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Finding rational solutions of rational systems of autonomous ODEs

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Abstract

In this paper we provide an algorithm to find explicitly rational solutions of a rational system of autonomous ordinary differential equations (ODEs) from its invariant algebraic curves. The method is based on the rational parametrization of the rational invariant algebraic curves and intensively using of linear fractional transformations between two proper rational parametrizations of the same algebraic curve.

1. Introduction

In this paper we consider the rational system of autonomous ordinary differential equations (ODEs)

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{1}$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$. A rational solution of this system is a pair of rational functions $(s(x), t(x))$ that satisfy the given system. Each rational solution of (1) represents a rational algebraic curve for which the rational solution itself is a rational parametrization of the curve. Such a rational algebraic curve is uniquely defined by its defining irreducible polynomial. This irreducible polynomial is also called the implicit representation of the algebraic curve. Therefore, one way to find a rational solution of (1) is first finding the implicit rational algebraic curves of the possible rational solutions and then choosing suitable parametrizations that satisfy the differential system (1).

In principle, there are several different parametrizations of the same rational algebraic curve. The key idea, which is used in this paper, is that two proper rational parametrizations of the same curve are tranformed to each other by a linear fractional transformation

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of the parameter. We derive a necessary and sufficient condition on this linear fractional transformation to get a rational solution from a rational parametrization of the rational algebraic curve.

The paper is consisting of the following sections. In Section 2 we recall the notion of invariant algebraic curves of a polynomial differential system, which is corresponding to the implicit representation of rational solutions. One can find such an invariant algebraic curve with a given upper bound for the degree of the curve by solving a system of algebraic equations on the coefficients of the invariant algebraic curve. In Section 3 we present an algorithm for finding explicitly rational solutions of a polynomial differential system from each invariant algebraic curve. Essentially, we construct an autonomous differential equation defining the transformation with respect to a proper parametrization of the invariant algebraic curve. Finally, in Section 4 we extend the previous algorithm to the rational system of ODEs. In fact, solving a rational system of ODEs can be reduced to finding invariant algebraic curves of a polynomial differential system and solving an autonomous differential equation defining the change of variable.

2. Finding invariant algebraic curves of a polynomial differential system

In this section we consider a special case of the rational system (1), namely, the polynomial differential system

$$\begin{cases} s' = P(s,t) \\ t' = Q(s,t) \end{cases} \tag{2}$$

where P and Q are polynomials in $\mathbb{K}[s,t]$ with constant coefficients.

Definition 2.1. An invariant algebraic curve of the polynomial differential system (2) is an algebraic curve $F(s,t) = 0$ such that

$$F_s P + F_t Q = FG, \tag{3}$$

for some polynomial $G \in \mathbb{K}[s,t]$, where $\mathbb{K}$ is the definition field of $F(s,t)$.

It turns out that the degree of the cofactor polynomial $G(s,t)$ is bounded by $\max\{\deg P, \deg Q\} - 1$. Therefore, if we know the degree of $F(s,t)$, then the coefficients of $F(s,t)$ can be found by equating the coefficients of the identity (3) and solving the system of quadratic polynomial equations on the coefficients of $F(s,t)$ and the coefficients of $G(s,t)$. In fact, $G(s,t)$ is uniquely defined by the quotient of the division F_sP+F_tQ by F. Thus we only need to solve a system of equations on the coefficients of F (see [Man93]). This observation makes the computation of invariant algebraic curves more effectively because one need not to involve more equations and variables from the coefficients of $G(s,t)$.

It is known that $F(s,t) = 0$ is an invariant algebraic curve of the system (2) if and only if each irreducible component of the curve $F(s,t) = 0$ is an invariant algebraic curve. Precisely, let $F(s,t) = \Pi_{i=1}^m F_i^{n_i}$ be the decomposition of $F(s,t)$ into irreducible factors. Then $F(s,t) = 0$ is an invariant algebraic curve of the system (2) with cofactor $G(s,t)$ if and only if the curves $F_i(s,t) = 0$ are invariant algebraic curves of the same system (2) with some cofactors G_i and $G = \sum_{i=1}^m n_i G_i$. Therefore, from now on we only consider the irreducible invariant algebraic curves of the system (2).

Let $R = \gcd(P, Q)$, $P_1 = P/R$ and $Q_1 = Q/R$. Then every invariant algebraic curve of the system

$$\begin{cases} s' = P_1(s,t) \\ t' = Q_1(s,t) \end{cases} \tag{4}$$

is an invariant algebraic curve of (2). Conversely, suppose that $F(s,t) = 0$ is an invariant algebraic curve of the system (2). Then

$$(F_s P_1 + F_t Q_1)R = FG$$

for some $G \in \mathbb{K}[s,t]$. Since $F(s,t)$ is irreducible, either $F|R$ or $F|(F_sP_1 + F_tQ_1)$. In the latter case, $F(s,t) = 0$ is an invariant algebraic curve of the system (4). In the first case, $F(s,t)$ is an irreducible factor of $R(s,t)$. Then no non-trivial parametrization of $F(s,t) = 0$ is a solution of the system (2) otherwise $P(s(x), t(x)) = 0 = Q(s(x), t(x))$ whereas $(s'(x), t'(x)) \neq (0,0)$.

Example 2.2. Consider the polynomial differential system

$$\begin{cases} s' = st \\ t' = s + t^2 \end{cases} \tag{5}$$

We ask for an invariant algebraic curve of degree 2 of the system (5). Take the graded lexicographic order with $s > t$. Let

$$F(s,t) \equiv s^2 + bst + ct^2 + ds + et + f.$$

Then

$$\begin{aligned} F_sP + F_tQ &= (2s + bt + d)st + (bs + 2ct + e)(s + t^2) \\ &= 2s^2t + 2bst^2 + (d + 2c)st + bs^2 + 2ct^3 + es + st^2. \end{aligned}$$

The remainder of the division $F_sP + F_tQ$ by F is

$$\left(2c - d - b^2\right) st + (e - bd)\, s - (2f + be)\, t - (e + bc)\, t^2 - bf.$$

Then the algebraic system of equations of the coefficients of $F(s,t)$ is

$$\begin{cases} 2c - d - b^2 = 0 \\ e - bd = 0 \\ 2f + be = 0 \\ e + bc = 0 \\ bf = 0. \end{cases} \tag{6}$$

Solving this system we obtain the following solution

$$\{b = 0, c = c, d = 2c, e = 0, f = 0\},$$

i.e.,

$$F(s,t) = s^2 + ct^2 + 2cs.$$

Using the same procedure if we ask for an invariant algebraic curve of degree 1, then we obtain

$$F(s,t) = s.$$

In this example we get a family of invariant algebraic curves of degree 2

$$s^2 + ct^2 + 2cs = 0,$$

depending on a parameter c and one single curve $s = 0$ of degree 1.

It is interesting to know what is the maximum number of possible parameters in the invariant algebraic curves of a polynomial system.

3. Finding rational solutions from invariant algebraic curves

In this section we provide an algorithm to find a rational solution $(s(x), t(x))$ of the polynomial differential system (2) from a given invariant algebraic curve, which can be found in the previous section. In fact, the invariant algebraic curves of the polynomial differential system (2) can be viewed as the trajectories of certain objects moving along the given vector field. Finding an explicit solution $(s(x), t(x))$ of the system (2) gives us a formula for representing the coordinates (s, t) of the object in terms of time x via the data of the velocity of the object at certain points.

Definition 3.1. A rational solution of the polynomial differential system (2) is called a *proper rational solution* iff it forms a proper rational parametrization of its corresponding invariant algebraic curve. An invariant algebraic curve of the system (2) is called a *rational invariant algebraic curve* iff it has a rational parametrization.

Theorem 3.2. *(1) If the system* (2) *has a rational solution, then it has only proper rational solutions.*

(2) Let $F(s, t) = 0$ be a rational invariant algebraic curve of the system (2) *defined by $P(s, t)$ and $Q(s, t)$. Let $(s(x), t(x))$ be an arbitrary proper rational parametrization of the curve $F(s, t) = 0$. If there is a linear fractional transformation $T(x) = \dfrac{ax + b}{cx + d}$ satisfying the autonomous differential equation*

$$T'(x) = \frac{P(s(T(x)), t(T(x)))}{s'(T(x))} = \frac{Q(s(T(x)), t(T(x)))}{t'(T(x))},$$

then a proper rational solution of the system (2) *defined by $F(s, t) = 0$ is given by*

$$\bar{s}(x) = s(T(x)), \quad \bar{t}(x) = t(T(x)).$$

Proof. (1). Let $(\bar{s}(x), \bar{t}(x))$ be a rational solution of the system (2), i.e.,

$$\begin{cases} \bar{s}'(x) = P(\bar{s}(x), \bar{t}(x)) \\ \bar{t}'(x) = Q(\bar{s}(x), \bar{t}(x)). \end{cases} \tag{7}$$

Let $F(s, t) = 0$ be the implicit equation of $(\bar{s}(x), \bar{t}(x))$. Let $(s(x), t(x))$ be a proper parametrization of $F(s, t) = 0$. By the relation between two rational parametrizations of the same algebraic curve (see e.g [SWPD08], Lemma 4.17), there exists a rational function $T(x)$ such that

$$\begin{cases} \bar{s}(x) = s(T(x)) \\ \bar{t}(x) = t(T(x)). \end{cases} \tag{8}$$

It follows from (7) that

$$\begin{cases} s'(T(x)) \cdot T'(x) = P(s(T(x)), t(T(x))) \\ t'(T(x)) \cdot T'(x) = Q(s(T(x)), t(T(x))). \end{cases} \tag{9}$$

Therefore,

$$T'(x) = \frac{P(s(T(x)), t(T(x)))}{s'(T(x))} = \frac{Q(s(T(x)), t(T(x)))}{t'(T(x))}.$$

Note that this is an autonomous differential equation with unknow $T(x)$. Moreover, this differential equation is of degree 1 with respect to $T'(x)$. Therefore, its rational solutions are linear fractional transformations (see e.g [FG06]). Hence $(\overline{s}(x), \overline{t}(x))$ is a proper rational solution.

(2). It follows from the above construction immediately. □

Algorithm 1. Input: $P(s,t), Q(s,t), F(s,t)$ such that

$$F_s P + F_t Q = FG$$

for some G.

Output: A rational solution (if any) of the system corresponding to $F(s,t) = 0$.

(1) `if` $F(s,t) = 0$ is not a rational curve, `then` return NO rational solution corresponding to $F(s,t) = 0$.
(2) `else` compute a proper rational parametrization $(s(x), t(x))$ of $F(s,t) = 0$.
(3) finding the rational solution of the autonomous differential equation

$$T'(x) = \frac{P(s(T(x)), t(T(x)))}{s'(T(x))}.$$

(4) `if` $T(x)$ is a (linear) rational function, `then` return

$$(s(T(x)), t(T(x)))$$

(5) `else` return NO rational solution corresponding to $F(s,t) = 0$.

Remark 3.3. 1. If $F(s,t) = 0$ is a rational invariant algebraic curve of the system (2), then for any non-trivial rational parametrization $(s(x), t(x))$ of $F(s,t) = 0$ we have

$$\frac{P(s(x), t(x))}{s'(x)} = \frac{Q(s(x), t(x))}{t'(x)}.$$

Therefore,

$$\frac{P(s(T(x)), t(T(x)))}{s'(T(x))} = \frac{Q(s(T(x)), t(T(x)))}{t'(T(x))}.$$

2. Let

$$T(x) = \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{K}$. Then

$$T'(x) = \frac{ad-bc}{(cx+d)^2} = \frac{(cT(x)-a)^2}{ad-bc}.$$

Therefore, the autonomous differential equation

$$T'(x) = \frac{P(s(T(x)), t(T(x)))}{s'(T(x))}$$

has a rational solution if and only if there exist $a, b, c, d \in \mathbb{K}$ such that

$$\frac{P(s(T(x)), t(T(x)))}{s'(T(x))} = \frac{(cT(x) - a)^2}{ad - bc}.$$

Example 3.4. Consider again the Example 2.2

$$\begin{cases} s' = st \\ t' = s + t^2. \end{cases}$$

We have found that

$$F(s, t) = Cs^2 + t^2 + 2s$$

is an invariant algebraic curve, where C is a constant parameter. We can check that $F(s, t) = 0$ is a rational algebraic curve with the proper parametrization given by

$$(s(x), t(x)) = \left(\frac{\sqrt{-C}x^2}{-2\sqrt{-C} + 2Cx}, \frac{-2\sqrt{-C}x + Cx^2}{-2\sqrt{-C} + 2Cx} \right).$$

We have

$$\frac{s(T(x)) \cdot t(T(x))}{s'(T(x))} = \frac{s(T(x)) + t(T(x))^2}{t'(T(x))} = \frac{1}{2}T(x)^2.$$

Solving the differential equation

$$T'(x) = \frac{1}{2}T(x)^2$$

we obtain

$$T(x) = -\frac{2}{x}.$$

By Theorem 3.2, the rational solution is

$$s(T(x)) = -\frac{2}{x(x - 2\sqrt{-C})}, \quad t(T(x)) = -\frac{2(x - \sqrt{-C})}{x(x - 2\sqrt{-C})}.$$

Remark 3.5. In Section 2 we have seen that it is sufficient to find the invariant algebraic curves of the polynomial system (4) instead of the system (2). In case P and Q have a non-trivial common factor, it makes the computation of invariant algebraic curves more simple because the degrees of P_1 and Q_1 are smaller than the degrees of P and Q.

Lemma 3.6. *Suppose that $F(s, t) = 0$ is a common rational invariant algebraic curve of both system (2) and (4). Assume that $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ are rational solutions, corresponding to the same invariant algebraic curve $F(s, t) = 0$, of (2) and (4) respectively. Then there exists a linear fractional transformation $T(x)$ such that*

$$\begin{cases} s_1(x) = s_2(T(x)) \\ t_1(x) = t_2(T(x)) \end{cases} \tag{10}$$

and

$$T'(x) = R(s_2(T(x)), t_2(T(x))) \tag{11}$$

where $R = \gcd(P, Q)$.

Therefore, solving the system (2) is reduced to solving the simpler system (4) and the autonomous differential equation (11).

4. Rational solutions of rational differential systems

In this section we extend the previous algorithm to the rational system of autonomous ODEs

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{12}$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$. The idea is that we will find a polynomial differential system which defines the same set of invariant algebraic curves of the system (12). Again if the invariant algebraic curve is rational, then we can find the rational solution of the system (12) from the rational parametrization of the given invariant algebraic curve.

Lemma 4.1. *Each rational solution of the rational differential system* (12) *defines a rational invariant algebraic curve of the polynomial differential system*

$$\begin{cases} s' = N_1(s,t) \cdot M_2(s,t) \\ t' = N_2(s,t) \cdot M_1(s,t). \end{cases} \tag{13}$$

Conversely, suppose that $F(s,t) = 0$ is a rational invariant algebraic curve of the system (13)*. Let $(s(x), t(x))$ be a rational parametrization of $F(s,t) = 0$. If $M_1(s(x), t(x)) \neq 0$ and $M_2(s(x), t(x)) \neq 0$, then*

$$s'(x) \cdot \frac{N_2(s(x), t(x))}{M_2(s(x), t(x))} = t'(x) \cdot \frac{N_1(s(x), t(x))}{M_1(s(x), t(x))}.$$

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the system (12). Let $F(s,t) = 0$ be its implicit equation. Then we have

$$F_s(s(x), t(x)) \cdot s'(x) + F_t(s(x), t(x)) \cdot t'(x) = 0.$$

Hence

$$F_s(s(x), t(x)) \cdot \frac{N_1(s(x), t(x))}{M_1(s(x), t(x))} + F_t(s(x), t(x)) \cdot \frac{N_2(s(x), t(x))}{M_2(s(x), t(x))} = 0.$$

It implies that

$$F_s(s(x),t(x)) \cdot N_1(s(x),t(x)) \cdot M_2(s(x),t(x)) + F_t(s(x),t(x)) \cdot N_2(s(x),t(x)) \cdot M_1(s(x),t(x)) = 0.$$

Therefore,

$$F_s(s,t) \cdot N_1(s,t) \cdot M_2(s,t) + F_t(s,t) \cdot N_2(s,t) \cdot M_1(s,t) = F(s,t) \cdot G(s,t)$$

for some $G \in \mathbb{K}[s,t]$. In other words, $F(s,t) = 0$ is a rational invariant algebraic curve of the polynomial differential system (13).

Conversely, if $F(s,t) = 0$ is an invariant algebraic curve of the system (13), then we have

$$F_s(s,t) \cdot N_1(s,t) \cdot M_2(s,t) + F_t(s,t) \cdot N_2(s,t) \cdot M_1(s,t) = F(s,t) \cdot G(s,t)$$

for some $G \in \mathbb{K}[s,t]$. Hence if $(s(x), t(x))$ is a parametrization of $F(s,t) = 0$, then

$$F_s(s(x),t(x)) \cdot N_1(s(x),t(x)) \cdot M_2(s(x),t(x)) + F_t(s(x),t(x)) \cdot N_2(s(x),t(x)) \cdot M_1(s(x),t(x)) = 0.$$

On the other hand,

$$F(s(x), t(x)) = 0$$

implies that

$$F_s(s(x), t(x)) \cdot s'(x) + F_t(s(x), t(x)) \cdot t'(x) = 0.$$

Since $(F_s(s(x), t(x)), F_t(s(x), t(x))) \neq (0, 0)$, it follows that

$$\begin{vmatrix} N_1(s(x), t(x)) \cdot M_2(s(x), t(x)) & N_2(s(x), t(x)) \cdot M_1(s(x), t(x)) \\ s'(x) & t'(x) \end{vmatrix} = 0.$$

Therefore, if $M_1(s(x), t(x)) \neq 0$ and $M_2(s(x), t(x)) \neq 0$, then

$$s'(x) \cdot \frac{N_2(s(x), t(x))}{M_2(s(x), t(x))} = t'(x) \cdot \frac{N_1(s(x), t(x))}{M_1(s(x), t(x))}.$$

This completes the proof of the lemma. $\square$

Remark 4.2. Let $(s(x), t(x))$ be a rational parametrization of $F(s, t) = 0$. The conditions $M_1(s(x), t(x)) \neq 0$ and $M_2(s(x), t(x)) \neq 0$ are equivalent to $F \not| M_1(s, t)$ and $F \not| M_2(s, t)$.

Intuitively, under certain conditions the two differential systems (12) and (13) should define the same set of invariant algebraic curves because we can forget for the moment about the length of the velocity vector of the object to find all its possible trajectories.

Definition 4.3. The system (13) is called the *associated polynomial differential system* of the rational differential system (12). By abuse of terminology each invariant algebraic curve of the associated polynomial differential system is also called an *invariant algebraic curve of the rational differential system* (12).

A similar theorem to Theorem 3.2 applies for the rational differential system (12) as follows.

Theorem 4.4. *(1) If the rational differential system* (12) *has a rational solution, then it has only proper rational solutions.*

(2) Let $F(s, t) = 0$ be a rational invariant algebraic curve of the rational differential system (12) *defined by $\dfrac{N_1(s, t)}{M_1(s, t)}$ and $\dfrac{N_2(s, t)}{M_2(s, t)}$ such that $F \not| M_1(s, t)$ and $F \not| M_2(s, t)$. Let $(s(x), t(x))$ be an arbitrary proper rational parametrization of the curve $F(s, t) = 0$. If there is a linear fractional transformation $T(x) = \dfrac{ax + b}{cx + d}$ satisfying the autonomous differential equation*

$$T'(x) = \frac{N_1(s(T(x)), t(T(x)))}{M_1(s(T(x)), t(T(x))) \cdot s'(T(x))} = \frac{N_2(s(T(x)), t(T(x)))}{M_2(s(T(x)), t(T(x))) \cdot t'(T(x))},$$

then a proper rational solution of the system (12) *defined by $F(s, t) = 0$ is given by*

$$\bar{s}(x) = s(T(x)), \quad \bar{t}(x) = t(T(x)).$$

Proof. The proof is completely the same as Theorem 3.2. Note that $F(s, t)$ does not divide $M_1(s, t)$ if and only if $M_1(s(x), t(x)) \neq 0$ for each rational parametrization $(s(x), t(x))$ of $F(s, t) = 0$. $\square$

From this theorem we need to find a rational solution of the autonomous differential equation

$$T'(x) = \frac{N_1(s(T(x)), t(T(x)))}{M_1(s(T(x)), t(T(x))) \cdot s'(T(x))}.$$

We prove in the next theorem that the rational solvability of this equation does not depend on the choice of the parametrization $\mathcal{P} = (s(x), t(x))$ of the curve $F(s,t) = 0$.

Theorem 4.5. *Let $F(s,t) = 0$ be a rational invariant algebraic curve of the rational differential system* (12) *defined by $\frac{N_1(s,t)}{M_1(s,t)}$ and $\frac{N_2(s,t)}{M_2(s,t)}$ such that $F \nmid M_1(s,t)$ and $F \nmid M_2(s,t)$. Let $\mathcal{P} = (s(x), t(x))$ and $\overline{\mathcal{P}} = (\overline{s}(x), \overline{t}(x))$ be two different proper parametrizations of $F(s,t) = 0$. Then two autonomous differential equations*

$$T'(x) = \frac{N_1(s(T(x)), t(T(x)))}{M_1(s(T(x)), t(T(x))) \cdot s'(T(x))} \tag{14}$$

and

$$\Phi'(x) = \frac{N_1(\overline{s}(\Phi(x)), \overline{t}(\Phi(x)))}{M_1(\overline{s}(\Phi(x)), \overline{t}(\Phi(x))) \cdot \overline{s}'(\Phi(x))} \tag{15}$$

have the same rational solvability in the sense that one of them has rational solutions if and only if the other one has. Moreover,

$$\mathcal{P} = \overline{\mathcal{P}} \circ \Phi \circ T^{-1}.$$

Proof. Suppose that (14) has a rational solution $T(x)$. Then the rational solution of the system (12) corresponding to $\mathcal{P}$ is $(s(T(x)), t(T(x)))$. Since $\overline{\mathcal{P}} = (\overline{s}(x), \overline{t}(x))$ is a rational parametrization of the same curve $F(s,t) = 0$, there exists a linear fractional transformation $\Phi(x)$ such that

$$\begin{cases} s(T(x)) = \overline{s}(\Phi(x)) \\ t(T(x)) = \overline{t}(\Phi(x)). \end{cases} \tag{16}$$

Hence

$$\begin{cases} s'(T(x)) \cdot T'(x) = \overline{s}'(\Phi(x)) \cdot \Phi'(x) \\ t'(T(x)) \cdot T'(x) = \overline{t}(\Phi(x)) \cdot \Phi'(x). \end{cases}$$

It follows that (14) has a rational solution $T(x)$ if and only if (15) has a rational solution $\Phi(x)$. Moreover, it is clear from (16) that $\mathcal{P} = \overline{\mathcal{P}} \circ \Phi \circ T^{-1}$. □

We may obtain two different rational general solutions corresponding to two different proper parametrizations. In the next theorem we will see that they are related to each other by a shifting of the variable.

Theorem 4.6. *Let $(s(x), t(x))$ and $(\overline{s}(x), \overline{t}(x))$ be rational solutions of the differential system* (12) *corresponding to the same rational invariant algebraic curve. Then there exists a constant c such that*

$$(s(x+c), t(x+c)) = (\overline{s}(x), \overline{t}(x)).$$

Proof. Since $(s(x), t(x))$ and $(\overline{s}(x), \overline{t}(x))$ are rational parametrizations of the same invariant algebraic curve, there exists a linear fractional transformation $T(x)$ such that

$$(\overline{s}(x), \overline{t}(x)) = (s(T(x)), t(T(x))).$$

Hence

$$\begin{cases} s'(T(x))T'(x) = \overline{s}'(x) = \dfrac{N_1(\overline{s}(x),\overline{t}(x))}{M_1(\overline{s}(x),\overline{t}(x))} = \dfrac{N_1(s(T(x)),t(T(x)))}{M_1(s(T(x)),t(T(x)))} = s'(T(x)) \\ t'(T(x))T'(x) = \overline{t}'(x) = \dfrac{N_2(\overline{s}(x),\overline{t}(x))}{M_2(\overline{s}(x),\overline{t}(x))} = \dfrac{N_2(s(T(x)),t(T(x)))}{M_2(s(T(x)),t(T(x)))} = t'(T(x)). \end{cases} \tag{17}$$

It follows that

$$T'(x) = 1.$$

Therefore, $T(x) = x + c$ for some constant c. In fact, we can compute the precise transformation from one solution into the other one as follows. Let $\mathcal{P} = (s(x), t(x))$ and $\overline{\mathcal{P}} = (\overline{s}(x), \overline{t}(x))$ as above. Then the transformation from $\mathcal{P}$ to $\overline{\mathcal{P}}$ is $\overline{\mathcal{P}} \circ \mathcal{P}^{-1}$ and from $\overline{\mathcal{P}}$ to $\mathcal{P}$ is $\mathcal{P} \circ \overline{\mathcal{P}}^{-1}$. □

Example 4.7. Let us consider the rational differential system

$$\begin{cases} s' = \dfrac{-2(-(t-1)^2+s^2)(t-1)^2}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^3 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{18}$$

The associated polynomial differential system is

$$\begin{cases} s' = -2(-(t-1)^2+s^2)(t-1)^2((t-1)^2+s^2)^2 \\ t' = -4(t-1)^3 s((t-1)^2+s^2)^2. \end{cases} \tag{19}$$

The set of invariant algebraic curves of degree less than or equal to 2 is

$$\{t-1=0, s+\sqrt{-1}(t-1)=0, s^2+t^2+(-1-a)t+a=0\}.$$

Obviously, $t = 1, s = c$ is a solution of the system for an arbitrary constant c. The invariant algebraic curve $s + \sqrt{-1}(t-1) = 0$ is rejected because the polynomial $s + \sqrt{-1}(t-1)$ is a factor of the denominator $(t-1)^2 + s^2$. It remains to consider the invariant algebraic curve

$$s^2 + t^2 + (-1-a)t + a = 0.$$

This is a family of curves depending on the parameter a. We compute a proper parametrization of this algebraic curve.

$$R := \left[\frac{-ai+ix+iax-ix^2}{1+a-2x}, \frac{a-x^2}{1+a-2x}\right].$$

Now we need to find a rational solution $T(x)$ of the autonomous differential equation

$$T' = \frac{-2(R[1]^2-(R[2]-1)^2)(R[2]-1)^2}{((R[2]-1)^2+R[1]^2)^2}\frac{1}{R[1]'}$$

i.e.,

$$T' = \frac{-2i(T-1)^2}{(a-1)^2}.$$

Therefore,

$$T(x) = 1 + \frac{(a-1)^2}{2ix}.$$

Now we subsitute $T(x)$ into $R[1]$ and $R[2]$ to obtain

$$s(x) = \frac{(2ix - a + 1)(a-1)^2}{4x(ix - a + 1)}, \quad t(x) = \frac{i(4x^2 + 4i(a-1)x + (a-1)^3)}{4x(ix - a + 1)}.$$

The equation

$$s^2 + t^2 - (1+a)t + a = 0$$

can be written as

$$s^2 + \left(t - \frac{a+1}{2}\right)^2 - \frac{(a-1)^2}{4} = 0.$$

This is a family of circles with center $\left(0, \frac{a+1}{2}\right)$ and radius $\frac{|a-1|}{2}$. The point $(0,1)$ is a fixed point of this family.

We can parametrize these circles over the field of real numbers by the line $t - 1 = xs$, namely

$$R = \left(\frac{(a-1)x}{1+x^2}, \frac{ax^2+1}{1+x^2}\right).$$

Then we find $T(x)$ such that

$$T' = \frac{-2T^2}{a-1}.$$

Hence

$$T(x) = \frac{a-1}{2x}.$$

Therefore, the solution is

$$s(x) = \frac{2(a-1)^2 x}{4x^2 + (a-1)^2}, \quad t(x) = \frac{a(a-1)^2 + 4x^2}{4x^2 + (a-1)^2}.$$

The linear transformation from the first solution into the second one is

$$T(x) = x + \frac{i(a-1)}{2}.$$

The linear transformation from the second solution into the first one is

$$T(x) = x - \frac{i(a-1)}{2}.$$

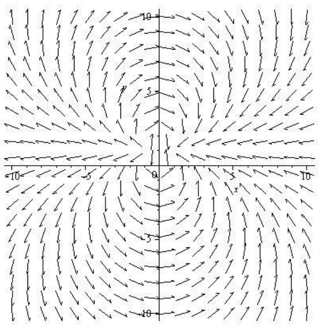

Fig. 1. The normal vector field of the rational system

Remark 4.8. (1) The invariant algebraic curve $s + \sqrt{-1}(t-1) = 0$ is just a factor of the denominator $((t-1)^2 + s^2)^2$ of the rational differential system. Therefore, we should exclude those invariant algebraic curves of the associated polynomial differential system that are factors of the denominators.

(2) Solving the system (18) can be performed as follows. First we solve a simpler polynomial system

$$\begin{cases} s' = -(t-1)^2 + s^2 \\ t' = 2(t-1)s. \end{cases} \tag{20}$$

to obtain

$$s(x) = \frac{ta - t - i}{(-ta + t + 2i)t}, \quad t(x) = \frac{-at^2 + t^2 + 2ti - 1}{(-ta + t + 2i)t}.$$

Let

$$R(s,t) = \frac{-2(t-1)^2}{((t-1)^2 + s^2)^2}$$

be the common rational factor of the right hand sides of the given system. We need to find a rational function $T(x)$ such that

$$T'(x) = R(s(T(x)), t(T(x))),$$

where $(s(x), t(x))$ is the above solution of the simpler system. In this example, we have to solve the autonomous differential equation

$$T'(x) = -\frac{2}{(a-1)^2}.$$

Hence

$$T(x) = -\frac{2x}{(a-1)^2}.$$

Finally, we get the solution

$$s(T(x)) = \frac{(2ix - a + 1)(a-1)^2}{4x(ix - a + 1)}, \quad t(T(x)) = \frac{i(4x^2 + 4i(a-1)x + (a-1)^3)}{4x(ix - a + 1)}.$$

5. Rational general solutions and the family of invariant algebraic curves

Definition 5.1. A rational solution $(s(x), t(x))$ of the system (12) is called a rational general solution iff for any $H \in \mathbb{K}[s,t]$,

$$H(s(x), t(x)) = 0 \Leftrightarrow H = 0.$$

Theorem 5.2. *Let $F(s,t,c) = 0$ be a family of rational invariant algebraic curves, depending on an arbitrary constant parameter c, of the differential system* (12). *Let $(s(x), t(x))$ be a rational solution of the differential system* (12) *corresponding to $F(s,t,c) = 0$. Then $(s(x), t(x))$ is a rational general solution of the system.*

Proof. Let $H(s,t)$ be a polynomial in $\mathbb{K}[s,t]$ such that $H(s(x), t(x)) = 0$. Then $H(s,t)$ is divisible by $F(s,t,c)$ for any constant parameter c, hence $H(s,t) = 0$. Therefore, $(s(x), t(x))$ is a rational general solution of the system (12). □

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