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Tomographic Reconstruction of Harmonic Functions

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I. Georgieva[*] C. Hofreither[†]

Abstract

We consider an algebraic method for reconstruction of harmonic functions via a finite number of values of its Radon projections. More precisely, for given values of some Radon projections, we seek a harmonic polynomial which matches these data exactly. In the present work, we focus mostly on the case where these measurements are taken along equally spaced chords of the unit circle. We present an efficient reconstruction algorithm which is robust with respect to noise in the input data and provide numerical examples.

1 Introduction

The Radon transform, named after Johann Radon who studied it in the early twentieth century, is the theoretical foundation for tomography methods for shape reconstruction of objects with non-homogeneous density. These methods were intensively studied in the 1960s and continue to find many applications in medicine, electronic microscopy, geology, biology, materials science, radiology, plasma investigations, finding defects in nuclear reactors, etc. Modern methods of tomography involve gathering projection data from multiple directions and applying this data into a tomographic reconstruction software algorithm processed by a computer. Generally, the output from these reconstruction procedures appears as 2D slice images. There exist different reconstruction algorithms: filtered back projection, iterative reconstruction, direct methods, etc. These procedures give inexact results: they represent a compromise between accuracy and computation time required. From the mathematical point of view, the problem is to recover a multivariate function using information given as line integrals of the unknown function.

An idea suggested by B. Bojanov is to incorporate additional knowledge about the function to be recovered into approximation methods. It is to be expected that this can improve the accuracy of the approximation while reducing the amount of input data required as well as the computational effort. In applications, such problem-specific knowledge is often provided in the form of a partial differential equation which the unknown satisfies.

In the present work, we concern ourselves with the simple case where the unknown is harmonic, i.e., satisfies the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$. This elliptic partial differential equation is important both as a model problem as well as in actual applications, like heat transport, diffusion problems or Stokes flow of incompressible fluids.

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2 Preliminaries and related work

Let $I(\theta, t)$ denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Figure 1).

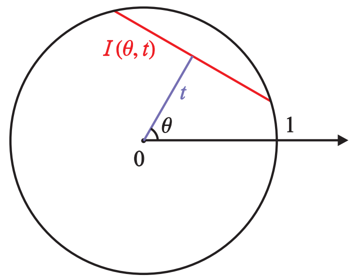


Figure 1: The chord $I(\theta, t)$ of the unit circle.

The chord $I(\theta, t)$ is parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \qquad \text{where } s \in (-\sqrt{1-t^2}, \sqrt{1-t^2}). \tag{1}$$

Definition 1. Let $f(x, y)$ be a real-valued bivariate function in the unit disk in $\mathbb{R}^2$. The *Radon projection* $\mathcal{R}_\theta(f; t)$ of f in direction θ is defined by the line integral

$$\mathcal{R}_\theta(f;t) := \int_{I(\theta,t)} f(\mathbf{x})\, d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

Johann Radon [22] showed in 1917 that a differentiable function f is uniquely determined by the values of its Radon transform,

$$f \;\mapsto\; \left\{\mathcal{R}_\theta(f;t) \,:\, -1 \le t \le 1,\ 0 \le \theta < \pi\right\}.$$

The problem of recovery of a polynomial from a finite number of values of its Radon transform may be viewed as a bivariate interpolation problem where the traditional point values are replaced by the means over chords of the unit circle.

In the following we formulate the problem of recovery of a polynomial from a finite number of values of its Radon transform. Essentially, this may be viewed as a bivariate interpolation problem where the traditional point values are replaced by the means over chords of the unit circle.

Let $\Pi_n^2 = \left\{\sum_{i+j\le n} \alpha_{ij} x^i y^j \,:\, \alpha_{ij} \in \mathbb{R}\right\}$ denote the space of real bivariate polynomials of total degree at most n. This space has dimension $\binom{n+2}{2}$. Assume that a set $\mathcal{I} = \left\{I_m = I(\theta_m, t_m) : m = 1, \dots, \binom{n+2}{2}\right\}$ of chords of the unit circle is given. Furthermore, to each chord $I \in \mathcal{I}$ a given value $\gamma_I \in \mathbb{R}$ is associated. Then, the aim is to find a polynomial $p \in \Pi_n^2$ such that

$$\int_I p(\mathbf{x})\, d\mathbf{x} = \gamma_I \qquad \forall I \in \mathcal{I}. \tag{2}$$

If this interpolation problem has a unique solution for every choice of values $\{\gamma_I\}$, then the scheme $\mathcal{I}$ of chords is called *regular*. The question of how to construct such regular schemes has been extensively studied. The first general result was given by Marr [19] in 1974, who proved that the set of chords connecting $n+2$ equally spaced points on the unit circle is regular for Π_n^2. A more general result for $\mathbb{R}^d$ and general convex domains was published by Hakopian [15] in 1982. Applied to the unit disk in $\mathbb{R}^2$, it states that even the chords connecting any $n+2$ distinct points on the unit circle form a regular scheme for Π_n^2.

Another family of regular schemes was provided by Bojanov and Georgieva [2]. They showed that a scheme consisting of $\binom{n+2}{2}$ chords partitioned into $n+1$ subsets such that the k-th subset consists of k parallel chords is regular for Π_n^2, provided that the distances t satisfy some additional conditions. Particular choices of suitable distances t were later given by Georgieva and Ismail [11] in terms of zeroes of Chebyshev polynomials of the second kind, as well as Georgieva and Uluchev [12] in terms of zeroes of Jacobi polynomials.

Bojanov and Xu [5] proposed a regular scheme consisting of $\binom{n+2}{2}$ chords partitioned into $2\lfloor (n+1)/2 \rfloor + 1$ equally spaced directions, such that in every direction there are $\lfloor n/2 \rfloor + 1$ parallel chords. The distances t of the chords are zeroes of Chebyshev polynomials of the second kind.

A mixed regular scheme which incorporates Radon projections and point evaluations on the unit circle was given by Georgieva, Hofreither, and Uluchev [10].

Many other mathematicians have worked on problems with applications in the mathematical foundations of computer tomography, among them [17, 6, 7, 8, 16, 18, 20]. Recovery of polynomials in two variables based on Radon projections is also considered in [1, 21, 3, 4, 13, 14].

3 Description of the method

Assume that we know *a priori* that the function to be interpolated is harmonic. Then it seems natural to work in the space $\mathcal{H}_n$ of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n+1$. Let a set of chords of the unit circle $\mathcal{I} = \{I_1, I_2, \ldots, I_{2n+1}\}$ be given. Furthermore, to each chord $I \in \mathcal{I}$ a given value $\gamma_I \in \mathbb{R}$ is associated. Then, the aim is to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\int_I p(\mathbf{x})\, d\mathbf{x} = \gamma_I \qquad \forall I \in \mathcal{I}. \tag{3}$$

Here the given values γ_I are the chord integrals corresponding to an unknown harmonic function u. The hope is that then p approximates u reasonably well.

We call $\mathcal{I}$ *regular* if the interpolation problem (3) has a unique solution for all given values $\{\gamma_I\}$.

The regular schemes which we work with were constructed with the help of methods from symbolic computation [9], and we briefly present them below.

Theorem 1 ([9]). *Let the chords $\mathcal{I}$ be given by $I_m = I(\theta_m, t)$, where the angles θ_m are equally spaced over the unit circle $(0, 2\pi)$ and $t \in (0,1)$ is not a zero of any Chebyshev polynomial of the second kind $U_1, \ldots, U_n$. Then the interpolation problem* (3) *has a unique solution in $\mathcal{H}_n$ for any given data $\{\gamma_I\}$.*

In fact, in [9], arbitrarily spaced angles θ_m were admitted, however in the present work we stick with equally spaced chords. Some possible configurations which satisfy the assumptions of Theorem 1 are shown in Figure 2.

The proof uses the following basis of the harmonic polynomials,

$$\phi_0(x,y) = 1, \quad \phi_{k,1}(x,y) = \mathrm{Re}(x + \mathbf{i}y)^k, \quad \phi_{k,2}(x,y) = \mathrm{Im}(x + \mathbf{i}y)^k,$$

which is equal, in polar coordinates, to

$$\phi_{k,1}(r,\theta) = r^k \cos(k\theta), \quad \phi_{k,2}(r,\theta) = r^k \sin(k\theta).$$

We have shown the following analogue to Marr's formula [19] in the harmonic case.

Lemma 2. *The Radon projections of the basis harmonic polynomials satisfy*

$$\int_{I(\theta,t)} \phi_{k,1}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\sin(k\theta).$$

Under the assumption of constant $t_m = t$, we can use this lemma to derive the following representation of the system matrix corresponding to (3): $A = CD$, where

$$C = \begin{pmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \cos(2\theta_1) & \sin(2\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(\theta_2) & \sin(\theta_2) & \cos(2\theta_2) & \sin(2\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\theta_{2n}) & \sin(\theta_{2n}) & \cos(2\theta_{2n}) & \sin(2\theta_{2n}) & \dots & \cos(n\theta_{2n}) & \sin(n\theta_{2n}) \\ 1 & \cos(\theta_{2n+1}) & \sin(\theta_{2n+1}) & \cos(2\theta_{2n+1}) & \sin(2\theta_{2n+1}) & \dots & \cos(n\theta_{2n+1}) & \sin(n\theta_{2n+1}) \end{pmatrix},$$

$$D = \mathrm{diag}(\alpha_0, \alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n)$$

with the column factors $\alpha_k = \frac{2}{k+1}\sqrt{1-t^2}U_k(t) > 0$.

The matrix C is the same as for the one-dimensional problem of interpolation with a trigonometric polynomial of degree n in $[0, 2\pi]$ at the points $\{\theta_1, \dots, \theta_{2n+1}\}$. We use the fact that C is invertible if and only if the angles θ_m are pairwise distinct to show that A is invertible in this case.

4 Inversion of the linear system

For the equally spaced angles $\theta_m = \frac{2\pi m}{2n+1}$ and $t \in (0,1)$ such that $U_k(t) \neq 0$ for all $k \in \{0, \dots, n\}$, we use the orthogonality of the columns of C to derive the explicit inverse

$$A^{-1} = \mathrm{diag}(\beta_0, \beta_1, \beta_1, \dots, \beta_n, \beta_n)C^\top,$$

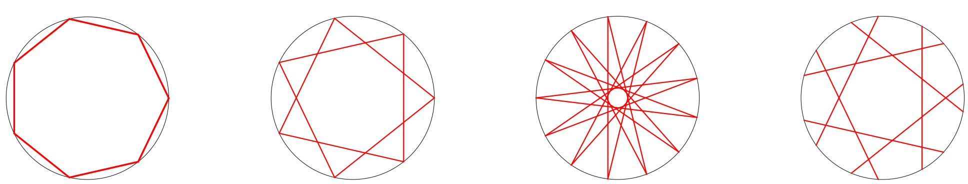


Figure 2: Regular schemes according to Theorem 1.

where the row factors

$$\beta_k = \begin{cases} \frac{1}{2(2n+1)}(\sqrt{1-t^2})^{-1} = \frac{1}{2n+1}\alpha_k^{-1}, & k = 0, \\ \frac{k+1}{2n+1}(\sqrt{1-t^2}U_k(t))^{-1} = \frac{2}{2n+1}\alpha_k^{-1}, & k \geq 1 \end{cases}$$

serve to normalize the orthogonal system formed by the columns of C.

Note that the action of the matrix $C^\top$ is essentially a discrete Fourier transform of the given data. This suggests an efficient algorithm for the solution of the linear system: using a suitable Fast Fourier Transform (FFT), we can compute the coefficients of the interpolating polynomial in slightly worse than linear ($\mathcal{O}(n)$) time. Having such a quasi-optimal solution method available is invaluable for practical large-scale problems.

5 Condition number

With the use of the explicit formula for A^{-1}, it is not difficult to compute the singular values of both A and A^{-1}. This allows us to obtain a uniform bound for the spectral condition number of A, to be precise,

$$\kappa_2(A) \leq 2\sqrt{2},$$

again under the assumption of equally spaced angles and constant t. The significance of this result is that the method is very stable with respect to errors in the input data, i.e., noise in the given measurements results in an error which is of the same order of magnitude. Indeed, our numerical examples confirm this.

6 Numerical Examples

6.1 Example 1

In this example, we restrict ourselves to the case where the chords $\mathcal{I}$ form a regular $(2n+1)$-sided convex polygon inscribed in the unit circle (cf. Figure 2, first picture), i.e., $I_m = I(\theta_m, t_m)$ with

$$\theta_m = \frac{2\pi m}{2n+1}, \quad t_m = t = \cos\frac{\pi}{2n+1} \qquad \text{for } m = 1, \ldots, 2n+1. \tag{4}$$

We approximate the harmonic function

$$u(x, y) = \exp(2y)\cos(2x)$$

by a harmonic polynomial $p \in \mathcal{H}_n$ given $2n+1$ values of its Radon projections taken along the edges of a regular $(2n+1)$-sided convex polygon (Figure 2, first picture), i.e., θ_m and t_m are chosen according to (4). In Figure 3, we display the function u as well as its interpolating polynomial of degree 12 (using information from 25 chords) and the resulting error. For Figure 6.1, we vary the degree of the interpolating polynomial and plot the resulting relative L_2-errors. We see that the error decreases exponentially with n, indicating that the smooth function u is being approximated with optimal order.

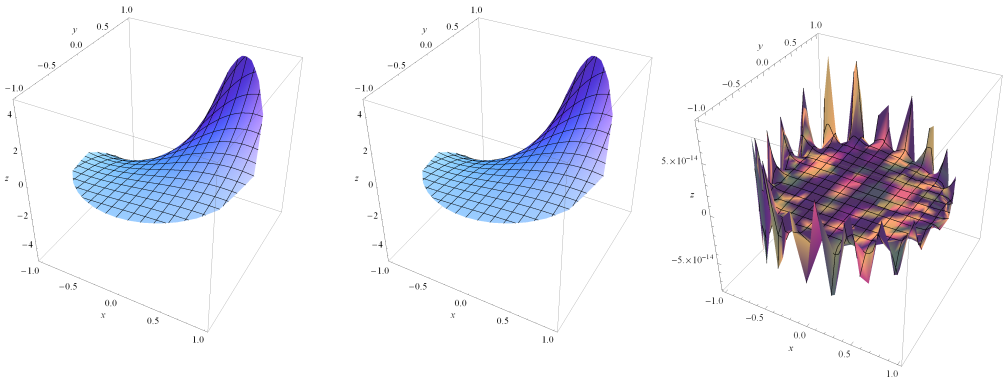


Figure 3: Example 1, $n = 12$: function u, interpolant p, error $u - p$

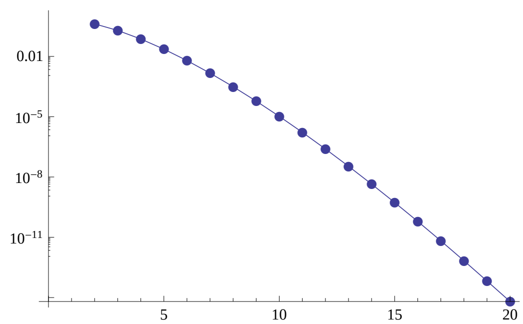


Figure 4: Example 1: errors. x-axis: degree of interpolating polynomial. y-axis: relative L_2-error

6.2 Example 2

We consider the same problem as in Example 1, but with artificially added measurement noise. For this, we add to the given values of the Radon projections random numbers from a normal distribution with zero mean and standard deviation ϵ. We perform three experiments with error levels $\epsilon \in \{10^{-3}, 10^{-6}, 10^{-9}\}$. The resulting relative errors in the reconstructed function are plotted in Figure 6.2. We see that the input function is reconstructed to the accuracy limit given by the noise level. No amplification of the noise or instabilities are observed.

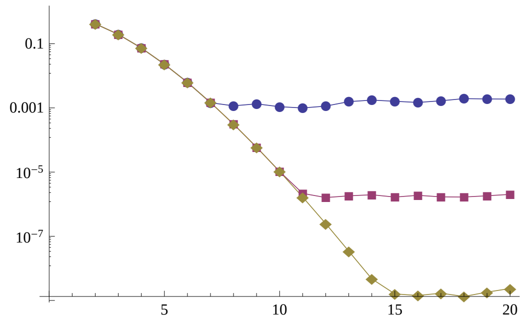
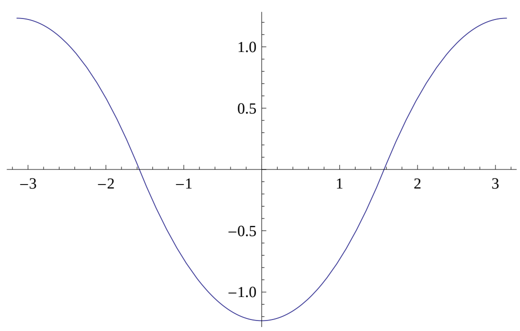


Figure 5: Example 2: errors with noisy data. Displayed are three experiments with noise levels of 10^{-3}, 10^{-6}, 10^{-9}. x-axis: degree of interpolating polynomial. y-axis: relative L_2-error

6.3 Example 3

We test our method on a function which is given by the harmonic extension of the quadratic spline $f(\theta)$, $-\pi \leq \theta \leq \pi$, where θ is the angle on the unit circle.

$$f(\theta) = \begin{cases} -\frac{1}{2}(\theta + \frac{\pi}{2})(\theta + \frac{3}{2}\pi), & -\pi \leq \theta < -\frac{\pi}{2}, \\ \frac{1}{2}(\theta - \frac{\pi}{2})(\theta + \frac{\pi}{2}), & -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}, \\ -\frac{1}{2}(\theta - \frac{\pi}{2})(\theta - \frac{3}{2}\pi), & \frac{\pi}{2} \leq \theta < \pi. \end{cases}$$



Note that $f(\theta)$ is a periodic C^1-function with discontinuous second derivative. The resulting harmonic function u has the series representation (in polar coordinates)

$$u(r, \theta) = \sum_{k=1}^{\infty} (-1)^k r^{2k-1} \frac{4 \cos((2k-1)\theta)}{(2k-1)^3 \pi}.$$

For our chords $\mathcal{I}$, we choose the edges of a regular $(2n+1)$-sided convex polygon (cf. Figure 2, first picture).

Figure 6 shows the relative L_2-errors for varying degree n of the interpolating polynomial. The last column of the table displays the ratio between successive errors. This rate of convergence approaches 8 and thus suggests that the interpolation error is of the order $\mathcal{O}(n^{-3})$.

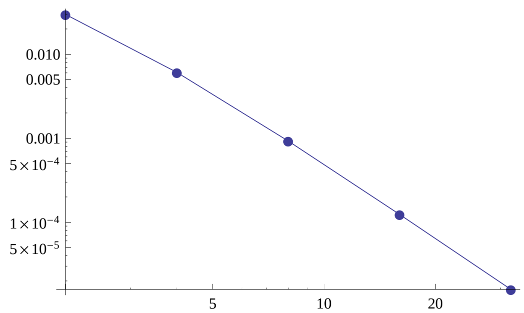


Figure 6: log-log-plot of the relative L_2 errors for varying degree n

n	relative L_2 error	rate
2	$2.97973 \cdot 10^{-2}$	—
4	$6.08456 \cdot 10^{-3}$	4.90
8	$9.26954 \cdot 10^{-4}$	6.56
16	$1.23962 \cdot 10^{-4}$	7.47
32	$1.58587 \cdot 10^{-5}$	7.82

Table 1: Relative L_2 errors for varying degree n

7 Conclusion and outlook

We have stated an interpolation problem for a harmonic function in the unit disk given certain values of its Radon projections and have provided an efficient algorithm for solving this problem in the case when the Radon projections are taken along equally spaced chords in the unit circle which is robust with respect to noise in the input data. Moreover, we are able to compute the coefficients of the interpolating polynomial in slightly worse than linear time. Our numerical experiments for recovery of functions which are C^∞ in the closed unit disk have shown exponential convergence.

In future work, we plan to derive cubature formulae for harmonic functions given Radon projection type of data and investigate error estimates for such interpolation methods and cubature rules. Some possible modifications to the problem (3) include the replacement of some of the chord integral conditions by different interpolation conditions, for instance some point values on the unit circle; the incorporation of a "too large" data set, $|\mathcal{I}| > \dim P$, via, e.g., least-squares minimization; the treatment of more general PDEs. For many problems, allowing the interpolation of functions satisfying an inhomogeneous partial differential equation of the form $\Delta u = f$ would be highly useful and is a possible subject of further work.

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