L_2 Error Estimates for a Nonstandard Finite Element Method on Polyhedral Meshes

Clemens Hofreither

DK-Report No. 2010-13

12 2010

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by

Austrian Science Fund (FWF)

Upper Austria

Communicated by: Ulrich Langer
Veronika Pillwein

DK sponsors:

- **Johannes Kepler University Linz** (JKU)
- **Austrian Science Fund** (FWF)
- **Upper Austria**

L_2 Error Estimates for a Nonstandard Finite Element Method on Polyhedral Meshes

Clemens Hofreither

December 3, 2010

Abstract

Recently, C. Hofreither, U. Langer and C. Pechstein have analyzed a nonstandard finite element method based on element-local boundary integral operators. The method is able to treat general polyhedral meshes and employs locally PDE-harmonic trial functions. In the previous work, the primal formulation of the method has been analyzed as a perturbed Galerkin scheme, obtaining H^1 error estimates. In this work, we pass to an equivalent mixed formulation and derive error estimates in the L_2-norm, which were so far not available. Many technical tools from our previous analysis remain applicable in this setting.

1 Introduction

In certain applications, it is advantageous to discretize partial differential equations (PDEs) on non-standard grids consisting of heterogeneous, non-simplicial elements and incorporating hanging nodes. For instance, in reservoir simulation, polygonal or polyhedral meshes are in common use (cf., e.g., [12]). In simulating drug diffusion through the human skin, tetrakaidecahedra (14-faced polyhedra) have been employed to model cells in the outermost skin layer, so-called corneocytes [7].

Previously established methods which are able to treat such generalized meshes are, among others, the Mimetic Finite Difference Method (see, e.g., [12] or [2]), special Mixed Finite Element Methods (see [10] and [11]), or the Discontinuous Galerkin Method (see, e.g., [6]). D. Copeland, U. Langer and D. Pusch have recently introduced a novel technique for treating boundary value problems on polyhedral meshes [5]. They have demonstrated that this new method works well for different classes of problems including diffusion problems, the Helmholtz equation and the Maxwell equations in the frequency domain (see also [4]). This approach employs locally PDE-harmonic trial functions, i.e. trial functions which are elementwise PDE-harmonic, and uses boundary element techniques to assemble the element stiffness matrices. For this reason, the new non-standard finite element method was also called BEM-based FEM.

First steps towards a rigorous analysis of this approach were done in [8], where the method was studied in the framework of a primal variational formulation with elementwise Dirichlet traces of the solution as its unknowns. The realization of this Galerkin method requires the inversion of the single layer potential operator in every element, which can typically only be done approximately. This implicates a "variational crime" in the form of an inexact bilinear form, and introduces a consistency error to the numerical scheme, making L_2 error estimates hard to obtain via standard techniques. In the present work, we show an alternate approach to the analysis via a mixed formulation having both Dirichlet and Neumann traces as its unknowns. Building upon the technical tools developed in our previous work [8], we will be able to recover the error estimates in the H^1-norm obtained therein as well as derive previously unavailable L_2 error estimates.

The remainder of this paper is organized as follows. In Section 2, we derive both the primal variational formulation and the equivalent mixed variational formulation, and discretize the latter. In Section 3, we formulate regularity assumptions for general polyhedral meshes, and state an approximation result on the skeletons of such meshes. Section 4 is devoted to the derivation of mesh-independent error estimates for the BEM-based FEM in both the H^1- and the L_2-norms. In the final Section 5, we draw some conclusion.

2 Formulations of a BEM-based FEM

2.1 The primal skeletal variational formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma = \partial\Omega$ its boundary, and let us consider the pure Dirichlet boundary value problem for the Poisson equation

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ on } \Gamma,$$

with $g \in H^{1/2}(\Gamma)$ and $f \in L_2(\Omega)$, as our model problem. The standard variational formulation is the following: find $u \in H^1(\Omega)$ such that the trace $\gamma_\Gamma^0 u$ of u on Γ equals g and the standard variational equation

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega). \tag{1}$$

holds.

We now consider a family of non-overlapping decompositions $(T_i)_{i=1}^N$ of Ω,

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{T}_i, \qquad T_i \cap T_j = \emptyset \quad \forall i \neq j.$$

We assume that each *element* T_i is an open Lipschitz polyhedron whose boundary $\Gamma_i = \partial T_i$ has a conforming triangulation $\mathcal{F}_i = \{\tau_{ij} \subset \Gamma_i\}_j$ composed of open triangles. We call such a decomposition $(T_i)_{i=1}^N$ a *polyhedral mesh* of Ω. We further assume that the elements are *matching* in the sense that, for all

$\tau_i \in \mathcal{F}_i$ and $\tau_j \in \mathcal{F}_j$, we have $\tau_i \cap \tau_j \neq \emptyset \Leftrightarrow \tau_i = \tau_j \in \mathcal{F}_i \cap \mathcal{F}_j$. In other words, boundary triangles from two neighboring elements should either be identical or not intersect at all.

For any suitable domain T, let

$$H^1_{\Delta,f}(T) := \left\{ u \in H^1(T) : \int_T \nabla u \cdot \nabla v \, dx = \int_T f v \, dx \quad \forall v \in H^1_0(T) \right\}$$

denote the manifold of weak local solutions of the Poisson equation.

Following McLean [13], we introduce the Dirichlet and Neumann trace operators

$$\gamma_i^0 = \gamma_{\Gamma_i}^0 : H^1(T_i) \to H^{1/2}(\Gamma_i) \quad \text{and} \quad \gamma_i^1 : H^1_{\Delta,f}(T_i) \to H^{-1/2}(\Gamma_i)$$

which satisfy, for all $u \in H^1_{\Delta,f}(T_i)$ and $v \in H^1(T_i)$, the Green's identity

$$\langle \gamma_i^1 u, \, \gamma_i^0 v \rangle = -\int_{T_i} f v \, dx + \int_{T_i} \nabla u \cdot \nabla v \, dx, \tag{2}$$

where $\langle \cdot \, , \, \cdot \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. Furthermore, we define the extension operators

$$\mathcal{H}_i^f : H^{1/2}(\Gamma_i) \to H^1_{\Delta,f}(T_i)$$

such that, for any $\phi \in H^{1/2}(\Gamma_i)$, its image $\mathcal{H}_i^f(\phi)$ is the uniquely defined element of $\mathcal{H}^f(T_i)$ having ϕ as its Dirichlet data. By a superposition argument, it is easy to see that $\mathcal{H}_i^f(\phi) = \mathcal{H}_i^f(0) + \mathcal{H}_i^0(\phi)$.

Finally, we introduce the *Dirichlet-to-Neumann maps*

$$S_i^f := \gamma_i^1 \circ \mathcal{H}_i^f \;\; : \;\; H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i),$$

and from the above we infer that

$$S_i^f(\phi) = \gamma_i^1(\mathcal{H}_i^f(0) + \mathcal{H}_i^0(\phi)) = S_i^f(0) + S_i^0(\phi). \tag{3}$$

Note that $\mathcal{H}_i := \mathcal{H}_i^0$ and $S_i := S_i^0$ are linear operators.

Let us $\Gamma_S := \bigcup_{i=1}^N \Gamma_i$ denote the *skeleton* of the mesh, and let us introduce the space $H^{1/2}(\Gamma_S)$ as the trace space of H^1-functions onto the skeleton. Furthermore, let $W = \{v \in H^{1/2}(\Gamma_S) : v|_\Gamma = 0\}$ be the space of all skeletal functions with vanishing boundary values. A discussion analogous to the one used to prove [8, Proposition 2.1] convinces us that the following two variational problems are equivalent:

- find $u_\Omega \in H^1(\Omega)$ such that $\gamma_\Gamma^0 u = g$ and

$$\int_\Omega \nabla u_\Omega \cdot \nabla v_\Omega = \int_\Omega f v \, dx \qquad \forall v \in H^1_0(\Omega);$$

- find $u \in H^{1/2}(\Gamma_S)$ such that $u|_\Gamma = g$ and

$$\sum_{i=1}^{N} \langle S_i^f(u_i),\, v_i\rangle = 0 \qquad \forall v \in W. \tag{4}$$

(Here and henceforth we adopt the notational convention $v_i = v|_{\Gamma_i}$ for skeletal functions.) The equivalence is to be understood in the sense that $u_i = \gamma_i^0 u_\Omega$, and $u_\Omega = \mathcal{H}_i^f(u_i)$ on every element T_i. In other words, u is the skeletal trace of the solution u_Ω, and u_Ω can be locally reconstructed as the extension of u.

The Green's identity (2) with the choice $u = \mathcal{H}_i^f(0)$ and $v = \mathcal{H}_i\phi$ for arbitrary $\phi \in H^{1/2}(\Gamma_i)$ yields

$$\langle S_i^f(0),\, \phi\rangle = -\int_{T_i} f\mathcal{H}_i\phi\, dx + \int_{T_i} \nabla\mathcal{H}_i^f(0)\cdot\nabla\mathcal{H}_i\phi\, dx = -\int_{T_i} f\mathcal{H}_i\phi\, dx. \tag{5}$$

Using relations (3) and (5), we may rewrite the variational problem (4) as

$$\sum_{i=1}^{N} \langle S_i u_i,\, v_i\rangle = \sum_{i=1}^{N} \int_{T_i} f\, \mathcal{H}_i v_i\, dx \qquad \forall v \in W,$$

We introduce the shorthand notation $\mathcal{H}_S : H^{1/2}(\Gamma_S) \to H^1(\Omega)$ for the piecewise harmonic extension from the skeleton to each element T_i. Also, for convenience, we identify the given Dirichlet data g with a suitable skeletal extension $g \in H^{1/2}(\Gamma_S)$, which always exists. We thus have the variational problem: find $u \in g + W$ with

$$\sum_{i=1}^{N} \langle S_i u_i,\, v_i\rangle = \int_\Omega f\, \mathcal{H}_S v\, dx \quad \forall v \in W. \tag{6}$$

2.2 The mixed skeletal variational formulation

The Dirichlet-to-Neumann map S_i has the representation

$$S_i u_i = D_i u_i + (\tfrac{1}{2}I + K_i')V_i^{-1}(\tfrac{1}{2}I + K_i)u_i$$

in terms of the boundary integral operators

$$\begin{aligned} V_i &: H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i), & K_i &: H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i), \\ K_i' &: H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i), & D_i &: H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i). \end{aligned}$$

They are called, in turn, the *single layer potential*, *double layer potential*, *adjoint double layer potential*, and *hypersingular* operators. Note that their definition requires the explicit knowledge of a fundamental solution of the differential operator in question. For details, we refer the reader to, e.g., McLean [13] or Steinbach [15].

We introduce the space of elementwise Neumann traces,

$$Z := \bigotimes_{i=1}^{N} H^{-1/2}(\Gamma_i).$$

In contrast to the space W, whose members are globally continuous on the skeleton, Z contains functions which are discontinuous and double-valued on inner triangles.

We now introduce the auxiliary variable $t := (t_i)_{i=1}^N \in Z$, with components $t_i = V_i^{-1}(\frac{1}{2}I + K_i)u_i$ for $i = 1, 2, \ldots, n$. Equivalently, $t_i \in H^{-1/2}(\Gamma_i)$ is determined by the local variational equation

$$\langle z_i,\, V_i t_i\rangle = \langle z_i,\, (\tfrac{1}{2}I + K_i)u_i\rangle \qquad \forall z_i \in H^{-1/2}(\Gamma_i).$$

With this, $S_i u_i = D_i u_i + (\frac{1}{2}I + K_i')t_i$, and hence we can write the following equivalent mixed formulation for (6): find $(u, t) \in X := W \times Z$ such that

$$\begin{aligned} a(u,v) + b(v,t) &= \langle F,\, v\rangle \qquad \forall v \in W, \\ -b(u,z) + c(z,t) &= \langle G,\, z\rangle \qquad \forall z \in Z, \end{aligned}$$

where

$$a(u,v) = \sum_{i=1}^{N}\langle D_i u_i,\, v_i\rangle, \quad b(v,t) = \sum_{i=1}^{N}\langle t_i,\, (\tfrac{1}{2}I + K_i)v_i\rangle, \quad c(z,t) = \sum_{i=1}^{N}\langle z_i,\, V_i t_i\rangle,$$
$$\langle F,\, v\rangle = \int_\Omega f\,\mathcal{H}_S v\,dx - a(g,v), \qquad \langle G,\, z\rangle = b(g,z).$$

With the combined bilinear form

$$\mathcal{A}((u,t),(v,z)) := a(u,v) + b(v,t) - b(u,z) + c(z,t),$$

we may write more compactly: find $(u,t) \in X$ such that

$$\mathcal{A}((u,t),(v,z)) = \langle F,\, v\rangle + \langle G,\, z\rangle \quad \forall (v,z) \in X. \tag{7}$$

2.3 Discretization

Let us recall that the elements $\{T_i\}$ are provided with the boundary triangulations $\{\mathcal{F}_i\}$, all of which match across elements. Therefore, $\mathcal{F} := \bigcup_i \mathcal{F}_i$ describes a triangulation of the skeleton Γ_S. With this, we introduce the trial spaces

$$\begin{aligned} W_h &:= \{v \in W : v|_\tau \in P^1(\tau)\ \forall \tau \in \mathcal{F}\}, \text{ and} \\ Z_h &:= \bigotimes_{i=1}^{N} Z_{h,i}, \text{ where } Z_{h,i} := \{z \in L_2(\Gamma_i) : z|_\tau \in P^0(\tau)\ \forall \tau \in \mathcal{F}_i\}. \end{aligned}$$

Here, $P^k(\tau)$ denotes the polynomial space of degree k on the triangle τ.

We discretize the variational formulation (7) by looking for some $(u_h, t_h) \in X_h := W_h \times Z_h \subset X$ such that

$$\mathcal{A}((u_h, t_h), (v_h, z_h)) = \langle F, v_h \rangle + \langle G, z_h \rangle \quad \forall (v_h, z_h) \in X_h. \tag{8}$$

In practice, the auxiliary variable t_h can be eliminated locally on each element, and only the primal unknowns u_h enter the linear system to be solved.

This discrete variational formulation is equivalent to a primal formulation where the Dirichlet-to-Neumann map S_i has been replaced with a symmetric approximation leading to a so-called variational crime. Based on Strang's Lemma, C. Hofreither, U. Langer and C. Pechstein provide a discretization error analysis with respect to the H^1-norm in [8]. Now, the detour via the mixed variational reformulation leads to the conforming Galerkin discretization (8) of (7). Therefore, we have the Galerkin orthogonality

$$\mathcal{A}((u - u_h, t - t_h), (v_h, z_h)) = 0 \quad \forall (v_h, z_h) \in X_h. \tag{9}$$

3 Mesh regularity

For general polyhedral meshes with arbitrary element shapes, we cannot use the standard technique of transforming to a reference element to obtain uniform approximation properties. In [8, Sect. 4.3], two generalized regularity assumptions on such meshes are given which substitute for more standard transformation-based regularity assumptions. For the sake of completeness we repeat these assumptions here.

Assumption 1. We assume that the polyhedral mesh $(T_i)_{i=1}^N$ satisfies the following conditions.

- There is a small, fixed integer uniformly bounding the number of boundary triangles of every element.
- Every element T_i has an auxiliary conforming, quasi-regular, tetrahedral triangulation with regularity parameters which are uniform across all elements, cf. [3].

Definition 1 (Uniform domain [9])**.** A bounded and connected set $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists a constant $C_U(D)$ such that any pair of points $x_1 \in D$ and $x_2 \in D$ can be joined by a rectifiable curve $\gamma(t) : [0,\, 1] \to D$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, such that the Euclidean arc length of γ is bounded by $C_U(D)\, |x_1 - x_2|$ and

$$\min_{i=1,2} |x_i - \gamma(t)| \leq C_U(D) \ \mathrm{dist}(\gamma(t),\, \partial D) \qquad \forall t \in [0, 1].$$

Any Lipschitz domain is also a uniform domain. In the following, for any Lipschitz domain D, we call the smallest constant $C_U(D)$ that complies with Definition 1 the *Jones parameter* of D.

The second parameter that we use is the constant in Poincaré's inequality. For a uniform domain D, let $C_P(D)$ be the smallest constant such that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L_2(D)} \leq C_P(D)\,\mathrm{diam}(D)\,|u|_{H^1(D)} \qquad \forall u \in H^1(D). \tag{10}$$

For convex domains D, one can show that $C_P(D) \leq 1/\pi$, cf. [1]. Estimates for star-shaped domains can be found in [16, 17].

Since each individual element T_i is Lipschitz, the Jones parameter $C_U(T_i)$ and the constant $C_P(T_i)$ in Poincaré's inequality are both bounded.

Assumption 2. We assume that there are constants $C_U^* > 0$ and $C_P^* > 0$ such that, for all $i \in \{1, \dots, N\}$,

$$\begin{aligned} C_U(T_i) &\leq C_U^*, \qquad & C_U(B_i \setminus \overline{T}_i) &\leq C_U^*, \\ C_P(T_i) &\leq C_P^*, \qquad & C_P(B_i \setminus \overline{T}_i) &\leq C_P^*, \end{aligned}$$

where B_i is a ball (or a suitable Lipschitz domain) enclosing T_i which satisfies $\mathrm{dist}(\partial B_i, \partial T_i) \geq \frac{1}{2}\,\mathrm{diam}(T_i)$.

In the following, we will assume that all polyhedral meshes we work with satisfy Assumption 1 and Assumption 2. We will call such meshes *regular*. Furthermore, we will generically use C to refer to constants which depend only on the regularity parameters from the two assumptions, and call such constants *uniform*.

For the convergence and approximation results that follow, we equip the space $X = W \times Z$ with the norm

$$\|(v, z)\|_X^2 := \|v\|_S^2 + \|z\|_V^2 := \sum_{i=1}^N \langle S_i v_i,\, v_i\rangle + \sum_{i=1}^N \langle V_i z_i,\, z_i\rangle.$$

Let $h := \max_i\{\mathrm{diam}\, T_i\}$ denote the *mesh size*. On regular meshes, we have the following approximation theorem.

Theorem 1. *Assume that the mesh $(T_i)_{i=1}^N$ is regular, i.e. Assumption 1 and Assumption 2 hold. If $w_\Omega \in H^2(\Omega)$ with piecewise linear boundary conditions g, and if $(\phi, \eta) \in (g, 0) + X$ denotes its skeletal Dirichlet and Neumann data, respectively, then*

$$\inf_{(\phi_h,\eta_h)\in(g,0)+X_h} \|(\phi - \phi_h, \eta - \eta_h)\|_X \leq C\, h\, |w_\Omega|_{H^2(\Omega)} \tag{11}$$

with a uniform constant C.

Proof. This theorem subsumes results on approximation of both Dirichlet and Neumann traces which were originally derived in [8]. These results were therein stated for the case where the function w_Ω to be approximated is the exact solution of (1), but inspecting the proofs makes it clear that only the property

$w_\Omega \in H^2(\Omega)$ is actually used. In particular, [8, Theorem 4.8] asserts that, under the above assumptions,

$$\inf_{\phi_h \in g + W_h} |||\phi - \phi_h|||_S \le C\, h\, |w_\Omega|_{H^2(\Omega)}.$$

Analogously, for the Neumann traces, [8, Theorem 4.11] states that on every element T_i,

$$\inf_{\eta_{h,i} \in Z_{h,i}} \|\eta_i - \eta_{h,i}\|_{V_i} \le C\, (\operatorname{diam} T_i)\, |w_\Omega|_{H^2(T_i)}.$$

Obtaining the statement is then a simple matter of combining these results. □

4 Error estimates

In this section, we provide error estimates for the discretized problem (8). Error estimates in skeletal function spaces, while inherently mesh-dependent, are an important intermediate result in the derivation of mesh-independent estimates, and are given first. Next we provide an error estimate in the H^1-norm which was already given in [8], but is here rederived using our new mixed variational framework. Finally, we present an estimate in the L_2-norm which constitutes the main new result of this paper.

4.1 Convergence on the skeleton

Theorem 2. *Let Assumption 1 and Assumption 2 be fulfilled. Then the discrete solution $(u_h, t_h) \in X_h$ of (8) is a quasi-optimal approximation to the solution $(u, t) \in X$ of (7). That is,*

$$|||(u - u_h, t - t_h)|||_X \le C \inf_{(v_h, z_h) \in X_h} |||(u - v_h, t - z_h)|||_X \tag{12}$$

with a uniform constant C.

Proof. The result is proved using Céa's Lemma. Hence, only uniform coercivity and boundedness of the bilinear form $\mathcal{A}$ need to be shown.

We take note of the spectral equivalence

$$\frac{1}{C}\langle S_i v_i,\, v_i\rangle \le \langle D_i v_i,\, v_i\rangle \le \langle S_i v_i,\, v_i\rangle \qquad \forall v \in H^{1/2}(\Gamma_i), \tag{13}$$

which is well-known in boundary integral operator theory [15]. Pechstein has shown in [14, Lemma 6.6] that $D_i \ge c^\star_{D,i}\, S_i$, where $c^\star_{D,i} = \frac{1}{2}\, C_E(B_i \setminus T_i)^{-2}\, (1 + C_P(B_i \setminus T_i)^2)^{-1}$, and the extension constant $C_E(B_i \setminus T_i)$ depends only on $C_U(B_i \setminus T_i) \le C_U^*$. Therefore, the constant $C \ge 1$ in (13) can be bounded explicitly in terms of C_P^* and C_U^* and is thus uniform.

Hence we obtain coercivity of the bilinear form $\mathcal{A}$ via

$$\begin{aligned}\mathcal{A}((v, z), (v, z)) &= \sum_i \langle D_i v_i,\, v_i\rangle + \sum_i \langle z_i,\, V_i z_i\rangle \\ &\ge \frac{1}{C}\sum_i \langle S_i v_i,\, v_i\rangle + \sum_i \langle z_i,\, V_i z_i\rangle \ge \frac{1}{C}|||(v, z)|||_X^2.\end{aligned}$$

In order to get upper bounds, we again use (13) as well as the Cauchy-Schwarz inequality for the symmetric and positive (semi-)definite forms $\langle \cdot, V_i \cdot \rangle$ and $\langle D_i \cdot, \cdot \rangle$ to see that

$$|a(u,v)| \le \|u\|_S \|v\|_S, \qquad |c(t,z)| \le \|t\|_V \|z\|_V.$$

By duality of the norms $\|\cdot\|_{V_i}$ and $\|\cdot\|_{V_i^{-1}}$, we get

$$\begin{aligned} b(v,t) &= \sum_i \langle t_i, (\tfrac{1}{2}I + K_i)v_i \rangle \le \sum_i \|t_i\|_{V_i} \|(\tfrac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \\ &\overset{(*)}{\le} C \sum_i \|t_i\|_{V_i} |v_i|_{S_i} \le C \|t\|_V \|v\|_S. \end{aligned}$$

The inequality marked with $(*)$ stems from the relation $\|(\frac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \le c_{K,i}(1 - c_{K,i})^{-1/2} |v_i|_{S_i}$ proved in [8, Equation (3.1)]. Pechstein [14] has shown that the contraction constants $c_{K,i}$ can be bounded explicitly in terms of C_P^* and C_U^*, and thus $C \ge 1$ is a uniform constant.

Combined, the above bounds yield

$$\begin{aligned} |\mathcal{A}((u,t),(v,z))| &\le C\big(\|u\|_S \|v\|_S + \|t\|_V \|v\|_S + \|u\|_S \|z\|_V + \|t\|_V \|z\|_V\big) \\ &= C(\|u\|_S + \|t\|_V)(\|v\|_S + \|z\|_V) \\ &\le 2\,C\, \|(u,t)\|_X \, \|(v,z)\|_X. \end{aligned} \qquad \square$$

While error estimates on the skeleton follow directly from this result and Theorem 1, they are inherently mesh-dependent and therefore of limited use. More interesting is the error within the domain with respect to the exact solution of (1), which will typically have additional regularity, say, $u_\Omega \in H^2(\Omega)$. Within a given element T_i, this error is given by

$$u_\Omega - \mathcal{H}_i^f(u_h + g) = \mathcal{H}_i^f(u + g) - \mathcal{H}_i^f(u_h + g) = \mathcal{H}_i(u - u_h),$$

and hence it is sufficient to bound the error $\mathcal{H}_S(u - u_h)$.

4.2 Convergence in the H^1-norm

From Green's identity, it is easy to see that

$$|\mathcal{H}_i \phi|^2_{H^1(T_i)} = \langle S_i \phi, \phi \rangle \qquad \forall \phi \in H^{1/2}(\Gamma_i).$$

Hence, with Theorem 2 and Theorem 1, it follows

$$|\mathcal{H}_S(u - u_h)|_{H^1(\Omega)} = \|u - u_h\|_S \le \|(u - u_h, t - t_h)\|_X \le C\, h\, |u_\Omega|_{H^2(\Omega)}.$$

4.3 Convergence in the L_2-norm

The proof of the error estimate in the L_2-norm proceeds by a standard Aubin-Nitsche duality argument. We assume that the adjoint to variational problem (1) is H^2-coercive and take the harmonic extension $\mathcal{H}_S(u-u_h)$ of the discretization error as the right-hand side in the adjoint variational problem. Then the solution $w \in H_0^1$ of the adjoint problem

$$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Omega \mathcal{H}_S(u-u_h)\, v \, dx \qquad \forall v \in H_0^1(\Omega) \tag{14}$$

even belongs to $H^2(\Omega)$ and satisfies the estimate

$$|w|_{H^2(\Omega)} \le C \, \|\mathcal{H}_S(u-u_h)\|_{L_2(\Omega)}. \tag{15}$$

Then the skeletal traces (ϕ, η), where $\phi_i := \gamma_i^0 w$, $\eta_i := \gamma_i^1 w$ for $i = 1, \dots, N$, satisfy the (adjoint) mixed skeletal variational formulation (7), i.e.,

$$\mathcal{A}((v,z),(\phi,\eta)) = \int_\Omega \mathcal{H}_S(u-u_h)\, \mathcal{H}_S v \, dx \qquad \forall (v,z) \in X.$$

In particular, with the choice $(v,z) = (u-u_h, t-t_h)$ and exploiting the Galerkin orthogonality (9) as well as uniform boundedness of $\mathcal{A}$, we get

$$\begin{aligned}
\|\mathcal{H}_S(u-u_h)\|_{L_2(\Omega)}^2 &= \mathcal{A}((u-u_h, t-t_h),(\phi,\eta)) \\
&= \mathcal{A}((u-u_h, t-t_h),(\phi-\phi_h, \eta-\eta_h)) \\
&\le C \, \|(u-u_h, t-t_h)\|_X \, \|(\phi-\phi_h, \eta-\eta_h)\|_X
\end{aligned}$$

for arbitrary $(\phi_h, \eta_h) \in X_h$. Taking the infimum over (ϕ_h, η_h) and using Theorem 2 and Theorem 1, we obtain

$$\|\mathcal{H}_S(u-u_h)\|_{L_2(\Omega)}^2 \le C \, h^2 \, |u_\Omega|_{H^2(\Omega)} \, |w|_{H^2(\Omega)}.$$

Using now estimate (15), we arrive at the L_2 error estimate

$$\|\mathcal{H}_S(u-u_h)\|_{L_2(\Omega)} \le C \, h^2 \, |u_\Omega|_{H^2(\Omega)}. \tag{16}$$

This proves our main theorem.

Theorem 3. *Let us assume that the assumptions of Theorem 1 are satisfied and that the adjoint problem* (14) *is H^2-coercive. If the solution u of the variational problem* (1) *belongs to $H^2(\Omega)$, then the quasi-optimal L_2 discretization error estimate* (16) *holds.*

5 Conclusions

The detour via a mixed variational formulation allows us to establish a quasi-optimal L_2 discretization error estimate for the BEM-based FE discretization of the diffusion equation on polyhedral meshes that was introduced by D. Copeland, U. Langer and D. Pusch in [5] and whose H^1-convergence has been analyzed in [8]. Numerical results confirming the $O(h^2)$ behaviour of the L_2 discretization error were already presented in [5] and [8] for two- and three-dimensional diffusion problems, respectively.

Acknowledgments

The author would like to thank Joachim Schöberl (TU Vienna) as well as Ulrich Langer and Clemens Pechstein (JKU Linz) for fruitful discussions. Both of the latter also provided invaluable support in the preparation of this paper. Furthermore, the support by the Austrian Science Fund (FWF) under grant DK W1214 is gratefully acknowledged.

References

[1] M. Bebendorf. A note on the Poincaré inequality for convex domains. *Z. Anal. Anwendungen*, 22(4):751–756, 2003.

[2] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(3):1872–1896, 2005.

[3] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 4 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1987.

[4] D. M. Copeland. Boundary-element-based finite element methods for helmholtz and maxwell equations on general polyhedral meshes. *Int. J. Appl. Math. Comput. Sci.*, 5(1):60–73, 2009.

[5] D.M. Copeland, U. Langer, and D. Pusch. From the boundary element method to local Trefftz finite element methods on polyhedral meshes. In M. Bercovier, M. J. Gander, R. Kornhuber, and O. Widlund, editors, *Domain Decomposition Methods in Science and Engineering XVIII*, volume 70 of *Lecture Notes in Computational Science and Engineering*, pages 315–322. Springer-Verlag, Heidelberg, Berlin, 2009.

[6] V. Dolejší, M. Feistauer, and V. Sobotíková. Analysis of the discontinuous Galerkin method for nonlinear convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 194(25-26):2709–2733, 2005.

[7] D. Feuchter, M. Heisig, and G. Wittum. A geometry model for the simulation of drug diffusion through the stratum corneum. *Computing and Visualization in Science*, 9:117–130, 2006.

[8] C. Hofreither, U. Langer, and C. Pechstein. Analysis of a non-standard finite element method based on boundary integral operators. DK-Report 10-05, Doctoral Program in Computational Mathematics, Johannes Kepler University Linz, Austria, 2010. `https://www.dk-compmath.jku.at/publications/dk-reports/dk-report-10-05/view`. Accepted for publication in the Electronic Transactions on Numerical Analysis (ETNA).

[9] P. W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.*, 147:71–88, 1981.

[10] Y. Kuznetsov. Mixed finite element method for diffusion-type equations on polygonal meshes with mixed cells. *J. Numer. Math.*, 14(4):305–315, 2006.

[11] Y. Kuznetsov. Mixed finite element methods on polyhedral meshes for diffusion equations. In R. Glowinski and P. Neittaanmäki, editors, *Computational Methods in Applied Sciences*, volume 16, pages 27–41. Springer Netherlands, 2008.

[12] Y. Kuznetsov, K. Lipnikov, and M. Shashkov. The mimetic finite difference method on polygonal meshes for diffusion-type problems. *Comput. Geosci.*, 8(4):301–324, 2004.

[13] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, UK, 2000.

[14] C. Pechstein. Shape-explicit constants for some boundary integral operators. DK-Report 09-11, Doctoral Program in Computational Mathematics, Johannes Kepler University Linz, Austria, 2009. https://www.dk-compmath.jku.at/publications/dk-reports/dk-report-09-11/view.

[15] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements.* Springer-Verlag, New York, 2008.

[16] A. Thrun. Über die Konstanten in Poincaréschen Ungleichungen. Master's thesis, Ruhr-Universität Bochum, Bochum, 2003. http://www.ruhr-uni-bochum.de/num1/files/theses/da_thrun.pdf.

[17] A. Veeser and R. Verfürth. Poincaré constants for finite element stars. Technical report, Fakultät für Mathematik, Ruhr-Universität Bochum, 2009. http://www.ruhr-uni-bochum.de/num1/files/reports/Poincare.pdf.

Technical Reports of the Doctoral Program

“Computational Mathematics”

2010

2010-01 S. Radu, J. Sellers: *Parity Results for Broken k-diamond Partitions and (2k+1)-cores* March 2010. Eds.: P. Paule, V. Pillwein

2010-02 P.G. Gruber: *Adaptive Strategies for High Order FEM in Elastoplasticity* March 2010. Eds.: U. Langer, V. Pillwein

2010-03 Y. Huang, L.X.Châu Ngô: *Rational General Solutions of High Order Non-autonomous ODEs* June 2010. Eds.: F. Winkler, P. Paule

2010-04 S. Beuchler, V. Pillwein, S. Zaglmayr: *Sparsity optimized high order finite element functions for H(div) on simplices* September 2010. Eds.: U. Langer, P. Paule

2010-05 C. Hofreither, U. Langer, C. Pechstein: *Analysis of a non-standard finite element method based on boundary integral operators* September 2010. Eds.: B. Jüttler, J. Schicho

2010-06 M. Hodorog, J. Schicho: *A symbolic-numeric algorithm for genus computation* September 2010. Eds.: B. Jüttler, R. Ramlau

2010-07 M. Hodorog, J. Schicho: *Computational geometry and combinatorial algorithms for the genus computation problem* September 2010. Eds.: B. Jüttler, R. Ramlau

2010-08 C. Koukouvinos, V. Pillwein, D.E. Simos, Z. Zafeirakopoulos: *A Note on the Average Complexity Analysis of the Computation of Periodic and Aperiodic Ternary Complementary Pairs* October 2010. Eds.: P. Paule, J. Schicho

2010-09 V. Pillwein, S. Takacs: *Computing smoothing rates of collective point smoothers for optimal control problems using symbolic computation* October 2010. Eds.: U. Langer, P. Paule

2010-10 T. Takacs, B. Jüttler: *Existence of Stiffness Matrix Integrals for Singularly Parameterized Domains in Isogeometric Analysis* November 2010. Eds.: J. Schicho, W. Zulehner

2010-11 Y. Huang, L.X.Châu Ngô: *Rational Solutions of a Rational System of Autonomous ODEs: Generalization to Trivariate Case and Problems* November 2010. Eds.: F. Winkler, P. Paule

2010-12 S. Béla, B. Jüttler: *Approximating Algebraic Space Curves by Circular Arcs* November 2010. Eds.: J. Schicho, F. Winkler

2010-13 C. Hofreither: *L_2 Error Estimates for a Nonstandard Finite Element Method on Polyhedral Meshes* December 2010. Eds.: U. Langer, V. Pillwein

The complete list since 2009 can be found at
`https://www.dk-compmath.jku.at/publications/`

Doctoral Program
“Computational Mathematics”

Director:

Prof. Dr. Peter Paule
Research Institute for Symbolic Computation

Deputy Director:

Prof. Dr. Bert Jüttler
Institute of Applied Geometry

Address:

Johannes Kepler University Linz
Doctoral Program “Computational Mathematics”
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-7174

E-Mail:

office@dk-compmath.jku.at

Homepage:

http://www.dk-compmath.jku.at