Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

# Interpolation of Harmonic Functions Based on Radon Projections

Irina Georgieva Clemens Hofreither

DK-Report No. 2012-11 10 2012

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by
Austrian Science Fund (FWF) Upper Austria

DK sponsors:

- **Johannes Kepler University Linz** (JKU)
- **Austrian Science Fund** (FWF)
- **Upper Austria**

# Interpolation of Harmonic Functions Based on Radon Projections

Irina Georgieva* Clemens Hofreither†

**Abstract**

We consider an algebraic method for reconstruction of a harmonic function in the unit disk via a finite number of values of its Radon projections. The approach is to seek a harmonic polynomial which matches given values of Radon projections along some chords of the unit circle. We prove an analogue of the famous Marr's formula for computing the Radon projection of the basis orthogonal polynomials in our setting of harmonic polynomials. Using this result, we show unique solvability for a family of schemes where all chords are chosen at equal distance to the origin. For the special case of chords forming a regular convex polygon, we prove error estimates on the unit circle and in the unit disk. We present an efficient reconstruction algorithm which is robust with respect to noise in the input data and provide numerical examples.

## 1 Introduction

Most methods for approximate reconstruction of a univariate function are based on sampling its values at a finite number of points, and the tools used are usually those of interpolation. This is a natural approach to approximation of univariate functions since a table of function values is a standard type of information that comes as output in practical problems and processes described by functions in one variable. Moreover, the Lagrange interpolation problem by polynomials is always solvable.

In the multivariate case, such an approach encounters serious difficulties. For example, the pointwise interpolation by multivariate polynomials is no longer possible for every choice of the nodes. Furthermore, there are many practical problems in which the information about the relevant function comes as a set of functionals which are not point evaluations. In many situations, a table of mean values of a function of $d$ variables on $(d-1)-$dimensional hyperplanes is considered to be the most natural type of data for multivariate functions. For

*Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev, Bl. 8, 1113, Sofia, Bulgaria, irina@math.bas.bg

†Doctoral College Computational Mathematics, Johannes Kepler University, Altenberger Str. 69, 4040 Linz, Austria, clemens.hofreither@dk-compmath.jku.at

instance, in tomography and electronic microscopy, often the data consist of values of linear integrals over line segments.

The work of the Austrian mathematician Johann Radon in the early twentieth century, in particular his results on the Radon transform later to be named after him [23], laid the theoretical foundation for tomography methods for shape reconstruction of objects with non-homogeneous density. These methods have many important practical applications, for example in medicine, radiology and geology.

From the mathematical point of view, the problem is to recover a multivariate function using information given as line integrals of the unknown function. Great efforts have been made to develop fast and effective high-accuracy algorithms for solving this problem. These methods are in general of two types: integral and algebraic. The integral methods are based on the Radon transform. Here all considerations are in continuous form and they come to discretization immediately before the implementation of the recovery algorithm. In the algebraic methods, discretization of the problem is carried out immediately, and the problem is then reduced to solving a linear or nonlinear system of equations. The approach described in the present paper falls into the latter category. More precisely, reconstruction of the unknown function is formulated as an interpolation problem where an approximate function is sought in an appropriate polynomial space such that it matches the given Radon projections.

An idea suggested by B. Bojanov is to incorporate additional knowledge about the function to be recovered into approximation methods. It is to be expected that this can improve the accuracy of the approximation while reducing the amount of input data required as well as the computational effort. In applications, such problem-specific knowledge is often provided in the form of a partial differential equation which the unknown satisfies. For the time being, we concern ourselves with the simple case where the unknown is harmonic, i.e., satisfies the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$. This elliptic partial differential equation is important both as a model problem as well as in actual applications, like heat transport, diffusion problems or Stokes flow of incompressible fluids.

The present work expands on the earlier article [9]. Therein, first results on interpolation of harmonic functions based on Radon projections along the sides of regular polygons were presented. Tools from symbolic computation were used intensively in the proofs.

Here we treat a more general setting. First of all, for a formula which gives integrals of certain harmonic basis polynomials and which was proved symbolically in [9] for a particular choice of parameters, we now give a general and analytic proof for arbitrary parameters. This result may be viewed as a harmonic analogue to the classic Marr's formula [20]. This allows us to generalize the existence and uniqueness theorem from [9] to a larger class of chord schemes.

For a special case, we perform a more detailed analysis of the resulting method. We obtain error estimates for the interpolation scheme on the unit circle and in the unit disk in the $L^2$- and maximum norms. Furthermore, we are able to show that the condition number of the matrix associated with the interpolation problem is uniformly bounded by a small constant independent of the degree of the interpolating polynomial $n$, indicating that the interpolation process is robust with respect to noise.

## 2 Preliminaries and related work

Let $D \subset \mathbb{R}^2$ denote the open unit disk and $\partial D$ the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Figure 1).

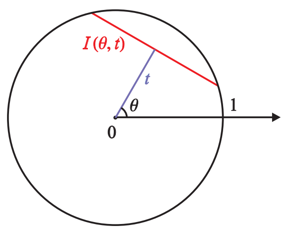



Figure 1: The chord $I(\theta, t)$ of the unit circle.

The chord $I(\theta, t)$ is parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \quad \text{where } s \in (-\sqrt{1-t^2}, \sqrt{1-t^2}). \quad (1)$$

**Definition 1.** Let $f(x, y)$ be a real-valued bivariate function in the unit disk $D$. The *Radon projection* $\mathcal{R}_\theta(f; t)$ of $f$ in direction $\theta$ is defined by the line integral

$$\mathcal{R}_\theta(f; t) := \int_{I(\theta,t)} f(\mathbf{x})\, d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

Johann Radon [23] showed in 1917 that a differentiable function $f$ is uniquely determined by the values of its Radon transform,

$$f \;\mapsto\; \left\{ \mathcal{R}_\theta(f; t) \,:\, -1 \le t \le 1,\ 0 \le \theta < \pi \right\}.$$

Further work in this area was done by John [18].

In the following we formulate the problem of recovery of a polynomial from a finite number of values of its Radon transform. Essentially, this may be viewed as a bivariate interpolation problem where the traditional point values are replaced by the means over chords of the unit circle.

Let $\Pi_n^2 = \left\{ \sum_{i+j \le n} a_{ij} x^i y^j \,:\, a_{ij} \in \mathbb{R} \right\}$ denote the space of real bivariate polynomials of total degree at most $n$. This space has dimension $\binom{n+2}{2}$. Assume that a set $\mathcal{I} = \left\{ I_m = I(\theta_m, t_m) : m = 1, \ldots, \binom{n+2}{2} \right\}$ of chords of the unit circle is given. Furthermore, to each chord $I \in \mathcal{I}$ a given value $\gamma_I \in \mathbb{R}$ is associated. Then, the aim is to find a polynomial $p \in \Pi_n^2$ such that

$$\int_I p(\mathbf{x})\, d\mathbf{x} = \gamma_I \qquad \forall I \in \mathcal{I}. \quad (2)$$

If this interpolation problem has a unique solution for every choice of values $\{\gamma_I\}$, then the scheme $\mathcal{I}$ of chords is called *regular*. The question of how to construct such regular schemes has been extensively studied. The first general result was given by Marr [20] in 1974, who proved that the set of chords connecting $n+2$ equally spaced points on the unit circle is regular for $\Pi_n^2$. A more general result for $\mathbb{R}^d$ and general convex domains was published by Hakopian [15] in 1982. Applied to the unit disk in $\mathbb{R}^2$, it states that even the chords connecting any $n+2$ distinct points on the unit circle form a regular scheme for $\Pi_n^2$.

Bojanov and Georgieva [2] constructed another family of schemes where $\binom{n+2}{2}$ chords are partitioned into $n+1$ subsets such that the $k$-th subset consists of $k$ parallel chords. They showed that these schemes yield a unique interpolation polynomial in $\Pi_n^2$ provided that the distances $t$ satisfy some additional conditions involving the Chebyshev polynomials of the second kind. Particular choices of suitable distances $t$ were later given by Georgieva and Ismail [11] in terms of zeroes of Chebyshev polynomials of the second kind, as well as Georgieva and Uluchev [12] in terms of zeroes of Jacobi polynomials.

Bojanov and Xu [5] proposed a regular scheme consisting of $\binom{n+2}{2}$ chords partitioned into $2\lfloor (n+1)/2 \rfloor + 1$ equally spaced directions, such that in every direction there are $\lfloor n/2 \rfloor + 1$ parallel chords. The distances $t$ of the chords are zeroes of Chebyshev polynomials of the second kind.

A mixed regular scheme which incorporates Radon projections and point evaluations on the unit circle was given by Georgieva, Hofreither, and Uluchev [10].

Many other mathematicians have worked on problems with applications in the mathematical foundations of computer tomography, among them [17, 6, 7, 8, 16, 19, 21]. Recovery of polynomials in two variables based on Radon projections is also considered in [1, 22, 3, 4, 13, 14].

# 3 Interpolation by harmonic polynomials

In this section, we state an interpolation problem for a harmonic function in the unit disk given values of its Radon projections over a set of chords and derive existence and uniqueness conditions for the corresponding interpolating harmonic polynomial.

If we know *a priori* that the function to be interpolated is harmonic, it is natural to work in the space

$$\mathcal{H}_n = \left\{ p \in \Pi_n^2 : \Delta p = 0 \right\}$$

of real bivariate harmonic polynomials of total degree at most $n$, which has dimension $2n+1$. Analogously to (2), we prescribe chords

$$\mathcal{I} = \{ I(\theta_m, t_m) : m = 1, \ldots, 2n+1 \}$$

of the unit circle and associated given values $\{\gamma_I\}$, and wish to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\int_I p(\mathbf{x})\, d\mathbf{x} = \gamma_I \qquad \forall I \in \mathcal{I}. \tag{3}$$

Again we call $\mathcal{I}$ *regular* if the interpolation problem (3) has a unique solution for all given values $\{\gamma_I\}$. In the following, we construct one family of such regular schemes.

We use the basis of the harmonic polynomials

$$\phi_0(x,y) = 1, \quad \phi_{k,1}(x,y) = \mathrm{Re}(x+\mathrm{i}y)^k, \quad \phi_{k,2}(x,y) = \mathrm{Im}(x+\mathrm{i}y)^k.$$

In polar coordinates, they have the representation

$$\phi_{k,1}(r,\theta) = r^k \cos(k\theta), \quad \phi_{k,2}(r,\theta) = r^k \sin(k\theta).$$

Expanding the harmonic polynomial $p$ in this basis,

$$p = p_0\phi_0 + \sum_{k=1}^{n} (p_{k,1}\phi_{k,1} + p_{k,2}\phi_{k,2}),$$

we obtain a linear system $A\underline{p} = \underline{\gamma}$ equivalent to (3) with the matrix

$$A = \begin{pmatrix} \int_{I_1} 1 & \int_{I_1} \phi_{1,1} & \cdots & \int_{I_1} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \int_{I_2} 1 & \int_{I_2} \phi_{1,1} & \cdots & \int_{I_2} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_{I_{2n+1}} 1 & \int_{I_{2n+1}} \phi_{1,1} & \cdots & \int_{I_{2n+1}} \phi_{n,1} & \int_{I_{2n+1}} \phi_{n,2} \end{pmatrix}. \tag{4}$$

The scheme $\mathcal{I}$ is regular if and only if $A$ is regular.

## 3.1 Analogue of Marr's formula

The following result, which gives a closed formula for the entries of the matrix $A$, can be considered a harmonic analogue to Marr's formula [20]. A special case of this harmonic version was first derived using tools from symbolic computation [9], and we now give an analytic proof in a more general setting.

**Theorem 1.** *The Radon projections of the basis harmonic polynomials are given by*

$$\int_{I(\theta,t)} \phi_{k,1}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\sin(k\theta),$$

*where $k \in \mathbb{N}$, $\theta \in \mathbb{R}$, $t \in (-1,1)$, and $U_k(t)$ is the $k$-th Chebyshev polynomial of second kind.*

*Proof.* Fix a chord $I(\theta, t)$ and let $\alpha = \arccos t$, and hence

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - t^2}.$$

Then the chord intersects the unit circle at points with the arguments $\theta \pm \alpha$. For computing the integral over the chord $I(\theta, t)$, we pass to the complex plane $\mathbb{C}$. There, the chord $I(\theta, t)$ is parameterized by $z_0 + sz$, $s \in (-1, 1)$, where $z_0 = te^{\mathrm{i}\theta}$ and $z = \mathrm{i}e^{\mathrm{i}\theta} \sin \alpha$. Let

$$\gamma(s) = z_0 + sz = \tilde{\gamma}_1(s) + \mathrm{i}\tilde{\gamma}_2(s) \in \mathbb{C}, \quad \tilde{\gamma}(s) = (\tilde{\gamma}_1(s), \tilde{\gamma}_2(s))^\top \in \mathbb{R}^2.$$

Note that

$$|\gamma'(s)| = |\tilde{\gamma}_1'(s) + \mathrm{i}\tilde{\gamma}_2'(s)| = \sqrt{\tilde{\gamma}_1'(s)^2 + \tilde{\gamma}_2'(s)^2} = |\tilde{\gamma}'(s)|.$$

Then

$$\begin{aligned}
\int_{I(\theta,t)} \phi_{k,1} \, d\mathbf{x} &= \int_{-1}^{1} \phi_{k,1}(\tilde{\gamma}(s)) \, |\tilde{\gamma}'(s)| \, ds \\
&= \int_{-1}^{1} \operatorname{Re} \left[ (\tilde{\gamma}_1(s) + \mathrm{i}\tilde{\gamma}_2(s))^k \right] \, |\gamma'(s)| \, ds \\
&= \int_{-1}^{1} \operatorname{Re} \left[ \gamma(s)^k \right] \, |\gamma'(s)| \, ds.
\end{aligned}$$

All of the integrals are real. However, a complex integral over the interval $(-1, 1)$ of the real axis is equivalent to a real integral over the same interval, and therefore we can write

$$\int_{-1}^{1} \operatorname{Re} \left[ \gamma(s)^k \right] \, |\gamma'(s)| \, ds = \operatorname{Re} \int_{-1}^{1} \gamma(s)^k \, |\gamma'(s)| \, ds,$$

where the right-hand side can now be viewed as a complex integral. Indeed, let $\gamma(s)^k = a(s) + \mathrm{i}b(s)$ with real functions $a, b$. Then

$$\begin{aligned}
\operatorname{Re} \int_{-1}^{1} \gamma(s)^k |\gamma'(s)| \, ds &= \operatorname{Re} \left[ \int_{-1}^{1} a(s) |\gamma'(s)| \, ds + \int_{-1}^{1} \mathrm{i}b(s) |\gamma'(s)| \, ds \right] \\
&= \int_{-1}^{1} a(s) |\gamma'(s)| \, ds = \int_{-1}^{1} \operatorname{Re} \left[ \gamma(s)^k \right] \, |\gamma'(s)| \, ds.
\end{aligned}$$

Since $|\gamma'(s)| = |z| = \sin\alpha$, we have

$$\begin{aligned}
\mathrm{Re}\int_{-1}^{1} \gamma(s)^k\,|\gamma'(s)|\,ds &= \sin\alpha\ \mathrm{Re}\int_{-1}^{1} e^{\mathrm{i}\theta k}(t+\mathrm{i}s\,\sin\alpha)^k\,ds \\
&= \sin\alpha\ \mathrm{Re}\left[\frac{e^{\mathrm{i}\theta k}}{(k+1)\mathrm{i}\sin\alpha}\left[(\cos\alpha+s\mathrm{i}\sin\alpha)^{(k+1)}\right]_{s=-1}^{1}\right] \\
&= \mathrm{Re}\left[\frac{1}{(k+1)}\frac{e^{\mathrm{i}\theta k}}{\mathrm{i}}(e^{\mathrm{i}(k+1)\alpha}-e^{-\mathrm{i}(k+1)\alpha})\right] \\
&= \mathrm{Re}\left[\frac{2}{(k+1)}e^{\mathrm{i}\theta k}\sin((k+1)\alpha)\right] \\
&= \mathrm{Re}\left[\frac{2}{(k+1)}e^{\mathrm{i}\theta k}U_k(t)\sqrt{1-t^2}\right] \\
&= \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta).
\end{aligned}$$

Above we used that, by definition, $U_k(\cos\alpha) = \frac{\sin((k+1)\alpha)}{\sin\alpha}$. The proof for $\phi_{k,2}$ is completely analogous. $\square$

## 3.2 Existence and uniqueness

Using Theorem 1, it is easy to see that the matrix $A$ given in (4) has the $2n+1$ columns

$$\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},\ \left\{\begin{pmatrix}U_k(t_1)\cos(k\theta_1)\\ \vdots\\ U_k(t_{2n+1})\cos(k\theta_{2n+1})\end{pmatrix},\begin{pmatrix}U_k(t_1)\sin(k\theta_1)\\ \vdots\\ U_k(t_{2n+1})\sin(k\theta_{2n+1})\end{pmatrix},k=1,\dots,n\right\},$$

and thus the interpolation scheme is regular if and only if these column vectors are linearly independent. For the case of constant distances $t_m = t$, this is easy to establish and leads to the following result.

**Theorem 2** (Existence and uniqueness)**.** *The interpolation problem* (3) *has a unique solution for the choice* $\mathcal{I} = \{I(\theta_m, t_m) : m = 1,\dots,2n+1\}$ *with*

$$0 \le \theta_1 < \theta_2 < \ldots < \theta_{2n+1} < 2\pi,$$

*while the distances* $t_m = t \in (-1,1)$ *are constant and* $t$ *is not a zero of any Chebyshev polynomial of the second kind* $U_1,\dots,U_n$.

See Figure 2 for some examples of schemes which satisfy the conditions of the above theorem.

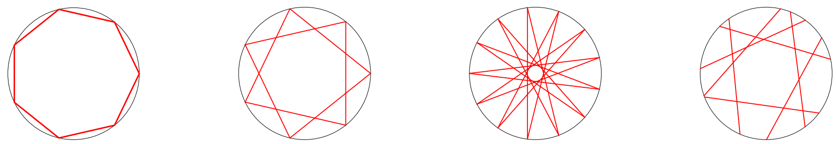


Figure 2: Some admissible schemes according to Theorem 2.

In the following proof and later on, we use the notations

$$\begin{aligned}
c(k) &:= (\cos(k\theta_1), \dots, \cos(k\theta_{2n+1}))^\top, \\
s(k) &:= (\sin(k\theta_1), \dots, \sin(k\theta_{2n+1}))^\top, \\
Q &:= \begin{pmatrix} | & | & | & \dots & | & | \\ 1 & c(1) & s(1) & \dots & c(n) & s(n) \\ | & | & | & \dots & | & | \end{pmatrix}, \\
\alpha_k &:= \frac{2}{k+1}\sqrt{1-t^2}U_k(t), \\
F &:= \operatorname{diag}(\alpha_0, \alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n).
\end{aligned}$$

*Proof.* Under the assumption of constant $t_m = t$ and using the harmonic basis introduced above, Theorem 1 yields the representation $A = QF$ of the system matrix corresponding to (3). Then, using the column-wise linearity of the determinant, we have $\det A = \alpha_0 \prod_{k=1}^n \alpha_k^2 \det Q$. Under the assumptions of the theorem, all the $\alpha_k$ are nonzero.

The functions $\{1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\}$ form a basis of the trigonometric polynomials of degree at most $n$. Indeed, the matrix $Q$ is the same as for the one-dimensional problem of interpolation with a trigonometric polynomial of degree $n$ in $[0, 2\pi]$ at the points $\{\theta_1, \dots, \theta_{2n+1}\}$. It is well-known that $Q$ is invertible if and only if the angles $\theta_m$ are pairwise distinct ([25]), which proves the theorem. $\square$

# 4 Inversion of the linear system for equispaced angles

For equally spaced angles, the columns of $Q$ are orthogonal. We exploit this fact to derive a simple representation of the inverse of the system matrix $A$.

**Theorem 3.** *Assume chords with equally spaced angles $\theta_m = \frac{2\pi m}{2n+1}$ and fixed distance $t_m = t \in (0,1)$, $m = 1, \dots, 2n+1$, such that $U_k(t) \neq 0$ for all $k \in \{0, \dots, n\}$. Then the inverse of $A$ is given by*

$$A^{-1} = EQ^\top,$$

*where*

$$E = \mathrm{diag}(\beta_0, \beta_1, \beta_1, \dots, \beta_n, \beta_n),$$
$$\beta_k = \begin{cases} \frac{1}{2(2n+1)}(\sqrt{1-t^2})^{-1} = \frac{1}{2n+1}\alpha_k^{-1}, & k = 0, \\ \\ \frac{k+1}{2n+1}(\sqrt{1-t^2}U_k(t))^{-1} = \frac{2}{2n+1}\alpha_k^{-1}, & k \geq 1. \end{cases}$$

The proof follows easily from the following lemma.

**Lemma 4.** *For equally spaced angles $\theta_m$, it holds*

- $c(k_1) \perp c(k_2)$ *and* $s(k_1) \perp s(k_2)$ *for* $k_1, k_2 \in \{0, \dots, n\}$ *and* $k_1 \neq k_2$,
- $c(k_1) \perp s(k_2)$ *for* $k_1, k_2 \in \mathbb{N}_0$,
- $|c(0)| = \sqrt{2n+1}$, $|s(0)| = 0$, $|c(k)| = |s(k)| = \sqrt{\frac{2n+1}{2}}$ *for* $k \geq 1$,
- $c(k) = c(k+l(2n+1))$ *and* $s(k) = s(k+l(2n+1))$ *for* $k \in \{0, \dots, n\}, l \in \mathbb{N}_0$,
- $c(k) = c((2n+1)-k)$ *and* $s(k) = -s((2n+1)-k)$ *for* $k \in \{0, \dots, n\}$.

*Proof.* The statements are easily proved by trigonometric identities or passing to the complex plane.

*Proof of Theorem 3.* Since $A = QF$, we need to show $EQ^\top QF = I$. The first three statements of the above lemma show that $Q^\top Q$ is the diagonal matrix

$$Q^\top Q = \mathrm{diag}\left\{2n+1, \frac{2n+1}{2}, \dots, \frac{2n+1}{2}\right\},$$

and the statement follows by definition of $\alpha_k$ and $\beta_k$. □

**Remark.** Note that the action of the matrix $Q^\top$ is essentially a discrete Fourier transform of the given data. This suggests an efficient algorithm for the solution of the linear system: using a suitable Fast Fourier Transform (FFT), we can compute the coefficients of the interpolating polynomial in $\mathcal{O}(n \log n)$ time.

# 5 Analysis

For the numerical analysis that follows, we make the stronger assumption that the chords form a regular convex $(2n+1)$-sided polygon inscribed in the unit circle; cf. Figure 2, first picture. We thus consider the sequence $\mathcal{I}^{(n)} = \{I(\theta_m^{(n)}, t^{(n)}) : m = 1, \dots, 2n+1\}$ of schemes with the angles and the distances, respectively,

$$\theta_m^{(n)} = \frac{2\pi m}{2n+1}, \quad t^{(n)} = \cos\frac{\pi}{2n+1}, \qquad \text{for } m = 1, \dots, 2n+1. \tag{5}$$

Furthermore, assume that the given data $\{\gamma_I\}$ are the Radon projections of some unknown harmonic function $u \in C^2(D)$. By Theorem 2, the resulting interpolation problem

$$\int_I p^{(n)}(\mathbf{x})\, d\mathbf{x} = \int_I u(\mathbf{x})\, d\mathbf{x} \qquad \forall I \in \mathcal{I}^{(n)}.$$

is uniquely solvable for every $n \in \mathbb{N}$. Thus we obtain a sequence of interpolating harmonic polynomials $p^{(n)} \in \mathcal{H}_n$. For ease of notation, we omit the superscript $(n)$ in most cases in the following but keep in mind the dependence on $n$.

### 5.1 Error estimate

In this section, we give error estimates for the interpolating polynomial $p^{(n)}$ in terms of the smoothness of the boundary data $f = u|_{\partial D}$. Being defined on the unit circle, $f$ can be written as a periodic function of the angle $\theta$. We will also rely on its Fourier series, i.e., let $(f_k)_{k\in\mathbb{Z}}$ be the Fourier coefficients of $f$ such that

$$f(\theta) = f_0 + \sum_{k=1}^{\infty} (f_k \cos(k\theta) + f_{-k} \sin(k\theta)). \tag{6}$$

For simplicity, we will assume that the Fourier series converges uniformly to $f$. In this case, the series of functions of $(r, \theta)$

$$f_0 + \sum_{k=1}^{\infty} (f_k r^k \cos(k\theta) + f_{-k} r^k \sin(k\theta))$$

converges uniformly on the unit disk, and its limit is a harmonic function, namely, $u$; cf. [24]. Introducing integer indices $k \in \mathbb{Z}$ for the basis polynomials,

$$\phi_k := \begin{cases} \phi_{k,1}, & \text{if } k \geq 0 \\ \phi_{-k,2}, & \text{if } k < 0, \end{cases}$$

this allows us to write the harmonic function $u$ as the uniformly convergent series

$$u = \sum_{k\in\mathbb{Z}} f_k \phi_k.$$

The error estimates are based on the following idea: due to linearity of the interpolation operator and the existence of a unique solution shown in Theorem 2, it is clear that the low frequency components, $f_k : |k| \leq n$, are reproduced exactly in the interpolating polynomial $p^{(n)}$. The question is therefore how the high frequency components, $f_k : |k| > n$, influence the interpolating polynomial. It turns out that every high frequency component $f_k$ is mirrored to a low frequency contribution $\delta_k f_{k'}$ with $|k'| < n$, where $\delta_k$ is a coefficient which we can bound uniformly. This effect is very similar to the aliasing phenomenon known from sampling sinusoidals at equal intervals. By imposing a decay condition on the high frequency components, we can thus obtain a bound on the total interpolation error.

Due to the above considerations, smoothness of $f$ will be quantified by the decay of its Fourier coefficients $f_k$ in the following. To be precise, we will require that the Fourier coefficients of $f$ decay like $|f_k| = \mathcal{O}(|k|^{-s})$ with some positive parameter $s$. Translated into a Sobolev space setting, this assumption means that $f$ lies in the Hilbert space $H^\ell(\partial D)$ with Sobolev parameter $\ell < 2s - 1$.

Before we proceed with the error estimate, we need a small technical lemma.

**Lemma 5.** *Let*

$$a(k,n) := \frac{k+1}{U_k(\cos\frac{\pi}{2n+1})}.$$

*For $n \in \mathbb{N}$, we have*

$$1 = a(0,n) \le a(1,n) \le \ldots \le a(n-1,n) \le a(n,n) \le 2.$$

*Proof.* By the definition of $U_n(t)$, we have that

$$a(k,n) = \frac{(k+1)\sin\frac{\pi}{2n+1}}{\sin\left[(k+1)\frac{\pi}{2n+1}\right]}.$$

All arguments to sin are in the range $(0,\pi)$, and thus the sines are all positive. In particular, $U_k(\cos\frac{\pi}{2n+1}) > 0$, and we use the well-known fact $|U_k(t)| \le k+1$ to conclude that $1 \le a(k,n)$. Rewriting further, we get

$$a(k,n) = \frac{(k+1)\frac{\pi}{2n+1}\frac{2n+1}{\pi}\sin\frac{\pi}{2n+1}}{\sin\left[(k+1)\frac{\pi}{2n+1}\right]} = \frac{\frac{\sin(x(n))}{x(n)}}{\frac{\sin(y(k,n))}{y(k,n)}} \tag{7}$$

with

$$x(n) = \frac{\pi}{2n+1}, \qquad y(k,n) = (k+1)\frac{\pi}{2n+1}.$$

We have $0 < y(k,n) < \pi$, and the function $y \mapsto \frac{\sin y}{y}$ is monotonically decreasing and positive in this interval. Since clearly $y(k,n) < y(k+1,n)$, this means that $\frac{\sin(y(k,n))}{y(k,n)}$ is monotonically decreasing in $k$, and thus $a(k,n)$ is monotonically increasing in $k$.

Since $x(n) \to 0$ as $n \to \infty$, then $\frac{\sin(x(n))}{x(n)} \to 1$. On the other hand, $y(n,n) \to \frac{\pi}{2}$, so $\frac{\sin(y(n,n))}{y(n,n)} \to \frac{2}{\pi}$. Using (7), this shows that $a(n,n) \xrightarrow{n\to\infty} \frac{\pi}{2}$.

In order to show monotonicity of $a(n,n)$, we rewrite again, yielding

$$a(n,n) = \frac{(n+1)\sin\frac{\pi}{2n+1}}{\sin(n+1)\frac{\pi}{2n+1}} = \frac{(n+1)2\sin\frac{\pi}{2(2n+1)}\cos\frac{\pi}{2(2n+1)}}{\sin(n+1)\frac{\pi}{2n+1}}.$$

Since $\sin(n+1)\frac{\pi}{2n+1} = \sin(2n+2)\frac{\pi}{4n+2} = \sin(\frac{\pi}{2} + \frac{\pi}{4n+2}) = \cos\frac{\pi}{4n+2}$, we get

$$a(n,n) = 2(n+1)\sin\frac{\pi}{4n+2}.$$

We want to show that $a(n,n) = (2n+2)\sin\frac{\pi}{4n+2}$ decreases when $n$ increases. Set $y := \frac{\pi}{4n+2}$. Then $a(n,n) = (\frac{\pi}{2y}+1)\sin y = f(y)$. As $n \to \infty$, $y$ decreases monotonically to 0, so in order to prove that $a(n,n)$ is decreasing, we have to prove that $f(y)$ is increasing. For this purpose we calculate the derivative

$$f'(y) = \frac{1}{2y^2}[-\pi\sin y + (\pi y + 2y^2)\cos y].$$

Then we put $g(y) = -\pi\sin y + (\pi y + 2y^2)\cos y$. We see that $g(0) = 0$ and $g'(y) = 4y\cos y + (\pi y + 2y^2)\sin y$. Since $0 < y < \frac{\pi}{2}$, then $g'(y) > 0$, $g(y) > g(0) = 0$ and hence $f'(y) > 0$. It follows that $a(n,n) \overset{n\to\infty}{\searrow} \frac{\pi}{2}$.

Finally it is easily computed that $a(1,1) = 2$, which, together with the monotonicity, proves the upper bound for $a(n,n)$. $\square$

**Lemma 6.** *Assume that $f = u|_{\partial D}$ has a uniformly convergent Fourier series (6) and its Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ decay like $|f_k| \le M|k|^{-s}$ with $M > 0$, $s > 1$. Let*

$$p^{(n)} = p_0^{(n)}\phi_0 + \sum_{k=1}^{n}(p_k^{(n)}\phi_k + p_{-k}^{(n)}\phi_{-k}) \in \mathcal{H}_n$$

*be the interpolating polynomial of degree $n$ obtained by our method. Then the error in the coefficients of the interpolating polynomial $p^{(n)}$ satisfies*

$$|f_k - p_k^{(n)}| \le MC_s n^{-s} \qquad \forall |k| \le n,$$

*where $C_s$ is a constant which depends only on $s$.*

*Proof.* In this proof, we use integer indices: for $k \in \mathbb{Z}$, we denote

$$h(k) := \begin{cases} c(k), & \text{if } k \ge 0 \\ s(-k), & \text{if } k < 0, \end{cases}$$

and write

$$\psi_k := \phi_k|_{\partial D} = \begin{cases} \cos(|k|\theta), & \text{if } k \ge 0 \\ \sin(|k|\theta), & \text{if } k < 0 \end{cases}$$

for the harmonic basis functions restricted to the unit circle. Under the assumptions, we then have

$$f = \sum_{k\in\mathbb{Z}} f_k\psi_k, \qquad u = \sum_{k\in\mathbb{Z}} f_k\phi_k.$$

Let $\Phi$ denote the operator which, for a given function $u$, computes its Radon projections along the $2n+1$ chords $I_m$, i.e.,

$$\begin{aligned} \Phi : C(D) &\to \mathbb{R}^{2n+1}, \\ u &\mapsto \left(\int_{I_1} u, \ldots, \int_{I_{2n+1}} u\right)^\top. \end{aligned}$$

With the help of Theorem 1 for the basis harmonic functions, we get

$$\Phi(\phi_k) = \frac{2}{|k|+1}\sqrt{1-t^2}U_{|k|}(t)h(k) = \alpha_{|k|}h(k) \qquad \forall k \in \mathbb{Z}. \tag{8}$$

We keep in mind that $p_k = p_k^{(n)}$ as well as most other terms above depend on the scheme and thus on $n$, but omit the superscript for ease of notation.

With the index function

$$i(k) = \begin{cases} 1 & \text{if } k = 0, \\ 2k, & \text{if } k > 0, \\ -2k+1, & \text{if } k < 0, \end{cases}$$

which maps indices of Fourier coefficients to column numbers of our matrix $A$, we have, for $k \in \{-n, \dots, n\}$,

$$p_k = [A^{-1}\Phi u]_{i(k)} = [A^{-1}\Phi \sum_{j\in\mathbb{Z}} f_j \phi_j]_{i(k)} = [A^{-1} \sum_{j\in\mathbb{Z}} f_j(\Phi\phi_j)]_{i(k)},$$

where the sum and the operator $\Phi$ may be exchanged because each component of $\Phi$ is just an integral and the sum is uniformly convergent by the assumption on $f$. Using (8) and that $A^{-1}$ is a matrix and thus a bounded operator, we obtain

$$\begin{aligned} p_k &= [A^{-1}\sum_{j\in\mathbb{Z}} f_j \alpha_{|j|} h(j)]_{i(k)} = \sum_{j\in\mathbb{Z}} f_j \alpha_{|j|} [A^{-1}h(j)]_{i(k)} \\ &= \sum_{j\in\mathbb{N}_0} f_j \alpha_j [A^{-1}c(j)]_{i(k)} + \sum_{j\in\mathbb{N}} f_{-j}\alpha_j [A^{-1}s(j)]_{i(k)}. \end{aligned}$$

For $k \in \{0, \dots, n\}$, the $i(k)$-th row of $Q^\top$ is $c(k)$, and thus we have from Theorem 3 and Lemma 4

$$[A^{-1}c(j)]_{i(k)} = [EQ^\top c(j)]_{i(k)} = \begin{cases} \beta_k |c(k)|^2, & \text{if } j = \mathbb{N}_0(2n+1)+k, \\ \beta_k |c(k)|^2, & \text{if } j = \mathbb{N}(2n+1)-k, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for $k \in \{1, \dots, n\}$, the $i(-k)$-th row of $Q^\top$ is $s(k)$, and hence

$$[A^{-1}s(j)]_{i(-k)} = [EQ^\top s(j)]_{i(-k)} = \begin{cases} \beta_k |s(k)|^2, & \text{if } j = \mathbb{N}_0(2n+1)+k, \\ -\beta_k |s(k)|^2, & \text{if } j = \mathbb{N}(2n+1)-k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the cosine coefficients $p_k$, $k \in \{0, \dots, n\}$, are

$$\begin{aligned} p_k &= \alpha_k \beta_k |c(k)|^2 f_k + \sum_{l=1}^{\infty} (\alpha_{l(2n+1)+k} \beta_k |c(k)|^2 f_{l(2n+1)+k}) \\ &\quad + \sum_{l=1}^{\infty} (\alpha_{l(2n+1)-k} \beta_k |c(k)|^2 f_{l(2n+1)-k}) \\ &= f_k + \sum_{l=1}^{\infty} \beta_k |c(k)|^2 (\alpha_{l(2n+1)+k} f_{l(2n+1)+k} + \alpha_{l(2n+1)-k} f_{l(2n+1)-k}) \end{aligned}$$

since $\alpha_k \beta_k |c(k)|^2 = 1$. For the sine coefficients $p_{-k}$, $k \in \{1, \dots, n\}$, we get analogously

$$p_{-k} = f_{-k} + \sum_{l=1}^{\infty} \beta_k |s(k)|^2 (\alpha_{l(2n+1)+k} f_{l(2n+1)+k} - \alpha_{l(2n+1)-k} f_{l(2n+1)-k}).$$

Using Lemma 5 and the fact $\|U_j\|_\infty = j + 1$, we have

$$\alpha_j \beta_k |c(k)|^2 = \alpha_j \beta_k |s(k)|^2 = \frac{U_j(t)}{U_k(t)} \frac{k+1}{j+1} \leq 1 \cdot \frac{k+1}{U_k(t)} \leq 2,$$

and we get, for any $k \in \{-n, \dots, n\}$,

$$|p_k - f_k| \leq \sum_{l=1}^{\infty} 2 \left( |f_{l(2n+1)+|k|}| + |f_{l(2n+1)-|k|}| \right).$$

Using the assumption that $|f_k| \leq M|k|^{-s}$, we have

$$\begin{aligned} |p_k - f_k| &\leq 2 \sum_{l=1}^{\infty} \left( M(l(2n+1) + |k|)^{-s} + M(l(2n+1) - |k|)^{-s} \right) \\ &= 2M \sum_{l=1}^{\infty} (l(2n+1))^{-s} \left[ \left( 1 + \frac{|k|}{l(2n+1)} \right)^{-s} + \left( 1 - \frac{|k|}{l(2n+1)} \right)^{-s} \right] \\ &\overset{(*)}{\leq} \frac{2M}{(2n+1)^s} \sum_{l=1}^{\infty} l^{-s} (1 + 2^s) \\ &\leq \frac{2M(1+2^s)}{(2n+1)^s} \left( 1 + \frac{1}{s-1} \right) \\ &= M \frac{2(1+2^s)}{2^s} \frac{s}{s-1} \frac{1}{(n+\frac{1}{2})^s} \leq M C_s n^{-s} \end{aligned}$$

with $C_s = \frac{2(1+2^s)}{2^s} \frac{s}{s-1}$. For $s \geq 2$, one can estimate $C_s \leq 5$.

In the estimate marked with $(*)$, we used the fact that

$$\left( 1 + \frac{|k|}{l(2n+1)} \right)^{-s} = \left( \frac{1}{1 + \frac{|k|}{l(2n+1)}} \right)^s \leq 1$$

and

$$\left(1-\frac{|k|}{l(2n+1)}\right)^{-s}=\left(\frac{1}{1-\frac{|k|}{l(2n+1)}}\right)^{s}\leq\left(\frac{l(2n+1)}{l(2n+1)-n}\right)^{s}\leq 2^{s},$$

because $l(2n+1)-n\geq n+1>\frac{2n+1}{2}$. $\square$

We can now use this result on the decay of the error in the Fourier coefficients in order to derive error estimates in function spaces. First, the following theorem states that the $L^2$-error on the boundary behaves like $\mathcal{O}(n^{-(s-1/2)})$. Indeed, this is the same order of convergence that the partial sums of the Fourier series exhibit. Since the latter are the $L^2$-best approximating trigonometric polynomials, our method has optimal convergence on the boundary in a certain sense. Clearly, for harmonic functions, due to the maximum principle the largest error is to be expected on the boundary.

**Theorem 7.** *Assume that $f=u|_{\partial D}$ has a uniformly convergent Fourier series (6) and its Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ decay like $|f_k|\leq M|k|^{-s}$ with $M>0$, $s>1$. Let $p^{(n)}\in\mathcal{H}_n$ be the interpolating polynomial of degree $n$ obtained by our method. Then the approximation error on the unit circle satisfies*

$$\|f-p^{(n)}\|_{L^2(\partial D)}\leq MCn^{-(s-1/2)}$$

*with a constant $C$ which depends only on $s$.*

*Proof.* Let $f_n^*$ be the best-approximating (in $L^2$) trigonometric polynomial to $f$ of degree $n$. Its coefficients are just the first $2n+1$ Fourier coefficients $(f_k)$, that is, $f_n^*=\sum_{|k|\leq n}f_k\phi_k|_{\partial D}$. Using the $L^2(\partial D)$-orthogonality of $(\phi_k|_{\partial D})_k$, we see that

$$\begin{aligned}\|f-p^{(n)}\|^2_{L^2(\partial D)}&=\|f-f_n^*+f_n^*-p^{(n)}\|^2_{L^2(\partial D)}\\&=\|f-f_n^*\|^2_{L^2(\partial D)}+\|f_n^*-p^{(n)}\|^2_{L^2(\partial D)}\\&=\|\sum_{|k|>n}f_k\phi_k\|^2_{L^2(\partial D)}+\|\sum_{|k|\leq n}(f_k-p_k^{(n)})\phi_k\|^2_{L^2(\partial D)}\\&=\sum_{|k|>n}f_k^2\|\phi_k\|^2_{L^2(\partial D)}+\sum_{|k|\leq n}(f_k-p_k^{(n)})^2\|\phi_k\|^2_{L^2(\partial D)}\\&\leq\pi\sum_{|k|>n}f_k^2+2\pi\sum_{|k|\leq n}(f_k-p_k^{(n)})^2.\end{aligned}$$

Using the assumption for the decay of $f_k$ to estimate the first term and Lemma 6 to estimate the second term we have

$$\begin{aligned}\|f-p^{(n)}\|^2_{L^2(\partial D)}&\leq M^2\pi\sum_{|k|>n}|k|^{-2s}+2M^2C_s^2\pi\sum_{|k|\leq n}n^{-2s}\\&\leq\frac{2\pi M^2}{2s-1}n^{-(2s-1)}+6\pi M^2C_s^2n^{-(2s-1)}=\mathcal{O}(n^{-(2s-1)}),\end{aligned}$$

where we used that $\sum_{k>n}k^{-\ell}\leq\frac{1}{(\ell-1)n^{\ell-1}}$. $\square$

**Lemma 8.** *For any harmonic function $u \in C^2(D)$ with boundary data $f = u|_{\partial D}$ for which the Fourier series* (6) *converges uniformly, we have*

$$\|u\|_{L^2(D)} \leq \|f\|_{L^2(\partial D)}.$$

*Proof.* Using the orthogonality of $\{\phi_k\}$ and $\{\phi_k|_{\partial D}\}$ as well as the facts

$$\|\phi_k\|^2_{L^2(D)} = \begin{cases} \pi, & k = 0, \\ \frac{\pi}{2|k|+2}, & k \neq 0, \end{cases} \quad \|\phi_k\|^2_{L^2(\partial D)} = \begin{cases} 2\pi, & k = 0, \\ \pi, & k \neq 0 \end{cases}$$

which are easy to prove by direct calculations, we have

$$\|u\|^2_{L^2(D)} = \|\sum_{k\in\mathbb{Z}} f_k\phi_k\|^2_{L^2(D)} = \pi f_0^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\pi}{2|k|+2} f_k^2$$

and

$$\|u\|^2_{L^2(\partial D)} = 2\pi f_0^2 + \pi \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k^2,$$

from which the statement follows. □

**Remark.** Under the assumptions of Theorem 7 and using Lemma 8, we immediately obtain an $L^2$ error estimate within the unit disk, namely,

$$\|u - p^{(n)}\|_{L^2(D)} = \mathcal{O}(n^{-(s-1/2)}).$$

Our experiments have however shown that this convergence rate does not seem to be optimal: in practice, we get an additional half order of $n$, i.e., $\mathcal{O}(n^{-s})$. How to prove this observation is still an open question.

Finally we prove an error estimate in the maximum norm. We remark that, on the unit circle, the order of convergence is the same as that of the partial sums of the Fourier series of $f$.

**Theorem 9.** *Under the assumptions of Theorem 7, we have*

$$\|u - p^{(n)}\|_\infty = \mathcal{O}(n^{-(s-1)}).$$

*Proof.* We first consider the error $\max_{\theta\in(-\pi,\pi)} |f(\theta) - p^{(n)}(\theta)|$ on the unit circle. If $f_n^*$ is the best $L^2$-approximating trigonometric polynomial of degree $n$ to $f$, i.e., the truncated Fourier series $f_n^* = \sum_{|k|\leq n} f_k\phi_k|_{\partial D}$, then, for any $\theta \in (-\pi, \pi)$, we have

$$\begin{aligned} |f(\theta) - p^{(n)}(\theta)| &\leq |f(\theta) - f_n^*(\theta)| + |f_n^*(\theta) - p^{(n)}(\theta)| \\ &\leq Cn^{-(s-1)} + \sum_{|k|\leq n} |f_k - p_k^{(n)}| \cdot (\max_{x\in\partial D} |\phi_k(x)|) = \mathcal{O}(n^{-(s-1)}), \end{aligned}$$

since the harmonic basis functions $\phi_k$ restricted to the unit circle can be written $\cos(k\theta)$ or $\sin(k\theta)$ and thus are bounded by 1. The statement for the entire disk $D$ follows by the maximum principle for harmonic functions. □

## 5.2 Condition number

**Theorem 10.** *If the chords $\mathcal{I}$ form a regular convex polygon, the spectral condition number of $A^{(n)} = A$ is bounded by*

$$\kappa_2(A) \leq 2\sqrt{2}$$

*independently of $n$.*

*Proof.* Recall that $A = QF$ and $A^{-1} = EQ^\top$. We then see that

$$A^\top A = FQ^\top A = FE^{-1}EQ^\top A = FE^{-1}A^{-1}A = FE^{-1}$$

is a diagonal matrix, and thus the singular values $\sigma_k$ of $A$ are given by

$$\sigma_k^2 = \frac{\alpha_k}{\beta_k} = \alpha_k^2 \cdot \frac{1}{\alpha_k \beta_k} = \frac{4}{(k+1)^2}(1-t^2)U_k^2(t) \cdot \begin{cases} 2n+1, & k=0, \\ \frac{2n+1}{2}, & \text{else.} \end{cases}$$

Lemma 5 shows that the $\sigma_k$ are monotonically decreasing. In particular, the largest and smallest singular values are

$$\begin{aligned} \sigma_{\max} &= \sigma_0 = 2\sqrt{2n+1}\sqrt{1-t^2}, \\ \sigma_{\min} &= \sigma_n = \sqrt{2}\frac{\sqrt{2n+1}}{n+1}\sqrt{1-t^2}|U_n(t)|. \end{aligned}$$

Using Lemma 5 to estimate, we get

$$\kappa_2(A) = \frac{\sigma_{\max}}{\sigma_{\min}} = \sqrt{2}\frac{n+1}{|U_n(t)|} \leq 2\sqrt{2}.$$

□

# 6 Numerical Experiments

## 6.1 Example 1

We approximate the harmonic function

$$u(x,y) = \arctan\frac{y+2}{x+2}$$

by a harmonic polynomial $p^{(n)} \in \mathcal{H}_n$ given $2n+1$ values of its Radon projections taken along the edges of a regular $(2n+1)$-sided convex polygon (Figure 2, first picture), i.e., $I_m = I(\theta_m, t)$ as in (5). In Figure 3, we display the graph of the error function $u - p^{(12)}$. For Figure 4, we vary the degree of the interpolating polynomial and plot the resulting relative $L_2$-errors. We see that the error decreases exponentially with $n$, indicating that the smooth function $u$ is being approximated with optimal order.

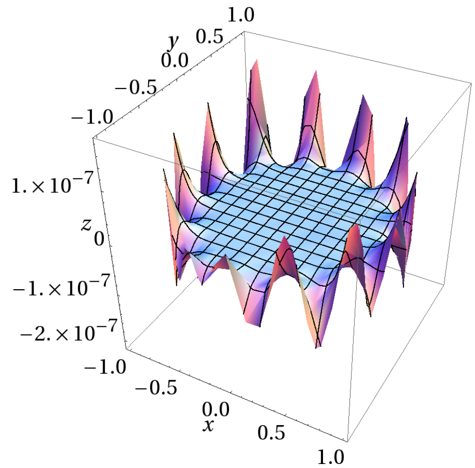



Figure 3: Example 1: error $u - p^{(12)}$

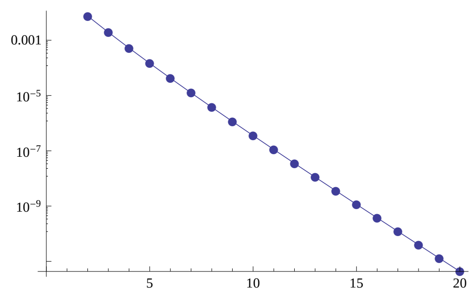



Figure 4: Example 1: errors. $x$-axis: degree of interpolating polynomial. $y$-axis: relative $L_2$-error

## 6.2 Example 2

We consider the same problem as in Example 1, but with artificially added measurement noise. For this, we add to the given values of the Radon projections random numbers from a normal distribution with zero mean and standard deviation $\epsilon$. We perform three experiments with error levels $\epsilon \in \{10^{-3}, 10^{-6}, 10^{-9}\}$. The resulting relative errors in the reconstructed function are plotted in Figure 5. We see that the input function is reconstructed to the accuracy limit given by the noise level. No amplification of the noise or instabilities are observed.

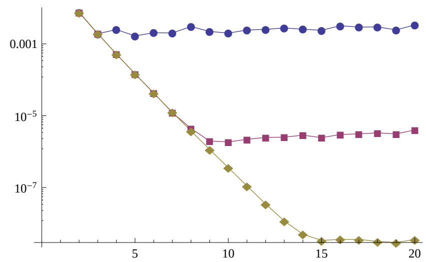
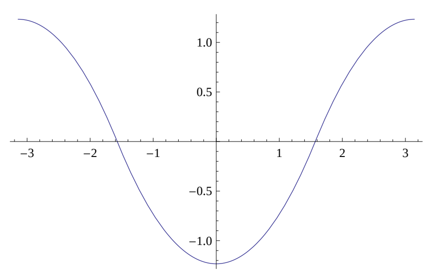



Figure 5: Example 2: errors with noisy data. Displayed are three experiments with noise levels of $10^{-3}$, $10^{-6}$, $10^{-9}$. x-axis: degree of interpolating polynomial. y-axis: relative $L_2$-error

## 6.3 Example 3

We test our method on a function which is given by the harmonic extension of the quadratic spline $f(\theta)$, $-\pi \le \theta \le \pi$, where $\theta$ is the angle on the unit circle.

$$f(\theta) = \begin{cases} -\frac{1}{2}(\theta + \frac{\pi}{2})(\theta + \frac{3}{2}\pi), & -\pi \le \theta < -\frac{\pi}{2}, \\ \frac{1}{2}(\theta - \frac{\pi}{2})(\theta + \frac{\pi}{2}), & -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ -\frac{1}{2}(\theta - \frac{\pi}{2})(\theta - \frac{3}{2}\pi), & \frac{\pi}{2} \le \theta < \pi. \end{cases}$$




Note that $f(\theta)$ is a periodic $C^1$-function with discontinuous second derivative. The resulting harmonic function $u$ has the series representation (in polar coordinates)

$$u(r, \theta) = \sum_{k=1}^{\infty} (-1)^k r^{2k-1} \frac{4 \cos((2k-1)\theta)}{(2k-1)^3 \pi}.$$

For our chords $\mathcal{I}$, we choose the edges of a regular $(2n+1)$-sided convex polygon (cf. Figure 2, first picture).

Figure 6 shows the relative $L_2$-errors for varying degree $n$ of the interpolating polynomial. The last column of the table displays the ratio between successive errors. This rate of convergence approaches 8 and thus suggests that the interpolation error is of the order $\mathcal{O}(n^{-3})$.

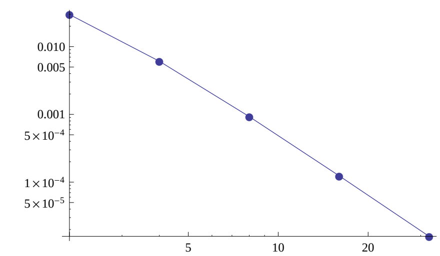



Figure 6: Example 3: log-log-plot of the relative $L_2$ errors for varying degree $n$.

| $n$ | relative $L_2$ error | rate |
|---|---|---|
| 2 | $2.97973 \cdot 10^{-2}$ | — |
| 4 | $6.08456 \cdot 10^{-3}$ | 4.90 |
| 8 | $9.26954 \cdot 10^{-4}$ | 6.56 |
| 16 | $1.23962 \cdot 10^{-4}$ | 7.47 |
| 32 | $1.58587 \cdot 10^{-5}$ | 7.82 |

Table 1: Example 3: Relative $L_2$ errors for varying degree $n$.

# 7 Conclusion and outlook

We have stated an interpolation problem for a harmonic function in the unit disk given certain values of its Radon projections. We have derived a formula for the Radon projections of certain harmonic basis polynomials which can be viewed as a harmonic analogue to the classic Marr's formula [20]. We then used this result to derive a general existence and uniqueness theorem for a class of chord schemes with constant distances $t$.

In the special case when the Radon projections are taken along chords with equally spaced angles, we are able to give an explicit formula for the inverse of the matrix corresponding to the interpolation problem and have performed a more detailed analysis of the resulting method. We have derived error estimates for the interpolation scheme on the unit circle and in the unit disk in the $L^2$- and maximum norms. The condition number of the matrix associated with the interpolation problem has been shown to be uniformly bounded by a small constant independent of the degree of the interpolating polynomial $n$.

We are able to compute the coefficients of the interpolating polynomial in slightly worse than linear time. Our numerical experiments for recovery of functions which are $C^\infty$ in the closed unit disk have shown exponential convergence. For functions with less smooth boundary data, we have observed convergence with a rate which corresponds to the analytically derived one.

In future work, we plan to derive cubature formulae for harmonic functions given Radon projection type of data and investigate error estimates for such interpolation methods and cubature rules. Some possible modifications to the problem (3) include the replacement of some of the chord integral conditions by different interpolation conditions, for instance some point values on the unit circle; smoothing over a too large data set, $|\mathcal{I}| > \dim \mathcal{H}_n$, via, e.g., least-squares minimization; the treatment of more general partial differential equations; allowing interpolation of functions satisfying an inhomogeneous partial differential equation of the form $\Delta u = f$.

## Acknowledgments

The authors acknowledge the support by Bulgarian National Science Fund, Grant DMU 03/17. The research of the first author was also supported by Bulgarian National Science Fund, Grant DDVU 0230/11. The research of the second author was funded by the Austrian Science Fund (FWF): W1214-N15, project DK04.

## References

[1] B. Bojanov and C. Draganova. Surface approximation by piecewise harmonic functions. In G. Nikolov and R. Uluchev, editors, *Constructive Theory of Functions, Sozopol 2010: In memory of Borislav Bojanov.* Prof. Marin Drinov Academic Publishing House, Sofia, 2011.

[2] B. Bojanov and I. Georgieva. Interpolation by bivariate polynomials based on Radon projections. *Studia Math.*, 162:141–160, 2004.

[3] B. Bojanov and G. Petrova. Numerical integration over a disc. A new Gaussian cubature formula. *Numer. Math.*, 80:39–59, 1998.

[4] B. Bojanov and G. Petrova. Uniqueness of the Gaussian cubature for a ball. *J. Approx. Theory*, 104:21–44, 2000.

[5] B. Bojanov and Y. Xu. Reconstruction of a bivariate polynomial from its Radon projections. *SIAM J. Math. Anal.*, 37:238–250, 2005.

[6] A.S. Cavaretta, C.A. Micchelli, and A. Sharma. Multivariate interpolation and the Radon transform, part I. *Math. Z.*, 174:263–279, 1980.

[7] A.S. Cavaretta, C.A. Micchelli, and A. Sharma. Multivariate interpolation and the Radon transform, part II. In *Quantitive Approximation*, pages 49–62, New York, 1980. Acad. Press.

[8] M.E. Davison and F.A. Grunbaum. Tomographic reconstruction with arbitrary directions. *Comm. Pure Appl. Math.*, 34:77–120, 1981.

[9] I. Georgieva, C. Hofreither, C. Koutschan, V. Pillwein, and T. Thanatipanonda. Harmonic interpolation based on Radon projections along the sides of regular polygons. 2011. To appear in CEJM. Also available as Technical Report 2011-12 in the series of the DK Computational

Mathematics Linz, https://www.dk-compmath.jku.at/publications/dk-reports/2011-10-20/view.

[10] I. Georgieva, C. Hofreither, and R. Uluchev. Interpolation of mixed type data by bivariate polynomials. In G. Nikolov and R. Uluchev, editors, *Constructive Theory of Functions, Sozopol 2010: In memory of Borislav Bojanov*, pages 93–107. Prof. Marin Drinov Academic Publishing House, Sofia, 2012. Also available as Technical Report 2010-14 in the series of the DK Computational Mathematics Linz, https://www.dk-compmath.jku.at/publications/dk-reports/2010-12-10/view.

[11] I. Georgieva and S. Ismail. On recovering of a bivariate polynomial from its Radon projections. In B. Bojanov, editor, *Constructive Theory of Functions, Varna 2005*, pages 127–134. Marin Drinov Academic Publishing House, Sofia, 2006.

[12] I. Georgieva and R. Uluchev. Smoothing of Radon projections type of data by bivariate polynomials. *J. Comput. Appl. Math.*, 215:167–181, 2008.

[13] I. Georgieva and R. Uluchev. Surface reconstruction and Lagrange basis polynomials. In I. Lirkov, S. Margenov, and J. Waśniewski, editors, *Large-Scale Scientific Computing 2007*, pages 670–678, Berlin, Heidelberg, 2008. Springer-Verlag.

[14] I. Georgieva and R. Uluchev. On interpolation in the unit disk based on both Radon projections and function values. In I. Lirkov, S. Margenov, and J. Waśniewski, editors, *Large-Scale Scientific Computing 2009*, pages 747–755, Berlin, Heidelberg, 2010. Springer-Verlag.

[15] H. Hakopian. Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type. *J. Approx. Theory*, 34:286–305, 1982.

[16] C. Hamaker and D.C. Solmon. The angles between the null spaces of $x$-rays. *J. Math. Anal. Appl.*, 62:1–23, 1978.

[17] A.K. Jain. *Fundamentals of Digital Image Processing*. Prentice Hall, 1989.

[18] F. John. Abhängigkeiten zwischen den Flächenintegralen einer stetigen Funktion. *Math. Anal.*, 111:541–559, 1935.

[19] B. Logan and L. Shepp. Optimal reconstruction of a function from its projections. *Duke Math. J.*, 42:645–659, 1975.

[20] R. Marr. On the reconstruction of a function on a circular domain from a sampling of its line integrals. *J. Math. Anal. Appl.*, 45:357–374, 1974.

[21] F. Natterer. *The Mathematics of Computerized Tomography*. Classics in Applied Mathematics 32. SIAM, 2001.

[22] G. Nikolov. Cubature formulae for the disk using Radon projections. *East journal on approximations*, 14:401–410, 2008.

[23] J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verch. Sächs. Akad.*, 69:262–277, 1917.

[24] J. L. Walsh. The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions. *Bull. Amer. Math. Soc*, 1929.

[25] A. Zygmund. *Trigonometric Series.* Cambridge University Press, third edition, 2002.

# Technical Reports of the Doctoral Program
# "Computational Mathematics"

## 2012

**2012-01** M.T. Khan: *Formal Semantics of* MiniMaple January 2012. Eds.: W. Schreiner, F. Winkler

**2012-02** M. Kollmann, W. Zulehner: *A Robust Preconditioner for Distributed Optimal Control for Stokes Flow with Control Constraints* January 2012. Eds.: U. Langer, R. Ramlau

**2012-03** W. Krendl, V. Simoncini, W. Zulehner: *Stability Estimates and Structural Spectral Properties of Saddle Point Problems* February 2012. Eds.: U. Langer, V. Pillwein

**2012-04** V. Pillwein, S. Takacs: *A local Fourier convergence analysis of a multigrid method using symbolic computation* April 2012. Eds.: M. Kauers, W. Zulehner

**2012-05** I. Georgieva, C. Hofreither: *Tomographic Reconstruction of Harmonic Functions* April 2012. Eds.: U. Langer, V. Pillwein

**2012-06** M.T. Khan: *Formal Semantics of a Specification Language for* MiniMaple April 2012. Eds.: W. Schreiner, F. Winkler

**2012-07** M. Borges-Quintana, M.A. Borges-Trenard, I. Márquez-Corbella and E. Martínez-Moro: *Computing coset leaders and leader codewords of binary codes* May 2012. Eds.: F. Winkler, V. Pillwein

**2012-08** S. Takacs and W. Zulehner: *Convergence analysis of all-at-once multigrid methods for elliptic control problems under partial elliptic regularity* June 2012. Eds.: U. Langer, R. Ramlau

**2012-09** C. Dönch and A. Levin: *Computation of the Strength of PDEs of Mathematical Physics and their Difference Approximations* July 2012. Eds.: F. Winkler, J. Schicho

**2012-10** S. Radu and J. Sellers: *Congruences Modulo Squares of Primes for Fu's Dots Bracelet Partitions* August 2012. Eds.: M. Kauers, V. Pillwein

**2012-11** I. Georgieva and C. Hofreither: *Interpolation of Harmonic Functions Based on Radon Projections* October 2012. Eds.: U. Langer, V. Pillwein

The complete list since 2009 can be found at
`https://www.dk-compmath.jku.at/publications/`

## Doctoral Program
## "Computational Mathematics"


**Director:**

Prof. Dr. Peter Paule
Research Institute for Symbolic Computation

**Deputy Director:**

Prof. Dr. Bert Jüttler
Institute of Applied Geometry

**Address:**

Johannes Kepler University Linz
Doctoral Program "Computational Mathematics"
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-6840

**E-Mail:**

office@dk-compmath.jku.at

**Homepage:**

http://www.dk-compmath.jku.at