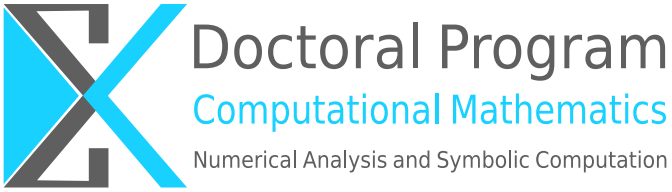

A Non-Standard Finite Element Method based on Boundary Integral Operators

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DK-Report No. 15 12 2011

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by

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A Non-Standard Finite Element Method based on Boundary Integral Operators

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Abstract. This paper provides an overview over our results on the construction and analysis of a non-standard finite element method that is based on the use of boundary integral operators for constructing the element stiffness matrices. This approach permits polyhedral element shapes as well as meshes with hanging nodes. We consider the diffusion equation and convection-diffusion-reaction problems as our model problems, but the method can also be generalized to more general problems like systems of partial differential equations. We provide a rigorous H^1- and L_2-error analysis of the method for smooth and non-smooth solutions. This a priori discretization error analysis is only done for the diffusion equation. However, our numerical results also show good performance of our method for convection-dominated diffusion problems.

Keywords: non-standard FEM, boundary integral operators, Trefftz method, polyhedral meshes, convection-diffusion-reaction problems

1 Introduction

We consider and analyze a non-standard finite element method (FEM) that was introduced by Copeland, Langer, and Pusch [4] and that is based on element-local boundary integral operators. This non-standard FEM permits polyhedral element shapes as well as meshes with hanging nodes. The method employs elementwise PDE-harmonic trial functions, and can thus be interpreted as a local Trefftz method. Indeed, E. Trefftz proposed to approximate the solution of the Dirichlet problem for the Laplace equation by means of a finite superposition of harmonic functions with unknown coefficients which are chosen in such a way that the Dirichlet boundary conditions are approximated in some weak sense [20]. In our approach, we do not explicitely use PDE-harmonic trial functions for computing the element stiffness matrices. Instead these local stiffness matrices are generated by a local boundary element method (BEM) since the intergrals over the polyhedral elements can be transformed to boundary integrals. This is

the reason why we call this method BEM-based FEM. This construction principle requires the explicit knowledge of the fundamental solution of the partial differential operator, but only locally in every polyhedral element. This allows us to solve PDEs with elementwise constant coefficients. The BEM-based FEM has its historical root not only in the Trefftz method, but also in the symmetric boundary domain decomposition method proposed by Hsiao and Wendland [12]. Moreover, the idea of decomposing the computational domain into smaller subdomains can already be found in Trefftz' paper and is of course the central topic of Hsiao's and Wendland's paper on domain decomposition methods.

In this paper we consider the diffusion equation and convection-diffusion-reaction problems as our model problems, but the method can also be generalized to more general problems like the Helmholtz equation and systems of PDEs including the linear elasticity system and the time-harmonic Maxwell equations with elementwise constant or, at least, elementwise smooth coefficients [4,3]. We review the results of the papers [9] and [8] which provided a rigorous H^1- and L_2-error analysis, respectively. This analysis is not trivial since the geometric properties of the polyhedral elements and variational crimes arising from the approximation of the Steklov-Poincaré operator must be handled appropriately. We generalize these discretization error estimates to non-smooth solutions by means of space interpolation technique. We mention that non-smooth solutions are typical for PDE problems with jumping coefficients. This a priori discretization error analysis is only done for the diffusion equation. However, first experiments published in [10] have already shown very good numerical performance for convection-dominated diffusion problems as well.

2 The BEM-based FEM for Convection-Diffusion-Reaction Problems

2.1 Skeletal variational formulation

We follow the approach taken in [9] for the Laplace equation in order to derive the so-called skeletal variational formulation for a convection-diffusion-reaction problem. We consider the Dirichlet boundary value problem

$$\begin{aligned} Lu = -\mathrm{div}(A\nabla u) + b\cdot\nabla u + cu = 0 \quad &\text{in } \Omega, \\ u = g \quad &\text{on } \partial\Omega \end{aligned} \tag{1}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with the boundary $\partial\Omega$ as our model problem. Here $A(x) \in \mathbb{R}^{3\times 3}$, $b(x) \in \mathbb{R}^3$, and $c(x) \in \mathbb{R}$ are the coefficient functions of the partial differential operator L, and g is the given Dirichlet data in $H^{1/2}(\partial\Omega)$. We assume that $A(x)$ is symmetric and uniformly positive, and that $c(x) \geq 0$. While we restrict ourselves to the pure Dirichlet problem here, the generalization to mixed Dirichlet/Neumann problems is straightforward.

The corresponding variational formulation reads as follows: find $u \in H^1(\Omega)$ with $\gamma_\Omega^0 u = g$ such that

$$\int_\Omega (A\nabla u\cdot\nabla v + b\cdot\nabla u\, v + cuv)\, dx = 0 \quad \forall\, v \in H_0^1(\Omega), \tag{2}$$

where $\gamma_\Omega^0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ refers to the Dirichlet trace operator from the domain Ω to its boundary and $H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma_\Omega^0 v = 0\}$. We require that the coefficients A, b, c are $L^\infty(\Omega)$ and that there exists a unique solution of (2).

Assume now that we are given a finite decomposition $\mathcal{T}$ of Ω into mutually disjoint Lipschitz polyhedra. As opposed to a standard finite element method, we do not require the existence of a reference element to which all elements $T \in \mathcal{T}$ can be mapped, but rather allow $\mathcal{T}$ to contain an arbitrary mixture of polyhedral element shapes. For reasons that will become clear later on, we require the coefficients $A(x)$, $b(x)$, and $c(x)$ to be piecewise constant with respect to the polyhedral mesh $\mathcal{T}$.

From the variational formulation and the density of $C_0^\infty(T)$ in $L_2(T)$, it follows that $\operatorname{div}(A\nabla u) = b \cdot \nabla u + cu \in L_2(T)$ for every element $T \in \mathcal{T}$, and we can conclude that the flux $A\nabla u$ is in $H(\operatorname{div}, T)$. Let n_T denote the outward unit normal vector on ∂T. Then the flux has a well-defined Neumann trace $\gamma_T^1 u := A\nabla u \cdot n_T \in H^{-1/2}(\partial T)$, also called the *conormal derivative* of u; cf. [6]. Moreover, we have the generalized Green's identity

$$\int_T A\nabla u \cdot \nabla v \, dx \;=\; -\int_T \operatorname{div}(A\nabla u) v \, dx + \langle \gamma_T^1 u, \, \gamma_T^0 v \rangle \qquad \forall v \in H^1(T). \tag{3}$$

Here and below, $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $H^{-1/2}(\partial T) \times H^{1/2}(\partial T)$ (the particular element boundary ∂T will always be clear by context). Inserting (3) into (2) and remembering that $Lu = 0$ in $L_2(T)$, we obtain

$$0 = \sum_{T\in\mathcal{T}} \int_T (A\nabla u \cdot \nabla v + b \cdot \nabla u \, v + cuv) \, dx = \sum_{T\in\mathcal{T}} \Big(\underbrace{\int_T Lu \, v \, dx}_{=0} + \langle \gamma_T^1 u, \, \gamma_T^0 v \rangle \Big).$$

Fix now some element $T \in \mathcal{T}$ and observe that $u|_T$ is the unique solution of the local problem

$$L\varphi = 0, \qquad \gamma_T^0 \varphi = \gamma_T^0 u.$$

Here again we must require that these local problems do indeed have unique solutions. If we denote by $S_T : H^{1/2}(\partial T) \to H^{-1/2}(\partial T)$ the *Steklov-Poincaré operator* or *Dirichlet-to-Neumann map* for this local problem, we therefore have $\gamma_T^1 u = S_T \gamma_T^0 u$, yielding the equation

$$\sum_{T\in\mathcal{T}} \langle S_T \gamma_T^0 u, \, \gamma_T^0 v \rangle = 0 \qquad \forall v \in H_0^1(\Omega).$$

The above formulation operates not on the functions u and v themselves, but only on their Dirichlet traces on the element boundaries ∂T. The idea is therefore to work with function spaces which are defined only on these boundaries. We call $\Gamma_S := \bigcup_{T\in\mathcal{T}} \partial T$ the *skeleton* of the mesh $\mathcal{T}$ and introduce a skeletal function space $W = H^{1/2}(\Gamma_S)$ consisting of the traces of all functions from $H^1(\Omega)$ on

Γ_S. We then search for a skeletal function $\hat{u} \in W$ which satisfies the Dirichlet boundary condition $\hat{u}|_{\partial\Omega} = g$ as well as the skeletal variational formulation

$$\sum_{T\in\mathcal{T}} \langle S_T \hat{u}|_{\partial T},\, \hat{v}|_{\partial T}\rangle = 0 \qquad \forall\, \hat{v} \in W_0 = \{\hat{v} \in W : \hat{v}|_{\partial\Omega} = 0\}\,. \tag{4}$$

This skeletal variational formulation is equivalent to the standard variational formulation (2) in the sense that the traces $\hat{u}|_{\partial T} \in H^{1/2}(\partial T)$ obtained from (4) match the traces $\gamma_T^0 u$ of the function $u \in H^1(\Omega)$ obtained from (2). Conversely, $u|_T$ can be recovered from $\hat{u}|_{\partial T}$ by solving a local Dirichlet problem in T, i.e., $u|_T = \mathcal{H}_T \hat{u}|_{\partial T}$ with the element-local harmonic extension operator $\mathcal{H}_T : H^{1/2}(\partial T) \to H^1(T)$. This motivates using (4) as a starting point for discretization in order to obtain a numerical method for (1).

We remark that another interpretation of (4) is that of a weak enforcement of the continuity of conormal derivatives on inter-element boundaries.

2.2 Boundary Integral Operators

Evaluating the Dirichlet-to-Neumann map S_T used above essentially corresponds to solving a local problem $L\varphi = 0$ on T with the given Dirichlet data and then obtaining the conormal derivative $\gamma_T^1 \varphi$ of its solution. These local problems are in general not analytically solvable, and we approximate their solutions by the Galerkin discretization of element-local boundary integral equations. In the following, we outline some standard results on boundary integral operators. A more detailed treatment of these topics can be found in, e.g., [13,15,17,18].

We say that a *fundamental solution* of the partial differential operator L is a function $G(x, y)$ such that $L_x G(x, y) = \delta(y - x)$, where δ is the Dirac δ-distribution and $x,\, y \in \mathbb{R}^d$. Fundamental solutions for L from (1) with constant coefficients A, b, c are given in [17]; in particular, in $\mathbb{R}^3$ and under the assumption $c + \|b\|_{A^{-1}}^2 \geq 0$, we have

$$G(x-y) = G(z) = \frac{1}{4\pi\sqrt{\det A}} \frac{\exp\left(b^\top A^{-1} z - \lambda \|z\|_{A^{-1}}\right)}{\|z\|_{A^{-1}}}\,,$$

where $\|x\|_{A^{-1}} = \sqrt{x^\top A^{-1} x}$ and $\lambda = \sqrt{c + \|b\|_{A^{-1}}^2}$.

More generally, we will assume that the coefficients A, b, c are constant only within each element. This leads to a potentially different fundamental solution in each element T, in the following denoted by $G_T(x, y)$, and allows us to treat PDEs with piecewise constant coefficients.

We now introduce the boundary integral operators

$$\begin{aligned} V_T &: H^{-1/2}(\partial T) \to H^{1/2}(\partial T), \quad K_T : H^{1/2}(\partial T) \to H^{1/2}(\partial T), \\ K_T' &: H^{-1/2}(\partial T) \to H^{-1/2}(\partial T), \quad D_T : H^{1/2}(\partial T) \to H^{-1/2}(\partial T), \end{aligned}$$

called, in turn, the *single layer potential*, *double layer potential*, *adjoint double layer potential*, and *hypersingular* operators. For sufficiently regular arguments,

they admit the integral representations

$$
\begin{aligned}
(V_T v)(y) &= \int_{\partial T} G_T(x,y) v(x)\, ds_x, \\
(K_T u)(y) &= \int_{\partial T} \widetilde{\gamma^1_{T,x}} G_T(x,y) u(x)\, ds_x, \\
(K'_T v)(y) &= \int_{\partial T} \gamma^1_{T,y} G_T(x,y) v(x)\, ds_x, \\
(D_T u)(y) &= -\gamma^1_{T,y} \int_{\partial T} \widetilde{\gamma^1_{T,x}} G_T(x,y) \big(u(x) - u(y)\big)\, ds_x,
\end{aligned}
$$

where $\gamma^1_{T,y}$ refers to the conormal derivative γ^1_T with respect to the variable y, and $\widetilde{\gamma^1_{T,x}}$ refers to the modified conormal derivative

$$
\widetilde{\gamma^1_T} u = \gamma^1_T u + (b \cdot n_T) \gamma^0_T u,
$$

associated with the adjoint problem, with respect to the variable x.

In the literature, we find two representations of the Steklov-Poincaré operator in terms of the boundary integral operators, namely,

$$
S_T \;=\; V_T^{-1}(\tfrac{1}{2} I + K_T) \;=\; D_T + (\tfrac{1}{2} I + K'_T) V_T^{-1} (\tfrac{1}{2} I + K_T). \tag{5}
$$

The first one is called the non-symmetric representation, and the latter is called the symmetric representation of the Steklov-Poincaré operator.

2.3 Discretization

Discretization of the Skeletal Function Space. We employ a Galerkin approach to the discretization of the skeletal variational formulation (4). To this end, we first discretize every element boundary ∂T by a conforming triangulation $\mathcal{F}_T$ composed of triangles. The number of triangular boundary elements per polyhedral element should be uniformly bounded. Furthermore, we assume that the element discretizations are *matching* in the sense that, for any two elements T_1 and T_2 having a common interface $\Gamma_{12} = \overline{T}_1 \cap \overline{T}_2 \neq \emptyset$, any triangle $\tau \in \mathcal{F}_{T_1}$ with $\tau \cap \Gamma_{12} \neq \emptyset$ should also belong to $\mathcal{F}_{T_2}$. In other words, inter-element boundaries must be triangulated identically in both elements. Finally, we require that the element meshes $\mathcal{F}_T$ are quasi-uniform and shape-regular in the usual sense, with constants which are uniform over all elements T.

This construction naturally gives us a triangulation $\mathcal{F} = \bigcup_{T \in \mathcal{T}} \mathcal{F}_T$ of the skeleton, on which we now construct a discrete trial space $W^h \subset W$ of piecewise (per boundary triangle $\tau \in \mathcal{F}$) linear and continuous functions. The space W^h is spanned by the nodal functions $\{\phi_i\}$, where i enumerates the vertices of the triangulation $\mathcal{F}$, and where each basis function ϕ_i has the value 1 in the i-th vertex, 0 in all other vertices, and is linear on each triangle $\tau \in \mathcal{F}$. The space $W^h_0 = W^h \cap W_0$ is constructed from W^h by excluding those basis functions which belong to vertices on $\partial\Omega$.

Assuming that the given Dirichlet data g is piecewise linear, we thus arrive at the following Galerkin equations as the discrete version of (4): find $u^h \in W^h$ such that $u^h|_{\partial\Omega} = g$ and

$$\sum_{T\in\mathcal{T}} \langle S_T u^h|_{\partial T},\, v^h|_{\partial T}\rangle = 0 \qquad \forall\, v^h \in W_0^h. \tag{6}$$

In the general case, the Dirichlet data g can be approximated piecewise linearly by interpolation (if continuous) or by L_2 projection onto the boundary triangles.

Discretization of the Dirichlet-to-Neumann Map. The boundary integral operator representation (5) of S_T contains the inverse of the single layer potential operator V_T, which is in general not computable exactly. Hence, we need to approximate the bilinear form $\langle S_T\cdot,\,\cdot\rangle$. To do this, we employ a mixed piecewise linear/piecewise constant scheme, where Dirichlet data is approximated piecewise linearly, while Neumann data is approximated by piecewise constants, as described in, e.g., [5,18,11].

Observe that, due to (5), the Neumann data corresponding to a function $u_T \in H^{1/2}(\partial T)$ can be written as

$$S_T u_T = D_T u_T + (\tfrac{1}{2}I + K_T')t_T \tag{7}$$

with $t_T = V_T^{-1}(\frac{1}{2}I + K_T)u_T \in H^{-1/2}(\partial T)$. Again referring to (5), we see that $t_T = S_T u_T$ is itself already the sought Neumann data. We will approximate S_T by first approximating t_T in a suitable space of piecewise constant functions and then substituting this approximation for t_T in (7).

Let $\{\phi_{T,i}\}$ denote the nodal piecewise linear functions restricted to the local mesh $\mathcal{F}_T$, where now i enumerates the vertices only of $\mathcal{F}_T$. Furthermore, introduce a space of piecewise constant boundary functions Z_T^h spanned by the basis $\{\psi_{T,k}\}$, where k enumerates the triangles $\tau \in \mathcal{F}_T$, such that $\psi_{T,k} \equiv 1$ on the k-th triangle and $\psi_{T,k} \equiv 0$ on all other triangles.

For any function $u_T \in H^{1/2}(\partial T)$, we compute the approximation $t_T^h \approx t_T$ of its Neumann data by the Galerkin projection of the equation $V_T t_T = (\frac{1}{2}I + K_T)u_T$ to the piecewise constant functions. In other words, we seek $t_T^h \in Z_T^h$ satisfying the variational equation

$$\langle \psi^h,\, V_T t_T^h\rangle = \langle \psi^h,\, (\tfrac{1}{2}I + K_T)u_T\rangle \qquad \forall\, \psi^h \in Z_T^h.$$

A computable approximation to $S_T u_T$ is then given by

$$\widetilde{S}_T u_T := D_T u_T + (\tfrac{1}{2}I + K_T')t_T^h,$$

and our fully discretized variational formulation takes the following form: find $u^h \in W^h$ such that $u^h|_{\partial\Omega} = g$ and

$$\sum_{T\in\mathcal{T}} \langle \widetilde{S}_T u^h|_{\partial T},\, v^h|_{\partial T}\rangle = 0 \qquad \forall\, v^h \in W_0^h. \tag{8}$$

The corresponding stiffness matrix is assembled, as in the FEM, from the contributions from element stiffness matrices. The latter are computed using the fully numerical integration technique described by Sauter and Schwab [17].

3 Discretization Error Analysis for the Diffusion Equation

There is no complete error analysis of the presented method for general elliptic operators L of the form (1). However, for the special case of the Laplace operator, rigorous error estimates in the H^1 and L_2 norms are given in [9] and [8], respectively. Even in the Laplacian case, the analysis is nontrivial. Among the main technical hurdles are the general polyhedral element shapes, which means that no reference element is available. This rules out the standard technique widely used in FEM analysis of transforming to the reference element, estimating a quantity of interest there, and transforming back. In particular, this complicates the derivation of Dirichlet and Neumann approximation properties for our discrete skeletal function spaces. An auxiliary regular tetrahedral decomposition of each polyhedral element T is used as an analytical tool to overcome these difficulties. Furthermore, the approximation of S_T by $\widetilde{S}_T$ is a "variational crime" leading to a consistency error which has to be treated either by Strang's lemma [9] or by passing to an equivalent mixed formulation [8]. In both cases, estimating the consistency error requires novel results on explicit bounds for boundary integral operators [16] and a notion of regularity for polyhedral elements which relies on uniform bounds for Poincaré constants and the so-called Jones parameter. The regularity assumptions used in the previous works are outlined below.

Assumption 1 *We assume that the polyhedral mesh $\mathcal{T}$ satisfies the following conditions.*

- *There is a small, fixed integer uniformly bounding the number of boundary triangles of every element.*
- *Every element $T \in \mathcal{T}$ has an auxiliary conforming, quasi-regular, tetrahedral triangulation (cf. [2]) with regularity parameters which are uniform across all elements.*

Definition 1 (Uniform domain [14]). *A bounded and connected set $D \subset \mathbb{R}^d$ is called a* uniform domain *if there exists a constant C_U such that any pair of points $x_1 \in D$ and $x_2 \in D$ can be joined by a rectifiable curve $\gamma(t) : [0, 1] \to D$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, such that the arc length of γ is bounded by $C_U|x_1 - x_2|$ and*

$$\min_{i=1,2} |x_i - \gamma(t)| \leq C_U \operatorname{dist}(\gamma(t), \partial D) \qquad \forall t \in [0, 1].$$

If D is a uniform domain, we denote the smallest such constant C_U by $C_U(D)$ and call it the Jones parameter *of D.*

Any Lipschitz domain is a uniform domain. However, its Jones parameter may be arbitrarily large.

Definition 2 (Poincaré constant). *For a uniform domain D, let $C_P(D)$ be the smallest constant such that*

$$\inf_{c \in \mathbb{R}} \|v - c\|_{L_2(D)} \leq C_P(D) \operatorname{diam}(D) |v|_{H^1(D)} \qquad \forall v \in H^1(D).$$

For convex domains D, one can show that $C_P(D) \le \pi^{-1}$, cf. [1]. Estimates for star-shaped domains can be found in [19,21].

Assumption 2 *We assume that there are constants $C_U^* > 0$ and $C_P^* > 0$ such that, for all $T \in \mathcal{T}$,*

$$C_U(T) \le C_U^*, \qquad C_U(B_T \setminus \overline{T}) \le C_U^*,$$
$$C_P(T) \le C_P^*, \qquad C_P(B_T \setminus \overline{T}) \le C_P^*,$$

where B_T is a ball (or a suitable Lipschitz domain) enclosing T which satisfies $\operatorname{dist}(\partial B_T, \partial T) \ge \frac{1}{2} \operatorname{diam}(T)$.

Under the assumptions stated above, we may now formulate the main results on the discretization error. Note that the discrete solution $u^h \in W^h$ of (8) is only defined on the skeleton. In order to compare it with the exact solution $u_\Omega \in H^1(\Omega)$ of (2) with traces $u \in W$, we use the harmonic extension operator $\mathcal{H}_T$. Within any element T, the error to be estimated is thus

$$u_\Omega - \mathcal{H}_T(u^h|_{\partial T}) = \mathcal{H}_T\left((u - u^h)|_{\partial T}\right),$$

or globally $\mathcal{H}_S(u - u^h)$ with the piecewise harmonic extension operator

$$\mathcal{H}_S : H^{1/2}(\Gamma_S) \to H^1(\Omega), \qquad \forall T \in \mathcal{T} : (\mathcal{H}_S v)|_T = \mathcal{H}_T(v|_{\partial T}).$$

Theorem 1 (H^1 error estimate, [9]). *Let $L = -\Delta$, and let the mesh $\mathcal{T}$ satisfy Assumptions 1 and 2. Assume further that the given Dirichlet data g is piecewise linear and that the exact solution u_Ω of (2) lies in $H^2(\Omega)$. With $u \in W$ the solution of the skeletal variational formulation (4) and $u^h \in W^h$ the solution of the discretized skeletal formulation (8), we have the error estimate*

$$|\mathcal{H}_S(u - u^h)|_{H^1(\Omega)} \le C \Big(\sum_{T \in \mathcal{T}} h_T^2 \, |u_\Omega|^2_{H^2(T)} \Big)^{1/2} \le C \, h \, |u_\Omega|_{H^2(\Omega)},$$

where the constant C depends only on the mesh regularity parameters, $h_T = \operatorname{diam} T$ denotes the element diameters, and $h = \max_T h_T$ denotes the mesh size.

Proof (Outline). The proof hinges on three results, namely, (i) a quasi-optimal bound for the discretization error in terms of the best approximation error for the exact Dirichlet and Neumann data, (ii) an approximation error estimate for the Dirichlet data, and (iii) an approximation error estimate for the Neumann data. All of these estimates need to be made explicit in terms of the mesh regularity parameters, and problem-adapted norms have to be used. The details may be found in [9].

Theorem 2 (L^2 error estimate, [8]). *Let the assumptions of Theorem 1 be satisfied. Assume further that the adjoint problem is H^2-coercive, i.e., that the solution $w \in H_0^1(\Omega)$ of*

$$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Omega \mathcal{H}_S(u - u^h) \, v \, dx \qquad \forall v \in H_0^1(\Omega)$$

lies in $H^2(\Omega)$ and satisfies the estimate

$$|w|_{H^2(\Omega)} \leq C \, \|\mathcal{H}_S(u - u^h)\|_{L_2(\Omega)}.$$

Then we have the quasi-optimal L_2 discretization error estimate

$$\|\mathcal{H}_S(u - u^h)\|_{L_2(\Omega)} \leq C \, h^2 \, |u_\Omega|_{H^2(\Omega)},$$

where the constant C depends only on the mesh regularity parameters.

Proof (Outline). Due to the consistency error introduced by approximating the bilinear form $\langle S_T \cdot, \cdot \rangle$, Galerkin orthogonality is violated and the usual Aubin-Nitsche technique is not available. This may be remedied by passing to an equivalent mixed formulation, searching for the unknowns $(u, \bigotimes_{T \in \mathcal{T}} t_T)$, which makes the error by the approximation $\widetilde{S}_T \approx S_T$ explicit and thus restores Galerkin orthogonality. Many of the technical tools used to prove Theorem 1 can then be reused. The details may be found in [8].

In many practical applications, we have to deal with heterogeneous coefficients. If the coefficients in the PDE (1) have jumps across interfaces, or if the computational domain is non-convex and non-smooth, or if there are changes in the boundary conditions, then we cannot expect a smooth solution u_Ω, i.e., the assumption $u_\Omega \in H^2(\Omega)$ is too restrictive. The following theorem provides a convergence rate estimate for this case too.

Theorem 3 (H^1 error estimate under reduced regularity assumptions). *Let the assumptions of Theorem 1 hold with the exception of the regularity assumption imposed on the exact solution, $u_\Omega \in H^2(\Omega)$. Instead, we only assume $u_\Omega \in H^{1+s}(\Omega)$ with some $0 < s \leq 1$. Then we have the error estimate*

$$\|\mathcal{H}_S(u - u^h)\|_{H^1(\Omega)} \leq C \, h^s \, \|u_\Omega\|_{H^{1+s}(\Omega)},$$

where the constant C again depends only on the mesh regularity parameters.

Proof (Outline). Theorem 1 together with Friedrichs' inequality implies

$$\|\mathcal{H}_S(u - u^h)\|_{H^1(\Omega)} \leq C \, h \, \|u_\Omega\|_{H^2(\Omega)}.$$

From the proof of Theorem 1, one easily obtains the stability estimate

$$\|\mathcal{H}_S(u - u^h)\|_{H^1(\Omega)} \leq C \, \|u_\Omega\|_{H^1(\Omega)}.$$

The statement then follows by a space interpolation argument (cf. [15]).

4 Numerical Results for a Convection-Diffusion Problem

The following example with strongly varying diffusion coefficients is adapted from [7]. We solve

$$\begin{aligned} -\mathrm{div}(A \nabla u) + (\beta, \beta, \beta)^\top \cdot \nabla u = 0 \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = (0,1)^3$ is the unit cube, $\beta = 50$, and

$$A(x,y,z) = \begin{cases} 10^4, & \frac{1}{3} < x,y,z < \frac{2}{3}, \\ 1, & \text{else}, \end{cases} \qquad g(x,y,z) = \begin{cases} 1, & z = 0, \\ 0, & \text{else}. \end{cases}$$

The non-symmetric linear systems resulting from either method are solved using GMRES, restarted every 500 iterations. Due to the strongly varying diffusion, GMRES takes a very high number of iterations to converge to a given accuracy, even on relatively small problems. A simple row scaling preconditioner can mitigate this problem ([7]), and we thus modify the linear system to be solved,

$$\underline{S}\,\underline{u}^h = \underline{f}^h \qquad \longrightarrow \qquad D\underline{S}\,\underline{u}^h = D\underline{f}^h,$$

where $\underline{S}$ is the stiffness matrix, $\underline{u}^h$ and $\underline{f}^h$ are the vectors corresponding to the discrete solution and right-hand side, respectively, and

$$D = \operatorname{diag}(1/\|\underline{S}_1\|_p, \ldots, 1/\|\underline{S}_n\|_p)$$

is a diagonal matrix containing the reciprocal p-norms of the rows of the stiffness matrix $\underline{S}$. In our experiments, we chose $p = 1$.

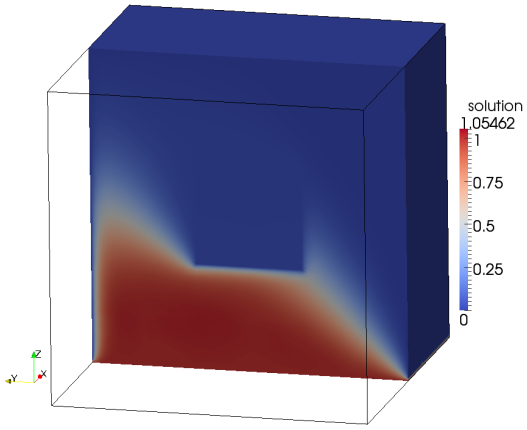

Fig. 1. Cross section through Ω at $x = 0.5$, computed by BEM-based FEM.

Figure 1 shows a solution computed using the BEM-based FEM for a mesh with 456 769 vertices and mesh size $h \approx 0.0232924$. Table 1 displays GMRES iteration numbers for a standard FEM using piecewise linear trial functions and the BEM-based FEM, without and with the row scaling preconditioner.

		degrees of freedom				
		199	1153	7921	59041	456769
without row scaling	FEM	59	178	469	6042	19597
	BEM-based FEM	57	123	442	5356	15756
with row scaling	FEM	42	87	135	229	440
	BEM-based FEM	50	87	129	215	376

Table 1. Iteration numbers using GMRES(500), without and with row scaling.

5 Conclusion and Outlook

We have summarized recent results on the so-called BEM-based FEM, giving a brief derivation of the method for a general elliptic partial differential equation and outlining the analysis in the Laplacian case for solutions of full as well as reduced regularity. We also have given new numerical results for a convection-diffusion benchmark problem.

The employed solution technique, namely, GMRES with a simple row scaling preconditioner, is clearly not optimal, and the development of optimal solvers will be a topic of future research. Furthermore, previous experiments in [10] have shown that, while stability of the method is superior to a standard piecewise linear FEM, additional stabilization is required for convection-dominated problems.

Acknowledgments. The support by the Austrian Science Fund (FWF) under grant DK W1214 is gratefully acknowledged.

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