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Abstract. In this paper, we present and discuss the results of our numerical studies of preconditioned MinRes methods for solving the optimality systems arising from the multiharmonic finite element approximations to time-periodic eddy current optimal control problems in different settings including different observation and control regions, different tracking terms as well as box constraints for the Fourier coefficients of the state and the control. These numerical studies confirm the theoretical results published by the first author in a recent paper.

Keywords: time-periodic eddy current optimal control problems, multiharmonic finite element discretization, MinRes solver, preconditioners.

1 Introduction

This work is devoted to the study of efficient solution procedures for the following time-periodic eddy current optimal control problem: Minimize the functional

$$\begin{aligned} J(\boldsymbol{y},\boldsymbol{u}) = \frac{1}{2}\int_{\Omega_1\times(0,T)} |\boldsymbol{y}-\boldsymbol{y_d}|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{dt} + \frac{\alpha}{2}\int_{\Omega_1\times(0,T)} |\,\mathbf{curl}\,\boldsymbol{y}-\boldsymbol{y_c}|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{dt} \\ + \frac{\lambda}{2}\int_{\Omega_2\times(0,T)} |\boldsymbol{u}|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{dt}, \end{aligned} \tag{1}$$

subject to the state equations

$$\begin{cases} \sigma\dfrac{\partial \boldsymbol{y}}{\partial t} + \mathbf{curl}(\nu\;\mathbf{curl}\,\boldsymbol{y}) = \boldsymbol{u}, & \text{in } \Omega\times(0,T), \\ \mathrm{div}(\sigma\boldsymbol{y}) = 0, & \text{in } \Omega\times(0,T), \\ \boldsymbol{y}\times\boldsymbol{n} = \boldsymbol{0}, & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{y}(0) = \boldsymbol{y}(T), & \text{in } \Omega, \end{cases} \tag{2}$$

where Ω is a bounded, simply connected Lipschitz domain with the boundary $\partial\Omega$. The domains Ω_1 and Ω_2 are non-empty Lipschitz subdomains of Ω, i.e.,

$\Omega_1, \Omega_2 \subset \Omega \subset \mathbb{R}^3$. The reluctivity $\nu \in L^\infty(\Omega)$ and the conductivity $\sigma \in L^\infty(\Omega)$ are supposed to be uniformly positive, i.e.,

$$0 < \nu_{\min} \leq \nu(\boldsymbol{x}) \leq \nu_{\max}, \quad \text{and} \quad 0 < \sigma_{\min} \leq \sigma(\boldsymbol{x}) \leq \sigma_{\max}, \qquad \boldsymbol{x} \in \Omega.$$

We mention that the electric conductivity σ vanishes in regions consisting of non-conducting materials. In order to fulfill the assumption made above on the uniform positivity of σ, one can replace $\sigma(x)$ by $\max\{\epsilon, \sigma(x)\}$ with some suitably chosen positive ϵ, see, e.g., [10,12] for more details. We here assume that the reluctivity ν is independent of $|\,\mathbf{curl}\,\boldsymbol{y}|$, i.e. we only consider linear eddy current problems. The regularization parameter $\lambda > 0$, the weight parameter $\alpha \geq 0$, and $\boldsymbol{y_d}$, $\boldsymbol{y_c} \in L_2((0,T), \boldsymbol{L_2}(\Omega))$ are given data, where $\boldsymbol{y_d}$ represents the desired state and $\boldsymbol{y_c}$ represents the desired **curl** of the state.

The problem setting (1)-(2) has been analyzed in [11,12], wherein, due to the time-periodic structure, a time-discretization in terms of a truncated Fourier series, also called multiharmonic approach, is used. In [12], we consider the special case of a fully distributed optimal control problem for tracking some $\boldsymbol{y_d}$ in the complete computational domain, i.e. $\Omega_1 = \Omega_2 = \Omega$ and $\alpha = 0$ in (1), whereas [11] is devoted to the various other settings including different observation and control regions, different tracking terms as well as box constraints for the Fourier coefficients of the state and the control. Similar optimal control problems for time-periodic parabolic equations and their numerical treatment by means of the multiharmonic Finite Element Method (FEM) have recently been considered in [9] and [8]. Other approaches to time-periodic parabolic optimal control problems have been discussed in [1]. There are many publications on optimal control problems with PDE constraints given by initial-boundary value problems for parabolic equations, see, e.g., [14] for a comprehensive presentation. There are less publications on optimal control problems where initial-boundary value problems for eddy current equations are consider as PDE constraints, see, e.g., [16,15] where one can also find interesting applications. The multiharmonic approach allows us to switch from the time domain to the frequency domain, and, therefore, to replace a time-dependent problem by a system of time-independent problems for the Fourier coefficients. Since we are here interested in studying robust solvers, this special time-discretization technique justifies the following assumption: Let us assume, that the desired states $\boldsymbol{y_d}$ and $\boldsymbol{y_c}$ are multiharmonic, i.e., $\boldsymbol{y_d}$ and $\boldsymbol{y_c}$ have the form of a truncated Fourier series:

$$\begin{aligned} \boldsymbol{y_d} &= \sum_{k=0}^{N} \boldsymbol{y^c_{d,k}} \cos(k\omega t) + \boldsymbol{y^s_{d,k}} \sin(k\omega t), \\ \boldsymbol{y_c} &= \sum_{k=0}^{N} \boldsymbol{y^c_{c,k}} \cos(k\omega t) + \boldsymbol{y^s_{c,k}} \sin(k\omega t). \end{aligned} \tag{3}$$

Consequently, the state $\boldsymbol{y}$ and the control $\boldsymbol{u}$ are multiharmonic as well, and, therefore, have a representation in terms of a truncated Fourier series with the same number of modes N, i.e.,

$$\begin{aligned} \boldsymbol{y} &= \sum_{k=0}^{N} \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}} \cos(k\omega t) + \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}} \sin(k\omega t), \\ \boldsymbol{u} &= \sum_{k=0}^{N} \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{c}} \cos(k\omega t) + \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{s}} \sin(k\omega t). \end{aligned} \tag{4}$$

Using the multiharmonic representation of $\boldsymbol{y_d}$, $\boldsymbol{y_c}$, $\boldsymbol{y}$ and $\boldsymbol{u}$, the minimization problem (1)-(2) can be state in the frequency domain: Minimize the functional

$$\begin{aligned} J_N = \frac{1}{2} \sum_{k=0}^{N} \Big[\sum_{j\in\{c,s\}} \Big[\int_{\Omega_1} |\boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{j}} - \boldsymbol{y}_{\boldsymbol{d,k}}^{\boldsymbol{j}}|^2 \mathrm{d}\boldsymbol{x} + \alpha \int_{\Omega_1} |\,\mathbf{curl}\, \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{j}} - \boldsymbol{y}_{\boldsymbol{c,k}}^{\boldsymbol{j}}|^2 \mathrm{d}\boldsymbol{x} \\ + \lambda \sum_{j\in\{c,s\}} \int_{\Omega_2} |\boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{j}}|^2 \mathrm{d}\boldsymbol{x} \Big]\Big], \end{aligned} \tag{5a}$$

subject to the state equation

$$\begin{cases} k\omega\, \sigma \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}} + \mathbf{curl}(\nu\ \mathbf{curl}\, \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}}) = \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{c}}, & \text{in } \Omega, k = 1, \dots, N, \\ -k\omega\, \sigma \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}} + \mathbf{curl}(\nu\ \mathbf{curl}\, \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}}) = \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{s}}, & \text{in } \Omega, k = 1, \dots, N, \\ \mathbf{curl}(\nu\ \mathbf{curl}\, \boldsymbol{y}_{\boldsymbol{0}}^{\boldsymbol{c}}) = \boldsymbol{u}_{\boldsymbol{0}}^{\boldsymbol{c}}, & \text{in } \Omega, \\ \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}} \times \boldsymbol{n} = \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \partial\Omega, k = 1, \dots, N, \\ \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{0}} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases} \tag{5b}$$

completed by the divergence constraints

$$\begin{cases} k\omega \operatorname{div}(\sigma \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}}) = 0, & \text{in } \Omega, k = 1, \dots, N, \\ k\omega \operatorname{div}(\sigma \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}}) = 0, & \text{in } \Omega, k = 1, \dots, N, \\ \operatorname{div}(\sigma \boldsymbol{y}_{\boldsymbol{0}}^{\boldsymbol{c}}) = 0, & \text{in } \Omega. \end{cases} \tag{5c}$$

Additionally, we add control constraints associated to the Fourier coefficients of the control $\boldsymbol{u}$, i.e.

$$\begin{aligned} \underline{\boldsymbol{u}}_{\boldsymbol{k}}^{\boldsymbol{c}} \le \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{c}} \le \overline{\boldsymbol{u}}_{\boldsymbol{k}}^{\boldsymbol{c}}, \quad \text{a.e. in } \Omega, k = 0, 1, \dots, N, \\ \underline{\boldsymbol{u}}_{\boldsymbol{k}}^{\boldsymbol{s}} \le \boldsymbol{u}_{\boldsymbol{k}}^{\boldsymbol{s}} \le \overline{\boldsymbol{u}}_{\boldsymbol{k}}^{\boldsymbol{s}}, \quad \text{a.e. in } \Omega, k = 1, \dots, N, \end{aligned} \tag{5d}$$

and state constraints associated to the Fourier coefficients of the state $\boldsymbol{y}$, i.e.

$$\begin{aligned} \underline{\boldsymbol{y}}_{\boldsymbol{k}}^{\boldsymbol{c}} \le \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{c}} \le \overline{\boldsymbol{y}}_{\boldsymbol{k}}^{\boldsymbol{c}}, \quad \text{a.e. in } \Omega, k = 0, 1, \dots, N, \\ \underline{\boldsymbol{y}}_{\boldsymbol{k}}^{\boldsymbol{s}} \le \boldsymbol{y}_{\boldsymbol{k}}^{\boldsymbol{s}} \le \overline{\boldsymbol{y}}_{\boldsymbol{k}}^{\boldsymbol{s}}, \quad \text{a.e. in } \Omega, k = 1, \dots, N. \end{aligned} \tag{5e}$$

This minimization problem is typically solved by deriving the corresponding optimality system, which is then discretized in space by means of the FEM. Since even the simple box constraints (5d)-(5e) give rise to nonlinear optimality systems, we apply a primal dual active set strategy (semi-smooth Newton) approach for their solution [5]. The resulting procedure is summarized in Algorithm 1.

Input: number of modes N, initial guesses $\boldsymbol{x}^{(k,0)} \in \mathbb{R}^n (k = 0, \ldots, N)$.
Output: approximate solution $\boldsymbol{x}^{(k,l)} \in \mathbb{R}^n (k = 0, \ldots, N)$.
for $k \leftarrow 0$ **to** N **do**
  Determine the active sets $\mathcal{E}^c_{k,0}$ and $\mathcal{E}^s_{k,0}$;
end
Set $l := 0$;
while *not converged* **do**
  for $k \leftarrow 0$ **to** N **do**
    Compute $\boldsymbol{b}_{\mathcal{E}}^{(k,l+1)}$, $\mathcal{A}_{\mathcal{E}}^{(k,l+1)}$;
    Solve $\mathcal{A}_{\mathcal{E}}^{(k,l+1)} \boldsymbol{x}^{(k,l+1)} = \boldsymbol{b}_{\mathcal{E}}^{(k,l+1)}$;
    Determine the active sets $\mathcal{E}^c_{k,l+1}$ and $\mathcal{E}^s_{k,l+1}$;
  end
  Set $l := l + 1$;
end

Algorithm 1: Primal dual active set strategy.

The specific structure of the Jacobi matrix $\mathcal{A}_{\mathcal{E}}^{(k,l+1)}$ depends on the actual computational setting. In our applications $\mathcal{A}_{\mathcal{E}}^{(k,l+1)}$ obtains either the form $\mathcal{A}_1$, cf. (6a), or the form $\mathcal{A}_2$, cf. (6b). It is clear, that the efficient and parameter-robust solution of the $(N+1)$ linear systems of equations at each semi-smooth Newton step are essential for the efficiency of the proposed method. For further details we refer to [11].

2 Parameter-robust and efficient solution procedures

In order to discretize the problems in space, we use the edge (Nédélec) finite element space $\mathcal{ND}_0^0(\mathcal{T}_h)$, that is a conforming finite element subspace of $\boldsymbol{H_0}(\mathbf{curl}, \Omega)$, and the nodal (Lagrange) finite element space $\mathcal{S}_0^1(\mathcal{T}_h)$, that is a conforming finite element subspace of $H_0^1(\Omega)$. Let $\{\boldsymbol{\varphi}_\mathbf{i}\}_{i=1,N_h}$ and $\{\psi_i\}_{i=1,M_h}$ denote the usual edge basis of $\mathcal{ND}_0^0(\mathcal{T}_h)$ and the usual nodal basis of $\mathcal{S}_0^1(\mathcal{T}_h)$, respectively. We are now in the position to define the following FEM matrices:

$$\begin{aligned}
(\boldsymbol{K_\nu})_{ij} &= (\nu\, \mathbf{curl}\, \boldsymbol{\varphi}_\mathbf{i}, \mathbf{curl}\, \boldsymbol{\varphi}_\mathbf{j})_{\mathbf{0},\Omega}, \\
(\boldsymbol{M_{\sigma,k\omega}})_{ij} &= k\omega(\sigma \boldsymbol{\varphi}_\mathbf{i}, \boldsymbol{\varphi}_\mathbf{j})_{\mathbf{0},\Omega}, \\
(\boldsymbol{M})_{ij} &= (\boldsymbol{\varphi}_\mathbf{i}, \boldsymbol{\varphi}_\mathbf{j})_{\mathbf{0},\Omega}, \\
(\boldsymbol{D_{\sigma,k\omega}})_{ij} &= k\omega(\sigma \boldsymbol{\varphi}_\mathbf{i}, \nabla \psi_j)_{\mathbf{0},\Omega},
\end{aligned}$$

where $(\cdot,\cdot)_{\mathbf{0},\Omega}$ denotes the inner product in $\boldsymbol{L_2}(\Omega)$. Throughout this paper we are repeatedly faced with the following two types of system matrices:

$$\mathcal{A}_1 = \begin{pmatrix} * & \mathbf{0} & \boldsymbol{K_\nu} & -\boldsymbol{M_{\sigma,k\omega}} \\ \mathbf{0} & * & \boldsymbol{M_{\sigma,k\omega}} & \boldsymbol{K_\nu} \\ \boldsymbol{K_\nu} & \boldsymbol{M_{\sigma,k\omega}} & -\lambda^{-1}* & \mathbf{0} \\ -\boldsymbol{M_{\sigma,k\omega}} & \boldsymbol{K_\nu} & \mathbf{0} & -\lambda^{-1}* \end{pmatrix} \tag{6a}$$

$$
\mathcal{A}_2 = \begin{pmatrix}
* & \mathbf{0} & \boldsymbol{K_\nu} & -\boldsymbol{M_{\sigma,k\omega}} & \mathbf{0} & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}}^T & \mathbf{0} \\
\mathbf{0} & * & \boldsymbol{M_{\sigma,k\omega}} & \boldsymbol{K_\nu} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}}^T \\
\boldsymbol{K_\nu} & \boldsymbol{M_{\sigma,k\omega}} & -\lambda^{-1}* & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\boldsymbol{M_{\sigma,k\omega}} & \boldsymbol{K_\nu} & \mathbf{0} & -\lambda^{-1}* & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}}^T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{D_{\sigma,k\omega}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{D_{\sigma,k\omega}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \boldsymbol{D_{\sigma,k\omega}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}. \tag{6b}
$$

Therein the placeholder $*$ stands for a symmetric and positive semi-definite matrix, that actually depends on the considered setting, cf. Table 1. The family of problems, that are related with the matrix type $\mathcal{A}_1$ and $\mathcal{A}_2$ are referred as *Formulation OC-FEM 1* and *Formulation OC-FEM 2*, respectively. In fact, the system matrix $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric and indefinite, and obtain a double- or three-fold saddle point structure, respectively. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric, the corresponding systems can be solved by a preconditioned Minimal Residual (MinRes) method, cf. [13]. Typically, the convergence rate of any iterative Krylov subspace method applied to the unpreconditioned system deteriorates with respect to the meshsize h, the parameters $k = 0, 1, \ldots, N$ and ω involved in the spectral time-discretization, and the problem parameters ν, σ and λ, cf. also Tables 2 and 3. Therefore, preconditioning is an important issue.

The proper choice of parameter-robust and efficient preconditioners has been addressed by the authors in [11,12]. While for equations with system matrices of type (6a), we propose to use the preconditioner

$$
\mathcal{C} := \operatorname{diag}\left(\sqrt{\lambda}\boldsymbol{F}, \sqrt{\lambda}\boldsymbol{F}, \frac{1}{\sqrt{\lambda}}\boldsymbol{F}, \frac{1}{\sqrt{\lambda}}\boldsymbol{F}\right), \tag{7}
$$

with the block $\boldsymbol{F} = \boldsymbol{K_\nu} + \boldsymbol{M_{\sigma,k\omega}} + 1/\sqrt{\lambda}\boldsymbol{M}$, for equations with system matrices of type (6b), we advise to use the preconditioner

$$
\mathcal{C}_M = \operatorname{diag}\left(\sqrt{\lambda}\boldsymbol{F}, \sqrt{\lambda}\boldsymbol{F}, \frac{1}{\sqrt{\lambda}}\boldsymbol{F}, \frac{1}{\sqrt{\lambda}}\boldsymbol{F}, \frac{1}{\sqrt{\lambda}}\boldsymbol{S_J}, \frac{1}{\sqrt{\lambda}}\boldsymbol{S_J}, \sqrt{\lambda}\boldsymbol{S_J}, \sqrt{\lambda}\boldsymbol{S_J}\right), \tag{8}
$$

where $\boldsymbol{S_J} = \boldsymbol{D_{\sigma,k\omega}}^T \boldsymbol{F}^{-1} \boldsymbol{D_{\sigma,k\omega}}$. In a MinRes setting, the quality of the preconditioners $\mathcal{C}$ and $\mathcal{C}_M$, used for the system matrices $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, is in general determined by the condition number κ_1 or κ_2 of the preconditioned system, defined as follows:

$$
\kappa_1 := \|\mathcal{C}^{-1}\mathcal{A}_1\|_{\mathcal{C}} \|\mathcal{A}_1^{-1}\mathcal{C}\|_{\mathcal{C}} \quad \text{and} \quad \kappa_2 := \|\mathcal{C}_M^{-1}\mathcal{A}_2\|_{\mathcal{C}_M} \|\mathcal{A}_2^{-1}\mathcal{C}_M\|_{\mathcal{C}_M}. \tag{9}
$$

In Table 1, we list the theoretical results, that have been derived for different settings of (5) in [11,12]. We especially want to point out, that the bounds for the condition numbers are at least uniform in the space discretization parameter h as well as the time discretization parameters ω and N. This has the important consequence, that the proposed preconditioned MinRes method converges within a few iterations, independent of the discretization parameters that are directly related to the size of the system matrices.

Table 1. Condition number estimates for different settings. Here (σ) denotes robustness with respect to $\sigma \in \mathbb{R}^+$.

	parameters	domains	equations	condition number estimate
I	$\alpha = 0$	$\Omega_1 = \Omega_2$	(5a)-(5b)	$\kappa_1 \leq \sqrt{3} \neq c(h, \omega, N, \sigma, \nu, \lambda)$
II	$\alpha = 0$	$\Omega_1 = \Omega_2$	(5a)-(5c)	$\kappa_2 \leq \sqrt{3}(1+\sqrt{5}) \neq c(h, \omega, N, \sigma, \nu, \lambda)$
III	$\alpha = \infty$	$\Omega_1 = \Omega_2$	(5a)-(5c)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma))$
IV	$\alpha = 0$	$\Omega_1 \neq \Omega_2$	(5a)-(5c)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma), \Omega_1, \Omega_2)$
V	$\alpha = 0$	$\Omega_1 = \Omega_2$	(5a)-(5d)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma), \text{index sets})$
VI	$\alpha = 0$	$\Omega_1 = \Omega_2$	(5a)-(5b) + (5e)	$\kappa_1 \leq c \neq c(h, \omega, N, \sigma, \nu, \lambda, \text{index sets})$

3 Numerical validation

The main aim of this paper is to verify the theoretical proven convergence rates by numerical experiments. We consider an academic test problem of the form (1)-(2) or rather (5) in the unit cube $\Omega = (0,1)^3$, and report on various numerical test for various computational settings and varying parameters. Since we are here only interested in the study of the robustness of the solver, it is obviously sufficient to consider the solution of the system corresponding to the block of the mode $k = 1$. The numerical results presented in this section were attained using ParMax[3]. We demonstrate the robustness of the block-diagonal preconditioners with respect to the involved parameters. Therefore, for the solution of the preconditioning equations arising from the diagonal blocks, we use the sparse direct solver UMFPACK[4] that is very efficient for several thousand unknowns in the case of three dimensional problems [2,3,4]. For numerical tests, where the diagonal blocks are realized by an auxiliary space preconditioner [6,7], we refer the reader to [10].

3.1 Test case I

Tables 2-5 provide the number of MinRes iterations needed for reducing the initial residual by a factor of 10^{-8}. These experiments demonstrate the independence of the MinRes convergence rate of the parameters ω, σ, λ and the mesh size h since the number of iterations is bounded by 28 for all computed constellations. We mention that varying ω also covers the variation of $k\omega$ in terms of k. Furthermore, in Table 2 and Table 3 we also report the number of unpreconditioned MinRes iterations, that are necessary for reducing the initial residual by a factor of 10^{-8}. The large number of iterations in the unpreconditioned case underline the importance of appropriate preconditioning.

[3] `http://www.numa.uni-linz.ac.at/P19255/software.shtml`
[4] `http://www.cise.ufl.edu/research/sparse/umfpack/`

Table 2. Formulation OC-FEM 1 for test case I. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMF-PACK for $\boldsymbol{F}$ (DOF = 2416, $\nu = \sigma = 1$), [$\cdot$] number of MinRes iterations without preconditioner.

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	7	7	7	7	7	7	7	7	7	6	4
	[587]	[587]	[586]	[587]	[587]	[587]	[587]	[591]	[485]	[263]	[116]
10^{-6}	21	21	21	21	21	21	20	12	6	4	4
	[373]	[373]	[373]	[373]	[373]	[373]	[373]	[263]	[116]	[114]	[114]
10^{-2}	20	20	20	20	20	20	20	12	6	4	4
	[1134]	[1134]	[1134]	[1136]	[1135]	[1134]	[227]	[114]	[114]	[114]	[114]
1	10	10	10	10	10	14	20	12	6	4	4
	[2349]	[2351]	[2349]	[2350]	[2350]	[2274]	[222]	[114]	[114]	[114]	[114]
10^{2}	6	6	6	6	8	10	20	12	6	4	4
	[2688]	[2681]	[2696]	[2667]	[3291]	[2494]	[224]	[114]	[114]	[114]	[114]
10^{6}	4	4	4	6	6	10	20	12	6	4	4
	[1152]	[1159]	[3434]	[4697]	[4867]	[2493]	[222]	[114]	[114]	[114]	[114]
10^{10}	2	4	4	4	4	10	20	12	6	4	4
	[1157]	[1163]	[4937]	[5881]	[4791]	[2501]	[224]	[114]	[114]	[114]	[114]

Table 3. Formulation OC-FEM 1 for test case I. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMF-PACK for $\boldsymbol{F}$ (DOF = 16736, $\nu = \sigma = 1$), [$\cdot$] number of MinRes iterations without preconditioner. [-] indicates that MinRes did not converge within 10000 iterations.

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	9	9	9	9	9	9	9	10	6	4	4
	[708]	[708]	[708]	[708]	[708]	[708]	[708]	[711]	[578]	[308]	[134]
10^{-6}	21	21	21	21	21	21	20	18	6	4	4
	[825]	[824]	[825]	[825]	[825]	[825]	[824]	[307]	[134]	[132]	[132]
10^{-2}	18	18	18	18	18	20	22	20	6	4	4
	[6698]	[6669]	[6696]	[6698]	[6690]	[6676]	[1095]	[132]	[132]	[132]	[132]
1	10	10	10	10	10	14	22	20	6	4	4
	[-]	[-]	[-]	[-]	[-]	[-]	[1094]	[132]	[132]	[132]	[132]
10^{2}	6	6	6	6	8	10	22	20	6	4	4
	[-]	[-]	[-]	[-]	[-]	[-]	[1094]	[132]	[132]	[132]	[132]
10^{6}	4	4	4	6	6	10	22	20	6	4	4
	[7365]	[7547]	[-]	[-]	[-]	[-]	[1094]	[132]	[132]	[132]	[132]
10^{10}	2	4	4	4	4	10	22	20	6	4	4
	[7381]	[1545]	[-]	[-]	[-]	[-]	[1094]	[132]	[132]	[132]	[132]

Table 4. Formulation OC-FEM 1 for test case I. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMF-PACK for $\boldsymbol{F}$ ($DOF = 124096$, $\nu = \sigma = 1$).

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	13	13	13	13	13	13	13	13	8	4	4
10^{-8}	21	21	21	21	21	21	21	17	8	4	4
10^{-6}	21	21	21	21	21	21	21	20	8	4	4
10^{-4}	20	20	20	20	20	20	28	22	8	4	4
10^{-2}	16	16	16	16	16	18	22	22	8	4	4
1	10	10	10	10	10	12	20	22	8	4	4
10^{2}	6	6	6	6	8	10	20	22	8	4	4
10^{4}	4	4	4	6	6	10	20	22	8	4	4
10^{6}	4	4	4	4	6	10	20	22	8	4	4
10^{8}	2	4	4	4	6	10	20	22	8	4	4
10^{10}	3	4	4	4	4	10	20	22	8	4	4

Table 5. Formulation OC-FEM 1 for test case I. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMF-PACK for $\boldsymbol{F}$ ($DOF = 124096$, $\omega = \sigma = 1$).

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	2	2	3	3	5	13	21	16	6	4	3
10^{-8}	2	2	3	4	7	21	20	10	4	4	3
10^{-6}	2	3	3	5	13	21	16	6	4	4	4
10^{-4}	2	3	4	7	21	20	10	6	4	4	4
10^{-2}	3	4	6	13	21	18	8	4	4	6	6
1	4	4	8	17	28	12	6	4	6	6	9
10^{2}	4	4	8	20	22	10	6	4	6	6	8
10^{4}	4	4	8	22	20	10	6	4	4	4	8
10^{6}	4	4	8	22	20	10	4	4	4	4	8
10^{8}	4	4	8	22	20	10	4	4	4	4	8
10^{10}	4	4	8	22	20	10	4	2	4	4	8

3.2 Test case II

Table 6 and Table 7 provide the number of MinRes iterations needed for reducing the initial residual by a factor 10^{-8}. These experiments demonstrate the independence of the MinRes convergence rate of the parameters ω, σ, λ and the mesh size h since the number of iterations is bounded by 88 for all computed constellations.

3.3 Test case III

Numerical results for the observation of the magnetic flux density are reported in Table 8-11. The robustness with respect to the space and time discretization parameters h and ω is demonstrated in Table 8. Table 9 and Table 10 describe the non-robust behavior with respect to the parameters λ and ν. In Table 11 we observe that for large mesh sizes, good iteration numbers are observed even for small λ. Nevertheless, for fixed λ, the iteration numbers are growing with respect to the involved degrees of freedom.

The next experiment demonstrates, that robustness with respect to the time discretization parameter ω cannot be achieved by using the preconditioner $\mathcal{C}$ in *Formulation OC-FEM 1*. In Table 12 the number of MinRes iteration needed for reducing the initial residual by a factor of 10^{-8} are tabled. In Table 13, the same experiment as in Table 9 is performed, but using *Formulation OC-FEM 1* instead of *Formulation OC-FEM 2*. Indeed, comparing Table 8 with Table 12 and Table 9 with Table 13 clearly shows, that it is essential to work with *Formulation OC-FEM 2*. Beside the robustness with respect to the frequency ω, that is related to the time discretization parameters, we additionally observe better iteration numbers with respect to the regularization parameter λ in the interesting region $0 < \lambda < 1$.

3.4 Test case IV

In this subsection we consider a numerical example with different observation and control domains Ω_1 and Ω_2, i.e., $\Omega_1 = \Omega = (0,1)^3$ and $\Omega_2 = (0.25, 0.75)^3$. Let us mention that we have to ensure, that Ω_1 and Ω_2 are resolved by the mesh. The corresponding numerical results are documented in Table 14-18. Robustness with respect to the space and time discretization parameters h and ω is demonstrated in Table 14. Table 15 describes the non-robust behavior with respect to the parameters λ and ν. Table 16 in combination with Table 18 indicates, that, for the *Formulation OC-FEM 1* in combination with the preconditioner $\mathcal{C}$, robustness with respect to the frequency ω, that is related to the time discretization parameters, cannot be obtained. Here, we want to mention, that the good iteration numbers observed in Table 16 are caused by the special choice of $\lambda = 1$.

Table 6. Formulation OC-FEM 2 for test case II. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 19652$, $\nu = \sigma = 1$).

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	21	19	19	17	17	17	17	17	12	8	10
10^{-6}	33	33	33	33	33	33	29	33	10	8	8
10^{-2}	22	22	22	22	26	31	34	32	14	12	10
1	12	13	14	14	14	14	24	22	10	8	8
10^{2}	11	11	13	13	13	18	34	32	14	12	10
10^{6}	13	13	13	17	21	28	56	50	22	14	14
10^{10}	31	34	34	23	33	42	80	78	30	20	16

Table 7. Formulation OC-FEM 2 for test case II. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 143748$, $\nu = \sigma = 1$).

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	27	27	25	25	25	25	25	25	16	8	10
10^{-6}	32	32	32	32	32	32	33	35	14	8	8
10^{-2}	20	20	20	20	23	29	35	34	16	12	10
1	12	12	14	14	14	14	24	26	12	8	8
10^{2}	11	11	13	13	13	18	34	34	16	12	10
10^{6}	13	13	15	17	21	30	58	60	24	16	14
10^{10}	46	61	65	23	33	42	88	88	38	24	16

Table 8. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 2 for test case III. Number of MinRes iterations for different values of ω and various DOF using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($\nu = \sigma = \lambda = 1$).

DOF	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
500	13	13	14	14	14	16	23	12	9	8	7
2916	11	12	13	13	13	15	29	16	10	8	8
19652	11	11	12	12	12	14	30	21	11	8	8
143748	11	11	12	12	12	14	28	27	13	8	8

Table 9. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 2 for test case III. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 19652$, $\sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	174	175	175	176	175	213	290	68	14	10	8
10^{-6}	146	146	146	146	177	215	58	12	8	6	8
10^{-2}	272	272	272	289	306	55	13	10	9	10	13
1	290	290	290	292	240	14	8	6	8	10	12
10^{2}	475	474	479	448	83	18	12	10	14	14	26
10^{6}	193	193	195	179	55	28	18	24	24	26	360
10^{10}	36	38	39	77	84	42	26	36	50	264	[-]

Table 10. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 2 for test case III. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 143748$, $\sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	325	324	326	326	327	411	505	65	14	10	8
10^{-6}	289	289	289	289	359	392	53	12	10	6	8
10^{-2}	543	543	543	561	523	52	13	10	8	11	15
1	543	544	541	564	325	14	8	6	8	10	14
10^{2}	948	949	941	861	79	18	12	10	14	14	36
10^{6}	688	688	680	377	55	30	18	22	26	40	[-]
10^{10}	56	56	55	91	88	42	26	38	54	[-]	[-]

Table 11. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 2 for test case III. Number of MinRes iterations for different values of λ and various DOF using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($\nu = \sigma = \omega = 1$).

DOF	λ										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
500	36	36	37	39	40	16	19	26	30	36	44
2916	115	113	121	121	55	15	18	24	28	38	44
19652	213	214	215	195	55	14	18	24	28	36	42
143748	411	402	392	265	52	14	18	24	30	36	42

Table 12. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 1 for test case III. Number of MinRes iterations for different values of ω and various DOF using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($\nu = \sigma = \lambda = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

DOF	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
392	4133	[-]	46	20	16	15	21	9	5	4	3
2416	[-]	[-]	64	29	15	13	27	12	6	4	4
16736	[-]	[-]	102	28	15	13	26	18	7	4	4
124096	[-]	[-]	28	13	12	26	24	9	5	4	4

Table 13. Observation of the magnetic flux density $\boldsymbol{B}$ in Formulation OC-FEM 1 for test case III. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 16736$, $\sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	739	901	1073	1140	1462	1153	1548	182	32	19	[-]
10^{-6}	357	361	357	385	478	607	96	17	10	9	18
10^{-2}	234	234	234	253	279	50	9	6	7	6	9
1	260	260	260	259	214	13	7	5	6	6	8
10^{2}	462	462	469	440	76	11	6	4	6	6	7
10^{6}	79	79	79	73	21	10	4	4	4	4	6
10^{10}	10	10	9	19	22	10	4	3	4	4	6

Table 14. Different control and observation domains in Formulation OC-FEM 2 for test case IV. Number of MinRes iterations for different values of ω and various DOF using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($\nu = \sigma = \lambda = 1$).

DOF	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
2916	19	19	20	21	23	30	30	22	12	8	8
19652	19	19	20	21	24	30	32	26	12	8	8
143748	19	19	19	21	23	29	32	28	14	10	8

Table 15. Different control and observation domains in Formulation OC-FEM 2 for test case IV. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 19652$, $\sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	1038	1006	661	3421	[-]	[-]	[-]	946	49	28	9
10^{-6}	342	344	363	843	6843	7142	619	256	26	9	8
10^{-2}	188	206	209	313	607	204	114	82	79	80	106
1	40	40	41	48	52	30	26	26	26	24	26
10^{2}	41	41	42	64	70	40	26	22	22	20	28
10^{6}	24	24	30	68	76	38	24	16	26	42	414
10^{10}	22	22	34	88	148	46	44	36	68	276	[-]

Table 16. Different control and observation domains in Formulation OC-FEM 1 for test case IV. Number of MinRes iterations for different values of ω and various DOF using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($\nu = \sigma = \lambda = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

DOF	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
2416	34	34	67	61	52	30	22	12	6	4	4
16736	32	33	82	67	51	30	22	20	6	4	4
124096	29	31	83	63	48	30	20	22	8	4	4

Table 17. Different control and observation domains in Formulation OC-FEM 1 for test case IV. Number of MinRes iterations for different values of ν and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 16736$, $\sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ν										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	34	34	36	66	2701	[-]	983	103	60	47	[-]
10^{-6}	31	32	32	87	2630	828	81	46	41	58	73
10^{-2}	29	37	37	66	169	61	43	39	37	43	47
1	19	20	22	29	39	30	25	23	22	21	24
10^{2}	10	10	11	20	22	13	12	12	11	10	10
10^{6}	6	6	6	20	22	10	6	6	6	6	6
10^{10}	4	4	6	20	22	10	4	4	4	4	6

Table 18. Different control and observation domains in Formulation OC-FEM 1 for test case IV. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 16736$, $\sigma = \nu = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	9338	9347	9346	9340	[-]	[-]	2630	66	11	6	4
10^{-6}	571	571	571	1075	983	828	169	20	6	4	4
10^{-2}	49	49	122	103	81	61	22	20	6	4	4
1	32	33	82	67	51	30	22	20	6	4	4
10^{2}	23	112	60	46	43	13	22	20	6	4	4
10^{6}	[-]	46	41	39	12	10	22	20	6	4	4
10^{10}	[-]	58	37	12	6	10	22	20	6	4	4

3.5 Test case VI

Numerical results for the case of state constraints imposed on the Fourier coefficients are presented in Table 19-20. Here we choose 15512 random points as the active sets $\mathcal{E}^c$ and $\mathcal{E}^s$ and solve the resulting Jacobi system. The dependence of the MinRes convergence rate on the Moreau-Yosida regularization parameter ε is demonstrated in Table 19 and Table 20. Table 21 clearly demonstrates the robustness with respect to the parameters λ and ω. We refer the reader to [11] for a detailed description of the treatment of state constraints via the Moreau-Yosida regularization. Furthermore, we mention that the presence of constrains imposed on the control Fourier coefficients finally results in (linearized) systems with system matrices having the same structure as the system matrix arising from the case of different observation and control domains.

4 Summary and Conclusion

We demonstrated in many numerical experiments that the preconditioners derived and analysed in [12] and [11] lead to parameter-robust and efficient solvers in many practically important cases. Therefore, we reported on a broad range of numerical experiments, that confirm the theoretical convergence rates. Consequently, the multiharmonic finite element discretization technique in combination with efficient and parameter-robust solvers leads to a very competitive method. Furthermore, we want to mention, that due to the decoupling nature of the frequency domain equations with respect to the individual modes, a parallelization of the proposed method is straightforward, cf. Algorithm 1.

Table 19. State constraints in Formulation OC-FEM 1 for test case VI. Number of MinRes iterations for different values of ε and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 16736$, $\nu = \sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ϵ										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	88	74	59	45	31	17	9	9	9	9	9
10^{-6}	992	801	612	421	220	36	21	21	21	21	21
10^{-2}	[-]	[-]	[-]	3259	351	29	20	20	20	20	20
1	[-]	[-]	[-]	3795	191	24	16	16	14	14	14
10^{2}	[-]	[-]	[-]	1619	120	13	12	10	10	10	10
10^{6}	[-]	[-]	5852	160	12	10	10	10	10	10	10
10^{10}	[-]	7681	162	12	10	10	10	10	10	10	10

Table 20. State constraints in Formulation OC-FEM 1 for test case VI. Number of MinRes iterations for different values of ε and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 124096$, $\nu = \sigma = \omega = 1$). [-] indicates that MinRes did not converge within 10000 iterations.

λ	ϵ										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	142	118	94	70	46	22	13	13	13	13	13
10^{-6}	3275	2602	1930	1241	372	35	21	21	21	21	21
10^{-2}	[-]	[-]	[-]	5482	383	29	18	18	18	18	18
1	[-]	[-]	[-]	5443	206	24	16	14	13	12	12
10^{2}	[-]	[-]	[-]	1836	124	13	12	10	10	10	10
10^{6}	[-]	[-]	6619	167	11	10	10	10	10	10	10
10^{10}	[-]	8883	167	11	10	10	10	10	10	10	10

Table 21. State constraints in Formulation OC-FEM 1 for test case VI. Number of MinRes iterations for different values of ω and λ using the EXACT version of the preconditioner with UMFPACK for $\boldsymbol{F}$ ($DOF = 124096$, $\nu = \sigma = \varepsilon = 1$).

λ	ω										
	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	22	22	22	22	22	22	22	22	12	6	4
10^{-6}	35	35	35	35	35	35	35	22	8	4	4
10^{-2}	30	30	30	30	30	29	22	22	8	4	4
1	20	20	20	20	20	24	20	22	8	4	4
10^{2}	16	16	16	16	18	13	20	22	8	4	4
10^{6}	13	13	14	18	12	10	20	22	8	4	4
10^{10}	13	13	16	12	6	10	20	22	8	4	4

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