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Abstract. Given the line integrals of a harmonic function on a finite set of chords of the unit circle, we consider the problem of fitting these Radon projections type of data by a harmonic polynomial in the unit disk. In particular, we focus on the overdetermined case where the amount of given data is greater than the dimension of the polynomial space. We prove sufficient conditions for existence and uniqueness of a harmonic polynomial fitting the data by using least squares method. Combining with recent results on interpolation with harmonic polynomials, we obtain an algorithm of practical application. We extend our results to fitting of more general mixed data consisting of both Radon projections and function values. Numerical results are presented and discussed.

Keywords: multivariate interpolation, Radon transform, harmonic polynomials, least-squares fitting

1 Introduction

There are many important problems in medicine, geophysics, biology, materials science, radiology, oceanography, and other sciences, where information about processes can only be obtained by nondestructive testing methods. Among the most successful techniques for reconstruction of objects with non-homogeneous density are tomographic imaging methods. Johann Radon and his results on the Radon transform [23] later to be named after him laid the mathematical foundation for this approach.

From the mathematical point of view, the problem is to recover a multivariate function using information given as line integrals of the unknown function. This problem has been intensively studied since the 1960s by different approaches [17,5,6,7,15,19,20,24,16] and continues to find a lot of applications. Various reconstruction algorithms have been developed: filtered backprojection, iterative reconstruction, direct methods, etc., and some are based on the inverse Radon transform (see [21] and the bibliography therein).

Another class of methods for function reconstruction use direct interpolation by multivariate polynomials [20,14,1,4,11,12,13,10]. Many results along these lines are due to a research group founded by Prof. Borislav Bojanov which studies approximation problems using Radon projections type of data (see also [2,3,22]). A key question in this approach is how to construct a regular set of line segments, i.e., in what manner to choose chords of the unit circle so that there exists a unique polynomial of a corresponding degree with preassigned Radon projections over the configuration of chords.

To improve the approximation accuracy and to reduce the amount of input data required as well as the computational effort, one could try to incorporate some characteristic about the function to be recovered into approximation methods. According to this concept, interpolation of a harmonic function by harmonic polynomials based on Radon projections was studied in [9], where tools from symbolic computation were used, and in [8], where an analytical proof in a more general setting was given.

In the present paper, we continue the investigation of approximating harmonic functions using Radon projections type of data. In particular, we focus on the overdetermined case where the amount of data is greater than the dimension of the polynomial space. We use a least-squares method to determine a harmonic polynomial which fits the given data.

It turns out that the least-squares fitting problem and the interpolation problem are closely related. In [12], it was shown for the non-harmonic case that existence and uniqueness of the least-squares fitting polynomial relies on a regularity property of a subset of the scheme of chords.

With a similar proof technique, we derive sufficient conditions for existence and uniqueness of the least-squares harmonic polynomial, making use of recent results on interpolation with harmonic polynomials. We also consider fitting more general mixed data consisting of both Radon projections and function values. A reconstruction algorithm is developed and tested and numerical results are presented in the last section.

2 Preliminaries and Related Work

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Figure 1). The chord $I(\theta, t)$ is parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \quad \text{where } s \in (-\sqrt{1-t^2}, \sqrt{1-t^2}).$$

Definition 1. *Let $f(x, y)$ be a real-valued bivariate function in the unit disk D. The* Radon projection $\mathcal{R}_\theta(f; t)$ *of f in direction θ is defined by the line integral*

$$\mathcal{R}_\theta(f; t) := \int_{I(\theta,t)} f(\mathbf{x})\, d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

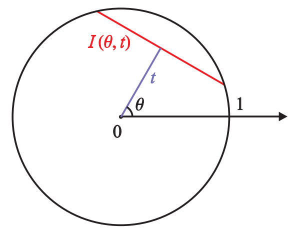

Fig. 1. The chord $I(\theta,t)$ of the unit circle.

Johann Radon [23] showed in 1917 that a differentiable function f is uniquely determined by the values of its Radon transform,

$$f \;\mapsto\; \left\{\mathcal{R}_\theta(f;t) \,:\, -1 \le t \le 1,\; 0 \le \theta < \pi\right\}.$$

Further works in this area are due to John [18], Solmon [24], and others.

2.1 Interpolation and Fitting by Bivariate Polynomials

A fundamental problem in our investigations is to recover a polynomial using a finite number of values of its Radon transform. Essentially, this may be viewed as a bivariate interpolation problem where the usual function values are replaced by means over chords of the unit circle.

Let $\Pi_n^2 = \left\{\sum_{i+j\le n} a_{ij}x^iy^j \,:\, a_{ij}\in\mathbb{R}\right\}$ denote the space of real bivariate polynomials of total degree at most n. This space has dimension $\binom{n+2}{2}$. Assume that a set $\mathcal{I} = \left\{I_m = I(\theta_m,t_m) \,:\, m=1,\ldots,\binom{n+2}{2}\right\}$ of chords of ∂D is given. Furthermore, to each chord $I_m\in\mathcal{I}$ a given value $\gamma_m\in\mathbb{R}$ is associated. Then, the aim is to find a polynomial $p\in\Pi_n^2$ such that

$$\mathcal{R}_{\theta_m}(p,t_m) = \int_{I_m} p(\mathbf{x})\,d\mathbf{x} = \gamma_m \qquad \forall I_m\in\mathcal{I}. \tag{1}$$

If this interpolation problem has a unique solution for every choice of values $\Gamma = \left\{\gamma_m,:\, m=1,\ldots,\binom{n+2}{2}\right\}$, then the scheme $\mathcal{I}$ of chords is called *regular*.

The question of how to construct such regular schemes has been extensively studied. The first general result was given by Marr [20] in 1974, who proved that the set of chords connecting $n+2$ equally spaced points on the unit circle is regular for Π_n^2. A more general result for $\mathbb{R}^d$ and general convex domains was published by Hakopian [14] in 1982.

Different families of regular schemes of chords of the unit circle were constructed by Bojanov and Georgieva [1], Bojanov and Xu [4], Georgieva and Ismail [11], Georgieva and Uluchev [12], A mixed regular scheme which incorporates Radon projections and function values at points on the unit circle was proposed by Georgieva, Hofreither, and Uluchev [10].

Georgieva and Uluchev [12] considered a least-squares fitting problem for the overdetermined case of Radon projections type of data with algebraic polynomials and proved existence and uniqueness of the fitting polynomial. The proof was based on the above cited previous interpolation results. Moreover, this least-squares fitting was extended to mixed type data of Radon projections and function values.

2.2 Interpolation by Harmonic Polynomials

If we know *a priori* that the function to be interpolated is harmonic, then it seems natural to work in the space $\mathcal{H}_n$ of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n+1$. Analogous to (1), we prescribe chords $\mathcal{I} := \{I(\theta_i, t_i) : \theta_i \in [0, \pi),\ t_i \in (-1, 1)\}_{i=1}^{2n+1}$ of the unit circle and associated given values $\Gamma = \{\gamma_i\}_{i=1}^{2n+1}$, and wish to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\mathcal{R}_{\theta_i}(p, t_i) = \int_{I(\theta_i, t_i)} p(\mathbf{x})\, d\mathbf{x} = \gamma_i, \qquad i = 1, \ldots, 2n+1. \tag{2}$$

Again we call $\mathcal{I}$ *regular* if the interpolation problem (2) has a unique solution for all given values Γ. In the following, we present one family of such regular schemes.

We use the following basis of the space of harmonic polynomials $\mathcal{H}_n$,

$$\begin{aligned} &h_0(x, y) = 1, \\ &h_{2k-1}(x, y) = \mathrm{Re}(x + \mathrm{i}y)^k, \quad h_{2k}(x, y) = \mathrm{Im}(x + \mathrm{i}y)^k, \quad k = 1, \ldots, n, \end{aligned}$$

with representation in polar coordinates

$$\begin{aligned} &h_0(r, \theta) = 1, \\ &h_{2k-1}(r, \theta) = r^k \cos(k\theta), \quad h_{2k}(r, \theta) = r^k \sin(k\theta), \quad k = 1, \ldots, n. \end{aligned}$$

Every harmonic polynomial p of degree less than or equal to n can be expanded in this basis,

$$p = \sum_{k=0}^{2n} p_k h_k,$$

where p_k are real coefficients.

The following result, which gives a closed formula for Radon projections of the basis harmonic polynomials can be considered a harmonic analogue to the famous Marr's formula [20]. A special case of this harmonic version was first derived using tools from symbolic computation [9]. Later, Georgieva and Hofreither [8] have given an analytic proof in a more general setting.

Lemma 1. *The Radon projections of the basis harmonic polynomials* $h_k, k \in \mathbb{N}$, *are given by*

$$\mathcal{R}_\theta(h_{2k-1}, t) = \int_{I(\theta,t)} h_{2k-1}(\mathbf{x})\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta),$$
$$\mathcal{R}_\theta(h_{2k}, t) = \int_{I(\theta,t)} h_{2k}(\mathbf{x})\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\sin(k\theta),$$

where $\theta \in \mathbb{R}$, $t \in (-1,1)$ *and* $U_k(t)$ *is the* k*-th degree Chebyshev polynomial of second kind.*

The above lemma plays a crucial role in proving regularity of a particular family of schemes $\mathcal{I}$ of chords.

Theorem 1 (Existence and uniqueness [9,8]). *The interpolation problem* (2) *has a unique solution for any set of chords* $\mathcal{I} = \{I(\theta_i, t_i)\}_{i=1}^{2n+1}$ *with*

$$0 \le \theta_1 < \theta_2 < \ldots < \theta_{2n+1} < 2\pi$$

and with constant distances $t_i = t \in (-1,1)$ *such that* t *is not a zero of any Chebyshev polynomial of the second kind* $U_1, \ldots, U_n$.

See Figure 2 for some examples of schemes which satisfy the conditions of the above theorem.

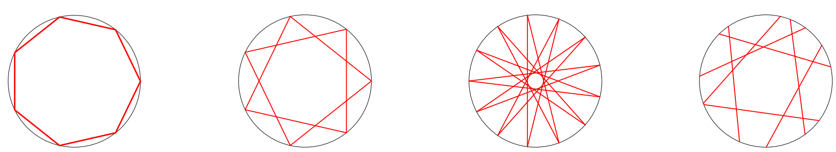

Fig. 2. Some admissible schemes according to Theorem 1.

3 Least-squares Fitting

Here we deal with the problem of fitting some given Radon projections of a harmonic function by a harmonic polynomial in the overdetermined case where the amount of data is greater than the dimension of the polynomial space. A least-squares method is used to determine a harmonic polynomial which fits the given data. The problem of least-squares fitting of Radon projections was first considered for the case of algebraic polynomials by Marr [20].

3.1 Radon Projections Type of Data

Let a set $\mathcal{I} := \{I(\theta_i, t_i) : \theta_i \in [0,\pi),\ t_i \in (-1,1)\}_{i=1}^N$ of N distinct chords of the unit circle ∂D, be given, and let $\Gamma := \{\gamma_i\}_{i=1}^N$ be the Radon projections of a harmonic function u along these chords, i.e.,

$$\mathcal{R}_{\theta_i}(u, t_i) = \gamma_i, \qquad i = 1, \ldots, N.$$

We regard the set of chords $\mathcal{I}$ and the set of values Γ generally as *data*. Finally, by $\Lambda := \{\lambda_i\}_{i=1}^N$ we denote a set of positive real numbers which we consider to be *weights* related to the corresponding Radon projections.

The least squares fitting problem is formulated as follows.

Given data $\mathcal{I}$ and Γ, and weights Λ, find a polynomial $p \in \mathcal{H}_n$, $N > 2n+1$, such that

$$\sum_{i=1}^N \lambda_i \big(\mathcal{R}_{\theta_i}(p, t_i) - \gamma_i\big)^2 \to \min . \tag{3}$$

Theorem 2. *Assume that data $\mathcal{I}$ and Γ, and weights Λ are given. Suppose that there exists a subset $J \subset \{1, 2, \ldots, N\}$, $|J| = 2n+1$, such that the interpolatory scheme of chords $\{I(\theta_\ell, t_\ell)\}_{\ell \in J}$ is regular. Then there exists a unique harmonic polynomial $p \in \mathcal{H}_n$ for which the minimum in (3) is attained.*

Proof. Suppose p is a harmonic polynomial of degree at most n. Then p can be represented in the form

$$p = \sum_{k=0}^{2n} p_k h_k .$$

Since the Radon projection for a fixed line segment is a linear functional it follows that

$$\mathcal{R}_{\theta_i}(p; t_i) = \sum_{k=0}^{2n} p_k \mathcal{R}_{\theta_i}(h_k; t_i), \quad i = 1, \ldots, N.$$

Hence, the problem (3) is equivalent to the problem

$$\Phi := \sum_{i=1}^N \lambda_i \left(\sum_{k=0}^{2n} p_k \mathcal{R}_{\theta_i}(h_k; t_i) - \gamma_i \right)^2 \to \min,$$

where Φ is a function of the coefficients $\{p_k\}$, $k = 0, 1, \ldots, 2n$.

Applying the necessary conditions for extrema

$$\frac{\partial \Phi}{\partial p_j} = 0, \qquad j = 0, \ldots, 2n,$$

we obtain the system of linear equations, for $j = 0, 1, \ldots, 2n$,

$$\sum_{k=0}^{2n} \left(\sum_{i=1}^N \lambda_i \mathcal{R}_{\theta_i}(h_k, t_i) \mathcal{R}_{\theta_i}(h_j, t_i) \right) p_k = \sum_{i=1}^N \lambda_i \gamma_i \mathcal{R}_{\theta_i}(h_j, t_i), \tag{4}$$

with respect to the coefficients $\{p_k\}$, $k = 0, 1, \dots, 2n$.

In order to prove that (4) has a unique solution for arbitrary set $\varGamma$ of Radon projections we consider the corresponding homogeneous system

$$\sum_{k=0}^{2n} \left(\sum_{i=1}^{N} \lambda_i \mathcal{R}_{\theta_i}(h_k, t_i) \mathcal{R}_{\theta_i}(h_j, t_i) \right) q_k = 0, \qquad j = 0, 1, \dots, 2n. \tag{5}$$

Using the linearity of the functionals $\mathcal{R}_{\theta_i}(\cdot, t_i)$, we get

$$\sum_{i=1}^{N} \lambda_i \mathcal{R}_{\theta_i} \Big(\sum_{k=0}^{2n} q_k h_k, t_i \Big) \mathcal{R}_{\theta_i}(h_j, t_i) = 0, \qquad j = 0, 1, \dots, 2n. \tag{6}$$

Denote

$$q := \sum_{k=0}^{2n} q_k h_k.$$

Let us note that q is a polynomial from $\mathcal{H}_n$. Then (5) may be rewritten as

$$\sum_{i=1}^{N} \lambda_i \mathcal{R}_{\theta_i}(q, t_i) \mathcal{R}_{\theta_i}(h_j, t_i) = 0, \qquad j = 0, 1, \dots, 2n.$$

We now sum all the equations of (6) multiplied by the corresponding q_j and obtain

$$\sum_{i=1}^{N} \lambda_i \big(\mathcal{R}_{\theta_i}(q, t_i) \big)^2 = 0.$$

Hence, by the positivity of the weights λ_i, we have

$$\mathcal{R}_{\theta_i}(q, t_i) = 0, \qquad i = 1, \dots, N.$$

Since there exists a subset $J \subset \{1, 2, \dots, N\}$, $|J| = 2n + 1$, such that the interpolatory scheme of chords $\big\{ I(\theta_\ell, t_\ell) \big\}_{\ell \in J}$ is regular, we conclude that $q \equiv 0$. Then

$$q_k = 0, \qquad k = 0, 1, \dots, 2n,$$

i.e., the homogeneous system (5) has only the zero solution.

Therefore the linear system (4) has a unique solution, and the theorem is proved. □

Remark 1. From the proof of Theorem 2, it can be seen that the coefficients $\{p_k\}_{k=0}^{2n}$ of the least-squares fitting polynomial $p = \sum_{k=0}^{2n} p_k h_k$ can be found as the solution of the following system of linear equations,

$$\sum_{k=0}^{2n} a_{jk} p_k = \sum_{i=1}^{N} b_{ji} \gamma_i, \qquad j = 0, \dots, 2n.$$

In short, the vector $\underline{p}$ of coefficients is determined by

$$A\underline{p} = B\Gamma$$

with the symmetric and positive definite matrix $A = (a_{jk})_{j,k=0}^{2n}$ and the rectangular matrix $B = (b_{ji})_{j=0,i=1}^{2n,\ N}$ having entries

$$a_{jk} = \sum_{i=1}^{N} \lambda_i \mathcal{R}_{\theta_i}(h_k, t_i)\mathcal{R}_{\theta_i}(h_j, t_i), \qquad b_{ji} = \lambda_i \mathcal{R}_{\theta_i}(h_j, t_i).$$

These matrix entries can be computed using the formulas in Lemma 1 for the Radon projections of the harmonic basis functions.

3.2 Mixed Type of Data

Now, we shall consider a fitting problem for *mixed type of data* – both Radon projections and function values. Namely, let the data $\mathcal{I}$ and Γ, and the weights Λ be given as above in Section 3.1. Additionally we take values $U := \{u_j\}_{j=1}^{M}$ of the harmonic function u at arbitrary points $X := \{\mathbf{x}_j\}_{j=1}^{M}$ in the closed unit disk $\overline{D}$, i.e.,

$$u(\mathbf{x}_j) = u_j, \qquad j = 1, \ldots, M.$$

In particular, the points X can be chosen only on the unit circle ∂D. Let $\Omega := \{\omega_j\}_{j=1}^{M}$ be given weights corresponding to the function values.

The least squares fitting problem for mixed type of data is formulated as follows: given

- the data $\mathcal{I}$ and Γ, and corresponding weights Λ;
- function values U at points X and weights Ω;

find a harmonic polynomial $p \in \mathcal{H}_n$, $N > 2n + 1$, such that

$$\sum_{i=1}^{N} \lambda_i \big(\mathcal{R}_{\theta_i}(p, t_i) - \gamma_i\big)^2 + \sum_{j=1}^{M} \omega_j \big(p(\mathbf{x}_j) - u_j\big)^2 \to \min . \tag{7}$$

Theorem 3. *Assume that mixed type of data $\mathcal{I}$, Γ, X, U, and weights Λ, Ω are given. Suppose that there exists a subset $J \subset \{1, 2, \ldots, N\}$, $|J| = 2n + 1$, such that the interpolatory scheme of chords $\{I(\theta_\ell, t_\ell)\}_{\ell \in J}$ is regular. Then there exists a unique harmonic polynomial $p \in \mathcal{H}_n$ for which the minimum in* (7) *is attained.*

Therefore including a regular interpolatory scheme from Section 2.2 into the set of chords $\mathcal{I}$ assures the uniqueness of the solution to the problem for mixed data. The proof of the theorem is similar to the proof of Theorem 2 and the coefficients of the least-squares minimizing polynomial can be computed by solving a linear system similar as in Remark 1.

4 Numerical Examples

4.1 Example 1

We approximate the harmonic function $u(x,y) = \exp(x)\cos(y)$ by a harmonic polynomial $p \in \mathcal{H}_n$ given $N = 2(2n+1)$ values of its Radon projections: $2n+1$ taken along the edges of a regular $(2n+1)$-sided convex polygon (Figure 2, first picture), and $2n+1$ along random chords. The weights are all set to 1. In Figure 3, we display the scheme of chords, the function u as well as the error $u-p$, where p is the least-squares fitting polynomial of degree $n = 7$ fitting information on 30 chords.

4.2 Example 2

We consider a similar problem as in Example 1, but in this case the weights are set to 1 for the chords forming a regular $(2n+1)$-sided convex polygon, and to 100 for the remaining $N-(2n+1)$ random chords.

In Figure 4, we plot the scheme of chords, and the error function $u-p$, where the degree of the least-squares fitting polynomial p is $n = 7$ and the number of chords is $N = 30$. No qualitative change in behavior from Example 1 is observed.

4.3 Example 3

We again approximate the harmonic function $u(x,y) = \exp(x)\cos(y)$, but consider the case of noisy data. We start with $2n+1$ chords forming a regular convex polygon and then add a variable number m of additional, randomly distributed chords such that we have a total of $N = 2n+1+m$ chords. To the exact Radon projections $\gamma_i \in \Gamma$ we add Gaussian random numbers with mean 0 and standard deviation $\epsilon = 10^{-2}$ so that we obtain noisy data $\tilde{\gamma}_i$. The weights are all set to 1. The degree of the least-squares fitting polynomial is kept at $n = 7$ throughout.

In Figure 5, we plot the relative L_2-errors $\|u-p\|_{L_2(D)}/\|u\|_{L_2(D)}$ for varying number of added chords m. We observe that for noisy data, additional pieces of data improve the approximation. In this particular case, it seems that after an initial slower decay, the error behaves approximately like $\mathcal{O}(m^{-0.8})$.

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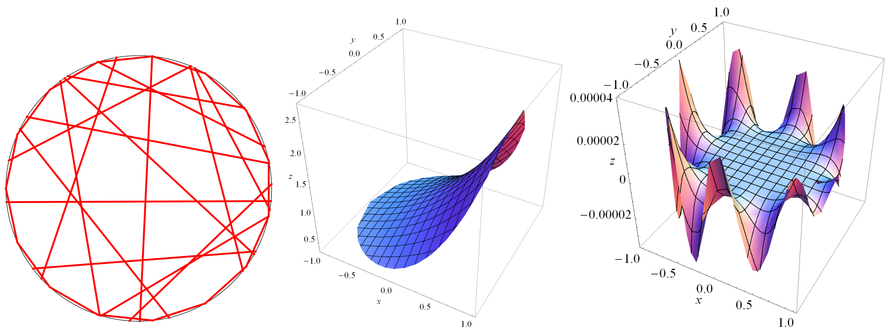

Fig. 3. Example 1, $n = 7, N = 30$: the scheme of chords, function u, error $u - p$

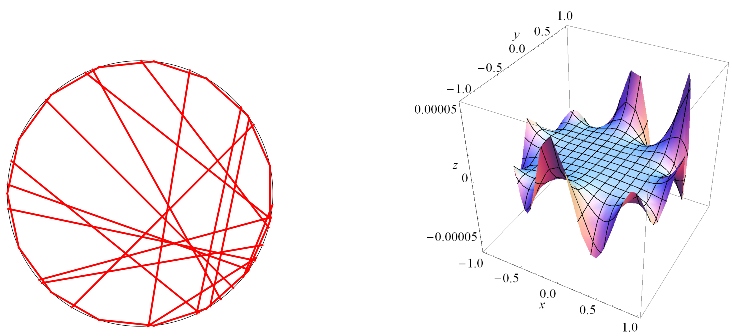

Fig. 4. Example 2, $n = 7, N = 30$: the scheme of chords, error $u - p$

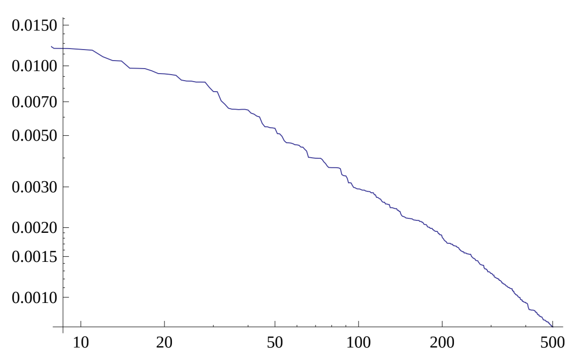

Fig. 5. Example 3: errors with noisy data, $n = 7$. x-axis: number m of additional chords. y-axis: relative L_2-error

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