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A preconditioned MinRes solver for time-periodic parabolic optimal control problems

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A PRECONDITIONED MINRES SOLVER FOR TIME-PERIODIC PARABOLIC OPTIMAL CONTROL PROBLEMS

MARKUS KOLLMANN AND MICHAEL KOLMBAUER

ABSTRACT. This work is devoted to the multiharmonic analysis of parabolic optimal control problems in a time-periodic setting. In contrast to previous approaches, we include the cases of different control and observation domains, the observation in certain energy spaces and the presence of control constraints. In all these cases we propose a new preconditioned MinRes solver for the frequency domain equations and show that this solver is robust with respect to the space and time discretization parameters as well as the involved "bad" model parameters of the state equation.

1. Introduction

This work is devoted to the development of efficient solution procedures for the following optimal control problem:

$$\min_{y,u} \frac{1}{2} \int_0^T \int_{\Omega_1} [D(y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t))]^2 \, d\boldsymbol{x} \, dt + \frac{\lambda}{2} \int_0^T \int_{\Omega_2} [u(\boldsymbol{x},t)]^2 \, d\boldsymbol{x} \, dt.$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial}{\partial t} y(\boldsymbol{x},t) - \nabla \cdot (\nu \nabla y(\boldsymbol{x},t)) = u(\boldsymbol{x},t) & \forall (\boldsymbol{x},t) \in \Omega \times (0,T), \\ y(\boldsymbol{x},t) = 0 & \forall (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall \boldsymbol{x} \in \bar{\Omega}, \end{cases}$$

and possible constraints imposed on the Fourier coefficients of the control (see Subsection 4.3).

Here Ω is a bounded Lipschitz domain. Ω_1 and Ω_2 are non-empty subsets of Ω, i.e. $\Omega_1, \Omega_2 \subset \Omega \subset \mathbb{R}^d (d = 1,2,3)$, D is either the identity or ∇, σ and ν are some positive model parameters, $\lambda > 0$ is a cost coefficient.

In order to derive a solution of this constrained minimization problem, we derive the first order optimality conditions. If there are additional constraints to the control u, the optimality system is nonlinear. We discretize in space and time by means of a multiharmonic finite element method (MH-FEM). Indeed, this method is very useful for solving problems in time-periodic settings (see [13] and the references therein).

The idea of using a multiharmonic approach as a non-standard discretization method in time and combining it with the finite element method for approximating the Fourier coefficients is not new and has been used by many authors in different applications (e.g. [1, 7, 9, 20, 22]). Indeed, in [2] and [13], a rigorous numerical analysis for the multiharmonic finite element method applied to the eddy current problem and the parabolic optimal control problem is provided.

The fast solution of the corresponding large linear system of finite element frequency domain equations is crucial for the competitiveness of this method. Hence

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appropriate (parameter-robust) preconditioning is an important issue. In [12] a general recipe for the construction of preconditioners for discrete linear operators that arise from a Galerkin approach is shown. There the resulting preconditioners are robust with respect to the choice of the bases for trial and test spaces. In order to construct parameter-robust preconditioners, we follow an abstract preconditioning strategy that has been rigorously analyzed in [23] and has already successfully applied to the multiharmonic finite element (MH-FEM) or boundary element method (MH-BEM) or their coupling (MH-FEM/BEM) in several works [5, 13, 14, 15, 16].

Indeed, in [13], time-periodic parabolic optimal control problems are analyzed and a parameter-robust preconditioned MinRes solver is constructed. In fact, the applicability of their approach for time-periodic parabolic optimal control problems is shown with some eminent restrictions, that limits the applicability of their solver to general problems, e.g. in computational electromagnetics:

1) The observation domain and the control domain have to coincide with the computational domain, i.e. $\Omega_1 = \Omega_2 = \Omega$.
2) The observation is restricted to be done in the L_2-norm, i.e. $D = Id$.
3) There are no control constraints involved.

The overcome of these drawbacks is not straightforward, since we have to cope with up to six discretization and model parameters, that impinge upon the convergence rate of any iterative method. Anyhow, in all these cases we construct an almost robust preconditioned MinRes method, meaning that the convergence rate is independent of all but one parameter. In fact it turns out, that a generalization of the parameter-robust block-diagonal preconditioner for the time-harmonic state equations as studied in [5] and [14] is also the right choice in this framework.

Since inequality constraints give rise to nonlinear optimality systems, following Herzog and Sachs [10], we apply a Newton-type approach for their solution. The saddle point problems arising at each Newton step are solved with the almost optimal preconditioned MinRes method.

The rest of the paper is organized as follows. In Section 2, we derive the optimality system of the model problem and discretize in space and time by means of the multiharmonic finite element method. Here we briefly recall the setting used in [13]. Section 3 is devoted to preconditioning of the resulting linear system of equations. After providing a theoretical basis according to a result by Zulehner [23], we apply the preconditioning technique to the model problem stated in Section 2. In Section 4, we generalize the results obtained in Section 3 and obtain an almost parameter-robust MinRes solver. In Section 5, we present numerical results confirming the rate estimates given in Section 4. Section 6 draws some conclusions.

2. The first model problem

As a first model problem we consider the time-periodic parabolic optimal control problem which was studied in [13] and briefly recall the setting used there. We derive a preconditioner (see Section 3), which is different to the one used in [13], but very useful for the extended model problems (see Section 4).

Therefore we consider the model problem:

$$\min_{y,u} \bar{\mathcal{J}}(y,u) = \min_{y,u} \frac{1}{2}\int_0^T \int_\Omega [y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt + \frac{\bar{\lambda}}{2}\int_0^T \int_\Omega [\bar{u}(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt$$

subject to the state equation

$$\begin{cases} \bar{\sigma}\dfrac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\bar{\nu}\nabla y(\boldsymbol{x},t)) = \bar{u}(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T), \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\bar{\Omega}. \end{cases}$$

Here $\bar{\lambda} > 0$ denotes the control cost parameter. Additionally, there are two coefficients $\bar{\nu} \in L^\infty(\Omega)$ and $\bar{\sigma} \in L^\infty(\Omega)$, fulfilling

$$0 < \bar{\nu}_{\min} \le \bar{\nu}(\mathbf{x}) \le \bar{\nu}_{\max}, \quad \text{and} \quad 0 \le \bar{\sigma}(\mathbf{x}) \le \bar{\sigma}_{\max}, \qquad \text{a.e in } \Omega.$$

In practical applications, e.g., for 2D eddy current problems in computational electromagnetics, $\bar{\sigma}(\cdot)$ corresponds to the conductivity and $\bar{\nu}(\cdot)$ corresponds to the reluctivity. By a simple scaling argument it can always be achieved that $\bar{\nu}_{\min} = 1$: For arbitrary $\bar{\nu}$ we scale the state equation with $\bar{\nu}_{\min}^{-1}$ to obtain the equivalent minimization problem:

(1)
$$\min_{y,u} \mathcal{J}(y,u) = \min_{y,u} \frac{1}{2}\int_0^T\int_\Omega [y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t)]^2\,d\boldsymbol{x}\,dt + \frac{\lambda}{2}\int_0^T\int_\Omega [u(\boldsymbol{x},t)]^2\,d\boldsymbol{x}\,dt$$

subject to the state equation

$$\begin{cases} \sigma\dfrac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\nu\nabla y(\boldsymbol{x},t)) = u(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T), \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\bar{\Omega}, \end{cases} \tag{2}$$

with the new parameters $\sigma = \frac{\bar{\sigma}}{\bar{\nu}_{\min}}$, $\nu = \frac{\bar{\nu}}{\bar{\nu}_{\min}}$, $\lambda = \bar{\lambda}\bar{\nu}_{\min}^2$ and the scaled state $u = \frac{\bar{u}}{\bar{\nu}_{\min}}$. In the remainder of this work we consider the scaled PDE-constraint minimization problem (1)-(2) with $\nu \ge 1$ a.e. in Ω.

2.1. **Optimality system.** In order to solve the optimal control problem (1)-(2), we formulate the corresponding optimality system. The Lagrange functional of the model problem is given by

$$\mathcal{L}(y,u,p) := \mathcal{J}(y,u) + \int_0^T\int_\Omega \Big(\sigma\frac{\partial y}{\partial t} - \nabla\cdot\big(\nu\nabla y\big) - u\Big)p\,d\boldsymbol{x}\,dt, \tag{3}$$

where p denotes the co-state. A stationary point (y,u,p) of the Lagrange functional is characterized by the following three conditions:

$$\begin{cases} \nabla_y\mathcal{L}(y,u,p) = 0, \\ \nabla_u\mathcal{L}(y,u,p) = 0, \\ \nabla_p\mathcal{L}(y,u,p) = 0. \end{cases} \tag{4}$$

We can eliminate the control u from the optimality system (4) using the second condition, i.e.

$$u(\boldsymbol{x},t) = \lambda^{-1}p(\boldsymbol{x},t). \tag{5}$$

By (3),(4) and (5), we obtain the following (classical) formulation of the optimality system:

(6)
$$\begin{cases} \sigma\frac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\nu\nabla y(\boldsymbol{x},t)) - \lambda^{-1}p(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\Omega\times(0,T) \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T) \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\bar\Omega \\ -\sigma\frac{\partial}{\partial t}p(\boldsymbol{x},t) - \nabla\cdot(\nu\nabla p(\boldsymbol{x},t)) + y(\boldsymbol{x},t) = y_d(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T) \\ p(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T) \\ p(\boldsymbol{x},T) = p(\boldsymbol{x},0) & \forall\boldsymbol{x}\in\bar\Omega. \end{cases}$$

The space-time variational formulation of (6) is the starting point of discretization in space and time (see [13]).

2.2. **Multiharmonic FEM.** Let us now assume that the desired state y_d is multiharmonic, i.e. y_d has the form

$$y_d(\boldsymbol{x},t) = \sum_{k=0}^{N} y^c_{d,k}(\boldsymbol{x})\cos(k\omega t) + y^s_{d,k}(\boldsymbol{x})\sin(k\omega t), \tag{7}$$

with some natural number N, frequency $\omega = 2\pi/T$ and amplitudes $y^c_{d,k}$ and $y^s_{d,k}$. Mention that in the case of a more general, but time-periodic desired state y_d, we can approximate y_d by a truncated Fourier series. The quality of the approximation depends on N and the smoothness of y_d, see [13].

Due to the linearity of the state equation, we obtain, that the state y and the co-state p are multiharmonic as well, and, therefore, they have the same structure:

$$y(\boldsymbol{x},t) = \sum_{k=0}^{N} y^c_{k}(\boldsymbol{x})\cos(k\omega t) + y^s_{k}(\boldsymbol{x})\sin(k\omega t),$$
$$p(\boldsymbol{x},t) = \sum_{k=0}^{N} p^c_{k}(\boldsymbol{x})\cos(k\omega t) + p^s_{k}(\boldsymbol{x})\sin(k\omega t).$$

Due to the linearity of the reduced optimality problem, the huge $(4N+2)\times(4N+2)$ system decouples into N 4×4 systems of partial differential equations for the two Fourier coefficients of each, the state y and the co-state p belonging to the mode k, and a 2×2 system of partial differential equations for the mode $k=0$. (Clearly we don't have to solve for p^s_0 and y^s_0, since $\sin(0\omega t) = 0$.) Hence, we solve the time-independent system of partial differential equations: Find the state $\mathbf{y} = (y^c_0, y^c_1, y^s_1, \dots, y^c_N, y^s_N) \in H^1_0(\Omega)^{2N+1}$ and the co-state $\mathbf{p} = (p^c_0, p^c_1, p^s_1, \dots, p^c_N, p^s_N) \in H^1_0(\Omega)^{2N+1}$, such that

$$\begin{cases} \omega k\sigma p^c_k - \nabla\cdot(\nu\,\nabla p^s_k) + y^s_k = y^s_{d,k}, & \text{in } \Omega, \\ -\omega k\sigma p^s_k - \nabla\cdot(\nu\,\nabla p^c_k) + y^c_k = y^c_{d,k}, & \text{in } \Omega, \\ -\omega k\sigma y^c_k - \nabla\cdot(\nu\,\nabla y^s_k) - \lambda^{-1}p^s_k = 0, & \text{in } \Omega, \\ \omega k\sigma y^s_k - \nabla\cdot(\nu\,\nabla y^c_k) - \lambda^{-1}p^c_k = 0, & \text{in } \Omega, \\ y^c_k = y^s_k = p^c_k = p^s_k = 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

for all modes $k = 0,1,\dots,N$. Since the whole problem decouples to a block-diagonal one corresponding to each mode k, for preconditioning purpose, it is enough to discuss the block for a fixed mode k. Therefore, for the rest of this work, we concentrate on solving a 4×4 block for a fixed mode k.

Deriving the usual space-variational formulation of (8) for some fixed $k \in \mathbb{N}$, we end up with the following variational problem (we omit the subindex k): Find $(y^c, y^s, p^c, p^s) \in H_0^1(\Omega)^4$, such that

$$\mathcal{B}((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s)) = \mathcal{F}((v^c, v^s, q^c, q^s)) \tag{9}$$

for all $(v^c, v^s, q^c, q^s) \in H_0^1(\Omega)^4$, where the left-hand side $\mathcal{B}$ is given by

$$\begin{aligned}\mathcal{B}((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s)) &= a((y^c, y^s), (v^c, v^s)) + b((v^c, v^s), (p^c, p^s)) \\ &\quad + b((y^c, y^s), (q^c, q^s)) - c((p^c, p^s), (q^c, q^s)),\end{aligned}$$

with the bilinear forms

$$\begin{aligned}a((y^c, y^s), (v^c, v^s)) &:= (y^c, v^c)_{L_2(\Omega)} + (y^s, v^s)_{L_2(\Omega)} \\ c((p^c, p^s), (q^c, q^s)) &:= \lambda^{-1}(p^c, q^c)_{L_2(\Omega)} + \lambda^{-1}(p^s, q^s)_{L_2(\Omega)} \\ b((y^c, y^s), (q^c, q^s)) &:= (\nu\nabla y^c, \nabla q^c)_{L_2(\Omega)} + (\nu\nabla y^s, \nabla q^s)_{L_2(\Omega)} \\ &\quad + k\omega(\sigma y^s, q^c)_{L_2(\Omega)} - k\omega(\sigma y^c, q^s)_{L_2(\Omega)}.\end{aligned}$$

The right-hand side $\mathcal{F}$ is given by

$$\mathcal{F}((v^c, v^s, q^c, q^s)) = (y_d^c, v^c)_{L_2(\Omega)} + (y_d^s, v^s)_{L_2(\Omega)}.$$

The finite element discretization (e.g. using piece-wise linear functions with the usual nodal basis) of each 4×4 block leads to a 4×4 matrix $\mathcal{A}$ of the following form:

$$\mathcal{A} = \begin{pmatrix} \mathbf{M} & \cdot & \mathbf{K}_{\boldsymbol{\nu}} & -\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} \\ \cdot & \mathbf{M} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} \\ \mathbf{K}_{\boldsymbol{\nu}} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & -\lambda^{-1}\mathbf{M} & \cdot \\ -\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} & \cdot & -\lambda^{-1}\mathbf{M} \end{pmatrix}. \tag{10}$$

Here the mass matrix $\mathbf{M}$, the conductivity matrix $\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}$ and the stiffness matrix $\mathbf{K}_{\boldsymbol{\nu}}$ arise from the finite element discretization of the corresponding bilinear forms

$$\mathbf{M} : (\cdot, \cdot)_{L_2(\Omega)}, \quad \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} : k\omega(\sigma\cdot, \cdot)_{L_2(\Omega)} \quad \text{and} \quad \mathbf{K}_{\boldsymbol{\nu}} : (\nu\nabla\cdot, \nabla\cdot)_{L_2(\Omega)}.$$

Hence, we have to solve a linear system of finite element equations of the form

$$\mathcal{A}\mathbf{w} = \mathbf{f}, \tag{11}$$

where the system matrix $\mathcal{A}$ is given by (10) and $\mathbf{f}$ is the finite element discretization of $\mathcal{F}$.

In fact, the system matrix $\mathcal{A}$ is symmetric and indefinite and obtains a double saddle-point structure. Since $\mathcal{A}$ is symmetric, the system can be solved by a Minimal Residual (MinRes) method [19]. Typically, the convergence rate of any iterative method deteriorates with respect to the meshsize h and the "bad" parameters $k\omega$, ν, σ and λ, if applied to the unpreconditioned system (11). Therefore, preconditioning is a challenging topic.

3. Preconditioning

In [13], the parameter-robust block-diagonal preconditioner

$$\mathcal{P} = \operatorname{diag}\left(D, D, \lambda^{-1}D, \lambda^{-1}D\right). \tag{12}$$

with $D = \sqrt{\lambda}\mathbf{K}_{\boldsymbol{\nu}} + \sqrt{\lambda}\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} + \mathbf{M}$ for the solution of (11) in a MinRes setting is proposed. This preconditioner yields robust convergence rates with respect to all six involved discretization and model parameters, i.e. h, k, ω, σ, ν and λ. Additionally, there is a rigorous condition number bound $\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}) \leq \sqrt{3}$. The application of $\mathcal{P}$ to more general problems, as mentioned in the introduction, is not straightforward. Since here, we are heading towards more practical applications, it turns out, that we have to pay a price in the sense, that we lose robustness with respect to the cost coefficient λ in order to get robustness with respect to modifications in the cost functional and/or the state equation. The idea is to use a block-diagonal preconditioner for the optimal control problem, which obtains high structural similarities with the parameter-robust preconditioner (12).

Furthermore, the cases handled in this work are not capable with the preconditioning theory used in [13]. Therefore, we use a generalized preconditioning result from Zulehner in [23], that is very useful for proving almost parameter-independent bounds in all these cases.

3.1. **Abstract preconditioning theory.** In this Subsection we briefly recall a result of Zulehner [23]. Let V and Q be Hilbert spaces with the inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$. The associated norms are given by $\|\cdot\|_V = \sqrt{(\cdot,\cdot)_V}$ and $\|\cdot\|_Q = \sqrt{(\cdot,\cdot)_Q}$. Furthermore let X be the product space $X = V \times Q$, equipped with the inner product $((v,q),(w,r))_X = (v,w)_V + (q,r)_Q$ and the associated norm $\|(v,q)\|_X = \sqrt{((v,q),(v,q))_X}$.

Consider a mixed variational problem in the product space $X = V \times Q$: Find $z = (w,r) \in X$, such that

$$\mathcal{A}(z,y) = \mathcal{F}(y), \quad \text{for all } y \in X$$

with

$$\mathcal{A}(z,y) = a(w,v) + b(v,r) + b(w,q) - c(r,q), \quad \text{and} \quad \mathcal{F}(y) = f(v) + g(q)$$

for $y = (v,q)$ and $z = (w,r)$. We introduce $B \in L(V,Q^*)$ and its adjoint $B^* \in L(Q,V^*)$ by

$$\langle Bw, q\rangle = b(w,q) \quad \text{and} \quad \langle B^* r, v\rangle = \langle Bv, r\rangle.$$

Furthermore, we denote by $\mathcal{A} \in L(X,X^*)$ the operator induced by

$$\langle \mathcal{A}x, y\rangle = \mathcal{A}(x,y).$$

The next theorem provides necessary and sufficient conditions for providing parameter independent bounds and can be found in Zulehner [23].

Theorem 1 ([23, Theorem 2.6]). *If there are constants* $\underline{c}_w$, $\underline{c}_r$, $\overline{c}_w$, $\overline{c}_r > 0$, *such that*

$$\underline{c}_w \|w\|_V^2 \le a(w,w) + \|Bw\|_{Q^*}^2 \le \overline{c}_w \|w\|_V^2, \quad \text{for all } w \in V \tag{13}$$

and

$$\underline{c}_r \|r\|_Q^2 \le c(r,r) + \|B^* r\|_{V^*}^2 \le \overline{c}_r \|r\|_Q^2, \quad \text{for all } r \in Q, \tag{14}$$

then

$$\underline{c}\|z\|_X \le \|\mathcal{A}x\|_{X^*} \le \overline{c}\|z\|_X, \quad \text{for all } z \in X \tag{15}$$

is satisfied with constants $\underline{c}$, $\overline{c} > 0$ *that depend only on* $\underline{c}_w$, $\overline{c}_w$, $\underline{c}_r$, $\overline{c}_r$.

As stated in [23, Remark 2], for the special case $c(\cdot,\cdot) = 0$, Theorem 1 simplifies to the classical Theorem of Brezzi.

3.2. **Preconditioning the MH-FEM matrices.** We use the preconditioning technique of the last subsection to analyze a preconditioner for the MH-FEM problem (10). Before we tackle the generalized problems, we demonstrate the application of Theorem 1 for the simple model problem (1)-(2). Later on, parts of this proof can be re-used in the generalized cases.

In order to construct a parameter-robust preconditioner, we start by choosing special norms in $H_0^1(\Omega)$. Therefore we introduce the non-standard norm $\|\cdot\|_{\mathcal{C}_1}$ in $H_0^1(\Omega)$ as follows:

$$\|u\|_{\mathcal{C}_1}^2 := (\nu\nabla u, \nabla u)_{L_2(\Omega)} + k\omega(\sigma u, u)_{L_2(\Omega)}.$$

Note, that this is a norm in $H_0^1(\Omega)$ even for the degenerated case $\sigma = 0$. This definition gives rise to non-standard norms $\|\cdot\|_{\mathcal{C}_2}$ and $\|\cdot\|_{\mathcal{C}}$ in the product spaces $H_0^1(\Omega)^2$ and $H_0^1(\Omega)^4$, respectively:

$$\|(y^c, y^s)\|_{\mathcal{C}_2}^2 := \sum_{j\in\{c,s\}} \left[(\nu\nabla y^j, \nabla y^j)_{L_2(\Omega)} + k\omega(\sigma y^j, y^j)_{L_2(\Omega)}\right]$$
$$\|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}}^2 := \|(y^c, y^s)\|_{\mathcal{C}_2}^2 + \|(p^c, p^s)\|_{\mathcal{C}_2}^2$$

The main result is summarized in the following Lemma, that claims that the inf-sup condition in the Theorem of Babuška-Aziz is fulfilled in this non-standard norm.

Lemma 1 (Robust estimates of standard case)**.** *We have*

$$\underline{c}\,\|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} \le \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}},$$
$$\overline{c}\,\|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} \ge \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}}$$

with constants $\underline{c}$, $\overline{c}$ independent of k, ω, ν and σ.

Proof. Using Friedrichs' inequality $\|y\|_0^2 \le C_F|y|_1^2$ for $y \in H_0^1(\Omega)$, we can show non-negativity and boundedness of $a(\cdot,\cdot)$:
(16)
$$0 \le a((y^c, y^s), (y^c, y^s)) = \|y^c\|_{L_2(\Omega)}^2 + \|y^s\|_{L_2(\Omega)}^2 \le C_F\left(|y^c|_{H^1(\Omega)}^2 + |y^s|_{H^1(\Omega)}^2\right)$$
$$\le C_F \sum_{j\in\{c,s\}} (\nu\nabla y^j, \nabla y^j)_{L_2(\Omega)} \le C_F\|(y^c, y^s)\|_{\mathcal{C}_2}^2.$$

Non-negativity and boundedness of $c(\cdot,\cdot)$ follows by an analogous procedure:
(17)
$$0 \le c((p^c, p^s), (p^c, p^s)) = \lambda^{-1}\left(\|p^c\|_{L_2(\Omega)}^2 + \|p^s\|_{L_2(\Omega)}^2\right)$$
$$\le \lambda^{-1}C_F\left(|p^c|_{H^1(\Omega)}^2 + |p^s|_{H^1(\Omega)}^2\right)$$
$$\le \lambda^{-1}C_F \sum_{j\in\{c,s\}} (\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} \le \lambda^{-1}C_F\|(p^c, p^s)\|_{\mathcal{C}_2}^2.$$

Finally, we show boundedness of $b(\cdot,\cdot)$,

$$b((y^c, y^s), (q^c, q^s)) \le \left(\sum_{j\in\{c,s\}} \left[(\nu\nabla y^j, \nabla y^j)_{L_2(\Omega)} + k\omega(\sigma y^j, y^j)_{L_2(\Omega)}\right]\right)^{1/2} \cdot$$
$$\left(\sum_{j\in\{c,s\}} \left[(\nu\nabla q^j, \nabla q^j)_{L_2(\Omega)} + k\omega(\sigma q^j, q^j)_{L_2(\Omega)}\right]\right)^{1/2}$$
$$= \|(y^c, y^s)\|_{\mathcal{C}_2}\|(q^c, q^s)\|_{\mathcal{C}_2},$$

and the inf-sup condition of $b(\cdot,\cdot)$: For the test function we use the special choice $(q^c, q^s) = (y^c + y^s, y^s - y^c)$ to obtain

$$b((y^c, y^s), (y^c, y^s)) + b((y^c, y^s), (y^s, -y^c)) \ge$$
$$\sum_{j\in\{c,s\}} \left[(\nu\nabla y^j, \nabla y^j)_{L_2(\Omega)} + k\omega(\sigma y^j, y^j)_{L_2(\Omega)}\right] = \|(y^c, y^s)\|_{\mathcal{C}_2}^2.$$

Note, that for this special choice we have $\|(q^c, q^s)\|_{\mathcal{C}_2} = \sqrt{2}\|(y^c, y^s)\|_{\mathcal{C}_2}$. Hence we obtain the estimate

$$\frac{1}{\sqrt{2}}\|(y^c, y^s)\|_{\mathcal{C}_2} \leq \sup_{0\neq(q^c,q^s)} \frac{b((y^c, y^s), (q^c, q^s))}{\|(q^c, q^s)\|_{\mathcal{C}_2}} \leq \|(y^c, y^s)\|_{\mathcal{C}_2}. \tag{18}$$

Since $b(\cdot,\cdot)$ is skew symmetric, the same estimate can be obtained for the adjoint setting:

$$\frac{1}{\sqrt{2}}\|(p^c, p^s)\|_{\mathcal{C}_2} \leq \sup_{0\neq(v^c,v^s)} \frac{b((v^c, v^s), (p^c, p^s))}{\|(v^c, v^s)\|_{\mathcal{C}_2}} \leq \|(p^c, p^s)\|_{\mathcal{C}_2}. \tag{19}$$

Therefore, combining (16) with (18) and (17) with (19) yields

$$\begin{aligned}
\frac{1}{2}\|(y^c, y^s)\|^2_{\mathcal{C}_2} &\leq a((y^c, y^s), (y^c, y^s)) + \left(\sup_{0\neq(q^c,q^s)} \frac{b((y^c, y^s), (q^c, q^s))}{\|(q^c, q^s)\|_{\mathcal{C}_2}}\right)^2 \\
&\leq (1 + C_F)\|(y^c, y^s)\|^2_{\mathcal{C}_2}, \\
\frac{1}{2}\|(p^c, p^s)\|^2_{\mathcal{C}_2} &\leq c((p^c, p^s), (p^c, p^s)) + \left(\sup_{0\neq(v^c,v^s)} \frac{b((v^c, v^s), (p^c, p^s))}{\|(v^c, v^s)\|_{\mathcal{C}_2}}\right)^2 \\
&\leq (1 + \lambda^{-1}C_F)\|(p^c, p^s)\|^2_{\mathcal{C}_2}.
\end{aligned}$$

Now the result follows with Theorem 1. □

Furthermore Lemma 1 remains valid for the finite element subspace of $H^1_0(\Omega)^4$, since we are working in a conforming Galerkin approach and consequently the proof can be repeated for the finite element functions step by step.

Hence it follows by the theorem of Babuška-Aziz, that there exists a unique solution of the corresponding variational problem (9), and that the solution continuously depends on the data, uniformly in k, ω, σ and ν. Hence we conclude, that the block-diagonal preconditioner

$$\mathcal{C} = \text{diag}\ (\mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}\ , \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}, \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}, \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}) \tag{20}$$

yields robust convergence rates with respect to the space discretization parameter h and the time discretization parameters k and ω, as well as to the model parameters σ and ν. Additionally, from Lemma 1, we immediately obtain that the spectral condition number of the preconditioned system can be estimated by a constant, i.e.

$$\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}) := \|\mathcal{C}^{-1}\mathcal{A}\|_{\mathcal{C}}\|\mathcal{A}^{-1}\mathcal{C}\|_{\mathcal{C}} \leq \frac{\overline{c}}{\underline{c}} \neq c(\nu, \sigma, \omega, k, h).$$

Clearly, if we use (20) instead of (12), we lose robustness with respect to the cost coefficient λ, but the block-diagonal preconditioner (20) is also applicable to the extensions discussed in the introduction. This is the topic of the next section.

4. Extensions to more practical applications

In this section, we extend the basic model problem (1)-(2) step by step to more involved problems. In all the three cases discussed in the introduction, the resulting system matrices (after an appropriate linearization) obtain high structural similarities to $\mathcal{A}$ as in (10). Therefore, we analyze the applicability of the block-diagonal preconditioner (20) for these general problems and study the robustness with respect to the involved parameters.

4.1. **Different control and observation domains.** In many practical applications, it makes no sense to locate the observation and/or control in the full computational domain Ω. Therefore, we assume that the observation and control domains Ω_1 and Ω_2 are Lipschitz domains and simply connected subdomains of the computational domain Ω, i.e. $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$. In order to deal with the different support of the observation and control, we define the prolongation operators P_i, $i = 1, 2$, by

$$\begin{aligned} &P_i : L_2(\Omega_i) \to L_2(\Omega), \\ &(P_i u, v)_{L_2(\Omega)} = (u, v)_{L_2(\Omega_i)}, \quad \forall u \in L_2(\Omega_i)\ \forall v \in L_2(\Omega), \end{aligned}$$

and the corresponding restriction operators P_i^*, $i = 1, 2$, by

$$\begin{aligned} &P_i^* : L_2(\Omega) \to L_2(\Omega_i), \\ &(P_i u, v)_{L_2(\Omega)} = (u, P_i^* v)_{L_2(\Omega_i)}, \quad \forall u \in L_2(\Omega_i)\ \forall v \in L_2(\Omega). \end{aligned}$$

Therefore, the constraint minimization problem reads as:

$$\min_{y,u} \frac{1}{2} \int_0^T \int_{\Omega_1} |P_1^* \left[y(\boldsymbol{x}, t) - y_d(\boldsymbol{x}, t) \right] |^2 d\boldsymbol{x}\, dt + \frac{\lambda}{2} \int_0^T \int_{\Omega_2} \left[u(\boldsymbol{x}, t) \right]^2 d\boldsymbol{x}\, dt$$

subject to the state equation

$$\begin{cases} \sigma(\boldsymbol{x}) \dfrac{\partial}{\partial t} y(\boldsymbol{x}, t) - \nabla \cdot (\nu(\boldsymbol{x}) \nabla y(\boldsymbol{x}, t)) = P_2 u(\boldsymbol{x}, t) & \forall (\boldsymbol{x}, t) \in \Omega \times (0, T), \\ y(\boldsymbol{x}, t) = 0 & \forall (\boldsymbol{x}, t) \in \partial\Omega \times (0, T), \\ y(\boldsymbol{x}, 0) = y(\boldsymbol{x}, T) & \forall \boldsymbol{x} \in \bar{\Omega}. \end{cases}$$

Here $y \in H_0^1(\Omega)$ and $u \in L_2(\Omega_2)$. We use the same approach as in Subsection 2.2. Again we can eliminate the control u and end up with a new bilinear form $\mathcal{B}_1$ for the reduced optimality-system for a fixed mode k, given by

$$\begin{aligned} \mathcal{B}_1((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s)) = a_1((y^c, y^s), (v^c, v^s)) + b((v^c, v^s), (p^c, p^s)) \\ + b((y^c, y^s), (q^c, q^s)) - c_1((p^c, p^s), (q^c, q^s)), \end{aligned}$$

with the modified bilinear forms

$$\begin{aligned} a_1((y^c, y^s), (v^c, v^s)) &:= (P_1^* y^c, P_1^* v^c)_{L_2(\Omega_1)} + (P_1^* y^s, P_1^* v^s)_{L_2(\Omega_1)} \quad \text{and} \\ c_1((p^c, p^s), (q^c, q^s)) &:= \lambda^{-1} (P_2^* p^c, P_2^* q^c)_{L_2(\Omega_2)} + \lambda^{-1} (P_2^* p^s, P_2^* q^s)_{L_2(\Omega_2)}. \end{aligned}$$

Remark 1. *Let us mention, that in the continuous setting we have*

$$(P_i^* y, P_i^* v)_{L_2(\Omega_i)} = (y, v)_{L_2(\Omega_i)}, \quad i = 1, 2.$$

In view of the discrete system it is more convenient to stay with the restriction and prolongation operators also in the continuous setting.

Using the $\mathcal{C}$ norm as defined in Subsection 3.2, we obtain the following result:

Lemma 2 (Robust estimates)**.** *We have*

$$\begin{aligned} \underline{c}_1 \, \|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} &\le \sup_{(v^c, v^s, q^c, q^s) \neq 0} \frac{\mathcal{B}_1((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}}, \\ \overline{c}_1 \, \|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} &\ge \sup_{(v^c, v^s, q^c, q^s) \neq 0} \frac{\mathcal{B}_1((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}} \end{aligned}$$

with constants $\underline{c}_1$, $\overline{c}_1$ independent of k, ω, ν, σ, Ω_1 and Ω_2.

Proof. The proof is basically the same as the proof of Lemma 1. The main differences are the lower and upper bounds for $a_1(\cdot,\cdot)$ and $c_1(\cdot,\cdot)$. Indeed, we have

$$0 \leq a_1((y^c,y^s),(y^c,y^s)) \leq a((y^c,y^s),(y^c,y^s)) \leq C_F \|(y^c,y^s)\|^2_{\mathcal{C}_2}$$

and

$$0 \leq c_1((p^c,p^s),(p^c,p^s)) \leq c((p^c,p^s),(p^c,p^s)) \leq \lambda^{-1} C_F \|(p^c,p^s)\|^2_{\mathcal{C}_2}.$$

This finishes the proof. □

Furthermore Lemma 2 remains valid for finite element subspaces of $H^1_0(\Omega)^4$, since we are working in a conforming Galerkin approach, and consequently the proof can be repeated step by step for the finite element functions. The finite element discretization (e.g. using piece-wise linear functions with the usual nodal basis) of each 4×4 block leads to a 4×4 matrix $\mathcal{A}_1$ of the following form:

$$\mathcal{A}_1 = \begin{pmatrix} \mathbf{M_1} & \cdot & \mathbf{K_\nu} & -\mathbf{M_{k\omega,\sigma}} \\ \cdot & \mathbf{M_1} & \mathbf{M_{k\omega,\sigma}} & \mathbf{K_\nu} \\ \mathbf{K_\nu} & \mathbf{M_{k\omega,\sigma}} & -\lambda^{-1}\mathbf{M_2} & \cdot \\ -\mathbf{M_{k\omega,\sigma}} & \mathbf{K_\nu} & \cdot & -\lambda^{-1}\mathbf{M_2} \end{pmatrix}.$$

Here the matrices $\mathbf{M_1}$ and $\mathbf{M_2}$ are reduced mass matrices, indeed

$$\mathbf{M_i} = \mathbf{P_i}(\mathbf{P_i}^T\mathbf{M}\mathbf{P_i})\mathbf{P_i}^T,$$

where the prolongation matrices $\mathbf{P_i}$, $i=1,2$, are the finite element discretization of P_i.

Remark 2. *In the special case $\Omega_1 = \Omega_2$, a modification of the parameter-robust preconditioner (12) can be used to obtain convergence-rates that are additionally independent of λ (see also* [16]*).*

4.2. **Observation in the energy norm.** In some practical applications, the quantity of interest in the observation domain is not the L_2-norm, but some energy norm, e.g. the H^1-semi norm. Therefore, we are dealing with the following problem:

$$\min_{y,u} \frac{1}{2}\int_0^T \int_\Omega [\nabla(y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t))]^2 \, d\boldsymbol{x}\, dt + \frac{\lambda}{2}\int_0^T\int_\Omega [u(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt$$

subject to the state equation

$$\begin{cases} \sigma(\boldsymbol{x})\dfrac{\partial}{\partial t} y(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y(\boldsymbol{x},t)) = u(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T), \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall \boldsymbol{x}\in\bar{\Omega}. \end{cases}$$

In order to derive the reduced optimality-system for a fixed mode k in the weak form, we use the same approach as in Subsection 2.2 and end up with a new bilinear form $\mathcal{B}_2$, given by

$$\begin{aligned}\mathcal{B}_2((y^c,y^s,p^c,p^s),(v^c,v^s,q^c,q^s)) &= a_2((y^c,y^s),(v^c,v^s)) + b((v^c,v^s),(p^c,p^s)) \\ &\quad + b((y^c,y^s),(q^c,q^s)) - c((p^c,p^s),(q^c,q^s)),\end{aligned}$$

with the modified bilinear form

$$a_2((y^c,y^s),(v^c,v^s)) := (\nabla y^c, \nabla v^c)_{L_2(\Omega)} + (\nabla y^s,\nabla v^s)_{L_2(\Omega)}.$$

Using the $\mathcal{C}$ norm as defined in Subsection 3.2, we obtain the following result:

Lemma 3 (Robust estimates)**.** *We have*

$$\underline{c}_2\|(y^c,y^s,p^c,p^s)\|_{\mathcal{C}} \le \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}_2((y^c,y^s,p^c,p^s),(v^c,v^s,q^c,q^s))}{\|(v^c,v^s,q^c,q^s)\|_{\mathcal{C}}},$$
$$\overline{c}_2\|(y^c,y^s,p^c,p^s)\|_{\mathcal{C}} \ge \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}_2((y^c,y^s,p^c,p^s),(v^c,v^s,q^c,q^s))}{\|(v^c,v^s,q^c,q^s)\|_{\mathcal{C}}}$$

with constants $\underline{c}_2$*,* $\overline{c}_2$ *independent of* k*,* ω*,* ν *and* σ*.*

Proof. The proof is basically the same as the proof of Lemma 1. The main differences are the lower and upper bounds for $a_2(\cdot,\cdot)$. We have

$$0 \le a_2((y^c,y^s),(y^c,y^s)) = |y^c|^2_{H^1(\Omega)} + |y^s|^2_{H^1(\Omega)} \le \|(y^c,y^s)\|^2_{\mathcal{C}_2}.$$

This finishes the proof. □

Furthermore, Lemma 3 remains also valid for finite element subspaces of $H^1_0(\Omega)^4$, since we are working in a conforming Galerkin approach, and consequently the proof can be repeated step by step for the finite element functions. The finite element discretization (e.g. using piece-wise linear functions with the usual nodal basis) of each 4×4 block leads to a 4×4 matrix $\mathcal{A}_2$ of the following form:

$$\mathcal{A}_2 = \begin{pmatrix} \mathbf{K} & \cdot & \mathbf{K}_{\boldsymbol{\nu}} & -\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} \\ \cdot & \mathbf{K} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} \\ \mathbf{K}_{\boldsymbol{\nu}} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & -\lambda^{-1}\mathbf{M} & \cdot \\ -\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} & \cdot & -\lambda^{-1}\mathbf{M} \end{pmatrix}.$$

Remark 3. *This case is of special interest if we want to observe the magnetic flux density* $\mathbf{B} = \mathbf{curl}\,\mathbf{y}$ *in eddy current optimal control problems (cf.* [16]*).*

4.3. **Control constraints.** In this generalized problem we add control constraints to the standard setting (1)-(2). This is done in a very specific way, namely by adding control constraints for each mode k to the according Fourier coefficients. Again we observe a decoupling with respect to the modes k. Due to the control constraints the optimality system obtains a nonlinear structure. In order to deal with the nonlinearity, following Herzog and Sachs [10], we apply a semi-smooth Newton approach on the theoretical basis of [11]. At each Newton step we have to solve a saddle point equation.

We consider the following optimal control problem with pointwise control constraints for the Fourier coefficients.

$$\min_{y,u} \frac{1}{2}\int_0^T\int_\Omega [y(\boldsymbol{x},t)-y_d(\boldsymbol{x},t)]^2\,d\boldsymbol{x}\,dt + \frac{\lambda}{2}\int_0^T\int_\Omega [u(\boldsymbol{x},t)]^2\,d\boldsymbol{x}\,dt$$

subject to the state equation

$$\begin{cases} \sigma(\boldsymbol{x})\dfrac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y(\boldsymbol{x},t)) = u(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T), \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\Omega, \end{cases}$$

with the control constraints associated to the Fourier coefficients

$$\underline{u}^c_k \le u^c_k \le \overline{u}^c_k, \quad \text{a.e. in } \Omega, k = 0,1,\ldots,N,$$
$$\underline{u}^s_k \le u^s_k \le \overline{u}^s_k, \quad \text{a.e. in } \Omega, k = 1,\ldots,N.$$

Here the Fourier coefficients are given by

$$u^c_k(\boldsymbol{x}) = \frac{2}{T}\int_0^T u(\boldsymbol{x},t)\cos(k\omega t)\,dt \quad\text{and}\quad u^s_k(\boldsymbol{x}) = \frac{2}{T}\int_0^T u(\boldsymbol{x},t)\sin(k\omega t)\,dt.$$

We mention, that the desired state y_d is multiharmonic, cf. (7). In this setting, it is more convenient to apply the multiharmonic approach to the constrained minimization problem before deriving the necessary and sufficient optimality conditions. In fact, the time discretization and the derivation of the optimality system commutes. Since the problem decouples with respect to the modes k, we again concentrate on one block for a fixed k and omit the subindex. Therefore we consider the following problem:

$$\min_{y^c,y^s,u^c,u^s} \mathcal{J}_d(y^c,y^s,u^c,u^s) = \min_{y^c,y^s,u^c,u^s} \sum_{j\in\{c,s\}} \frac{1}{2}\int_\Omega \left[y^j - y_d^j\right]^2 d\boldsymbol{x} + \frac{\lambda}{2}\int_\Omega \left[u^j\right]^2 d\boldsymbol{x},$$

subject to the state equation

$$\begin{cases} k\omega\sigma(\boldsymbol{x})y^s(\boldsymbol{x}) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y^c(\boldsymbol{x})) = u^c(\boldsymbol{x}) & \forall \boldsymbol{x}\in\Omega,\\ -k\omega\sigma(\boldsymbol{x})y^c(\boldsymbol{x}) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y^s(\boldsymbol{x})) = u^s(\boldsymbol{x}) & \forall \boldsymbol{x}\in\Omega,\\ y^c(\boldsymbol{x}) = y^s(\boldsymbol{x}) = 0 & \forall \boldsymbol{x}\in\partial\Omega, \end{cases}$$

with the control constraints associated to the Fourier coefficients

$$\begin{aligned} \underline{u}^c \leq & u^c \leq \overline{u}^c, \quad \text{a.e. in } \Omega,\\ \underline{u}^s \leq & u^s \leq \overline{u}^s, \quad \text{a.e. in } \Omega. \end{aligned}$$

Continuous version. The corresponding Lagrange functional is given by

$$\begin{aligned} &\mathcal{L}(y^c,y^s,u^c,u^s,p^c,p^s,\xi^{c,+},\xi^{s,+},\xi^{c,-},\xi^{s,-}) := \mathcal{J}_d(y^c,y^s,u^c,u^s)\\ &\quad + k\omega(\sigma y^s,p^c)_{L_2(\Omega)} + (\nu\nabla y^c,\nabla p^c)_{L_2(\Omega)} - (u^c,p^c)_{L_2(\Omega)}\\ &\quad - k\omega(\sigma y^c,p^s)_{L_2(\Omega)} + (\nu\nabla y^s,\nabla p^s)_{L_2(\Omega)} - (u^s,p^s)_{L_2(\Omega)}\\ &\quad + \sum_{j\in\{c,s\}} \left[(\xi^{j,+},u^j-\overline{u}^j)_{L_2(\Omega)} + (\xi^{j,-},\underline{u}^j - u^j)_{L_2(\Omega)}\right]. \end{aligned}$$

The first order necessary and sufficient optimality conditions are given by

$$(21)\quad \begin{cases} -k\omega(\sigma p^s,v^c)_{L_2(\Omega)} + (\nu\nabla p^c,\nabla v^c)_{L_2(\Omega)} + (y^c,v^c)_{L_2(\Omega)} = (y_d^c,v^c)_{L_2(\Omega)},\\ k\omega(\sigma p^c,v^s)_{L_2(\Omega)} + (\nu\nabla p^s,\nabla v^s)_{L_2(\Omega)} + (y^s,v^s)_{L_2(\Omega)} = (y_d^s,v^s)_{L_2(\Omega)},\\ \lambda(u^c,w^c)_{L_2(\Omega)} - (p^c,w^c)_{L_2(\Omega)} + (\xi^c,w^c)_{L_2(\Omega)} = 0,\\ \lambda(u^s,w^s)_{L_2(\Omega)} - (p^s,w^s)_{L_2(\Omega)} + (\xi^s,w^s)_{L_2(\Omega)} = 0,\\ k\omega(\sigma y^s,q^c)_{L_2(\Omega)} + (\nu\nabla y^c,\nabla q^c)_{L_2(\Omega)} - (u^c,q^c)_{L_2(\Omega)} = 0,\\ -k\omega(\sigma y^c,q^s)_{L_2(\Omega)} + (\nu\nabla y^s,\nabla q^s)_{L_2(\Omega)} - (u^s,q^s)_{L_2(\Omega)} = 0,\\ \xi^c - \max(0,\xi^c + C(u^c-\overline{u}^c)) - \min(0,\xi^c - C(\underline{u}^c - u^c)) = 0, \quad \text{a.e. in } \Omega,\\ \xi^s - \max(0,\xi^s + C(u^s-\overline{u}^s)) - \min(0,\xi^s - C(\underline{u}^s - u^s)) = 0, \quad \text{a.e. in } \Omega, \end{cases}$$

with Lagrange multipliers p^c, p^s, $\xi^c = \xi^{c,+} - \xi^{c,-}$ and $\xi^s = \xi^{s,+} - \xi^{s,-}$ and test functions v^c, v^s, q^c, q^s, w^c, w^s and some positive constant C.

This system is nonlinear, but due to [11], the last two equations in (21) enjoy the Newton differentiability, at least for $C = \lambda$. In order to solve this system, we use the primal-dual active set method as introduced in [11]. This method is equivalent to a semi-smooth Newton method. The strategy proceeds as follows: Given an iterate $(y_l^c, y_l^s, u_l^c, u_l^s, p_l^c, p_l^s, \xi_l^c, \xi_l^s)$, the active sets are determined by

$$\begin{aligned} \mathcal{E}_l^{c,+} &= \{\boldsymbol{x}\in\Omega : \xi_l^c + C(u_l^c - \overline{u}^c) > 0\},\\ \mathcal{E}_l^{c,-} &= \{\boldsymbol{x}\in\Omega : \xi_l^c - C(\underline{u}^c - u_l^c) < 0\},\\ \mathcal{E}_l^{s,+} &= \{\boldsymbol{x}\in\Omega : \xi_l^s + C(u_l^s - \overline{u}^s) > 0\},\\ \mathcal{E}_l^{s,-} &= \{\boldsymbol{x}\in\Omega : \xi_l^s - C(\underline{u}^s - u_l^s) < 0\}, \end{aligned}$$

and the inactive sets are determined by $\mathcal{I}_l^c = \Omega \backslash (\mathcal{E}_l^{c,+} \cup \mathcal{E}_l^{c,-})$ and $\mathcal{I}_l^s = \Omega \backslash (\mathcal{E}_l^{s,+} \cup \mathcal{E}_l^{s,-})$. The Newton step for the solution of (21), given in terms of the new iterate, reads as follows:

$$\begin{pmatrix} I & \cdot & \cdot & \cdot & K & -I_{\sigma,k\omega} & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & I_{\sigma,k\omega} & K & \cdot & \cdot \\ \cdot & \cdot & \lambda I & \cdot & -I & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \lambda I & \cdot & -I & \cdot & I \\ K & I_{\sigma,k\omega} & -I & \cdot & \cdot & \cdot & \cdot & \cdot \\ -I_{\sigma,k\omega} & K & \cdot & -I & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & C\chi_{\mathcal{E}_l^c} & \cdot & \cdot & \cdot & \chi_{\mathcal{I}_l^c} & \cdot \\ \cdot & \cdot & \cdot & C\chi_{\mathcal{E}_l^s} & \cdot & \cdot & \cdot & \chi_{\mathcal{I}_l^s} \end{pmatrix} \begin{pmatrix} y_{l+1}^c \\ y_{l+1}^s \\ u_{l+1}^c \\ u_{l+1}^s \\ p_{l+1}^c \\ p_{l+1}^s \\ \xi_{l+1}^c \\ \xi_{l+1}^s \end{pmatrix} = \begin{pmatrix} y_d^c \\ y_d^s \\ 0 \\ 0 \\ 0 \\ 0 \\ C(\chi_{\mathcal{E}_l^{c,+}} \overline{u}^c + \chi_{\mathcal{E}_l^{c,-}} \underline{u}^c) \\ C(\chi_{\mathcal{E}_l^{s,+}} \overline{u}^s + \chi_{\mathcal{E}_l^{s,-}} \underline{u}^s) \end{pmatrix}.$$

The operators I, $I_{\sigma,k\omega}$ and K correspond to the identity, a weighted identity and the differential operator $-\Delta$, respectively. The symbol χ denotes the characteristic function with respect to the set denoted in the subscript. Next we use that the restriction of ξ^j, $j \in \{c, s\}$, to the inactive sets $\mathcal{I}^j$ is zero, and hence this variable can be eliminated from the system. Therefore only $\xi^j_{\mathcal{E}^j}$, the restriction of ξ^j to the active sets $\mathcal{E}^j$, enter the system. The Newton system then attains the equivalent symmetric saddle point form (we omit the the Newton iteration index l):

$$\begin{pmatrix} I & \cdot & \cdot & \cdot & K & -I_{\sigma,k\omega} & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & I_{\sigma,k\omega} & K & \cdot & \cdot \\ \cdot & \cdot & \lambda I & \cdot & -I & \cdot & \chi_{\mathcal{E}^c} & \cdot \\ \cdot & \cdot & \cdot & \lambda I & \cdot & -I & \cdot & \chi_{\mathcal{E}^s} \\ K & I_{\sigma,k\omega} & -I & \cdot & \cdot & \cdot & \cdot & \cdot \\ -I_{\sigma,k\omega} & K & \cdot & -I & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \chi_{\mathcal{E}^c} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \chi_{\mathcal{E}^s} & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} y^c \\ y^s \\ u^c \\ u^s \\ p^c \\ p^s \\ \xi^c_{\mathcal{E}^c} \\ \xi^s_{\mathcal{E}^s} \end{pmatrix} = \begin{pmatrix} y_d^c \\ y_d^s \\ 0 \\ 0 \\ 0 \\ 0 \\ \chi_{\mathcal{E}^{c,+}} \overline{u}^c + \chi_{\mathcal{E}^{c,-}} \underline{u}^c \\ \chi_{\mathcal{E}^{s,+}} \overline{u}^s + \chi_{\mathcal{E}^{s,-}} \underline{u}^s \end{pmatrix}.$$

In the following we concentrate on the efficient and robust solution of the Newton system at each step. According to the strategy presented in Section 3, we start with the definition of non-standard norms in the product spaces $H_0^1(\Omega)^2 \times L_2(\Omega)^2$ and $H_0^1(\Omega)^2 \times L_2(\mathcal{E}^c) \times L_2(\mathcal{E}^s)$:

$$\|(y^c, y^s, u^c, u^s)\|^2_{\mathcal{D}_1} := \|(y^c, y^s)\|^2_{\mathcal{C}_2} + \sum_{j \in \{c,s\}} \|u^j\|^2_{L_2(\Omega)} \quad \text{and}$$

$$\|(p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|^2_{\mathcal{D}_2} := \|(p^c, p^s)\|^2_{\mathcal{C}_2} + \sum_{j \in \{c,s\}} \|\xi^j_{\mathcal{E}^j}\|^2_{L_2(\mathcal{E}^j)}.$$

These non-standard norms give rise to a norm in the product space $(H_0^1(\Omega)^2 \times L_2(\Omega)^2) \times (H_0^1(\Omega)^2 \times L_2(\mathcal{E}^c) \times L_2(\mathcal{E}^s))$:

$$\|(y^c, y^s, u^c, u^s, p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|^2_{\mathcal{D}} := \|(y^c, y^s, u^c, u^s)\|^2_{\mathcal{D}_1} + \|(p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|^2_{\mathcal{D}_2}.$$

According to the notation in Section 3 we introduce the bilinear forms

$$a((y^c, y^s, u^c, u^s), (v^c, v^s, w^c, w^s)) := \sum_{j \in \{c,s\}} \left[(y^j, v^j)_{L_2(\Omega)} + \lambda (u^j, w^j)_{L_2(\Omega)} \right] \quad \text{and}$$

$$\begin{aligned} b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) &:= \sum_{j \in \{c,s\}} \left[(w^j, \xi^j_{\mathcal{E}^j})_{L_2(\mathcal{E}^j)} - (w^j, p^j)_{L_2(\Omega)} \right] \\ + (\nu \nabla v^c, \nabla p^c)_{L_2(\Omega)} + (\nu \nabla v^s, \nabla p^s)_{L_2(\Omega)} &+ k\omega (\sigma v^s, p^c)_{L_2(\Omega)} - k\omega (\sigma v^c, p^s)_{L_2(\Omega)}. \end{aligned}$$

Now we consider the computed bilinear form

$$\begin{aligned} &\tilde{\mathcal{B}}_3((y^c, y^s, u^c, u^s, p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s}), (v^c, v^s, w^c, w^s, q^c, q^s, \zeta^c_{\mathcal{E}^c}, \zeta^s_{\mathcal{E}^s})) := \\ &\quad a((y^c, y^s, u^c, u^s), (v^c, v^s, w^c, w^s)) + b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) \\ &\quad + b((y^c, y^s, u^c, u^s), (q^c, q^s, \zeta^c_{\mathcal{E}^c}, \zeta^s_{\mathcal{E}^s})). \end{aligned}$$

Note, that in this setting we have $c(\cdot, \cdot) = 0$.

Lemma 4 (Robust estimates)**.** *We have*

$$
\begin{aligned}
&\underline{\tilde{c}}_3 \|(y^c, y^s, u^c, u^s, p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|_{\mathcal{D}} \leq \\
&\quad \sup \frac{\tilde{\mathcal{B}}_3((y^c, y^s, u^c, u^s, p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s}), (v^c, v^s, w^c, w^s, q^c, q^s, \zeta^c_{\mathcal{E}^c}, \zeta^s_{\mathcal{E}^s}))}{\|(v^c, v^s, w^c, w^s, q^c, q^s, \zeta^c_{\mathcal{E}^c}, \zeta^s_{\mathcal{E}^s})\|_{\mathcal{D}}} \\
&\quad \leq \overline{\tilde{c}}_3 \|(y^c, y^s, u^c, u^s, p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|_{\mathcal{D}}
\end{aligned}
$$

with constants $\underline{\tilde{c}}_3$, $\overline{\tilde{c}}_3$ *independent of* k, ω, ν, σ, $\mathcal{E}^c$ *and* $\mathcal{E}^s$*.*

Proof. Throughout this proof we use a generic constant c that may depend on λ, but is independent of h, k, ω, σ, ν, $\mathcal{E}^c$ and $\mathcal{E}^s$. We use Theorem 1 for the special case $c(\cdot,\cdot) = 0$. Boundedness of $a(\cdot,\cdot)$ follows easily by using Cauchy-Schwarz' and Friedrichs' inequalities:

$$
\begin{aligned}
&a((y^c, y^s, u^c, u^s), (v^c, v^s, w^c, w^s)) \\
&\quad \leq c \sum_{j\in\{c,s\}} \left[\|y^j\|_{L_2(\Omega)} \|v^j\|_{L_2(\Omega)} + \|u^j\|_{L_2(\Omega)} \|w^j\|_{L_2(\Omega)} \right] \\
&\quad \leq c \left(\sum_{j\in\{c,s\}} \|y^j\|^2_{L_2(\Omega)} + \|u^j\|^2_{L_2(\Omega)} \right)^{1/2} \left(\sum_{j\in\{c,s\}} \|v^j\|^2_{L_2(\Omega)} + \|w^j\|^2_{L_2(\Omega)} \right)^{1/2} \\
&\quad \leq c \left(\sum_{j\in\{c,s\}} |y^j|^2_{H^1(\Omega)} + \|u^j\|^2_{L_2(\Omega)} \right)^{1/2} \left(\sum_{j\in\{c,s\}} |v^j|^2_{H^1(\Omega)} + \|w^j\|^2_{L_2(\Omega)} \right)^{1/2} \\
&\quad \leq c \|(y^c, y^s, u^c, u^s)\|_{\mathcal{D}_1} \|(v^c, v^s, w^c, w^s)\|_{\mathcal{D}_1}.
\end{aligned}
$$

To verify the coercivity of $a(\cdot,\cdot)$, let (y^c, y^s, u^c, u^s) be in the kernel of $b(\cdot,\cdot)$. Then, in particular,

$$
\begin{cases}
(\nu\nabla y^c, \nabla q^c)_{L_2(\Omega)} + k\omega(\sigma y^s, q^c)_{L_2(\Omega)} = (u^c, q^c)_{L_2(\Omega)} \\
(\nu\nabla y^s, \nabla q^s)_{L_2(\Omega)} - k\omega(\sigma y^c, q^s)_{L_2(\Omega)} = (u^s, q^s)_{L_2(\Omega)}
\end{cases}
$$

holds for all $(q^c, q^s) \in H^1_0(\Omega)^2$. In particular, for the choice $q^c = y^c + y^s$ and $q^s = y^s - y^c$, we obtain the a priori estimate

$$
\|(y^c, y^s)\|^2_{\mathcal{C}_2} \leq c(\|u^c\|^2_{L_2(\Omega)} + \|u^s\|^2_{L_2(\Omega)}),
$$

where again c is a generic constant. This implies

$$
\begin{aligned}
a((y^c, y^s, u^c, u^s), (y^c, y^s, u^c, u^s)) &= \sum_{j\in\{c,s\}} \left[\|y^j\|^2_{L_2(\Omega)} + \lambda \|u^j\|^2_{L_2(\Omega)} \right] \\
&\geq c \left(\|u^c\|^2_{L_2(\Omega)} + \|u^s\|^2_{L_2(\Omega)} + \|(y^c, y^s)\|^2_{\mathcal{C}_2} \right) \\
&= c \|(y^c, y^s, u^c, u^s)\|^2_{\mathcal{D}_1}.
\end{aligned}
$$

Boundedness of $b(\cdot,\cdot)$ follows directly by using Cauchy's inequality several times:

$$
\begin{aligned}
&b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) \\
&\quad \leq c \|(v^c, v^s)\|_{\mathcal{C}_2} \|(p^c, p^s)\|_{\mathcal{C}_2} + \sum_{j\in\{c,s\}} \left[\|w^j\|_{L_2(\mathcal{E}^j)} \|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} + \|w^j\|_{L_2(\Omega)} \|p^j\|_{L_2(\Omega)} \right] \\
&\quad \leq c \left(\|(v^c, v^s)\|_{\mathcal{C}_2} + \sum_{j\in\{c,s\}} \|w^j\|_{L_2(\Omega)} \right) \left(\|(p^c, p^s)\|_{\mathcal{C}_2} + \sum_{j\in\{c,s\}} \|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} \right) \\
&\quad \leq c \|(v^c, v^s, w^c, w^s)\|_{\mathcal{D}_1} \|(p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})\|_{\mathcal{D}_2}.
\end{aligned}
$$

The inf-sup condition for $b(\cdot,\cdot)$ can be obtained as follows: We start by choosing $(v^c, v^s, w^c, w^s) = (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})$ (by extending $\xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s}$ by zero outside of $\mathcal{E}^c$ and $\mathcal{E}^s$, respectively). Then we obtain the estimate

$$
\begin{aligned}
&b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) \\
&= \sum_{j\in\{c,s\}} \left[\|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} - (\xi^j_{\mathcal{E}^j}, p^j)_{L_2(\Omega)} + (\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} \right] \\
&\geq \sum_{j\in\{c,s\}} \left[\|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} - \|\xi^j_{\mathcal{E}^j}\|_{L_2(\Omega)} \|p^j\|_{L_2(\Omega)} + (\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} \right] \\
&\geq \sum_{j\in\{c,s\}} \left[\|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} - \frac{1}{2}\|\xi^j_{\mathcal{E}^j}\|^2_{L_2(\Omega)} - \frac{1}{2}\|p^j\|^2_{L_2(\Omega)} + (\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} \right] \\
&\geq \sum_{j\in\{c,s\}} \left[\frac{1}{2}\|\xi^j_{\mathcal{E}^j}\|_{L_2(\mathcal{E}^j)} - \frac{C_F}{2}(\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} + (\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)} \right].
\end{aligned}
\tag{22}
$$

Furthermore, for the choice $(v^c, v^s, w^c, w^s) = (p^s, -p^c, 0, 0)$, we have

$$
b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) = \sum_{j\in\{c,s\}} k\omega(\sigma p^j, p^j)_{L_2(\Omega)}. \tag{23}
$$

Finally, for the choice $(v^c, v^s, w^c, w^s) = C_F/2(p^c, p^s, 0, 0)$, we obtain

$$
b((v^c, v^s, w^c, w^s), (p^c, p^s, \xi^c_{\mathcal{E}^c}, \xi^s_{\mathcal{E}^s})) = \sum_{j\in\{c,s\}} \frac{C_F}{2}(\nu\nabla p^j, \nabla p^j)_{L_2(\Omega)}. \tag{24}
$$

So combining the estimates (22)-(24), gives the inf-sup condition for $b(\cdot,\cdot)$ with a constant independent of k, ω, σ, ν, $\mathcal{E}^c$ and $\mathcal{E}^s$. This completes the proof. □

Discrete version. After discretization in space and time, for a fixed mode k, we are dealing with the following constrained minimization problem (again we omit the subscript k):

$$
\min_{(\mathbf{y^c}, \mathbf{y^s}, \mathbf{u^c}, \mathbf{u^s})} \frac{1}{2} \sum_{j\in\{c,s\}} \left[(\mathbf{y^j} - \mathbf{y^j_d})^T \mathbf{M} (\mathbf{y^j} - \mathbf{y^j_d}) + \lambda {\mathbf{u^j}}^T \mathbf{M}\mathbf{u^j} \right]
$$

subject to the state equation

$$
\begin{cases}
\mathbf{K}_{\boldsymbol{\nu}}\mathbf{y^c} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}\mathbf{y^s} = \mathbf{M}\mathbf{u^c}, \\
-\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}\mathbf{y^c} + \mathbf{K}_{\boldsymbol{\nu}}\mathbf{y^s} = \mathbf{M}\mathbf{u^s}, \\
\underline{\mathbf{u}}^{\mathbf{c}} \leq \mathbf{u^c} \leq \overline{\mathbf{u}}^{\mathbf{c}} \quad \text{(componentwise)}, \\
\underline{\mathbf{u}}^{\mathbf{s}} \leq \mathbf{u^s} \leq \overline{\mathbf{u}}^{\mathbf{s}} \quad \text{(componentwise)}.
\end{cases}
$$

We can use the same approach as used in the continuous setting. Additionally we eliminate the control variables $\mathbf{u^c}$ and $\mathbf{u^s}$ and the Lagrange multipliers for the inequality constraints $\boldsymbol{\xi}^{\mathbf{c}}_{\boldsymbol{\mathcal{E}}^{\mathbf{c}}}$ and $\boldsymbol{\xi}^{\mathbf{s}}_{\boldsymbol{\mathcal{E}}^{\mathbf{s}}}$ from the resulting system of linear equations and end up with the following problem to be solved at each Newton step l:

$$
\underbrace{\begin{pmatrix}
\mathbf{M} & \cdot & \mathbf{K}_{\boldsymbol{\nu}} & -\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} \\
\cdot & \mathbf{M} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} \\
\mathbf{K}_{\boldsymbol{\nu}} & \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & -\lambda^{-1}\mathbf{M}_{\mathcal{E}^c} & \cdot \\
-\mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}} & \mathbf{K}_{\boldsymbol{\nu}} & \cdot & -\lambda^{-1}\mathbf{M}_{\mathcal{E}^s}
\end{pmatrix}}_{\mathcal{A}_3}
\begin{pmatrix} \mathbf{y^c} \\ \mathbf{y^s} \\ \mathbf{p^c} \\ \mathbf{p^s} \end{pmatrix}
=
\begin{pmatrix} \mathbf{M}\mathbf{y^c_d} \\ \mathbf{M}\mathbf{y^c_d} \\ \mathbf{r^c} \\ \mathbf{r^s} \end{pmatrix}. \tag{25}
$$

Here, for $j \in \{c, s\}$, $\mathbf{M}_{\mathcal{E}^j}$ are the matrices corresponding to the active index set $\mathcal{E}^j$, i.e.

$$
\mathbf{M}_{\mathcal{E}^j} = \mathbf{M} - \mathbf{P}_{\mathcal{E}^j} \left(\mathbf{P}^T_{\mathcal{E}^j} \mathbf{M^{-1}} \mathbf{P}_{\mathcal{E}^j} \right)^{-1} \mathbf{P}^T_{\mathcal{E}^j},
$$

and $\mathbf{r^j}$ are parts of the right-hand sides

$$\mathbf{r^j} = \mathbf{P}_{\boldsymbol{\mathcal{E}}^{\mathrm{j}}}(\mathbf{P}_{\boldsymbol{\mathcal{E}}^{\mathrm{j}}}{}^T\mathbf{M}^{-1}\mathbf{P}_{\boldsymbol{\mathcal{E}}^{\mathrm{j}}})^{-1}(\mathbf{P}_{\boldsymbol{\mathcal{E}}+,\mathrm{j}}{}^T\overline{\mathbf{u}}^{\mathbf{j}} + \mathbf{P}_{\boldsymbol{\mathcal{E}}-,\mathrm{j}}{}^T\underline{\mathbf{u}}^{\mathbf{j}}).$$

Here, for $j \in \{c, s\}$, $\mathbf{P}_{\boldsymbol{\mathcal{E}}^{\mathrm{j}}}{}^T$ is a rectangular matrix consisting of those rows, which belong to the active indices, and similarly for $\mathbf{P}_{\boldsymbol{\mathcal{E}}\pm,\mathrm{j}}{}^T$. For the reduced system, an analogon to Lemma 4 holds. Let $\mathcal{B}_3$ denote the bilinear form corresponding to the continuous form of the reduced system (25), then the following result holds.

Lemma 5 (Robust estimates). *We have*

$$\underline{c}_3\|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} \le \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}_3((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}}$$
$$\overline{c}_3\|(y^c, y^s, p^c, p^s)\|_{\mathcal{C}} \ge \sup_{(v^c,v^s,q^c,q^s)\neq 0} \frac{\mathcal{B}_3((y^c, y^s, p^c, p^s), (v^c, v^s, q^c, q^s))}{\|(v^c, v^s, q^c, q^s)\|_{\mathcal{C}}}$$

with constants $\underline{c}_3$, $\overline{c}_3$ independent of k, ω, ν, σ, $\mathcal{E}^c$ and $\mathcal{E}^s$.

Proof. cf. proof of Lemma 4. □

4.4. **Condition number estimate and MinRes convergence-rate.** Consequently, since Lemma 2, Lemma 3 and Lemma 5 are also valid for finite element functions, we immediately obtain that the spectral condition number of the preconditioned systems can be estimated by constants c_i, i.e.

Proposition 1. *For $i = 1, 2, 3$, we have a parameter-independent condition number bound:*

$$\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}_i) := \|\mathcal{C}^{-1}\mathcal{A}_i\|_{\mathcal{C}}\|\mathcal{A}_i^{-1}\mathcal{C}\|_{\mathcal{C}} \le \frac{\overline{c}_i}{\underline{c}_i} \neq c_i(\nu, \sigma, \omega, k, h),$$

with constants $\underline{c}_i$, $\overline{c}_i$ and consequently c_i independent of k, ω, ν, σ and h.

Using the convergence rate estimate of the MinRes method (e.g. [8]), we finally arrive at the following theorem.

Theorem 2 (Robust solver). *For $i = 1, 2, 3$, the MinRes method applied to the preconditioned systems converges. At the m-th iteration, the preconditioned residual $\mathbf{r^m} = \mathcal{C}^{-1}\mathbf{f} - \mathcal{C}^{-1}\mathcal{A}_i\mathbf{w^m}$ is bounded as*

$$\left\|\mathbf{r^{2m}}\right\|_{\mathcal{C}} \le \frac{2q^m}{1+q^{2m}}\left\|\mathbf{r^0}\right\|_{\mathcal{C}} \quad \textit{where} \quad q = \frac{\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}_i) - 1}{\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}_i) + 1} \le \frac{c_i + 1}{c_i - 1}. \tag{26}$$

Proof. This result directly follows from [8] and Proposition 1. □

Therefore the number of MinRes iterations required for reducing the initial error by some fixed factor $\epsilon \in (0, 1)$ is independent of the space and time discretization parameter h and $k\omega$ and the involved model parameters ν and σ. In the nonlinear setting, the convergence is also independent of the active index sets $\mathcal{E}^c$ and $\mathcal{E}^s$.

Remark 4 (Combinations). *The results obtained in Theorem 2 are also valid for arbitrary combinations of the three generalizations derived in Section 4.*

Remark 5 (Friedrichs' inequality). *The proofs of Lemma 2 and Lemma 4 heavily rely on the existence of Friedrichs' inequality. Indeed, thinking about practical applications, e.g. in computational electromagnetics, the existence of a Friedrichs' type inequality in* $\mathbf{H}(\mathbf{curl})$ *is a serious requirement. For the special case $\sigma > 0$ and $k \ge 1$, the results of Subsection 4.1 and Subsection 4.3 can also be obtained without the need for a Friedrich type inequality. This can be seen in the following way: By a scaling of the state equation with the factor $\sigma_{\min}^{-1}$, we obtain an equivalent minimization problem:*

$$\min_{y,u} \mathcal{J}(y, u) = \min_{y,u} \frac{\omega}{2}\int_0^T \int_\Omega [y(\boldsymbol{x}, t) - y_d(\boldsymbol{x}, t)]^2 \, d\boldsymbol{x}\, dt + \frac{\tilde{\lambda}}{2\omega}\int_0^T \int_\Omega [\tilde{u}(\boldsymbol{x}, t)]^2 \, d\boldsymbol{x}\, dt$$

subject to the state equation

$$\begin{cases} \tilde{\sigma}\dfrac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\tilde{\nu}\nabla y(\boldsymbol{x},t)) = \tilde{u}(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in\Omega\times(0,T), \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\partial\Omega\times(0,T), \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\Omega, \end{cases}$$

with the new parameters $\tilde{\sigma} = \frac{\bar{\sigma}}{\bar{\sigma}_{\min}}$, $\tilde{\nu} = \frac{\bar{\nu}}{\bar{\sigma}_{\min}}$, $\tilde{\lambda} = \bar{\lambda}\bar{\sigma}_{\min}^2\omega^2$ *and the scaled state* $\tilde{u} = \frac{\bar{u}}{\bar{\sigma}_{\min}}$. *This scaling gives* $\tilde{\sigma}_{\min} = 1$ *and therefore the estimate* $\omega(u,v)_{L_2(\Omega)} \leq k\omega(\sigma u, v)_{L_2(\Omega)}$. *Consequently the block-diagonal preconditioner (20) yields almost robust convergence rates. Let us mention, that in this setting the block corresponding to* $k = 0$ *has to be treated separately.*

Remark 6 (Realization of the diagonal blocks)**.** *The application of the preconditioner* $\mathcal{C}$ *as given in (20) requires an robust and efficient evaluation of the inverse applied to a given vector. Observe that the diagonal blocks* $D := \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\mathbf{k}\boldsymbol{\omega},\boldsymbol{\sigma}}$ *of* $\mathcal{C}$ *are the stiffness matrices representing the bilinear form* $d(y,v) = (\nu\nabla y, \nabla v)_{L_2(\Omega)} + k\omega(\sigma y, v)_{L_2(\Omega)}$. *For the practical realization, these diagonal blocks of the theoretical preconditioner* $\mathcal{C}$ *are replaced by appropriate, easily realizable preconditioners* $\tilde{D}$ *such that* $\tilde{D} \sim D$, *using the following notation:*

Let $M, N \in \mathbb{R}^{n\times n}, n \in \mathbb{N}$ *be two symmetric matrices. Then* $M \sim N$ *if and only if there exist constants* $\underline{c}, \overline{c}$ *such that*

$$\underline{c}\mathbf{x}^T M\mathbf{x} \leq \mathbf{x}^T N\mathbf{x} \leq \overline{c}\mathbf{x}^T M\mathbf{x} \quad \forall\mathbf{x}\in\mathbb{R}^n.$$

We call M *and* N *spectrally equivalent. If* M *and* N *depend on some parameters, then we additionally assume that the constants* $\underline{c}, \overline{c}$ *are independent of those parameters.*

The construction of such blocks $\tilde{D}$ *can be done by techniques as multigrid, multilevel or domain decomposition methods for the second order elliptic differential operator represented by the bilinear form* $d(\cdot,\cdot)$, *see, e.g.* [4, 18, 21, 17, 6]. *The practical block diagonal preconditioner is then given by*

$$\tilde{\mathcal{C}} = diag\left(\tilde{D}, \tilde{D}, \tilde{D}, \tilde{D}\right).$$

The spectral equivalence of the diagonal blocks $\tilde{D} \sim D$ *implies the spectral equivalence of the preconditioners* $\tilde{\mathcal{C}} \sim \mathcal{C}$ *with the same parameter independent constants. So, the practical block diagonal preconditioner yields again robust convergence rates (see also the discussion in* [13]*).*

5. Numerical results

All the numerical experiments are done for the two-dimensional case ($d = 2$). The computational domain is the unit square, i.e. $\Omega = (0,1)^2$. The problems are discretized using piecewise linear and continuous polynomials on a triangulation of Ω. We perform several experiments for the three cases discussed in Section 4 for various parameter settings ω, k, σ, ν, λ and h. We provide condition numbers of the preconditioned systems and the number of MinRes iteration required for reducing the $\mathcal{C}$-norm of the preconditioned initial residual by a factor 10^{-8}. In all the Tables presented in this Section, l denotes the number of refinements (corresponding to the mesh size $h = 2^{-l}$) and DOF is the total number of degrees of freedom. We perform our numerical test for constant coefficients ν and σ. Due to the scaling argument, we always have $\nu = 1$ and therefore we do not have to test for robustness with respect to ν. Nevertheless, for completeness, we report some numerical results for the unscaled problem.

As discussed in Remark 6, the theoretical preconditioner is not realized exactly. Each application of each block of the preconditioner $\mathcal{C}$ consists of one V-cycle. The coarsest level contains four triangles obtained by connecting the two diagonals and therein an exact solver is applied. We use the symmetric Gauss-Seidel with one pre-smoothing and one post-smoothing step as smoothing sweeps in the V-cycle.

5.1. **Different control and observation domains.** Table 1, 2, 3, 4, 5 and 6 provide numerical results for the problem stated in Subsection 4.1 for various settings of $k\omega\sigma$, λ, ν and h. The observation domain is chosen to be $\Omega_1 = \{(x,y) : (y \geq -x+0.5) \wedge (y \geq x-0.5) \wedge (y \leq -x+1.5) \wedge (y \leq x+0,5)\}$ and the control domain is chosen to be $\Omega_2 = (0.25, 0.75) \times (0.25, 0.75)$. Therefore we have $\Omega_2 \subsetneq \Omega_1 \subsetneq \Omega$. Note that the subdomains Ω_1 and Ω_2 have to be resolved by the mesh.

Table 1. Condition numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	1.16	1.16	1.16	1.16	1.2	1.38	1.09	1.09
3	516	1.18	1.18	1.18	1.18	1.23	1.69	1.09	1.22
4	2 052	1.21	1.21	1.21	1.21	1.24	1.84	1.09	1.09
5	8 196	1.25	1.25	1.25	1.25	1.25	1.45	1.09	1.09
6	32 772	1.3	1.3	1.3	1.3	1.29	1.49	1.09	1.09
7	131 076	1.3	1.3	1.3	1.3	1.29	1.56	1.09	1.09

Table 2. Iteration numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	12	12	12	12	13	12	10	10
3	516	14	13	14	14	16	18	10	10
4	2 052	16	16	16	16	16	22	12	12
5	8 196	16	16	16	16	16	22	12	12
6	32 772	16	16	16	16	16	22	12	12
7	131 076	18	18	18	18	18	24	12	12

Table 3. Condition numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	1.09	1.09	1.36	1.2	1.15	1.15	1.15
3	516	1.09	1.09	1.32	1.23	1.42	1.42	1.42
4	2 052	1.1	1.1	1.45	1.25	1.41	1.41	1.41
5	8 196	1.1	1.1	1.45	1.25	1.28	1.28	1.28
6	32 772	1.1	1.1	1.48	1.29	1.29	1.29	1.29
7	131 076	1.1	1.1	1.53	1.29	1.3	1.3	1.3

5.2. **Observation in H^1-semi norm.** Table 7, 8, 9, 10, 11 and 12 provide numerical results for the problem stated in Subsection 4.2 for various settings of $k\omega\sigma$, λ, ν and h.

TABLE 4. Iteration numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^4	10^8	10^{12}
2	132	10	10	12	13	12	12	12
3	516	10	10	18	16	14	14	14
4	2 052	12	10	22	16	16	16	16
5	8 196	12	12	22	16	16	16	16
6	32 772	12	12	22	16	16	16	16
7	131 076	12	12	24	18	18	18	18

TABLE 5. Condition numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ						
		10^{-3}	10^{-2}	10^{-1}	1	10^4	10^8	10^{12}
2	132	>500	20.21	1.56	1.2	1.22	1.22	1.22
3	516	>500	21.57	1.57	1.23	1.25	1.25	1.25
4	2 052	>500	21.85	1.57	1.25	1.26	1.26	1.26
5	8 196	>500	21.92	1.57	1.25	1.59	1.59	1.59
6	32 772	>500	21.94	1.57	1.29	1.29	1.29	1.29
7	131 076	>500	21.96	1.57	1.29	1.29	1.29	1.29

TABLE 6. Iteration numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ						
		10^{-3}	10^{-2}	10^{-1}	1	10^4	10^8	10^{12}
2	132	103	38	15	13	13	13	13
3	516	163	39	17	16	15	15	15
4	2 052	171	40	17	16	16	16	16
5	8 196	166	40	18	16	16	16	16
6	32 772	157	36	18	16	16	16	16
7	131 076	148	36	18	18	18	18	18

TABLE 7. Condition numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^4	10^8	10^{12}
2	132	2.83	2.83	2.83	2.83	2.91	2.62	2.62	2.62
3	516	2.91	2.91	2.91	2.91	2.96	2.62	2.62	2.62
4	2 052	3.07	3.07	3.07	3.07	3.06	2.62	2.62	2.62
5	8 196	3.2	3.2	3.2	3.2	3.19	2.62	2.62	2.62
6	32 772	3.26	3.26	3.26	3.26	3.24	2.62	2.62	2.62
7	131 076	3.27	3.27	3.27	3.27	3.25	2.62	2.62	2.62

5.3. **Control constraints for Fourier coefficients.** Table 13, 14, 15, 16, 17 and 18 provide numerical results for the problem stated in Subsection 4.3 for various settings of $k\omega\sigma$, λ, ν and h. For the active sets, we choose a randomly distributed set for the cosine active sets. For the sine active sets we choose all degrees of freedom in the upper half of the computational domain.

Table 8. Iteration numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	11	11	11	11	12	13	12	12
3	516	14	14	14	14	14	18	12	12
4	2 052	16	16	16	16	16	24	14	14
5	8 196	16	16	16	16	17	30	14	14
6	32 772	17	17	17	17	17	32	14	14
7	131 076	17	17	17	17	17	33	14	14

Table 9. Condition numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	2.62	2.62	2.62	2.91	2.83	2.83	2.83
3	516	2.62	2.62	2.62	2.96	2.91	2.91	2.91
4	2 052	2.62	2.62	3.32	3.06	3.07	3.07	3.07
5	8 196	2.62	2.62	3.36	3.19	3.2	3.2	3.2
6	32 772	2.62	2.62	3.44	3.24	3.26	3.26	3.26
7	131 076	2.62	2.62	3.67	3.25	3.27	3.27	2.27

Table 10. Iteration numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	12	12	13	12	11	11	11
3	516	12	12	18	14	14	14	14
4	2 052	14	14	24	16	16	16	16
5	8 196	14	14	30	17	16	16	16
6	32 772	14	14	32	17	17	17	17
7	131 076	14	14	33	17	17	17	17

Table 11. Condition numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ							
		10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10^{4}	10^{8}	10^{12}
2	132	>500	67.74	7.01	2.62	2.91	3.05	3.05	3.05
3	516	>500	68.18	7.04	2.77	2.96	3.12	3.12	3.12
4	2 052	>500	68.28	7.24	2.98	3.06	3.15	3.15	3.15
5	8 196	>500	68.47	7.88	3.16	3.19	3.17	3.17	3.17
6	32 772	>500	69.47	8.35	3.23	3.24	3.18	3.18	3.18
7	131 076	>500	70.13	8.57	3.26	3.25	3.19	3.19	3.19

6. Conclusion and Outlook

Summary and conclusion. We demonstrate, that a generalization of the parameter-robust preconditioner for the forward problem can be used for the optimal control as well to obtain almost robust convergence rates for the solution of the discretized optimality system. The convergence rates of the MinRes method applied to the preconditioned saddle point equations is independent of the space and time

TABLE 12. Iteration numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ							
		10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10^{4}	10^{8}	10^{12}
2	132	39	37	26	14	12	12	12	12
3	516	91	63	29	16	14	16	16	16
4	2 052	155	71	29	18	16	16	16	16
5	8 196	186	73	31	18	17	17	17	17
6	32 772	202	73	31	18	17	17	17	17
7	131 076	202	73	29	18	17	17	17	17

TABLE 13. Condition numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	1.17	1.17	1.17	1.17	1.21	1.14	1.09	1.09
3	516	1.2	1.2	1.2	1.2	1.25	1.55	1.09	1.09
4	2 052	1.22	1.22	1.22	1.22	1.4	1.67	1.09	1.09
5	8 196	1.27	1.27	1.27	1.27	1.27	1.81	1.1	1.1
6	32 772	1.29	1.29	1.29	1.29	1.29	1.5	1.1	1.1
7	131 076	1.31	1.31	1.31	1.31	1.3	1.56	1.1	1.1

TABLE 14. Iteration numbers ($\lambda = 1$, $\nu = 1$)

l	DOF	$k\omega\sigma$							
		0	10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	12	12	12	12	14	12	10	10
3	516	14	14	14	14	16	18	10	10
4	2 052	16	16	16	16	16	22	12	12
5	8 196	16	16	16	16	16	22	12	12
6	32 772	16	16	16	16	16	22	12	12
7	131 076	18	18	18	18	18	24	12	12

TABLE 15. Condition numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	1.09	1.09	1.4	1.21	1.17	1.2	1.17
3	516	1.09	1.09	1.33	1.25	1.2	1.2	1.2
4	2 052	1.09	1.09	1.46	1.4	1.22	1.22	1.22
5	8 196	1.1	1.1	1.46	1.27	1.27	1.27	1.27
6	32 772	1.1	1.1	1.5	1.29	1.29	1.29	1.29
7	131 076	1.1	1.1	1.56	1.3	1.31	1.31	1.31

discretization parameters h and $k\omega$ as well as the model parameters ν and σ of the state equation. Additionally, we gain robustness with respect to modifications in the minimization functional $\mathcal{J}$, the observation and control domains Ω_1 and Ω_2 and the presence of control constraints to the Fourier coefficients. Despite that, we have to pay the price, that the condition number depends on the cost coefficient λ.

Table 16. Iteration numbers ($k\omega\sigma = 1$, $\lambda = 1$)

l	DOF	ν						
		10^{-12}	10^{-8}	10^{-4}	1	10^{4}	10^{8}	10^{12}
2	132	10	10	12	14	12	12	12
3	516	10	10	18	16	14	14	14
4	2 052	12	12	22	16	16	16	16
5	8 196	12	12	22	16	16	16	16
6	32 772	12	12	22	16	16	16	16
7	131 076	12	12	24	18	18	18	18

Table 17. Condition numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ						
		10^{-3}	10^{-2}	10^{-1}	1	10^{4}	10^{8}	10^{12}
2	132	>500	17.22	1.53	1.21	1.24	1.34	1.24
3	516	>500	13.77	1.45	1.25	1.27	1.27	1.27
4	2 052	>500	13.65	1.44	1.4	1.28	1.28	1.28
5	8 196	>500	12.26	1.43	1.27	1.28	1.28	1.28
6	32 772	>500	12.21	1.42	1.29	1.29	1.29	1.29
7	131 076	>500	12.18	1.42	1.23	1.3	1.3	1.3

Table 18. Iteration numbers ($k\omega\sigma = 1$, $\nu = 1$)

l	DOF	λ						
		10^{-3}	10^{-2}	10^{-1}	1	10^{4}	10^{8}	10^{12}
2	132	107	36	15	14	12	12	12
3	516	174	38	17	16	15	15	15
4	2 052	176	39	17	16	16	16	16
5	8 196	172	37	17	16	16	16	16
6	32 772	158	37	18	16	16	16	16
7	131 076	154	35	18	18	18	18	18

Hence, for very small λ the convergence-rate deteriorates to 1. This behavior can also be seen in the numerical experiments.

Altogether the presented MinRes solver shows great potential towards the robust solution of practical parabolic time-periodic optimal control problems.

Outlook. The preconditioned MinRes solvers presented in this paper can be generalized to eddy current optimal control problems studied in [16], as well as to symmetrically coupled finite element and boundary element discretization, as done in [15]. The treatment of non-linear parabolic problems, i.e. $\nu = \nu(\mathbf{x}, |\nabla y|)$, and the treatment of control constraints of the form $\underline{u} \leq u(\mathbf{x}, t) \leq \overline{u}$ in $\Omega \times (0, T)$ as well as state constraints is more involved. These nonlinearities lead to a coupling of the Fourier coefficients (see [2, 3] for eddy current problems). Anyway, the preconditioners proposed in this paper can also be useful for these cases.

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