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A Robust Preconditioned-MinRes-Solver for Distributed Time-Periodic Eddy Current Optimal Control Problems

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A ROBUST PRECONDITIONED-MINRES-SOLVER FOR DISTRIBUTED TIME-PERIODIC EDDY CURRENT OPTIMAL CONTROL PROBLEMS

MICHAEL KOLMBAUER AND ULRICH LANGER

Abstract. This work is devoted to distributed optimal control problems for time-periodic eddy current problems. We apply the multiharmonic approach to the optimality system and construct a new preconditioned MinRes solver for the system of frequency domain equations. We show that this solver is robust with respect to the space discretization and time discretization parameters as well as all involved "bad" parameters like the conductivity, the reluctivity, and the regularization parameters.

1. Introduction

The multiharmonic finite element method or harmonic-balanced finite element method has been used by many authors in different applications (e.g. [3, 9, 12, 26, 37]). Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation.

Following this strategy, Copeland et al. [6, 7], Bachinger et al. [4, 5], and Kolmbauer and Langer [18, 19] applied harmonic and multiharmonic approaches to parabolic initial-boundary value problems and the eddy current problem. Indeed, in [19] a MinRes solver for the solution of time-harmonic eddy current problems was constructed, that is robust with respect to both the discretization parameter h and all involved parameters like frequency, conductivity and reluctivity.

The aim of this work is to generalize these ideas of combining the multiharmonic approach and the finite element method to optimal control problems. To the authors best knowledge this is the first approach of using the multiharmonic FEM for optimal control problems with PDE constraints. We mention that there exist Fourier series approaches for optimal control problems, controlled by a simple time-dependent ordinary differential equation (e.g. [8, 22] and the references therein).

The fast solution of the corresponding large linear system of finite element equations is crucial for the competitiveness of this method. Hence appropriate (parameter-robust) preconditioning is an important issue. Deriving the optimality system of the optimal control problem naturally results in a saddle point system. A new technique of parameter robust preconditioning of saddle point problems was introduced by Schöberl and Zulehner in [32] and generalized by Zulehner in [38]. We use this technique to construct a parameter-robust preconditioned MinRes solver for our huge system of algebraic equations resulting from the multiharmonic finite element discretization.

The outline of this work is the following. We start by stating our model problem. In order to ensure unique solvability of the state equation, namely the eddy current

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problem, we have to perform a regularization. Before considering the general multiharmonic approach, we have a look at the time-harmonic problem. Here we are able to construct a parameter-robust preconditioner for the special case of constant conductivity by an interpolation technique. For the case of piecewise constant conductivity including also non-conductive regions, we also propose a parameter robust preconditioner. This preconditioner is inspired by the preconditioner derived for the case of constant conductivity, but the analysis is based on other tools. Finally, we apply our new preconditioner to the general time-periodic optimal control problem. The paper concludes with the presentation of our first numerical results and with a conclusion in which we also discuss the possible generalization of these results to non-linear eddy current problems and to additional constraints imposed onto the state and the control.

2. An Optimal Control Problem with Distributed Control

In this work we consider distributed optimal control problems of the form: Find the state $\mathbf{y}$ and the control $\mathbf{u}$ that minimizes the cost functional

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_{\Omega\times(0,T)} |\mathbf{y}-\mathbf{y_d}|^2 d\mathbf{x}dt + \frac{\lambda}{2}\int_{\Omega\times(0,T)} |\mathbf{u}|^2 d\mathbf{x}dt \tag{1}$$

subject to the state equation

$$\begin{cases} \sigma\dfrac{\partial}{\partial t}\mathbf{y} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y}) = \mathbf{u}, & \text{in } \Omega\times(0,T), \\ \mathbf{y}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T), \\ \mathbf{y} = \mathbf{y_0}, & \text{on } \Omega\times\{0\}. \end{cases} \tag{2}$$

Here $\mathbf{y_d}$ is the desired state and $\lambda > 0$ is a regularization parameter. We assume, that $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain. The reluctivity $\nu = \nu(\mathbf{x})$ is supposed to be independent of $|\mathbf{curl}\ \mathbf{u}|$, i.e. we assume that the eddy current problem (2) is linear. The conductivity σ is piecewise constant and zero in non-conducting regions. We assume that the control $\mathbf{u}$ is weakly divergence free. Bachinger et al. [4] provide existence and uniqueness results for linear and non-linear eddy current problems in appropriate gauged spaces. Then the optimal control problem is also uniquely solvable [35]. Later we will consider the time-harmonic and the time-periodic cases where we look for the steady-state solution of the eddy-current problem (2). These cases of state equation were also considered in [4] where even non-linear eddy current problems were analyzed.

2.1. **Regularization.**

Eddy current problems are essentially different for conducting ($\sigma > 0$) and non-conducting regions ($\sigma = 0$). In order to gain uniqueness in the non-conducting regions, the state equation (2) has to be regularized. Candidates are elliptic, conductivity and exact regularizations. Since, for the preconditioning purpose, all of them can be handled in the same framework, we start with introducing formal regularization operators ($i = 1, 2, 3$). For completeness, we also include the case without any regularization ($i = 0$).

$$\mathcal{R}_i(\sigma) := \begin{cases} \sigma, & i=0 \\ \sigma, & i=1 \\ \max(\sigma,\varepsilon), & i=2 \\ \sigma, & i=3 \end{cases}, \quad \mathcal{Q}_i(\mathbf{y}) := \begin{cases} \mathbf{0}, & i=0 \quad \text{(no regularization)} \\ \mathbf{Q}(\mathbf{y}), & i=1 \quad \text{(exact)} \\ \mathbf{0}, & i=2 \quad \text{(parabolic)} \\ \varepsilon\mathbf{y}, & i=3 \quad \text{(elliptic)} \end{cases}$$

Here $\varepsilon > 0$ is a small regularization parameter. The operator $\mathbf{Q}$ is chosen in such a way that it ensures the coercivity of the resulting bilinear form and on the other hand $\mathbf{Q}(\mathbf{y})$ vanishes at the solution. The exact regularization technique is based on a Helmholtz-projection (for details see [19]). We mention, that for the exact regularization technique, no additional error is introduced (see [20]), while for the parabolic and elliptic regularization technique, we have to deal with an additional error of order $\mathcal{O}(\varepsilon)$ (see [4] and [28]).

Remark 1. *The exact regularization operator ($i = 1$) is defined by the regularization term*

$$\int_\Omega \nabla P\mathbf{u} \cdot \nabla P\mathbf{v} \; \mathrm{d}\mathbf{x}$$

added to the variational formulation of (2), where P denotes the Helmholtz projection.

Hence the regularized problem can be stated as follows:

$$\min \frac{1}{2} \int_{\Omega\times(0,T)} |\mathbf{y} - \mathbf{y_d}|^2 d\mathbf{x}dt + \frac{\lambda}{2} \int_{\Omega\times(0,T)} |\mathbf{u}|^2 d\mathbf{x}dt \tag{3}$$

subject to the state equation

$$\begin{cases} \mathcal{R}_i(\sigma)\dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y}) + \mathcal{Q}_i(\mathbf{y}) = \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y} = \mathbf{y_0}, & \text{on } \Omega \times \{0\}. \end{cases} \tag{4}$$

Applying any regularization technique $i = 1, 2, 3$, we have unique solvability of (4) in the full space.

2.2. **Optimality System.**

In order to solve our minimization problem, we formulate the optimality system called also Karush-Kuhn-Tucker system (see e.g. [35]). Therefore, we formally consider the Lagrangian functional

$$\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) := \mathcal{J}(\mathbf{y},\mathbf{u}) + \int_{\Omega\times(0,T)} \left(\mathcal{R}_i(\sigma)\frac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y}) + \mathcal{Q}_i(\mathbf{y}) - \mathbf{u} \right) \cdot \mathbf{p}\, d\mathbf{x}dt.$$

Deriving the necessary optimality conditions

$$\text{Find } \mathbf{y},\mathbf{u},\mathbf{p} : \quad \begin{cases} \nabla_{\mathbf{p}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0} \\ \nabla_{\mathbf{y}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0} \\ \nabla_{\mathbf{u}}\mathcal{L}(\mathbf{y},\mathbf{u},\mathbf{p}) = \mathbf{0} \end{cases}$$

yields the following system of partial differential equations: Find the state $\mathbf{y}$, the co-state $\mathbf{p}$ and the control $\mathbf{u}$, such that

$$\begin{cases} \mathcal{R}_i(\sigma)\dfrac{\partial}{\partial t}\mathbf{y} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y}) + \mathcal{Q}_i(\mathbf{y}) - \mathbf{u} = \mathbf{0}, & \text{in } \Omega \times (0,T) \\ -\mathcal{R}_i(\sigma)\dfrac{\partial}{\partial t}\mathbf{p} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{p}) + \mathcal{Q}_i(\mathbf{p}) + \mathbf{y} - \mathbf{y_d} = \mathbf{0}, & \text{in } \Omega \times (0,T) \\ \lambda\mathbf{u} - \mathbf{p} = \mathbf{0}, & \text{in } \Omega \times (0,T) \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T) \\ \mathbf{p} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T) \\ \mathbf{y} = \mathbf{y_0}, & \text{on } \Omega \times \{0\} \\ \mathbf{p} = \mathbf{0}, & \text{on } \Omega \times \{T\} \end{cases}$$

From the third equation we observe that $\mathbf{u} = \lambda^{-1}\mathbf{p}$, and hence we can eliminate the control. Therefore we end up with the following reduced optimality system

$$
\left\{
\begin{aligned}
\mathcal{R}_i(\sigma)\frac{\partial}{\partial t}\mathbf{y} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y}) + \mathcal{Q}_i(\mathbf{y}) - \lambda^{-1}\mathbf{p} &= \mathbf{0}, && \text{in } \Omega \times (0,T) \\
-\mathcal{R}_i(\sigma)\frac{\partial}{\partial t}\mathbf{p} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{p}) + \mathcal{Q}_i(\mathbf{p}) + \mathbf{y} &= \mathbf{y_d}, && \text{in } \Omega \times (0,T) \\
\mathbf{y} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega \times (0,T) \\
\mathbf{p} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega \times (0,T) \\
\mathbf{y} &= \mathbf{y_0}, && \text{on } \Omega \times \{0\} \\
\mathbf{p} &= \mathbf{0}, && \text{on } \Omega \times \{T\}.
\end{aligned}
\right. \tag{5}
$$

In the usual manner we derive a space-time variational formulation, and this is the starting point of our discretization in time and space.

3. Time-harmonic FEM and preconditioning

For the time being, we assume that our desired state $\mathbf{y_d}$ is given by a time-harmonic excitation with the frequency $\omega = 2\pi/T$ and the amplitudes $\mathbf{y_d^c}$ and $\mathbf{y_d^s}$, i.e.

$$\mathbf{y_d}(\mathbf{x},t) = \mathbf{y_d^c}(\mathbf{x})\cos(\omega t) + \mathbf{y_d^s}(\mathbf{x})\sin(\omega t).$$

Therefore, we can assume that the state $\mathbf{y}$, the co-state $\mathbf{p}$ and the control $\mathbf{u}$ are time-harmonic as well, with the same base frequency ω:

$$
\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \mathbf{u^c}(\mathbf{x})\cos(\omega t) + \mathbf{u^s}(\mathbf{x})\sin(\omega t), \\
\mathbf{p}(\mathbf{x},t) &= \mathbf{p^c}(\mathbf{x})\cos(\omega t) + \mathbf{p^s}(\mathbf{x})\sin(\omega t), \\
\mathbf{y}(\mathbf{x},t) &= \mathbf{y^c}(\mathbf{x})\cos(\omega t) + \mathbf{y^s}(\mathbf{x})\sin(\omega t).
\end{aligned} \tag{6}
$$

The Fourier coefficients of the control $\mathbf{u}$ and the co-state $\mathbf{p}$ are obviously aligned by the relation $\mathbf{u^c}(\mathbf{x}) = \lambda^{-1}\mathbf{p^c}(\mathbf{x})$ and $\mathbf{u^s}(\mathbf{x}) = \lambda^{-1}\mathbf{p^s}(\mathbf{x})$.

Remark 2. *In fact, the time-harmonic case may be not so relevant for practical applications. Nevertheless, it allows us to demonstrate the construction of a parameter-robust preconditioner for the resulting system matrix, that can later be applied to the more interesting multiharmonic or time-periodic case in a straight forward manner. Since the initial condition are not relevant in this time-harmonic setting, where we look for the steady-state solution, they will be neglected. We mention that the homogeneous boundary conditions for the state and the co-state can be replaced by non-homogeneous but harmonic boundary conditions.*

Using the time-harmonic representations (6), we can state our optimality system (5) in the frequency domain as follows: Find the states $\mathbf{y^s}$ and $\mathbf{y^c}$ and the co-states $\mathbf{p^s}$ and $\mathbf{p^c}$, such that

$$
\left\{
\begin{aligned}
-\omega\mathcal{R}_i(\sigma)\mathbf{y^s} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y^c}) + \mathcal{Q}_i(\mathbf{y^c}) - \lambda^{-1}\mathbf{p^c} &= \mathbf{0}, && \text{in } \Omega \\
\omega\mathcal{R}_i(\sigma)\mathbf{y^c} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y^s}) + \mathcal{Q}_i(\mathbf{y^s}) - \lambda^{-1}\mathbf{p^s} &= \mathbf{0}, && \text{in } \Omega \\
\omega\mathcal{R}_i(\sigma)\mathbf{p^s} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{p^c}) + \mathcal{Q}_i(\mathbf{p^c}) + \mathbf{y^c} &= \mathbf{y_d^c}, && \text{in } \Omega \\
-\omega\mathcal{R}_i(\sigma)\mathbf{p^c} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{p^s}) + \mathcal{Q}_i(\mathbf{p^s}) + \mathbf{y^s} &= \mathbf{y_d^s}, && \text{in } \Omega \\
\mathbf{y^c} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega \\
\mathbf{y^s} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega \\
\mathbf{p^c} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega \\
\mathbf{p^s} \times \mathbf{n} &= \mathbf{0}, && \text{on } \partial\Omega.
\end{aligned}
\right. \tag{7}
$$

The finite element discretization of the variational formulation of (7) with edge elements, introduced by Nédélec in [23, 24], yields the following system of linear algebraic equations for defining the finite element coefficient vectors approximating the Fourier coefficients of the state $\mathbf{y^c}$ and $\mathbf{y^s}$ and the co-state $\mathbf{p^c}$ and $\mathbf{p^s}$:

$$
\underbrace{\begin{pmatrix} \mathbf{M} & & \mathbf{K} & \mathbf{M}_{\sigma,\omega} \\ & \mathbf{M} & -\mathbf{M}_{\sigma,\omega} & \mathbf{K} \\ \mathbf{K} & -\mathbf{M}_{\sigma,\omega} & -\lambda^{-1}\mathbf{M} & \\ \mathbf{M}_{\sigma,\omega} & \mathbf{K} & & -\lambda^{-1}\mathbf{M} \end{pmatrix}}_{=:\mathcal{A}} \begin{pmatrix} \underline{\mathbf{y}}^{\mathbf{c}} \\ \underline{\mathbf{y}}^{\mathbf{s}} \\ \underline{\mathbf{p}}^{\mathbf{c}} \\ \underline{\mathbf{p}}^{\mathbf{s}} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{y}}_{\mathbf{d}}^{\mathbf{c}} \\ \underline{\mathbf{y}}_{\mathbf{d}}^{\mathbf{s}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{8}
$$

where the mass matrix $\mathbf{M}$, the conductivity matrix $\mathbf{M}_{\sigma,\omega}$ and the stiffness matrix $\mathbf{K}$ arise from the edge finite element discretization of the following bilinear forms:

$$
\begin{aligned}
\mathbf{M}: &\quad \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \qquad \mathbf{M}_{\sigma,\omega}: \quad \omega\int_\Omega \mathcal{R}_i(\sigma)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\\
\mathbf{K}: &\quad \int_\Omega \nu\,\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathcal{Q}_i(\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

The right-hand side vectors $\underline{\mathbf{y}}_{\mathbf{d}}^{\mathbf{c}}$ and $\underline{\mathbf{y}}_{\mathbf{d}}^{\mathbf{s}}$ are defined by the linear forms

$$
\int_\Omega \mathbf{y}_{\mathbf{d}}^{\mathbf{c}}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \quad \text{and} \quad \int_\Omega \mathbf{y}_{\mathbf{d}}^{\mathbf{s}}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},
$$

respectively, where $\mathbf{u}$ and $\mathbf{v}$ are vector functions from $H_0(\mathbf{curl})$. In our numerical experiments presented in Section 5, we use lowest-order tetrahedral edge elements for the finite element discretization.

Hence, we have to solve a linear system of finite element equations of the form

$$
\mathcal{A}\mathbf{w} = \mathbf{f}, \tag{9}
$$

where the system matrix $\mathcal{A}$ and the right-hand-side $\mathbf{f}$ are given by (8). Typically the condition number of the system matrix $\mathcal{A}$ behaves like

$$
\mathcal{O}\left(\frac{\omega}{h^2}\frac{\max(\sigma)}{\max(\varepsilon,\min(\sigma))}\frac{\max(\nu)}{\min(\nu)}\frac{1}{\lambda}\right),
$$

and therefore, we expect a very bad convergence rate if any iterative method without preconditioner is applied to (9). Hence appropriate preconditioning is an important issue .

3.1. **MinRes Preconditioning.** Since the resulting system matrix $\mathcal{A}$ is symmetric and indefinite, the corresponding system (9) of finite element equations can be solved by a preconditioned MinRes method [25]. Our ingredients for the construction of a parameter-robust preconditioner are, on the one hand, a constructive preconditioning strategy based on space interpolation proposed by Zulehner [38], and, on the other hand, the introduction of a non-standard norm in $\mathbf{H}(\mathbf{curl})$ and the theorem of Babuška-Aziz [2]. The former technique works in the case of constant conductivity, whereas the latter one also works for piecewise constant conductivity.

Choosing any regularization technique of the previous section, we end up with a structured system matrix, where the particular form of $\mathbf{M}_{\sigma,\omega}$ and $\mathbf{K}$ depends on the choice of the regularization technique. Anyhow, in all three cases the matrices $\mathbf{M}_{\sigma,\omega}$ and $\mathbf{K}$ are at least positive semi-definite and this is enough for our analysis. Hence we deal with all of them ($i = 1, 2, 3$) in the same framework.

We explore the block-saddle point structure of our system matrix $\mathcal{A}$ that can be rewritten in the form

$$\mathcal{A} = \begin{pmatrix} \mathbf{M} & & \mathbf{K} & \mathbf{M}_{\sigma,\omega} \\ & \mathbf{M} & -\mathbf{M}_{\sigma,\omega} & \mathbf{K} \\ \mathbf{K} & -\mathbf{M}_{\sigma,\omega} & -\lambda^{-1}\mathbf{M} & \\ \mathbf{M}_{\sigma,\omega} & \mathbf{K} & & -\lambda^{-1}\mathbf{M} \end{pmatrix} =: \begin{pmatrix} \mathbf{A} & \mathbf{B^T} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix},$$

where the blocks $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are defined by the relations

$$\mathbf{A} = \begin{pmatrix} \mathbf{M} & \\ & \mathbf{M} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} \mathbf{K} & -\mathbf{M}_{\sigma,\omega} \\ \mathbf{M}_{\sigma,\omega} & \mathbf{K} \end{pmatrix} \quad \mathbf{C} = \frac{1}{\lambda}\mathbf{A}.$$

Here, $\mathbf{A}$ and $\mathbf{C}$ are positive definite, and $\mathbf{B}$ is at least positive semi-definite. We mention that, for the exact and elliptic regularized problems, $\mathbf{B}$ is even positive definite. With this setting, our problem fits into the general framework proposed by Zulehner [38]. In the next subsection, we recall this approach that is based on space interpolation.

3.1.1. *Abstract Preconditioning Strategy.* For constructing a block-diagonal preconditioner, we want to use the general interpolation framework proposed by Zulehner for saddle point problems [38]. Hence, for the time being, let our system matrix $\mathcal{A}$ be given by the 2×2 symmetric and indefinite block matrix of the form

$$\mathcal{A} = \begin{pmatrix} \mathbf{A} & \mathbf{B^T} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}.$$

We quote the following theorem that can be found in [38, Section 3].

Theorem 1 ([38])**.** *Let $\mathbf{A}$, $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{R}$ be symmetric and positive definite matrices in $\mathbb{R}^{n\times n}$, where*

$$\mathbf{S} = \mathbf{C} + \mathbf{B}\mathbf{A}^{-1}\mathbf{B^T} \quad \textit{and} \quad \mathbf{R} = \mathbf{A} + \mathbf{B^T}\mathbf{C}^{-1}\mathbf{B}.$$

denote the negative Schur complements of $\mathcal{A}$. Then, for the matrix

$$\mathcal{C} = \begin{pmatrix} [\mathbf{A},\mathbf{R}]_{\frac{1}{2}} & \\ & [\mathbf{S},\mathbf{C}]_{\frac{1}{2}} \end{pmatrix},$$

the norm equivalence inequalities

$$c_1\|\mathbf{x}\|_{\mathcal{C}} \le \|\mathcal{A}\mathbf{x}\|_{\mathcal{C}^{-1}} \le c_2\|\mathbf{x}\|_{\mathcal{C}} \quad \forall \mathbf{x} \in \mathbb{R}^n$$

are satisfied with positive constants c_1 and c_2 which are independent of any involved parameters.

In the previous theorem the interpolation of matrices $[\cdot,\cdot]_{\frac{1}{2}}$ is defined by the relation

$$[\mathbf{M},\mathbf{N}]_{\frac{1}{2}} := \mathbf{M}^{\frac{1}{2}} \left(\mathbf{M}^{-\frac{1}{2}}\mathbf{N}\mathbf{M}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{M}^{\frac{1}{2}}.$$

In the following computations, we take advantage of the spectral equivalence inequalities

$$\frac{1}{\sqrt{2}}(\mathbf{C} + [\mathbf{C},\mathbf{R}]_{\frac{1}{2}}) \le [\mathbf{C},\mathbf{C}+\mathbf{R}]_{\frac{1}{2}} \le \mathbf{C} + [\mathbf{C},\mathbf{R}]_{\frac{1}{2}}$$

and the identities

$$[\mathbf{A},\lambda\mathbf{R}]_{\frac{1}{2}} = [\lambda\mathbf{R},\mathbf{A}]_{\frac{1}{2}} = \sqrt{\lambda}[\mathbf{R},\mathbf{A}]_{\frac{1}{2}}$$

which are obviously valid for all $\lambda \in \mathbb{R}^+$ and for any positive definite matrices $\mathbf{C},\mathbf{R}$ and $\mathbf{A}$.

3.1.2. *The Case of Constant Conductivity σ.* Before we turn to our general model problem, we start with analyzing the special case of constant conductivity, i.e.

$$\sigma(\mathbf{x}) = \sigma \in \mathbb{R}^+ \quad \forall \mathbf{x} \in \Omega.$$

In this special setting, we have $\mathbf{M}_{\sigma,\omega} = \omega\sigma\mathbf{M}$ with the constant positive conductivity σ. Due to this special structure, this case is much easier to handle. We mention that in this special case no regularization has to be applied (i.e. $i = 0$), since the state equation is well-posed anyway. Nevertheless, we stay with our notation in order to be consistent throughout this paper.

Our system (8) fulfills the requirements of Theorem 1. Following the strategy of the previous section, we have to compute the negative Schur complements $\mathbf{S}$ and $\mathbf{R}$ of our system matrix $\mathcal{A}$:

$$\begin{aligned}
\mathbf{S} &= \begin{pmatrix} (\lambda^{-1} + \omega^2\sigma^2)\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K} & \\ & (\lambda^{-1} + \omega^2\sigma^2)\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K} \end{pmatrix} \\
\mathbf{R} &= \lambda\mathbf{S}.
\end{aligned}$$

Introducing the notation $\mathbf{D} := (\lambda^{-1} + \omega^2\sigma^2)\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K}$, we have to evaluate the following interpolation in order to construct a parameter-robust preconditioner:

$$\begin{aligned}
\mathcal{C} = \begin{pmatrix} [\mathbf{A},\mathbf{R}]_{\frac{1}{2}} & \\ & [\mathbf{S},\mathbf{C}]_{\frac{1}{2}} \end{pmatrix} &= \begin{pmatrix} [\mathbf{M},\lambda\mathbf{D}]_{\frac{1}{2}} & & & \\ & [\mathbf{M},\lambda\mathbf{D}]_{\frac{1}{2}} & & \\ & & [\mathbf{D},\lambda^{-1}\mathbf{M}]_{\frac{1}{2}} & \\ & & & [\mathbf{D},\lambda^{-1}\mathbf{M}]_{\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\lambda}[\mathbf{M},\mathbf{D}]_{\frac{1}{2}} & & & \\ & \sqrt{\lambda}[\mathbf{M},\mathbf{D}]_{\frac{1}{2}} & & \\ & & \frac{1}{\sqrt{\lambda}}[\mathbf{M},\mathbf{D}]_{\frac{1}{2}} & \\ & & & \frac{1}{\sqrt{\lambda}}[\mathbf{M},\mathbf{D}]_{\frac{1}{2}} \end{pmatrix}.
\end{aligned}$$

In the following calculation, we use the notation $\mathbf{L} \sim \mathbf{N}$ for the spectral equivalence which means that there exist constants c_1 and c_2 independent of all involved parameters such that

$$c_1\mathbf{x}^T\mathbf{L}\mathbf{x} \le \mathbf{x}^T\mathbf{N}\mathbf{x} \le c_2\mathbf{x}^T\mathbf{L}\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Fortunately, the original interpolation of 2×2 block matrices reduces to a simpler one, namely

$$\begin{aligned}
[\mathbf{M},\mathbf{D}]_{\frac{1}{2}} &= [\mathbf{M}, (\lambda^{-1} + \omega^2\sigma^2)\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K}]_{\frac{1}{2}} \\
&\sim \sqrt{\lambda^{-1} + \omega^2\sigma^2}\mathbf{M} + [\mathbf{M}, \mathbf{K}\mathbf{M}^{-1}\mathbf{K}]_{\frac{1}{2}} \\
&= \sqrt{\lambda^{-1} + \omega^2\sigma^2}\mathbf{M} + \mathbf{M}^{\frac{1}{2}}\left(\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-1}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\right)^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}} \\
&= \sqrt{\lambda^{-1} + \omega^2\sigma^2}\mathbf{M} + \mathbf{M}^{\frac{1}{2}}\left(\left(\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}} \\
&= \sqrt{\lambda^{-1} + \omega^2\sigma^2}\mathbf{M} + \mathbf{K} \\
&\sim (\sqrt{\lambda}^{-1} + \omega\sigma)\mathbf{M} + \mathbf{K}.
\end{aligned}$$

The last spectral equivalences are due to the simple inequalities

$$x^2 + y^2 \le (x + y)^2 \le 2(x^2 + y^2)$$

which are true for all $x, y \ge 0$ in the sense of matrix functions of symmetric positive definite matrices. In this special case we are lucky, because we can really calculate the square root and write it in a close and nice form. This finishes the construction of the block-diagonal preconditioner for the time-harmonic case with constant conductivity σ. Now, from Theorem 1, we obtain, that the condition number of the preconditioned system can be estimated by a constant c that is independent of the meshsize h and all involved parameters λ, ω, ν and σ, i.e.

$$\kappa(\mathcal{C}^{-1}\mathcal{A}) := \|\mathcal{C}^{-1}\mathcal{A}\|_{\mathcal{C}} \|\mathcal{A}^{-1}\mathcal{C}\|_{\mathcal{C}} \leq c. \tag{10}$$

The case of constant conductivity is not really interesting in practical applications. Nevertheless, this special case give us the decisive hint how to choose the preconditioner even in the practically more relevant non-constant case.

3.1.3. *The Case of Piecewise Constant Conductivity σ.* For piecewise constant conductivity, we can also apply Theorem 1 for constructing a parameter-robust preconditioner, at least, theoretically. From the practical point of view, we fail, because we cannot derive a closed and nice expression for the interpolation. Nevertheless, as mentioned before, from the constant case we can make a good guess for a parameter-robust block-diagonal preconditioner even for the piecewise constant case. Consequently, following the approach in [19], we can give a proof that shows the robustness of our guessed preconditioner in a non-constructive way.

Exploring the structural similarities to the previous section, our guess for the block-diagonal preconditioner is

$$\mathcal{C} := \begin{pmatrix} \sqrt{\lambda}\mathbf{D} & & & \\ & \sqrt{\lambda}\mathbf{D} & & \\ & & \frac{1}{\sqrt{\lambda}}\mathbf{D} & \\ & & & \frac{1}{\sqrt{\lambda}}\mathbf{D} \end{pmatrix}, \tag{11}$$

where the block $\mathbf{D}$ is given by

$$\mathbf{D} := \mathbf{K} + \mathbf{M}_{\sigma,\omega} + \frac{1}{\sqrt{\lambda}}\mathbf{M}.$$

For proving robustness of the proposed block-diagonal preconditioner, we follow the approach in [19]. We switch back to the variational level. Based on our guess (11), we introduce a non-standard norm in $\mathbf{H}_0(\mathbf{curl})$. This non-standard norm gives rise to a non-standard norm in $\mathbf{H}_0(\mathbf{curl})^4$. Using this special norm, we can verify the so-called inf-sup condition, and, finally, we can apply the Theorem of Babuška-Aziz to our indefinite variational problem.

The variational problem reads as follows: Find $(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c}) \in \mathbf{H_0}(\mathbf{curl})^4$ such that

$$\mathcal{A}\left((\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c}), (\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c})\right) = \int_\Omega \mathbf{y^s_d} \cdot \mathbf{v^s} \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{y^c_d} \cdot \mathbf{v^c} \mathrm{d}\mathbf{x} \tag{12}$$

for all testfunctions $(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c}) \in \mathbf{H_0}(\mathbf{curl})^4$. Here, the symmetric but indefinite bilinear form $\mathcal{A}(\cdot,\cdot)$, belonging to our reduced optimality system (7), is given by

$$\begin{aligned}
&\mathcal{A}\left((\mathbf{y^c}, \mathbf{y^s}, \mathbf{p^c}, \mathbf{p^s}), (\mathbf{v^c}, \mathbf{v^s}, \mathbf{q^c}, \mathbf{q^s})\right) \\
&:= (\mathbf{y^c}, \mathbf{v^c})_0 + ((\nu\, \mathbf{curl}\, \mathbf{p^c}, \mathbf{curl}\, \mathbf{v^c})_0 + (\mathcal{Q}_i(\mathbf{p^c}), \mathbf{v^c})_0) + \omega(\mathcal{R}_i(\sigma)\mathbf{p^s}, \mathbf{v^c})_0 \\
&\quad + (\mathbf{y^s}, \mathbf{v^s})_0 - \omega(\mathcal{R}_i(\sigma)\mathbf{p^c}, \mathbf{v^s})_0 + ((\nu\, \mathbf{curl}\, \mathbf{p^s}, \mathbf{curl}\, \mathbf{v^s})_0 + (\mathcal{Q}_i(\mathbf{p^s}), \mathbf{v^s})_0) \\
&\quad ((\nu\, \mathbf{curl}\, \mathbf{y^c}, \mathbf{curl}\, \mathbf{q^c})_0 + (\mathcal{Q}_i(\mathbf{y^c}), \mathbf{q^c})_0) - \omega(\mathcal{R}_i(\sigma)\mathbf{y^s}, \mathbf{q^c})_0 - \lambda^{-1}(\mathbf{p^c}, \mathbf{q^c})_0 \\
&\quad \omega(\mathcal{R}_i(\sigma)\mathbf{y^c}, \mathbf{q^s})_0 + ((\nu\, \mathbf{curl}\, \mathbf{y^s}, \mathbf{curl}\, \mathbf{q^s})_0 + (\mathcal{Q}_i(\mathbf{y^s}), \mathbf{q^s})_0) - \lambda^{-1}(\mathbf{p^s}, \mathbf{q^s})_0.
\end{aligned}$$

Due to our guess for the preconditioner (11), we introduce the non-standard norm $\|\cdot\|_{\mathcal{C}_1}$ in $\mathbf{H}_0(\mathbf{curl})$ as follows:

$$\|\mathbf{y}\|^2_{\mathcal{C}_1} = (\nu\, \mathbf{curl}\, \mathbf{y}, \mathbf{curl}\, \mathbf{y})_0 + (\mathcal{Q}_i(\mathbf{y}), \mathbf{y})_0 + \omega(\mathcal{R}_i(\sigma)\mathbf{y}, \mathbf{y})_0 + \frac{1}{\sqrt{\lambda}}(\mathbf{y}, \mathbf{y})_0.$$

Note that the regularization terms $\mathcal{Q}_i$ and $\mathcal{R}_i$ for $i = 1, 2, 3$ ensure that this norm is well defined even in non-conducting regions. This definition gives rise to a non-standard norm $\|\cdot\|_{\mathcal{C}}$ in the product space $\mathbf{H}_0(\mathbf{curl})^4$:

$$\|(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c})\|_{\mathcal{C}}^2 = \sqrt{\lambda}\|\mathbf{y^s}\|_{\mathcal{C}_1}^2 + \sqrt{\lambda}\|\mathbf{y^c}\|_{\mathcal{C}_1}^2 + \frac{1}{\sqrt{\lambda}}\|\mathbf{p^s}\|_{\mathcal{C}_1}^2 + \frac{1}{\sqrt{\lambda}}\|\mathbf{p^c}\|_{\mathcal{C}_1}^2.$$

The main result is summarized in the following theorem that claims that the inf-sup condition in the Theorem of Babuška-Aziz is fulfilled with parameter-independent constants $1/\sqrt{3}$ and 1.

Theorem 2. *The bilinear form $\mathcal{A}(\cdot,\cdot)$ fulfills the inequalities*

$$\begin{aligned} \frac{1}{\sqrt{3}}\|(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c})\|_{\mathcal{C}} &\le \sup_{0\neq(\mathbf{v^s},\mathbf{v^c},\mathbf{q^s},\mathbf{q^c})} \frac{\mathcal{A}((\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c}),(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c}))}{\|(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c})\|_{\mathcal{C}}} \\ \|(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c})\|_{\mathcal{C}} &\ge \sup_{0\neq(\mathbf{v^s},\mathbf{v^c},\mathbf{q^s},\mathbf{q^c})} \frac{\mathcal{A}((\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c}),(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c}))}{\|(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c})\|_{\mathcal{C}}} \\ &. \end{aligned} \tag{13}$$

for all $(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c}) \in \mathbf{H}_0(\mathbf{curl})^4$.

Proof. Boundedness follows from reapplication of Cauchy's inequality with appropriate scaling of the parameter λ. Indeed, we get the estimates

$$\begin{aligned} &\left|(\mathbf{y^c},\mathbf{v^c})_0 + (\mathbf{y^s},\mathbf{v^s})_0 - \lambda^{-1}(\mathbf{p^c},\mathbf{q^c})_0 - \lambda^{-1}(\mathbf{p^s},\mathbf{q^s})_0\right| \le \\ &\le \|\mathbf{y^c}\|_0\|\mathbf{v^c}\|_0 + \|\mathbf{y^s}\|_0\|\mathbf{v^s}\|_0 + \lambda^{-1}\|\mathbf{p^c}\|_0\|\mathbf{q^c}\|_0 + \lambda^{-1}\|\mathbf{p^s}\|_0\|\mathbf{q^s}\|_0 \\ &\le \left(\|\mathbf{y^c}\|_0^2 + \|\mathbf{y^s}\|_0^2 + \lambda^{-1}\|\mathbf{p^c}\|_0^2 + \lambda^{-1}\|\mathbf{p^s}\|_0^2\right)^{\frac{1}{2}} \left(\|\mathbf{v^c}\|_0^2 + \|\mathbf{v^s}\|_0^2 + \lambda^{-1}\|\mathbf{q^c}\|_0^2 + \lambda^{-1}\|\mathbf{q^s}\|_0^2\right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &|-\omega(\mathcal{R}_i(\sigma)\mathbf{p^c},\mathbf{v^s})_0 + \omega(\mathcal{R}_i(\sigma)\mathbf{p^s},\mathbf{v^c})_0 + \omega(\mathcal{R}_i(\sigma)\mathbf{y^c},\mathbf{q^s})_0 - \omega(\mathcal{R}_i(\sigma)\mathbf{y^s},\mathbf{q^c})_0| \le \\ &\le \omega\frac{1}{\sqrt[4]{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{p^c}\|_0\sqrt[4]{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{v^s}\|_0 + \omega\frac{1}{\sqrt[4]{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{p^s}\|_0\sqrt[4]{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{v^c}\|_0 \\ &+ \omega\sqrt[4]{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{y^c}\|_0\frac{1}{\sqrt[4]{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{q^s}\|_0 + \omega\sqrt[4]{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{y^s}\|_0\frac{1}{\sqrt[4]{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{q^c}\|_0 \\ &\le \left(\omega\frac{1}{\sqrt{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{p^c}\|_0^2 + \omega\frac{1}{\sqrt{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{p^s}\|_0^2 + \omega\sqrt{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{y^c}\|_0^2 + \omega\sqrt{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{y^s}\|_0^2\right)^{\frac{1}{2}} \\ &\left(\omega\sqrt{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{v^s}\|_0^2 + \omega\sqrt{\lambda}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{v^c}\|_0^2 + \omega\frac{1}{\sqrt{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{q^s}\|_0^2 + \omega\frac{1}{\sqrt{\lambda}}\|\sqrt{\mathcal{R}_i(\sigma)}\mathbf{q^c}\|_0^2\right)^{\frac{1}{2}}. \end{aligned}$$

The remaining eight terms can be estimated in the same fashion as the last one. Adding up all expressions and using Cauchy once more, we arrive at the boundedness of the bilinear form $\mathcal{A}(\cdot,\cdot)$ in the $\mathcal{C}$-norm with the constant 1.
The lower estimate can be attained by choosing

$$\begin{aligned} \mathbf{v^c} &= \mathbf{y^c} + \frac{1}{\sqrt{\lambda}}\mathbf{p^c} + \frac{1}{\sqrt{\lambda}}\mathbf{p^s}, & \mathbf{v^s} &= \mathbf{y^s} + \frac{1}{\sqrt{\lambda}}\mathbf{p^s} - \frac{1}{\sqrt{\lambda}}\mathbf{p^c}, \\ \mathbf{q^c} &= -\mathbf{p^c} + \sqrt{\lambda}\mathbf{y^c} - \sqrt{\lambda}\mathbf{y^s}, \quad \text{and} & \mathbf{q^s} &= -\mathbf{p^s} + \sqrt{\lambda}\mathbf{y^s} - \sqrt{\lambda}\mathbf{y^c}. \end{aligned}$$

Note that, for this special choice, we have

$$\|(\mathbf{v^s}, \mathbf{v^c}, \mathbf{q^s}, \mathbf{q^c})\|_{\mathcal{C}}^2 = 3\|(\mathbf{y^s}, \mathbf{y^c}, \mathbf{p^s}, \mathbf{p^c})\|_{\mathcal{C}}^2.$$

TABLE 1. Theoretical bounds for reducing the initial residual by 10^{-k}.

k	4	6	8	10	12
max iteration	16	24	30	38	44

Indeed, inserting this special choice into our bilinear form $\mathcal{A}$, we obtain

$$\mathcal{A}((\mathbf{y^c},\mathbf{y^s},\mathbf{p^c},\mathbf{p^s}),(\mathbf{y^c},\mathbf{y^s},-\mathbf{p^c},-\mathbf{p^s})) = \|\mathbf{y^s}\|_0^2 + \|\mathbf{y^c}\|_0^2 + \lambda^{-1}\|\mathbf{p^s}\|_0^2 + \lambda^{-1}\|\mathbf{p^c}\|_0^2$$

$$\mathcal{A}((\mathbf{y^c},\mathbf{y^s},\mathbf{p^c},\mathbf{p^s}),(\frac{1}{\sqrt{\lambda}}\mathbf{p^c},\frac{1}{\sqrt{\lambda}}\mathbf{p^s},\sqrt{\lambda}\mathbf{y^c},\sqrt{\lambda}\mathbf{y^s})) = \sum_{j\in\{s,c\}}$$

$$\left[\sqrt{\lambda}\left((\nu\,\mathbf{curl}\,\mathbf{y^j},\mathbf{curl}\,\mathbf{y^j})_0 + (\mathcal{Q}_i(\mathbf{y^j}),\mathbf{y^j})_0\right) + \frac{1}{\sqrt{\lambda}}\left((\nu\,\mathbf{curl}\,\mathbf{p^j},\mathbf{curl}\,\mathbf{p^j})_0 + (\mathcal{Q}_i(\mathbf{p^j}),\mathbf{p^j})_0\right)\right]$$

$$\mathcal{A}((\mathbf{y^c},\mathbf{y^s},\mathbf{p^c},\mathbf{p^s}),(\frac{1}{\sqrt{\lambda}}\mathbf{p^c},-\frac{1}{\sqrt{\lambda}}\mathbf{p^c},-\sqrt{\lambda}\mathbf{y^s},\sqrt{\lambda}\mathbf{y^c}))$$

$$= \omega\sum_{j\in\{s,c\}}\left[\frac{1}{\sqrt{\lambda}}(\mathcal{R}_i(\sigma)\mathbf{p^j},\mathbf{p^j})_0 + \sqrt{\lambda}(\mathcal{R}_i(\sigma)\mathbf{y^j},\mathbf{y^j})_0\right],$$

and, consequently,

$$\frac{\mathcal{A}((\mathbf{y^s},\mathbf{y^c},\mathbf{p^s},\mathbf{p^c}),(\mathbf{v^s},\mathbf{v^c},\mathbf{q^s},\mathbf{q^c}))}{\|(\mathbf{v^s},\mathbf{v^c},\mathbf{q^s},\mathbf{q^c})\|_{\mathcal{C}}} = \frac{1}{\sqrt{3}}\|(\mathbf{y^s},\mathbf{y^c},\mathbf{p^s},\mathbf{p^c}\|_{\mathcal{C}}.$$

This concludes our proof. □

From the inequalities (13) and the theorem of Babuška-Aziz, we immediately conclude that there exists a unique solution of the corresponding variational problem (12), and that the solution continuously depends on the data, uniformly on all involved parameters.
Furthermore, the inequalities (13) remain valid for the Nédélec finite element subspaces of $\mathbf{H}_0(\mathbf{curl})^4$ since the proof can be repeated for the finite element functions step by step !

3.1.4. *MinRes Convergence.* From Theorem 2 we immediately obtain that the spectral condition number of the preconditioned system can be estimated by the constant $\sqrt{3}$ that is obviously independent of the meshsize h and all involved parameters λ, ω, ν, σ and ε, i.e. we have

$$\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}) := \|\mathcal{C}^{-1}\mathcal{A}\|_{\mathcal{C}}\|\mathcal{A}^{-1}\mathcal{C}\|_{\mathcal{C}} \le \sqrt{3}. \tag{14}$$

Using the convergence rate estimate of the MinRes method (e.g. [11]), we finally arrive at the following theorem.

Theorem 3 (Robust and optimal solver)**.** *The MinRes method applied to the preconditioned system converges. At the m-th iteration, the preconditioned residual* $\mathbf{r^m} = \mathcal{C}^{-1}\mathbf{f} - \mathcal{C}^{-1}\mathcal{A}\mathbf{w^m}$ *is bounded as*

$$\left\|\mathbf{r^{2m}}\right\|_{\mathcal{C}} \le \frac{2q^m}{1+q^{2m}}\left\|\mathbf{r^0}\right\|_{\mathcal{C}} \quad \textit{where} \quad q = \frac{\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A})-1}{\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A})+1} \le \frac{\sqrt{3}-1}{\sqrt{3}+1}. \tag{15}$$

Proof. This result directly follows from [11] and Theorem 2. □

Therefore the number of MinRes iterations required for reducing the initial error by some fixed factor $\delta \in (0,1)$ is independent of the discretization parameter h and all the involved parameters λ, ω, ν, σ and ε. The predicted theoretical bounds for the number of MinRes iterations needed for reducing the norm of the initial residual by a given tolerance $\delta = 10^{-k}$ are listed in Table 1, cf. also the numerical results presented in Section 5.

In our numerical experiments of Section 5, we use the sparse direct solver PARDISO for solving the preconditioning equations with the system matrix $\mathbf{D} := \mathbf{K} + \mathbf{M}_{\sigma,\omega} + \lambda^{-1/2}\mathbf{M}$. However, in large-scale computations, the diagonal blocks $\mathbf{D}$ of the preconditioner $\mathcal{C}$ have to be replaced by an easy "invertible" and robust SPD preconditioner $\tilde{\mathbf{D}}$ such that the spectral equivalence inequalities

$$\tilde{c}_1\, \mathbf{x}^T \tilde{\mathbf{D}} \mathbf{x} \le \mathbf{x}^T \mathbf{D} \mathbf{x} \le \tilde{c}_2\, \mathbf{x}^T \tilde{\mathbf{D}} \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{16}$$

are valid with positive spectral equivalence constants $\tilde{c}_1$ and $\tilde{c}_2$ which should be independent of the space discretization and all other "bad" parameters including the jumps in the reluctivity and conductivity. Now, the real preconditioner

$$\tilde{\mathcal{C}} := \begin{pmatrix} \sqrt{\lambda}\tilde{\mathbf{D}} & & & \\ & \sqrt{\lambda}\tilde{\mathbf{D}} & & \\ & & \frac{1}{\sqrt{\lambda}}\tilde{\mathbf{D}} & \\ & & & \frac{1}{\sqrt{\lambda}}\tilde{\mathbf{D}} \end{pmatrix}, \tag{17}$$

obviously fulfills the spectral equivalence inequalities

$$\tilde{c}_1\, \mathbf{x}^T \tilde{\mathcal{C}} \mathbf{x} \le \mathbf{x}^T \mathcal{C} \mathbf{x} \le \tilde{c}_2\, \mathbf{x}^T \tilde{\mathcal{C}} \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{4n} \tag{18}$$

with the same spectral equivalence constants $\tilde{c}_1$ and $\tilde{c}_2$ as in (16). The spectral equivalence inequalities (18) together with the condition number estimate (14) yield the estimate

$$\kappa_{\tilde{\mathcal{C}}}(\tilde{\mathcal{C}}^{-1}\mathcal{A}) := \|\tilde{\mathcal{C}}^{-1}\mathcal{A}\|_{\tilde{\mathcal{C}}} \|\mathcal{A}^{-1}\tilde{\mathcal{C}}\|_{\tilde{\mathcal{C}}} \le \tilde{c} = \sqrt{3}\,(\tilde{c}_2/\tilde{c}_1) \tag{19}$$

for the condition number of $\tilde{\mathcal{C}}^{-1}\mathcal{A}$ with respect to the $\tilde{\mathcal{C}}$ energy norm.

Indeed, on the one hand, we have

$$\|\tilde{\mathcal{C}}^{-1}\mathcal{A}\|^2_{\tilde{\mathcal{C}}} = \sup_{\mathbf{x}} \frac{(\tilde{\mathcal{C}}^{-1}\mathcal{A}\mathbf{x}, \mathcal{A}\mathbf{x})}{(\tilde{\mathcal{C}}\mathbf{x}, \mathbf{x})} \le \tilde{c}_2^2 \sup_{\mathbf{x}} \frac{(\mathcal{C}^{-1}\mathcal{A}\mathbf{x}, \mathcal{A}\mathbf{x})}{(\mathcal{C}\mathbf{x}, \mathbf{x})} = \tilde{c}_2^2 \|\mathcal{C}^{-1}\mathcal{A}\|^2_{\mathcal{C}}.$$

On the other hand, using the substitution $\mathbf{x} = \mathcal{C}^{-1}\mathbf{y}$, we get the estimate

$$\|\mathcal{A}^{-1}\tilde{\mathcal{C}}\|^2_{\tilde{\mathcal{C}}} = \sup_{\mathbf{x}} \frac{(\tilde{\mathcal{C}}\mathcal{A}^{-1}\tilde{\mathcal{C}}\mathbf{x}, \mathcal{A}^{-1}\tilde{\mathcal{C}}\mathbf{x})}{(\tilde{\mathcal{C}}\mathbf{x}, \mathbf{x})} = \sup_{\mathbf{y}} \frac{(\tilde{\mathcal{C}}\mathcal{A}^{-1}\mathbf{y}, \mathcal{A}^{-1}\mathbf{y})}{(\tilde{\mathcal{C}}^{-1}\mathbf{y}, \mathbf{y})} \le \tilde{c}_1^{-2} \|\mathcal{A}^{-1}\mathcal{C}\|^2_{\mathcal{C}}.$$

which proves our condition number estimate (19).

Now it is clear that the results of Theorem 3 remain valid with $\mathcal{C}$ replaced by $\tilde{\mathcal{C}}$ and $\sqrt{3}$ replaced by $\tilde{c}$. In Remark 4, we discuss some candidates for $\tilde{\mathbf{D}}$.

4. Multiharmonic FEM and Preconditioning

In general application we are dealing with a non-harmonic desired state $\mathbf{y_d}$. By approximating any non-harmonic desired state by a multiharmonic excitation in terms of a truncated Fourier series

$$\mathbf{y_d} = \sum_{k=0}^{N} \mathbf{y^c_{d,k}} \cos(k\omega t) + \mathbf{y^s_{d,k}} \sin(k\omega t)$$

it follows, that due to the linearity of the state equation, the state $\mathbf{y}$, the co-state $\mathbf{p}$ and the control $\mathbf{u}$ have the same structure:

$$\mathbf{y} = \sum_{k=0}^{N} \mathbf{y^c} \cos(k\omega t) + \mathbf{y^s} \sin(k\omega t)$$

$$\mathbf{p} = \sum_{k=0}^{N} \mathbf{p^c} \cos(k\omega t) + \mathbf{p^s} \sin(k\omega t)$$

$$\mathbf{u} = \sum_{k=0}^{N} \mathbf{u^c} \cos(k\omega t) + \mathbf{u^s} \sin(k\omega t)$$

Due to the linearity of the reduced optimality problem, the huge $(4N+2)\times(4N+2)$ system decouples into N 4×4 systems of partial differential equations for the two Fourier coefficients of each, the state $\mathbf{y}$ and the co-state $\mathbf{p}$ belonging to the mode k, and an 2×2 system of partial differential equations for the mode $k = 0$. (Clearly we don't have to solve for the $\mathbf{p^s_0}$ and $\mathbf{u^s_0}$, since $\sin(0\omega t) = 0$.) Hence we have to solve the following decoupled reduced optimality system in the frequency domain: Find the state $\mathbf{y} \in \mathbf{H}(\mathbf{curl})^{2N+1}$ and the co-state $\mathbf{p} \in \mathbf{H}(\mathbf{curl})^{2N+1}$, given by

$$\mathbf{y} = (\mathbf{y^c_0}, \mathbf{y^c_1}, \mathbf{y^s_1}, \ldots, \mathbf{y^c_N}, \mathbf{y^s_N})$$
$$\mathbf{p} = (\mathbf{p^c_0}, \mathbf{p^c_1}, \mathbf{p^s_1}, \ldots, \mathbf{p^c_N}, \mathbf{p^s_N}),$$

such that

$$\text{(20)} \quad \begin{cases} -\omega k \mathcal{R}_i(\sigma)\mathbf{y^s} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{y^c_k}) + \mathcal{Q}_i(\mathbf{y^c_k}) - \lambda^{-1}\mathbf{p^c_k} = \mathbf{0}, & \text{in } \Omega \\ \omega k \mathcal{R}_i(\sigma)\mathbf{y^c_k} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{y^s_k}) + \mathcal{Q}_i(\mathbf{y^s_k}) - \lambda^{-1}\mathbf{p^s_k} = \mathbf{0}, & \text{in } \Omega \\ \omega k \mathcal{R}_i(\sigma)\mathbf{p^s_k} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{p^c_k}) + \mathcal{Q}_i(\mathbf{p^c_k}) + \mathbf{y^c_k} = \mathbf{y^c_{d,k}}, & \text{in } \Omega \\ -\omega k \mathcal{R}_i(\sigma)\mathbf{p^c_k} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{p^s_k}) + \mathcal{Q}_i(\mathbf{p^s_k}) + \mathbf{y^s_k} = \mathbf{y^s_{d,k}}, & \text{in } \Omega \\ \mathbf{y^c_k} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \\ \mathbf{y^s_k} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \\ \mathbf{p^c_k} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \\ \mathbf{p^s_k} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases}$$

Remark 3 (Initial conditions)**.** *Since we solve for all modes $k = 0, \ldots, N$, there is no degree of freedom left for fulfilling the initial and end conditions $\mathbf{y}(\cdot, 0) = \mathbf{y_0}$ and $\mathbf{p}(\cdot, T) = 0$. Hence we are only allowed to claim periodic initial conditions $\mathbf{y}(\cdot, 0) = \mathbf{y}(\cdot, T)$ and $\mathbf{p}(\cdot, 0) = \mathbf{p}(\cdot, T)$ that are automatically fulfilled by our periodic time discretization in terms of a multiharmonic approach.*

The finite element discretization of each 4×4 block ($k = 1, \ldots, N$) leads to a 4×4 block-matrix $\mathcal{A}_k$ that formally has the same structure as $\mathcal{A}$ in (8) with ω replaced by $k\omega$.

$$\mathcal{A}_k = \begin{pmatrix} \mathbf{M} & & \mathbf{K} & \mathbf{M}_{\sigma,\mathbf{k}\omega} \\ & \mathbf{M} & -\mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K} \\ \mathbf{K} & -\mathbf{M}_{\sigma,\mathbf{k}\omega} & -\lambda^{-1}\mathbf{M} & \\ \mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K} & & -\lambda^{-1}\mathbf{M} \end{pmatrix}$$

So our $(4N + 2) \times (4N + 2)$ decoupled block-diagonal system matrix $\mathcal{B}$ has the following from

$$\mathcal{B} = \operatorname{diag}(\mathcal{A}_k)_{k=0,\ldots,N}$$

where the block corresponding to the mode $k = 0$ is given by

$$\mathcal{A}_0 = \begin{pmatrix} \mathbf{M} & \mathbf{K} \\ \mathbf{K} & -\lambda^{-1}\mathbf{M} \end{pmatrix}.$$

4.1. **Preconditioning.** The preconditioning of the 4×4 blocks for the modes $k = 1, \ldots, N$ can be done analogous to the time-harmonic case in Section 3.1. Defining the expression

$$\mathbf{D_k} := \mathbf{K} + \mathbf{M}_{\sigma,\mathbf{k}\omega} + \frac{1}{\sqrt{\lambda}}\mathbf{M}$$

our block-diagonal preconditioner for each block corresponding to the Fourier mode $k = 1, \ldots, N$ reads as

$$\mathcal{C}_k := \begin{pmatrix} \sqrt{\lambda}\mathbf{D_k} & & & \\ & \sqrt{\lambda}\mathbf{D_k} & & \\ & & \frac{1}{\sqrt{\lambda}}\mathbf{D_k} & \\ & & & \frac{1}{\sqrt{\lambda}}\mathbf{D_k} \end{pmatrix}. \tag{21}$$

Analogous to the time-harmonic case, we obtain the mesh and parameter independent condition number estimate

$$\kappa(\mathcal{C}_k^{-1}\mathcal{A}_k) \leq \sqrt{3}.$$

For the block $\mathcal{A}_0$ we can construct a preconditioner in the same manner. Following the strategy of the previous section, we have to compute the negative Schur complements $\mathbf{S}$ and $\mathbf{R}$ of our system matrix $\mathcal{A}_0$.

$$\mathbf{S} = \lambda^{-1}\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K}$$
$$\mathbf{R} = \lambda\mathbf{S}$$

We have to evaluate the following interpolation in order to construct a parameter-robust preconditioner.

$$\mathcal{C}_0 = \begin{pmatrix} [\mathbf{A}, \mathbf{R}]_{\frac{1}{2}} & \\ & [\mathbf{S}, \mathbf{C}]_{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda}[\mathbf{M}, \mathbf{S}]_{\frac{1}{2}} & \\ & \frac{1}{\sqrt{\lambda}}[\mathbf{S}, \mathbf{M}]_{\frac{1}{2}} \end{pmatrix}$$

Hence we have to compute the interpolation

$$\begin{aligned} [\mathbf{M}, \mathbf{S}]_{\frac{1}{2}} = [\mathbf{S}, \mathbf{M}]_{\frac{1}{2}} &= \mathbf{M}^{\frac{1}{2}} \left(\mathbf{M}^{-\frac{1}{2}}(\lambda^{-1}\mathbf{M} + \mathbf{K}\mathbf{M}^{-1}\mathbf{K})\mathbf{M}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{M}^{\frac{1}{2}} \\ &\sim \frac{1}{\sqrt{\lambda}}\mathbf{M} + \mathbf{M}^{\frac{1}{2}} \left(\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-1}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{M}^{\frac{1}{2}} = \frac{1}{\sqrt{\lambda}}\mathbf{M} + \mathbf{K}. \end{aligned}$$

Consequently we obtain the block-diagonal preconditioner

$$\mathcal{C}_0 = \begin{pmatrix} \mathbf{M} + \sqrt{\lambda}\mathbf{K} & \\ & \frac{1}{\lambda}\mathbf{M} + \frac{1}{\sqrt{\lambda}}\mathbf{K} \end{pmatrix}.$$

Hence using the Theorem of Babuška-Aziz for the block-diagonal preconditioner $\mathcal{C}_0$ we obtain a condition number estimate (see also [19])

$$\kappa(\mathcal{C}_0^{-1}\mathcal{A}_0) \leq \sqrt{2}.$$

Remark 4. *As was already discussed in Subsection 3.1.4, in large-scale computations the diagonal blocks* $\mathbf{D_k}$ *have to be replaced by easy "invertible" SPD preconditioner* $\tilde{\mathbf{D}}_\mathbf{k}$ *which are spectrally equivalent to* $\mathbf{D_k}$ *in a robust and optimal sense. We mention that the diagonal blocks* $\mathbf{D_k} = \mathbf{K} + \mathbf{M}_{\sigma,\mathbf{k}\omega} + \frac{1}{\sqrt{\lambda}}\mathbf{M}$ *obviously arise from the finite element discretization of the boundary value problem*

$$\begin{aligned} \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{y}) + \mathcal{Q}_i(\mathbf{y}) + \left(k\omega\mathcal{R}_i(\sigma) + \frac{1}{\sqrt{\lambda}}\right)\mathbf{y} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{y} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Candidates for robust and (almost) optimal (with respect to the complexity) preconditioners are multigrid preconditioners [1], *auxiliary space preconditioners* [14, 36],

Table 2. Number of MinRes iterations for reducing the initial residual by $\delta = 10^{-6}$ ($\omega = 1$).

DOF	$\log_{10}\lambda$	-10	-8	-6	-4	-2	0	2	4	6	8	10
16736	$h = 0.125$	7	13	15	16	14	9	8	6	6	6	6
124096	$h = 0.0625$	9	15	16	16	12	7	6	6	6	6	6

Table 3. Number of MinRes iterations for reducing the initial residual by $\delta = 10^{-6}$ ($\lambda = 1$).

DOF	$\log_{10}\omega$	-10	-8	-6	-4	-2	0	2	4	6	8	10
16736	$h = 0.125$	5	5	5	5	5	9	15	14	6	4	2
124096	$h = 0.0625$	5	5	5	5	5	7	13	16	6	4	2

Table 4. Number of MinRes iterations for reducing the initial residual by $\delta = 10^{-6}$ ($DOF = 16736$).

		$\log_{10}\omega$				
		-2	0	2	4	6
$\log_{10}\lambda$	-10	7	7	7	7	3
	-8	13	13	13	9	5
	-6	15	15	14	11	6
	-4	16	16	20	14	6
	-2	12	14	16	14	6
	0	5	9	15	14	6
	2	6	8	16	14	6

and domain decomposition (DD) preconditioners [33, 15, 34]. *Indeed, the DD preconditioner proposed by Hu and Zou* [15] *together with the results of the same authors on weighted Helmholtz decompositions* [16] *allow us to construct a robust and almost optimal DD preconditioners* $\tilde{\mathbf{D}}_{\mathbf{k}}$ *for* $\mathbf{D}_{\mathbf{k}}$.

5. First Numerical Results

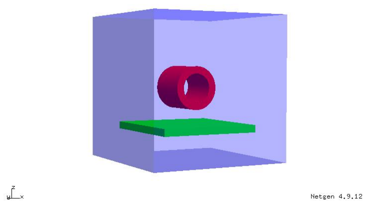
In order to confirm our theoretical results numerically, we report on our first numerical tests for an academic example, namely for the simple time-harmonic case discussed in Section 3. The numerical results presented in this section were attained using ParMax [27]. First, we demonstrate the robustness of the block-diagonal preconditioner with respect to the regularization parameter λ and the frequency ω. Therefore, for the solution of the preconditioning equations arising from the diagonal blocks, we use the sparse direct solver PARDISO that is very efficient for several hundred thousand unknowns in the case of three-dimensional problems [29, 30].

5.1. **Constant Conductivity.** Table 2, Table 3 and Table 4 provide the number of MinRes iterations needed for reducing the initial residual by a factor of 10^{-6} for different λ, ω and h. These numerical experiments were performed for a three-dimensional linear problem on the unit-cube, discretized by tetrahedra for the case $\nu = \sigma$. These experiments demonstrate the independence of the MinRes convergence rate on the regularization parameter and the meshsize since the number of iterations is bounded by 20 for all computed constellations that is less than the theoretical bound 24 given ind Table 1.

TABLE 5. Number of MinRes iterations for reducing the initial residual by $\delta = 10^{-6}$ ($\mathcal{C}$-norm) in the case of different meshes and cost parameters λ.

DOF	$h_{\min}$	$h_{\max}$	-8	-6	-4	-2	0	2	4	6	8
183724	0.00992389	0.385854	15	15	13	12	8	10	6	6	6
272220	0.00933755	0.49344	15	15	15	12	10	10	8	8	8

5.2. **Piecewise Constant Conductivity.** In Table 5, we report the results for the case of piecewise constant conductivity on three different meshes with varying cost parameter λ running from 10^{-8} to 10^{8} ($\log_{10}\lambda$ scale in Table 5). Our computational domain consist of air (blue), an iron shield (green) and a coil (red). We apply the conductivity regularization, hence the conductivity σ is given by (22).



$$\sigma = \begin{cases} 10^{6} & \text{in iron and the coil} \\ 10^{-3} & \text{in air} \end{cases}. \tag{22}$$

The results are obtained for the setting $\nu = \omega = 1$. We mention that the mesh is highly adapted to the geometry and to the eddy current boundary layers as is indicated by the minimal mesh size $h_{\min}$ and the maximal mesh size $h_{\max}$. Again, the numerical results show the robustness of our preconditioner.

6. GENERALIZATION

In practical applications the source or control $\mathbf{u}$ is prescribed by some electric current in a coil and not in the whole domain Ω as used in the model problem (1)-(2). Consequently the control $\mathbf{u}$ vanishes outside of the latter mentioned region. The previous analysis can easily be generalized to this case as well, where the control $\mathbf{u}$ is prescribed only in a subset Ω_d of the computational domain Ω. For mathematical treatment, the domain Ω_d can be characterized by a non-negative indicator function τ, i.e.

$$\tau(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_d \\ 0, & \mathbf{x} \in \Omega \backslash \Omega_d \end{cases}.$$

Due to technical reasons concerning the construction of the parameter-robust preconditioner, the observation $\mathbf{y} - \mathbf{y_d}$ has also be restricted to Ω_d. Therefore we deal with the generalized optimal control problem: Find the state $\mathbf{y}$ and the control $\mathbf{u}$ that minimizes the generalized cost functional

$$J_\tau(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_{\Omega \times (0,T)} \tau(\mathbf{x}) |\mathbf{y} - \mathbf{y_d}|^2 d\mathbf{x} dt + \frac{\lambda}{2} \int_{\Omega \times (0,T)} \tau(\mathbf{x}) |\mathbf{u}|^2 d\mathbf{x} dt,$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial}{\partial t} \mathbf{y} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{y}) = \tau(\mathbf{x}) \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega. \end{cases}$$

The analysis of the parameter-robust preconditioner is also valid for this more general cost functional leading to a block-diagonal preconditioner (11), with the scaled (symmetric and positive definite) blocks

$$\mathbf{D_k} := \mathbf{K} + \mathbf{M}_{\sigma,\mathbf{k}\omega} + \frac{1}{\sqrt{\lambda}}\mathbf{M}_\tau$$

for $k = 0, \ldots, N$ at the diagonal. Here $\mathbf{M}_\tau$ corresponds to the finite element discretization of

$$\mathbf{M}_\tau : \quad \int_\Omega \tau(\mathbf{x})\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

Again we obtain the condition number estimate $\sqrt{3}$.

Remark 5. *We mention that for the mode corresponding to $k = 0$, the diagonal block $\mathbf{D_0}$ is not positive definite in the case of parabolic regularization ($i = 2$). Therefore, in this special regime, we have to add a regularization operator for the zero mode, i.e. replacing $\tau(\mathbf{x})$ by $\mathcal{R}_2(\tau(\mathbf{x}))$ in $\mathcal{A}_0$ and $\mathbf{D_0}$.*

7. Conclusion and Outlook

The method developed in this work shows great potential for solving distributed optimal control problems for time-harmonic and time-periodic eddy current problems in an efficient and optimal way. The key points of our method are the usage of a non-standard time discretization technique in terms of a truncated Fourier series, and the construction of parameter-independent solvers for the resulting system of equations in the frequency domain. The theory developed in this paper establishes a theoretical estimate of the convergence rate of MinRes as a solver when our proposed preconditioner is applied. Numerical experiments confirm this convergence rate estimate. Due to the natural decoupling of the frequency domain equations an efficient parallel implementation of the solution procedure is straight-forward.

In some applications, it is reasonable to add so-called box constraints for the control $\mathbf{u}$ or / and the state $\mathbf{y}$ to an optimal control problem like (1) – (2) or (3) – (4). In the standard approach these constraints can be handled by a simple projection to the box [21] leading to a non-linear optimality system that can be solved by superlinearly convergent, semi-smooth Newton methods [13, 17]. Unfortunately, in the multiharmonic approach, box constraints for $\mathbf{u}$ or / and $\mathbf{y}$ cannot be handled in such an easy way. However, box constraints for their Fourier coefficients can be treated by such a projection ! Methods for treating box constraints for the control $\mathbf{u}$ or / and the state $\mathbf{y}$ in the multiharmonic setting are certainly the penalty or barrier methods [31, 10]. The same holds for treating initial conditions. In both cases, the equations for different modes are coupled via these penalty or barrier terms added to the cost functional. In non-linear eddy current problems we have a similar coupling of the Fourier coefficients belonging to different modes [4, 5]. Anyway, the preconditoners proposed and analyzed in this paper can be useful for all these cases too.

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