_i}, i = 1, 2, 3$. *Conversely, suppose that* $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ *is a rational invariant algebraic space curve of the polynomial system* (9) *of autonomous ODEs, where* H_1 *and* H_2 *are irreducible. Let* $(s_1(x), s_2(x), s_3(x))$ *be a rational parametrization of* $\boldsymbol{Z}(H_1, H_2)$. *If* $V_i(s_1(x), s_2(x), s_3(x)) \neq 0$ *for all* $i = 1, 2, 3$, *and* $H_{2s_3}(s_1(x), s_2(x), s_3(x)) \neq 0$, *then*

$$\begin{cases} s_1'(x) \cdot \dfrac{U_2(s_1(x), s_2(x), s_3(x))}{V_2(s_1(x), s_2(x), s_3(x))} = s_2'(x) \cdot \dfrac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))} \\ s_1'(x) \cdot \dfrac{U_3(s_1(x), s_2(x), s_3(x))}{V_3(s_1(x), s_2(x), s_3(x))} = s_3'(x) \cdot \dfrac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))} \\ s_2'(x) \cdot \dfrac{U_3(s_1(x), s_2(x), s_3(x))}{V_3(s_1(x), s_2(x), s_3(x))} = s_3'(x) \cdot \dfrac{U_2(s_1(x), s_2(x), s_3(x))}{V_2(s_1(x), s_2(x), s_3(x))}. \end{cases} \tag{10}$$

Proof. Let $(s_1(x), s_2(x), s_3(x))$ be a rational solution of the rational system (1) of autonomous ODEs and $\{H_1(s_1, s_2), H_2(s_1, s_2, s_3)\}$ the basis of the implicit ideal $\mathcal{ID}$ under the lexicographical order $s_1 < s_2 < s_3$ determined by the parametric space curve $(s_1(x), s_2(x), s_3(x))$. Then

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1(x), s_2(x)) \cdot s_i'(x) = 0 \\ \sum_{i=1}^{3} H_{2s_i}(s_1(x), s_2(x), s_3(x)) \cdot s_i'(x) = 0. \end{cases}$$

It follows that

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1(x), s_2(x)) \cdot \dfrac{U_i(s_1(x), s_2(x), s_3(x))}{V_i(s_1(x), s_2(x), s_3(x))} = 0 \\ \sum_{i=1}^{3} H_{2s_i}(s_1(x), s_2(x), s_3(x)) \cdot \dfrac{U_i(s_1(x), s_2(x), s_3(x))}{V_i(s_1(x), s_2(x), s_3(x))} = 0. \end{cases}$$

By clearing the denominators, we have

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1(x), s_2(x)) \cdot U_i(s_1(x), s_2(x), s_3(x)) \cdot W_i(s_1(x), s_2(x), s_3(x)) = 0 \\ \sum_{i=1}^{3} H_{2s_i}(s_1(x), s_2(x), s_3(x)) \cdot U_i(s_1(x), s_2(x), s_3(x)) \cdot W_i(s_1(x), s_2(x), s_3(x)) = 0, \end{cases}$$

where

$$W_i(s_1(x), s_2(x), s_3(x)) = \frac{\operatorname{lcm}(V_1(s_1(x), s_2(x), s_3(x)), V_2(s_1(x), s_2(x), s_3(x)), V_3(s_1(x), s_2(x), s_3(x)))}{V_i(s_1(x), s_2(x), s_3(x))}.$$

Therefore,

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1, s_2) \cdot U_i(s_1, s_2, s_3) \cdot W_i(s_1, s_2, s_3) = H_1 W_{1,1} + H_2 W_{1,2} \\ \sum_{i=1}^{3} H_{2s_i}(s_1, s_2, s_3) \cdot U_i(s_1, s_2, s_3) \cdot W_i(s_1, s_2, s_3) = H_1 W_{2,1} + H_2 W_{2,2}, \end{cases}$$

for some $W_{i,j} \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$. It follows from Definition 1 that $\boldsymbol{Z}(H_1, H_2)$ is a rational invariant algebraic space curve of trivariate polynomial system (9) of autonomous ODEs.

Conversely, if $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ is a rational invariant algebraic space curve of the polynomial system (9) of autonomous ODEs, then we have

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1, s_2) \cdot U_i(s_1, s_2, s_3) \cdot W_i(s_1, s_2, s_3) = H_1 W_{1,1} + H_2 W_{1,2} \\ \sum_{i=1}^{3} H_{2s_i}(s_1, s_2, s_3) \cdot U_i(s_1, s_2, s_3) \cdot W_i(s_1, s_2, s_3) = H_1 W_{2,1} + H_2 W_{2,2}, \end{cases}$$

for some $W_{i,j} \in \bar{\mathbb{Q}}[s_1, s_2, s_3]$. Let $(s_1(x), s_2(x), s_3(x))$ be a parametrization of $\boldsymbol{Z}(H_1, H_2)$, then

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1(x), s_2(x)) \cdot U_i(s_1(x), s_2(x), s_3(x)) \cdot W_i(s_1(x), s_2(x), s_3(x)) = 0 & (11) \\ \sum_{i=1}^{3} H_{2s_i}(s_1(x), s_2(x), s_3(x)) \cdot U_i(s_1(x), s_2(x), s_3(x)) \cdot W_i(s_1(x), s_2(x), s_3(x)) = 0. & (12) \end{cases}$$

On the other hand, it follows from $H_1(s_1(x), s_2(x)) = 0$ and $H_2(s_1(x), s_2(x), s_3(x)) = 0$ that

$$\begin{cases} \sum_{i=1}^{2} H_{1s_i}(s_1(x), s_2(x)) \cdot s_i'(x) = 0 & (13) \\ \sum_{i=1}^{3} H_{2s_i}(s_1(x), s_2(x), s_3(x)) \cdot s_i'(x) = 0. & (14) \end{cases}$$

Note that $(H_{1s_1}(s_1(x), s_2(x)), H_{1s_2}(s_1(x), s_2(x)) \neq (0, 0)$, i.e. the system determined by (11) and (13) has the non-zero solution. It follows that the determinant of coefficients matrix is equal to 0, i.e.

$$\begin{vmatrix} U_1(s_1(x), s_2(x), s_3(x))W_1(s_1(x), s_2(x), s_3(x)) & U_2(s_1(x), s_2(x), s_3(x))W_2(s_1(x), s_2(x), s_3(x)) \\ s_1'(x) & s_2'(x) \end{vmatrix} = 0. \tag{15}$$

Now, we need to prove

$$\begin{vmatrix} U_1(s_1(x), s_2(x), s_3(x))W_1(s_1(x), s_2(x), s_3(x)) & U_3(s_1(x), s_2(x), s_3(x))W_3(s_1(x), s_2(x), s_3(x)) \\ s_1'(x) & s_3'(x) \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} U_2(s_1(x), s_2(x), s_3(x))W_2(s_1(x), s_2(x), s_3(x)) & U_3(s_1(x), s_2(x), s_3(x))W_3(s_1(x), s_2(x), s_3(x)) \\ s_2'(x) & s_3'(x) \end{vmatrix} = 0. \tag{16}$$

In fact, for the system determined by (12) and (14), we consider the following two cases.

1. $U_3(s_1(x), s_2(x), s_3(x)) = 0$ and $s_3'(x) = 0$. The conclusion is obvious.

2. $U_3(s_1(x), s_2(x), s_3(x)) \neq 0$ or $s_3'(x) \neq 0$. In this case, it is equivalent to consider the following system

$$\begin{cases} \sum_{i=1}^{2} H_{2s_i}(s_1(x), s_2(x), s_3(x)) U_i(s_1(x), s_2(x), s_3(x)) W_i(s_1(x), s_2(x), s_3(x)) \\ \quad = -H_{2s_3}(s_1(x), s_2(x), s_3(x)) U_3(s_1(x), s_2(x), s_3(x)) W_3(s_1(x), s_2(x), s_3(x)) \\ \sum_{i=1}^{2} H_{2s_i}(s_1(x), s_2(x), s_3(x)) s_i'(x) = -H_{2s_3}(s_1(x), s_2(x), s_3(x)) s_3'(x). \end{cases} \tag{17}$$

According to $H_{2s_3}(s_1(x), s_2(x), s_3(x)) \neq 0$ and the condition (15), we claim that

$$(H_{2s_1}(s_1(x), s_2(x), s_3(x)), H_{2s_2}(s_1(x), s_2(x), s_3(x))) \neq (0, 0).$$

In fact, if $H_{2s_1}(s_1(x), s_2(x), s_3(x)) = H_{2s_2}(s_1(x), s_2(x), s_3(x)) = 0$, then

$$\begin{cases} H_{2s_3}(s_1(x), s_2(x), s_3(x)) U_3(s_1(x), s_2(x), s_3(x)) W_3(s_1(x), s_2(x), s_3(x)) = 0 \\ H_{2s_3}(s_1(x), s_2(x), s_3(x)) s_3'(x) = 0. \end{cases}$$

It follows that $H_{2s_3}(s_1(x), s_2(x), s_3(x)) = 0$. This is a contradiction. Therefore , $(s_1(x), s_2(x), s_3(x))$ satisfies (16) according to (15).

Note that $V_i(s_1(x), s_2(x), s_3(x)) \neq 0$ for all $i = 1, 2, 3$. It follows from (15) and (16) that

$$\begin{cases} s_1'(x) \cdot \dfrac{U_2(s_1(x), s_2(x), s_3(x))}{V_2(s_1(x), s_2(x), s_3(x))} = s_2'(x) \cdot \dfrac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))} \\ s_1'(x) \cdot \dfrac{U_3(s_1(x), s_2(x), s_3(x))}{V_3(s_1(x), s_2(x), s_3(x))} = s_3'(x) \cdot \dfrac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))} \\ s_2'(x) \cdot \dfrac{U_3(s_1(x), s_2(x), s_3(x))}{V_3(s_1(x), s_2(x), s_3(x))} = s_3'(x) \cdot \dfrac{U_2(s_1(x), s_2(x), s_3(x))}{V_2(s_1(x), s_2(x), s_3(x))}. \end{cases}$$

The Lemma is proved. □

Remark 11. The conditions $V_i(s_1(x), s_2(x), s_3(x)) \neq 0$ and $H_{2s_3}(s_1(x), s_2(x), s_3(x)) \neq 0$ mean $V_i, H_{2s_3} \notin \langle H_1(s_1, s_2), H_2(s_1, s_2, s_3) \rangle$.

It can be seen from Lemma 10 that the differential systems (1) and (9) define the same set of invariant algebraic space curves under certain conditions. Therefore, the system (9) is called the *associated polynomial differential system* of the rational system (1) of autonomous ODEs. Each invariant algebraic space curve of the associated polynomial differential system is also called an *invariant algebraic space curve of the rational system of autonomous ODEs.*

Based on the previous preparation, we have the following theorem for computing the rational solutions of rational system of autonomous ODEs.

Theorem 12. *(a) Any rational solution of rational system* (1) *of autonomous ODEs is a proper rational solution.*
(b) Let $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ be a rational invariant algebraic space curve of the rational system (1) *of autonomous ODEs such that $V_i, H_{2s_3} \notin \langle H_1, H_2 \rangle$ for all $i = 1, 2, 3$, and $(s_1(x), s_2(x), s_3(x))$ be an arbitrary proper rational parametrization of the space curve $\boldsymbol{Z}(H_1, H_2)$. Then the system* (1) *has a proper rational solution*

$$\widehat{s}_1(x) = s_1(T(x)), \quad \widehat{s}_2(x) = s_2(T(x)), \quad \widehat{s}_3(x) = s_3(T(x))$$

corresponding to $\boldsymbol{Z}(H_1, H_2)$ *if and only if there exists a linear rational transformation* $T(x) = \frac{ax+b}{cx+d}$ *which is a rational solution of the following autonomous differential equation*

$$T'(x) = \begin{cases} \dfrac{U_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), s_3(T(x))) \cdot s_1'(T(x))}, & \text{if } s_1'(x) \neq 0 \\ \dfrac{U_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_2(s_1(T(x)), s_2(T(x)), s_3(T(x))) \cdot s_2'(T(x))}, & \text{if } s_2'(x) \neq 0 \\ \dfrac{U_3(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_3(s_1(T(x)), s_2(T(x)), s_3(T(x))) \cdot s_3'(T(x))}, & \text{if } s_3'(x) \neq 0. \end{cases} \tag{18}$$

Proof. The proof of (a) is similar to that of Theorem 7. (b) is obvious by the constructive proof of (a). □

Now the problem for computing the rational solution of rational system (1) of autonomous ODEs is reduced to finding a rational solution of the autonomous differential equation (18). In fact, we have the following theorem about the linear rational solvability of the equation (18).

Theorem 13. *Let* $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ *be a rational invariant algebraic space curve of the rational system* (1) *of autonomous ODEs such that* $V_i, H_{2s_3} \notin \langle H_1, H_2 \rangle$ *for all* $i = 1, 2, 3$. *If* $\mathcal{P} = (s_1(x), s_2(x), s_3(x))$ *and* $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ *are two different proper parametrizations of* $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$, *then*

$$T_1'(x) = \frac{U_i(s_1(T_1(x)), s_2(T_1(x)), s_3(T_1(x)))}{V_i(s_1(T_1(x)), s_2(T_1(x)), s_3(T_1(x))) \cdot s_i'(T_1(x))} \tag{19}$$

has a linear rational solution $T_1(x)$ *if and only if*

$$T_2'(x) = \frac{U_i(\widehat{s}_1(T_2(x)), \widehat{s}_2(T_2(x)), \widehat{s}_3(T_2(x)))}{V_i(\widehat{s}_1(T_2(x)), \widehat{s}_2(T_2(x)), \widehat{s}_3(T_2(x))) \cdot s_i'(T_2(x))} \tag{20}$$

has a linear rational solution $T_2(x)$. *Moreover,* $\mathcal{P} = \widehat{\mathcal{P}} \circ T_2 \circ T_1^{-1}$.

Proof. Assume that $T_1(x)$ is a linear rational solution of (19). Then the rational solution of the system (1) corresponding to $\boldsymbol{Z}(H_1(s_1, s_2), H_2(s_1, s_2, s_3))$ is $(s_1(T_1(x)), s_2(T_1(x)), s_3(T_1(x)))$. According to Theorem 12(a), $(s_1(T_1(x)), s_2(T_1(x)), s_3(T_1(x)))$ is a proper rational solution. As $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ is a proper parametrization of the same space curve $\boldsymbol{Z}(H_1, H_2)$, there exists a linear rational function $T_2(x)$, such that

$$\begin{cases} s_1(T_1(x)) = \widehat{s}_1(T_2(x)) \\ s_2(T_1(x)) = \widehat{s}_2(T_2(x)) \\ s_3(T_1(x)) = \widehat{s}_3(T_2(x)). \end{cases} \tag{21}$$

It follows that

$$\begin{cases} s_1'(T_1(x)) \cdot T_1'(x) = \widehat{s}_1'(T_2(x)) \cdot T_2'(x) \\ s_2'(T_1(x)) \cdot T_1'(x) = \widehat{s}_2'(T_2(x)) \cdot T_2'(x) \\ s_3'(T_1(x)) \cdot T_1'(x) = \widehat{s}_3'(T_2(x)) \cdot T_2'(x). \end{cases}$$

Therefore, $T_2(x)$ is a linear rational solution of (20), and visa versa. Furthermore, it is obvious from (21) that $\mathcal{P} = \widehat{\mathcal{P}} \circ T_2 \circ T_1^{-1}$. □

It can be seen from Theorem 13 that the solvability of the linear rational solution of differential equation (18) does not depend on the choice of the proper parametrization of the invariant algebraic space curve. According to Theorem 13, we know that it is possible to get two different rational solutions from two different proper parametrizations of the same invariant algebraic space curve. In fact, they are related to each other by a shifting of the variable.

Theorem 14. *Let* $(s_1(x), s_2(x), s_3(x))$ *and* $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ *be two rational solutions of the trivariate rational system* (1) *of autonomous ODEs corresponding to the same invariant algebraic space curve. Then there exists a constant* c *such that*

$$(s_1(x+c), s_2(x+c), s_3(x+c)) = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)).$$

Proof. According to Theorem 12(a) and Definition 5, $(s_1(x), s_2(x), s_3(x))$ and $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ are two proper parametrizations of the same invariant algebraic space curve. Therefore, there exists a linear rational function $T(x)$ such that

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = (s_1(T(x)), s_2(T(x)), s_3(T(x))).$$

It follows that

$$s_i'(T(x))T'(x) = \widehat{s}_i'(x) = \frac{U_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))}{V_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))} = \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))} = s_i'(T(x)).$$

Assume that $s_1'(x) \neq 0$ without loss of generality, then $T'(x) = 1$. By solving this differential equation, we have $T(x) = x + c$ for some constant c. The theorem is proved. □

Remark 15. In fact, the transformation from one solution into another one in Theorem 14 can be computed. Let $\mathcal{P} = (s_1(x), s_2(x), s_3(x))$ and $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$. By considering these two rational solutions as the proper parametrizations, then $\widehat{\mathcal{P}} \circ \mathcal{P}^{-1}$ is the transformation from $\mathcal{P}$ to $\widehat{\mathcal{P}}$ and $\mathcal{P} \circ \widehat{\mathcal{P}}^{-1}$ is the transformation from $\widehat{\mathcal{P}}$ to $\mathcal{P}$, where $*^{-1}$ represents the inverse of the parametrization.

5 Conclusion and future work

In this paper, we generalize the method in [11] for computing the rational solutions of the bivariate rational system of autonomous ODEs to the case of the trivariate rational system of autonomous ODEs. According to the birational correspondence between space curve and plane curve, we extend the properties of rational parametrization for the plane curve to the space curve. Our method is mainly based on the proper parametrization of invariant algebraic space curve and the linear rational transformation between two proper parametrizations of the same algebraic space curve. It can compute the explicitly rational solutions corresponding to the invariant algebraic space curves of trivariate rational system of autonomous ODEs. Moreover, the relationship is studied between different rational solutions corresponding to the same invariant algebraic space curve.

The following are several problems arising in the process of the generalization for finding the rational general solution of high order non-autonomous ODEs.

(a) Note that the generalization for reducing the problem in [9] is mainly based on the proper parametrization of hypersurfaces. Because not every unirational hypersurface can be properly parametrized [3], we only consider hypersurfaces which have a proper parametrization. In fact, the theorem of Lüroth is very relevant in the curve case, since it implies that unirationality is equivalent to rationality. This equivalence also holds for surfaces, the relevant theorem is due to Castelnuovo [4,16]. But this equivalence does not hold for hypersurfaces in dimension greater than 3. It means even when the base field is the complex number field $\mathbb{C}$ (indeed the algebraically closed field), there exist improper parametric equations that do not have a proper reparametrization. Therefore, the problem is how to decide whether a given rational algebraic hypersurface has a proper parametrization? Does there exist a method for deciding the rationality of some special hypersurfaces? Are there interesting families of rational hypersurfaces? Actually, the method for checking whether a given parametrization of hypersurfaces is proper has been presented by Gao in [8].

(b) For the first order non-autonomous ODEs, Ngô and Winkler [14] used the degree bound of the invariant algebraic curves of the associated system in the non-dicritical case by the result in [5]. This degree bound ensures the termination of the algorithm in the case of non-dicritical singularities. In fact, the degree bound of H_i may be derived according to Proposition 21 in [9] for the linear system of autonomous ODEs. This proposition is a generalization of Ngô and Winkler's conclusion in [12,13]. But for the nonlinear system of autonomous ODEs, it is still an open problem for getting the degree bound of invariant algebraic space curves.

(c) When we compute the invariant algebraic space curve $\boldsymbol{Z}(H_1, H_2)$ with the fixed degree by using the undetermined coefficient method, it is necessary to choose the "good" pairs of (H_1, H_2) before our computation. Here, "good" means its corresponding system of equations has a solution. For example, we only use the H_1' and H_2' with the form (5) when the invariant algebraic space curve $\boldsymbol{Z}(H_1', H_2')$ satisfies $\deg(H_1') = 2$ and $\deg(H_2') = 1$ in Example 2. In fact, the other pairs of (H_1, H_2) listed in Remark 3 are not very useful because their corresponding systems of equations on the coefficients of H_i' have no solution. If it is difficult to choose all the "good" pairs of (H_1, H_2), can we select some pairs before our computation such that their corresponding systems of equations are likely to have solutions? How many "usable" pairs exist? Furthermore, if we get some solutions from these usable pairs of (H_1, H_2), it is also necessary to choose the solutions such that $\boldsymbol{Z}(H_1, H_2)$ is a rational space curve and H_i is irreducible. In fact, it is possible to get the same invariant algebraic space curve from different solutions. The problem is how to discard the solutions which don't satisfy the conditions more efficiently?

(d) In fact, the undetermined coefficient method for computing the invariant algebraic space curve is not efficient because it needs to solve the system of equations. With the increase of the degree, it is more difficult to solve the obtained system of equations. Therefore, it would be better to find other methods for computing the rational invariant algebraic space curves directly.

These are the problems we encounter in the generalization to the trivariate (and also multivariate) case. It would be very interesting to develop methods for solving some of the above problems in the future.

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