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Functional a posteriori error estimates for parabolic time-periodic boundary value problems

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FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR PARABOLIC TIME-PERIODIC BOUNDARY VALUE PROBLEMS

U. LANGER, S. REPIN, AND M. WOLFMAYR

Abstract. The paper is concerned with parabolic time-periodic boundary value problems which are of great theoretical interest and arise in different practical applications. The multiharmonic finite element method is well adapted to this class of parabolic problems. We study properties of multiharmonic approximations, and derive guaranteed and fully computable bounds of approximation errors. For this purpose, we use the functional a posteriori error estimation techniques earlier introduced by Sergey Repin.

1. Introduction

This work is devoted to the a posteriori error analysis of parabolic time-periodic boundary value problems in connection with their multiharmonic finite element discretization. More precisely, all functions are expanded into Fourier series, truncated and the Fourier coefficients are approximated by the finite element method (FEM). This so-called multiharmonic FEM (MhFEM) or harmonic-balanced FEM was successfully used for the simulation of electromagnetic devices described by nonlinear eddy current problems with harmonic excitations, see, e.g., [26, 1, 2, 5] and the references therein. Later, this discretization technique has been applied to linear time-periodic parabolic boundary value and optimal control problems [10, 11, 17, 20, 25] and to linear time-periodic eddy current problems and the corresponding optimal control problems [13, 14, 15]. The functional a posteriori error estimation techniques, which we use, are based on the works by Repin, see, e.g., the papers on parabolic problems [22, 8] as well as on optimal control problems [6, 7], the books [23, 21] and the references therein. In particular, our a posteriori error analysis uses the techniques close to the one suggested in [22], but the analysis contains essential changes. In the MhFEM setting, we are able to establish inf-sup and sup-sup conditions from which we deduce existence and uniqueness of the solution to the parabolic time-periodic problems by applying the theorem of Babuška and Aziz. We deduce fully computable error bounds, which to the best of our knowledge are new. Indeed, the a posteriori error analysis presented in this paper leads to guaranteed upper bounds that are very valuable for the evaluation of quality of the multiharmonic finite element solution. The functional a posteriori error analysis provides these bounds via majorants the minimization of which delivers the discrete solutions as well. This work is a starting point for the construction of a so-called *adaptive multiharmonic finite element method (AMhFEM)*, whose analysis and implementation is currently subject of ongoing work. In the case of linear time-periodic parabolic problems, the computations of the Fourier coefficients corresponding to every single mode $k = 0, 1 \dots$ are decoupled. Hence, we can use different meshes independently generated by adaptive finite element approximations to the Fourier coefficients for different modes. Then, by prescribing certain bounds, we can finally filter out the Fourier coefficients, which are important for the (numerical) solution of the problem. Altogether, such an AMhFEM yields complete adaptivity in space and time.

The paper provides a detailed functional a posteriori error analysis of a parabolic time-periodic boundary value problem which is discretized by means of the MhFEM. In particular, the paper is organized as follows: In Section 2, we discuss two space-time variational formulations for parabolic time-periodic boundary value problems (which are equivalent if the source term belongs to L_2). These problems form the basis of the MhFEM, which is considered in Section 3. Section 4 is

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devoted to the derivation of functional a posteriori error estimates adapted to problems in question. We derive two types of estimates including their multiharmonic setting.

2. A parabolic time-periodic boundary value problem

Let $Q_T := \Omega \times (0,T)$ denote the space-time cylinder and $\Sigma_T := \Gamma \times (0,T)$ its mantle boundary, where $\Omega \subset \mathbb{R}^d$, $d = \{1,2,3\}$, is a bounded Lipschitz domain with the boundary Γ, and $(0,T)$ is a given time interval. The following parabolic time-periodic boundary value problem is considered:

$$\sigma(\boldsymbol{x})\,\partial_t u(\boldsymbol{x},t) - \operatorname{div}(\nu(\boldsymbol{x})\,\nabla u(\boldsymbol{x},t)) = f(\boldsymbol{x},t) \qquad (\boldsymbol{x},t) \in Q_T, \tag{1}$$

$$u(\boldsymbol{x},t) = 0 \qquad (\boldsymbol{x},t) \in \Sigma_T, \tag{2}$$

$$u(\boldsymbol{x},0) = u(\boldsymbol{x},T) \qquad \boldsymbol{x} \in \overline{\Omega}, \tag{3}$$

where $f(\boldsymbol{x},t)$ is a given function in $L^2(Q_T)$, and $\sigma(\cdot)$ and $\nu(\cdot)$ satisfy the assumptions

$$0 < \underline{\sigma} \le \sigma(\boldsymbol{x}) \le \overline{\sigma}, \qquad 0 < \underline{\nu} \le \nu(\boldsymbol{x}) \le \overline{\nu}, \qquad \boldsymbol{x} \in \Omega. \tag{4}$$

In order to study the parabolic time-periodic boundary value problem (1)-(3), we will derive space-time variational formulations in Sobolev spaces of functions in the space-time cylinder Q_T using the approach similar to that used by Ladyzhenskaya et al., see [18, 19]. Let the Sobolev spaces $H^{1,0}(Q_T) = \{u \in L^2(Q_T) : \nabla u \in [L^2(Q_T)]^d\}$ and $H^{1,1}(Q_T) = \{u \in L^2(Q_T) : \nabla u \in [L^2(Q_T)]^d, \partial_t u \in L^2(Q_T)\}$ be equipped with the norms

$$\|u\|_{H^{1,0}(Q_T)} := \left(\int_{Q_T} \left(u(\boldsymbol{x},t)^2 + |\nabla u(\boldsymbol{x},t)|^2 \right) d\boldsymbol{x}\, dt \right)^{1/2},$$

$$\|u\|_{H^{1,1}(Q_T)} := \left(\int_{Q_T} \left(u(\boldsymbol{x},t)^2 + |\nabla u(\boldsymbol{x},t)|^2 + |\partial_t u(\boldsymbol{x},t)|^2 \right) d\boldsymbol{x}\, dt \right)^{1/2},$$

where $\nabla = \nabla_{\boldsymbol{x}}$ and ∂_t denote the generalized derivatives. The Sobolev space $H^{0,1}(Q_T) = \{u \in L^2(Q_T) : \partial_t u \in L^2(Q_T)\}$ is defined analogously. Furthermore, the boundary and time-periodicity conditions are included by defining the Sobolev spaces

$$\begin{aligned}
H_0^{1,0}(Q_T) &= \{u \in H^{1,0}(Q_T) : u = 0 \text{ on } \Sigma_T\},\\
H_0^{1,1}(Q_T) &= \{u \in H^{1,1}(Q_T) : u = 0 \text{ on } \Sigma_T\},\\
H_{per}^{0,1}(Q_T) &= \{u \in H^{0,1}(Q_T) : u(\boldsymbol{x},0) = u(\boldsymbol{x},T) \text{ for almost all } \boldsymbol{x} \in \Omega\},\\
H_{per}^{1,1}(Q_T) &= \{u \in H^{1,1}(Q_T) : u(\boldsymbol{x},0) = u(\boldsymbol{x},T) \text{ for almost all } \boldsymbol{x} \in \Omega\},\\
H_{0,per}^{1,1}(Q_T) &= \{u \in H_0^{1,1}(Q_T) : u(\boldsymbol{x},0) = u(\boldsymbol{x},T) \text{ for almost all } \boldsymbol{x} \in \Omega\}.
\end{aligned}$$

For ease of notation, all inner products and norms in L^2 are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, if they are related to the whole space-time domain Q_T. If they are associated with the spatial domain Ω, then we write $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, which denote the standard inner products and norms of the space $L^2(\Omega)$. The symbols $(\cdot,\cdot)_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ denote the standard inner products and norms of $H^1(\Omega)$.

The functions used in our analysis will be typically presented in the form of Fourier series, i.e.,

$$v(\boldsymbol{x},t) = v_0^c(\boldsymbol{x}) + \sum_{k=1}^{\infty} \left(v_k^c(\boldsymbol{x}) \cos(k\omega t) + v_k^s(\boldsymbol{x}) \sin(k\omega t) \right) \tag{5}$$

with the Fourier coefficients

$$v_0^c(\boldsymbol{x}) = \frac{1}{T} \int_0^T v(\boldsymbol{x},t)\, dt,$$

$$v_k^c(\boldsymbol{x}) = \frac{2}{T} \int_0^T v(\boldsymbol{x},t) \cos(k\omega t)\, dt, \qquad v_k^s(\boldsymbol{x}) = \frac{2}{T} \int_0^T v(\boldsymbol{x},t) \sin(k\omega t)\, dt,$$

where T and $\omega = 2\pi/T$ denote the periodicity and the frequency, respectively. Moreover, we define additional function spaces, see [20], in order to derive a symmetric variational formulation of problem (1)-(3). The function spaces $H^{0,\frac{1}{2}}_{per}(Q_T)$, $H^{1,\frac{1}{2}}_{per}(Q_T)$ and $H^{1,\frac{1}{2}}_{0,per}(Q_T)$ are defined by

$$\begin{aligned} H^{0,\frac{1}{2}}_{per}(Q_T) &= \{u \in L^2(Q_T) : \left\|\partial_t^{1/2} u\right\| < \infty\}, \\ H^{1,\frac{1}{2}}_{per}(Q_T) &= \{u \in H^{1,0}(Q_T) : \left\|\partial_t^{1/2} u\right\| < \infty\}, \\ H^{1,\frac{1}{2}}_{0,per}(Q_T) &= \{u \in H^{1,\frac{1}{2}}_{per}(Q_T) : u = 0 \text{ on } \Sigma_T\}, \end{aligned}$$

respectively, where $\left\|\partial_t^{1/2} u\right\|$ is defined in the Fourier space by the relation

$$\left\|\partial_t^{1/2} u\right\|^2 := |u|^2_{H^{0,\frac{1}{2}}(Q_T)} := \frac{T}{2}\sum_{k=1}^{\infty} k\omega \|\boldsymbol{u}_k\|^2_\Omega, \tag{6}$$

where $\boldsymbol{u}_k := (u_k^c, u_k^s)$ for all $k \in \mathbb{N}$. These spaces are equipped with the scalar products

$$\left(\partial_t^{1/2} u, \partial_t^{1/2} v\right) := \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\boldsymbol{u}_k, \boldsymbol{v}_k)_\Omega, \qquad \left(\sigma\partial_t^{1/2} u, \partial_t^{1/2} v\right) := \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma\boldsymbol{u}_k, \boldsymbol{v}_k)_\Omega. \tag{7}$$

The seminorm and the norm of the space $H^{1,\frac{1}{2}}_{per}(Q_T)$ are defined by the relations

$$|u|^2_{H^{1,\frac{1}{2}}(Q_T)} = \|\nabla u\|^2 + \|\partial_t^{1/2} u\|^2 = T\,\|\nabla u_0^c\|^2_\Omega + \frac{T}{2}\sum_{k=1}^{\infty}\left(k\omega\|\boldsymbol{u}_k\|^2_\Omega + \|\nabla \boldsymbol{u}_k\|^2_\Omega\right)$$

and

$$\begin{aligned} \|u\|^2_{H^{1,\frac{1}{2}}(Q_T)} &= \|u\|^2 + |u|^2_{H^{1,\frac{1}{2}}(Q_T)} \\ &= T\left(\|u_0^c\|^2_\Omega + \|\nabla u_0^c\|^2_\Omega\right) + \frac{T}{2}\sum_{k=1}^{\infty}\left((1+k\omega)\|\boldsymbol{u}_k\|^2_\Omega + \|\nabla \boldsymbol{u}_k\|^2_\Omega\right), \end{aligned}$$

respectively. Let us define

$$\begin{aligned} v^\perp(\boldsymbol{x}, t) &:= \sum_{k=1}^{\infty}\left(-v_k^c(\boldsymbol{x})\sin(k\omega t) + v_k^s(\boldsymbol{x})\cos(k\omega t)\right) \\ &= \sum_{k=1}^{\infty} \underbrace{(v_k^s(\boldsymbol{x}), -v_k^c(\boldsymbol{x}))}_{=:(-\boldsymbol{v}_k^\perp)^T} \cdot \begin{pmatrix} \cos(k\omega t) \\ \sin(k\omega t) \end{pmatrix}. \end{aligned} \tag{8}$$

Lemma 1. *The identities*

$$\left(\sigma\partial_t^{1/2} u, \partial_t^{1/2} v\right) = \left(\sigma\partial_t u, v^\perp\right) \quad \text{and} \quad \left(\sigma\partial_t^{1/2} u, \partial_t^{1/2} v^\perp\right) = \left(\sigma\partial_t u, v\right) \tag{9}$$

are valid for all $u \in H^{0,1}_{per}(Q_T)$ *and* $v \in H^{0,\frac{1}{2}}_{per}(Q_T)$.

Proof. Using the definition of the σ-weighted scalar product in (7) and inserting the Fourier expansions of

$$\partial_t u(\boldsymbol{x}, t) := \sum_{k=1}^{\infty}[k\omega\, u_k^s(\boldsymbol{x})\cos(k\omega t) - k\omega\, u_k^c(\boldsymbol{x})\sin(k\omega t)]$$

as well as (8) into the inner products, we obtain

$$\begin{aligned} \left(\sigma\partial_t^{1/2} u, \partial_t^{1/2} v\right) &= \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma\boldsymbol{u}_k, \boldsymbol{v}_k)_\Omega = \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma\boldsymbol{u}_k^\perp, \boldsymbol{v}_k^\perp)_\Omega \\ &= \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma(-\boldsymbol{u}_k^\perp), (-\boldsymbol{v}_k^\perp))_\Omega = \left(\sigma\partial_t u, v^\perp\right) \end{aligned}$$

with $\boldsymbol{u}_k^\perp = (-u_k^s, u_k^c)^T$ for all $k \in \mathbb{N}$, and

$$\big(\sigma\partial_t^{1/2}u, \partial_t^{1/2}v^\perp\big) = \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma\boldsymbol{u}_k, \boldsymbol{v}_k^\perp)_\Omega = \frac{T}{2}\sum_{k=1}^{\infty} k\omega(\sigma(-\boldsymbol{u}_k^\perp), \boldsymbol{v}_k)_\Omega = \big(\sigma\partial_t u, v\big).$$

□

Hence, the following orthogonality relations hold:

$$\begin{aligned} &\big(\sigma\partial_t u, u\big) = 0 \quad \text{and} \quad \big(\sigma u^\perp, u\big) = 0 \qquad \forall\, u \in H^{0,1}_{per}(Q_T), \\ &\big(\sigma\partial_t^{1/2}u, \partial_t^{1/2}u^\perp\big) = 0 \quad \text{and} \quad \big(\nu\nabla u, \nabla u^\perp\big) = 0 \qquad \forall\, u \in H^{1,\frac{1}{2}}_{per}(Q_T), \end{aligned} \tag{10}$$

where, e.g.,

$$\big(\nu\nabla u, \nabla u^\perp\big) = \sum_{k=1}^{\infty}(\nu\nabla\boldsymbol{u}_k, \nabla\boldsymbol{u}_k^\perp)_\Omega = 0 \qquad \forall\, u \in H^{1,\frac{1}{2}}_{per}(Q_T)$$

with $\nabla\boldsymbol{u}_k := ((\nabla u_k^c)^T, (\nabla u_k^s)^T)^T$ and $\nabla\boldsymbol{u}_k^\perp := (-(\nabla u_k^s)^T, (\nabla u_k^c)^T)^T$ for all $k \in \mathbb{N}$. The identity

$$\int_0^T \kappa\,\partial_t^{1/2}v^\perp\,dt = -\int_0^T \partial_t^{1/2}\kappa^\perp\,v\,dt \qquad \forall\,\kappa, v \in H^{0,\frac{1}{2}}_{per}(Q_T) \tag{11}$$

is also defined in the Fourier space yielding the definitions

$$\big(\kappa, \partial_t^{1/2}v\big) := \frac{T}{2}\sum_{k=1}^{\infty}(k\omega)^{1/2}(\boldsymbol{\kappa}_k, \boldsymbol{v}_k)_\Omega \tag{12}$$

as well as

$$\partial_t^{1/2}\kappa(\boldsymbol{x}, t) := \sum_{k=1}^{\infty}(k\omega)^{1/2}\left(\kappa_k^c(\boldsymbol{x})\cos(k\omega t) + \kappa_k^s(\boldsymbol{x})\sin(k\omega t)\right)$$

and

$$\partial_t^{1/2}\kappa^\perp(\boldsymbol{x}, t) := \sum_{k=1}^{\infty}(k\omega)^{1/2}\left(-\kappa_k^s(\boldsymbol{x})\cos(k\omega t) + \kappa_k^c(\boldsymbol{x})\sin(k\omega t)\right).$$

Hence,

$$\begin{aligned} \big(\kappa, \partial_t^{1/2}v^\perp\big) &= \frac{T}{2}\sum_{k=1}^{\infty}(k\omega)^{1/2}(\boldsymbol{\kappa}_k, \boldsymbol{v}_k^\perp)_\Omega = -\big(\partial_t^{1/2}\kappa, v^\perp\big), \\ \big(\kappa, \partial_t^{1/2}v^\perp\big) &= \frac{T}{2}\sum_{k=1}^{\infty}(k\omega)^{1/2}(\boldsymbol{\kappa}_k, \boldsymbol{v}_k^\perp)_\Omega = \frac{T}{2}\sum_{k=1}^{\infty}(k\omega)^{1/2}(-\boldsymbol{\kappa}_k^\perp, \boldsymbol{v}_k)_\Omega \\ &= -\frac{T}{2}\sum_{k=1}^{\infty}(k\omega)^{1/2}(\boldsymbol{\kappa}_k^\perp, \boldsymbol{v}_k)_\Omega = -\big(\partial_t^{1/2}\kappa^\perp, v\big) \end{aligned}$$

and all these identities coincide with the identities (9) in Lemma 1. In order to derive the space-time variational formulation of the parabolic time-periodic problem (1)-(3), the parabolic partial differential equation (1) is multiplied by a test function $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$, integrated over the space-time cylinder Q_T, and after integration by parts with respect to the space and time variables, the following "symmetric" space-time variational formulation of the parabolic time-periodic boundary value problem (1)-(3) is obtained: Given $f \in L^2(Q_T)$, find $u \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ such that

$$a(u, v) = \int_{Q_T} f(\boldsymbol{x}, t)\,v(\boldsymbol{x}, t)\,d\boldsymbol{x}\,dt \qquad \forall\, v \in H^{1,\frac{1}{2}}_{0,per}(Q_T) \tag{13}$$

with the space-time bilinear form

$$a(u, v) = \int_{Q_T}\Big(\sigma(\boldsymbol{x})\partial_t^{1/2}u(\boldsymbol{x}, t)\,\partial_t^{1/2}v^\perp(\boldsymbol{x}, t) + \nu(\boldsymbol{x})\nabla u(\boldsymbol{x}, t)\cdot\nabla v(\boldsymbol{x}, t)\Big)d\boldsymbol{x}\,dt, \tag{14}$$

where all functions are given in their Fourier series expansion in time, i.e., everything has to be understood in the sense of (6) and (7). In particular, this Fourier series approach makes sense due to the time-periodicity condition (for u and v).

3. Multiharmonic finite element approximation

Inserting the Fourier series ansatz (5) into (13) and exploiting the orthogonality of the functions $\cos(k\omega t)$ and $\sin(k\omega t)$ with respect to the inner product $(\cdot,\cdot)_{L^2(0,T)}$, we arrive at the following variational formulation corresponding to every single mode $k \in \mathbb{N}$: Given $\boldsymbol{f}_k \in (L^2(\Omega))^2$, find $\boldsymbol{u}_k \in \mathbb{V} := V \times V = (H^1_0(\Omega))^2$ such that

$$\int_\Omega \left(\nu(\boldsymbol{x})\nabla \boldsymbol{u}_k(\boldsymbol{x}) \cdot \nabla \boldsymbol{v}_k(\boldsymbol{x}) + k\omega\, \sigma(\boldsymbol{x})\boldsymbol{u}_k(\boldsymbol{x}) \cdot \boldsymbol{v}_k^\perp(\boldsymbol{x})\right) d\boldsymbol{x} = \int_\Omega \boldsymbol{f}_k(\boldsymbol{x}) \cdot \boldsymbol{v}_k(\boldsymbol{x})\, d\boldsymbol{x} \tag{15}$$

for all $\boldsymbol{v}_k \in \mathbb{V}$. In the case $k = 0$, we obtain the following variational formulation: Given $f_0^c \in L^2(\Omega)$, find $u_0^c \in V = H^1_0(\Omega)$ such that

$$\int_\Omega \nu(\boldsymbol{x})\nabla u_0^c(\boldsymbol{x}) \cdot \nabla v_0^c(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f_0^c(\boldsymbol{x})\, v_0^c(\boldsymbol{x})\, d\boldsymbol{x} \tag{16}$$

for all $v_0^c \in V$. The space $\mathbb{V} = (H^1_0(\Omega))^2$ for the Fourier coefficients is equipped with the norm $\|\boldsymbol{u}_k\|^2_{1,\Omega} = \|\boldsymbol{u}_k\|^2_\Omega + \|\nabla \boldsymbol{u}_k\|^2_\Omega$. Note that the relation $\|\boldsymbol{u}_k^\perp\|^2_\Omega = \|\boldsymbol{u}_k\|^2_\Omega$ is valid. The variational problems (15) and (16) have a unique solution due to the Babuška-Aziz theorem, see [25]. In order to numerically solve the problems, the Fourier series are truncated at a finite index N and the unknown Fourier coefficients $\boldsymbol{u}_k = (u_k^c, u_k^s)^T \in \mathbb{V}$ are approximated by finite element functions $\boldsymbol{u}_{kh} = (u_{kh}^c, u_{kh}^s)^T \in \mathbb{V}_h = V_h \times V_h \subset \mathbb{V}$. The space $\mathbb{V}_h = V_h \times V_h$ is a finite element space, where $V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ with the standard nodal basis $\{\varphi_i(\boldsymbol{x}) = \varphi_{ih}(\boldsymbol{x}) : i = 1, 2, \ldots, n_h\}$ and h denotes the usual discretization parameter such that $n = n_h = \dim V_h = O(h^{-d})$. We use continuous, piecewise linear functions on the finite elements on a regular triangulation $\mathcal{T}_h$ to construct the finite element subspace V_h and its basis, see, e.g., [3, 4, 9, 24]. Let us assume that the parameter σ is positive. Hence, the following saddle point system is obtained:

$$\begin{pmatrix} k\omega M_{h,\sigma} & -K_{h,\nu} \\ -K_{h,\nu} & -k\omega M_{h,\sigma} \end{pmatrix} \begin{pmatrix} \underline{u}_k^s \\ \underline{u}_k^c \end{pmatrix} = \begin{pmatrix} -\underline{f}_k^c \\ -\underline{f}_k^s \end{pmatrix}, \tag{17}$$

which has to be solved with respect to the nodal parameter vector $\underline{u}_k^j = (u_{k,i}^j)_{i=1,\ldots,n} \in \mathbb{R}^n$ of the finite element approximation

$$u_{kh}^j(\boldsymbol{x}) = \sum_{i=1}^n u_{k,i}^j\, \varphi_i(\boldsymbol{x})$$

to the unknown Fourier coefficients $u_k^j(\boldsymbol{x})$ with $j \in \{c, s\}$. The matrices $K_{h,\nu}$ and $M_{h,\sigma}$ correspond to the weighted stiffness matrix and weighted mass matrix, respectively. Their entries are computed by the formulas

$$K_{h,\nu}^{ij} = \int_\Omega \nu\, \nabla\varphi_i \cdot \nabla\varphi_j\, d\boldsymbol{x} \qquad \text{and} \qquad M_{h,\sigma}^{ij} = \int_\Omega \sigma\, \varphi_i\, \varphi_j\, d\boldsymbol{x}$$

with $i, j = 1, \ldots, n$, whereas

$$\underline{f}_k^c = \left[\int_\Omega f_k^c\, \varphi_j\, d\boldsymbol{x}\right]_{j=1,\ldots,n} \quad \text{and} \quad \underline{f}_k^s = \left[\int_\Omega f_k^s\, \varphi_j\, d\boldsymbol{x}\right]_{j=1,\ldots,n}.$$

In the case $k = 0$, the following linear system arising from the variational problem (16) is obtained:

$$K_{h,\nu}\, \underline{u}_0^c = \underline{f}_0^c. \tag{18}$$

Fast and robust solvers for the linear systems (17) and (18) can be found in [12, 16, 20, 25]. From the solutions of systems (17) and (18), the multiharmonic finite element approximation

$$u_{Nh}(\boldsymbol{x}, t) = u_{0h}^c(\boldsymbol{x}) + \sum_{k=1}^N \left(u_{kh}^c(\boldsymbol{x}) \cos(k\omega t) + u_{kh}^s(\boldsymbol{x}) \sin(k\omega t)\right) \tag{19}$$

to the exact solution $u(\boldsymbol{x},t)$ can be easily reconstructed. In the next section, we will present an a posteriori error analysis for the error between the unknown solution u and its multiharmonic finite element approximation u_{Nh}.

4. Functional a posteriori error estimates

In order to derive functional a posteriori error estimates, we first present inf-sup and sup-sup conditions for the space-time bilinear form (14):

Lemma 2 (Langer and Wolfmayr [20]). *The space-time bilinear form $a(\cdot,\cdot)$ defined by (14) satisfies the following inf-sup and sup-sup conditions:*

$$\mu_1 \|u\|_{H^{1,\frac{1}{2}}(Q_T)} \leq \sup_{0\neq v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} \leq \mu_2 \|u\|_{H^{1,\frac{1}{2}}(Q_T)} \tag{20}$$

for all $u \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ with positive constants $\mu_1 = \min\{\frac{\underline{\nu}}{C_F^2+1}, \underline{\sigma}\}$ and $\mu_2 = \max\{\overline{\sigma},\overline{\nu}\}$, where C_F is the constant coming from the Friedrichs inequality.

Remark 1. *Since the condition $u = 0$ is imposed on the whole boundary, we can easily find an upper bound of C_F. Indeed, $C_F(\Omega) \leq C_F(\hat{\Omega})$ if $\hat{\Omega} \supset \Omega$. Since for such domains as rectangles or balls the Friedrichs constants are known, we can obtain the required estimate.*

Lemma 3. *The space-time bilinear form $a(\cdot,\cdot)$ defined by (14) meets the following inf-sup and sup-sup conditions:*

$$\mu_1 |u|_{H^{1,\frac{1}{2}}(Q_T)} \leq \sup_{0\neq v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{|v|_{H^{1,\frac{1}{2}}(Q_T)}} \leq \mu_2 |u|_{H^{1,\frac{1}{2}}(Q_T)} \tag{21}$$

for all $u \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ with positive constants $\mu_1 = \min\{\underline{\nu},\underline{\sigma}\}$ and $\mu_2 = \max\{\overline{\sigma},\overline{\nu}\}$.

The proof of Lemma 3 follows the proof of Lemma 2, see [25]. Note that, due to the Friedrichs inequality, $|\cdot|_{H^{1,\frac{1}{2}}(Q_T)}$ is an equivalent norm.

4.1. **Error majorant of the first type.** Let a function η be an approximation of u. First, we assume that η is a bit more regular than u. More precisely, we set $\eta \in H^{1,1}_{0,per}(Q_T)$. This is of course true for the multiharmonic finite element approximation u_{Nh}, which will later play the role of η. The goal now is to deduce a computable upper bound of the error $e := u - \eta$ in $H^{1,\frac{1}{2}}_{0,per}(Q_T)$. Relation (13) implies the integral identity

$$\begin{aligned} &\int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2}(u-\eta)\,\partial_t^{1/2}v^{\perp} + \nu(\boldsymbol{x})\nabla(u-\eta)\cdot\nabla v\Big) d\boldsymbol{x}\,dt \\ &\qquad = \int_{Q_T} \Big(f\,v - \sigma(\boldsymbol{x})\partial_t^{1/2}\eta\,\partial_t^{1/2}v^{\perp} - \nu(\boldsymbol{x})\nabla\eta\cdot\nabla v\Big)\,d\boldsymbol{x}\,dt, \end{aligned} \tag{22}$$

which is valid for all $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$. Let

$$\mathcal{F}_\eta(v) := \int_{Q_T} \Big(f\,v - \sigma(\boldsymbol{x})\partial_t^{1/2}\eta\,\partial_t^{1/2}v^{\perp} - \nu(\boldsymbol{x})\nabla\eta\cdot\nabla v\Big)\,d\boldsymbol{x}\,dt.$$

It is easy to see that $\mathcal{F}_\eta(v)$ is a linear functional defined on $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$. Now (22) can be rewritten in the form

$$a(e,v) = \mathcal{F}_\eta(v). \tag{23}$$

Hence, getting an upper bound of the error is reduced to finding the quantities

$$\sup_{0\neq v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} \qquad \text{or} \qquad \sup_{0\neq v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,\frac{1}{2}}(Q_T)}}. \tag{24}$$

In order to find them, we reconstruct the functional $\mathcal{F}_\eta(v)$ using the identity

$$\big(\sigma\partial_t^{1/2}\eta, \partial_t^{1/2}v^\perp\big) = \big(\sigma\partial_t\eta, v\big) \qquad \forall\, \eta \in H^{1,1}_{0,per}(Q_T) \quad \forall\, v \in H^{1,\frac{1}{2}}_{0,per}(Q_T), \tag{25}$$

which follows from (9) and the identity

$$\int_\Omega \operatorname{div}\boldsymbol{\tau}\, v\, d\boldsymbol{x} = -\int_\Omega \boldsymbol{\tau}\cdot\nabla v\, d\boldsymbol{x},$$

which is valid for any $v \in H^1_0(\Omega)$ and any

$$\boldsymbol{\tau} \in H(\operatorname{div}_{\boldsymbol{x}}, Q_T) := \{\boldsymbol{\tau} \in [L^2(Q_T)]^d : \operatorname{div}_{\boldsymbol{x}}\boldsymbol{\tau}(\cdot, t) \in L^2(\Omega) \text{ for a.e. } t \in (0,T)\}.$$

For ease of notation, the index $\boldsymbol{x}$ in $\operatorname{div}_{\boldsymbol{x}}$ will be omitted, i.e., $\operatorname{div} = \operatorname{div}_{\boldsymbol{x}}$ denotes the generalized spatial divergence. Using the Cauchy-Schwarz inequality leads to

$$\begin{aligned}\mathcal{F}_\eta(v) &= \int_{Q_T}\Big(f\, v - \sigma(\boldsymbol{x})\partial_t\eta\, v + \operatorname{div}\boldsymbol{\tau}\, v + (\boldsymbol{\tau} - \nu(\boldsymbol{x})\nabla\eta)\cdot\nabla v\Big)\, d\boldsymbol{x}\, dt \\ &\le \|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|\|v\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\|\nabla v\|,\end{aligned} \tag{26}$$

where

$$\mathcal{R}_1(\eta,\boldsymbol{\tau}) := \sigma\partial_t\eta - \operatorname{div}\boldsymbol{\tau} - f \qquad \text{and} \qquad \mathcal{R}_2(\eta,\boldsymbol{\tau}) := \boldsymbol{\tau} - \nu\nabla\eta.$$

Applying the Friedrichs inequality in the space-time cylinder Q_T, i.e.,

$$\begin{aligned}\|\nabla u\|^2 &= \int_{Q_T}|\nabla u|^2\, d\boldsymbol{x}\, dt = T\,\|\nabla u_0^c\|^2_\Omega + \frac{T}{2}\sum_{k=1}^\infty\|\nabla\boldsymbol{u}_k\|^2_\Omega \\ &\ge \frac{1}{C_F^2}\left(T\,\|u_0^c\|^2_\Omega + \frac{T}{2}\sum_{k=1}^\infty\|\boldsymbol{u}_k\|^2_\Omega\right) = \frac{1}{C_F^2}\|u\|^2,\end{aligned} \tag{27}$$

yields

$$\begin{aligned}\mathcal{F}_\eta(v) &\le \|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|\|v\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\|\nabla v\| \\ &\le \|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|C_F\,\|\nabla v\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\|\nabla v\| \\ &\le (C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|)\,\|\nabla v\|.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{0\ne v\in H^{1,\frac12}_{0,per}(Q_T)}\frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,\frac12}(Q_T)}} &\le \sup_{0\ne v\in H^{1,\frac12}_{0,per}(Q_T)}\frac{(C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|)\,\|\nabla v\|}{|v|_{H^{1,\frac12}(Q_T)}} \\ &= \sup_{0\ne v\in H^{1,\frac12}_{0,per}(Q_T)}\frac{(C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|)\,\|\nabla v\|}{(\|\nabla v\|^2 + \|\partial_t^{1/2}v\|^2)^{1/2}} \\ &\le C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\end{aligned} \tag{28}$$

and using the left inequality of (21), i.e.,

$$|u-\eta|_{H^{1,\frac12}(Q_T)} \le \frac{1}{\mu_1}\sup_{0\ne v\in H^{1,\frac12}_{0,per}(Q_T)}\frac{a(u-\eta, v)}{|v|_{H^{1,\frac12}(Q_T)}} = \frac{1}{\mu_1}\sup_{0\ne v\in H^{1,\frac12}_{0,per}(Q_T)}\frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,\frac12}(Q_T)}},$$

yields the following result:

Theorem 1. *Let $\eta \in H^{1,1}_{0,per}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ satisfy (21). Then,*

$$|u-\eta|_{H^{1,\frac12}(Q_T)} \le \frac{1}{\mu_1}\left(C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\right) =: \mathcal{M}^{\oplus}_{|\cdot|}(\eta,\boldsymbol{\tau}), \tag{29}$$

where $\mu_1 = \min\{\underline{\nu}, \underline{\sigma}\}$ and $\boldsymbol{\tau} \in H(\operatorname{div}, Q_T)$.

Theorem 1 presents an estimate of $|e|_{H^{1,\frac{1}{2}}}$. We can also deduce an upper bound of the full $H^{1,\frac{1}{2}}$-norm. Indeed,

$$\begin{aligned}\mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|\|v\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\|\nabla v\| \\ &\leq \left(\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2\right)^{1/2}\left(\|v\|^2+\|\nabla v\|^2\right)^{1/2}.\end{aligned}$$

In view of (20), we obtain

$$\begin{aligned}\sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} &\leq \sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\left(\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2\right)^{1/2}\left(\|v\|^2+\|\nabla v\|^2\right)^{1/2}}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} \\ &\leq \left(\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2\right)^{1/2}.\end{aligned}$$

Altogether, we deduce a similar estimate for $\|e\|_{H^{1,\frac{1}{2}}}$:

Theorem 2. *Let $\eta \in H^{1,1}_{0,per}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ satisfy (20). Then,*

$$\|u-\eta\|_{H^{1,\frac{1}{2}}(Q_T)} \leq \frac{1}{\mu_1}\left(\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2\right)^{1/2} =: \mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau}), \tag{30}$$

where $\boldsymbol{\tau} \in H(\mathrm{div}, Q_T)$ and now $\mu_1 = \min\{\frac{\underline{\nu}}{C_F^2+1}, \underline{\sigma}\}$.

The functionals $\mathcal{M}^{\oplus}_{|\cdot|}(\eta,\boldsymbol{\tau})$ and $\mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau})$ present guaranteed and computable upper bounds of the error in $H^{1,\frac{1}{2}}$-norm. Henceforth, we call them error majorants.

Remark 2. *It is easy to see that the majorants are nonnegative functionals vanishing if and only if $\eta = u$ and $\boldsymbol{\tau} = \nu\nabla u$. Indeed, if $\mathcal{R}_1(\eta,\boldsymbol{\tau}) = 0$ and $\mathcal{R}_2(\eta,\boldsymbol{\tau}) = 0$, then*

$$\begin{aligned}&\sigma\partial_t\eta - div\,\boldsymbol{\tau} = f,\\ &\boldsymbol{\tau} = \nu\nabla\eta.\end{aligned}$$

Since $\eta \in H^{1,1}_{0,per}(Q_T)$ is a periodic function and satisfies the Dirichlet condition on Σ_T, it is the solution. On the other hand, $\mathcal{R}_i(u,\nu\nabla u) = 0$, $i = 1,2$.

Since $f \in L^2(Q_T)$, it can be expanded into a Fourier series. Moreover, we choose our approximation η of the solution u as well as the vector-valued function $\boldsymbol{\tau}$ to be truncated Fourier series, i.e.,

$$\begin{aligned}\eta(\boldsymbol{x},t) &= \eta_0^c(\boldsymbol{x}) + \sum_{k=1}^N \left(\eta_k^c(\boldsymbol{x})\cos(k\omega t) + \eta_k^s(\boldsymbol{x})\sin(k\omega t)\right),\\ \boldsymbol{\tau}(\boldsymbol{x},t) &= \boldsymbol{\tau}_0^c(\boldsymbol{x}) + \sum_{k=1}^N \left(\boldsymbol{\tau}_k^c(\boldsymbol{x})\cos(k\omega t) + \boldsymbol{\tau}_k^s(\boldsymbol{x})\sin(k\omega t)\right),\end{aligned} \tag{31}$$

where all Fourier coefficients are from the space $L^2(\Omega)$ and are defined by the relations

$$\begin{aligned}\eta_0^c(\boldsymbol{x}) &= \frac{1}{T}\int_0^T \eta(\boldsymbol{x},t)\,dt, & \boldsymbol{\tau}_0^c(\boldsymbol{x}) &= \frac{1}{T}\int_0^T \boldsymbol{\tau}(\boldsymbol{x},t)\,dt,\\ \eta_k^c(\boldsymbol{x}) &= \frac{2}{T}\int_0^T \eta(\boldsymbol{x},t)\cos(k\omega t)\,dt, & \boldsymbol{\tau}_k^c(\boldsymbol{x}) &= \frac{2}{T}\int_0^T \boldsymbol{\tau}(\boldsymbol{x},t)\cos(k\omega t)\,dt,\\ \eta_k^s(\boldsymbol{x}) &= \frac{2}{T}\int_0^T \eta(\boldsymbol{x},t)\sin(k\omega t)\,dt, & \boldsymbol{\tau}_k^s(\boldsymbol{x}) &= \frac{2}{T}\int_0^T \boldsymbol{\tau}(\boldsymbol{x},t)\sin(k\omega t)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}\partial_t \eta(\boldsymbol{x},t) &= \sum_{k=1}^{N} \left(k\omega\, \eta_k^s(\boldsymbol{x}) \cos(k\omega t) - k\omega\, \eta_k^c(\boldsymbol{x}) \sin(k\omega t)\right),\\ \nabla\eta(\boldsymbol{x},t) &= \nabla\eta_0^c(\boldsymbol{x}) + \sum_{k=1}^{N} \left(\nabla\eta_k^c(\boldsymbol{x})\, \cos(k\omega t) + \nabla\eta_k^s(\boldsymbol{x})\, \sin(k\omega t)\right),\\ \operatorname{div}\boldsymbol{\tau}(\boldsymbol{x},t) &= \operatorname{div}\boldsymbol{\tau}_0^c(\boldsymbol{x}) + \sum_{k=1}^{N} \left(\operatorname{div}\boldsymbol{\tau}_k^c(\boldsymbol{x})\, \cos(k\omega t) + \operatorname{div}\boldsymbol{\tau}_k^s(\boldsymbol{x})\, \sin(k\omega t)\right)\end{aligned}$$

and the $L^2(Q_T)$-norms of the functions

$$\mathcal{R}_1(\eta,\boldsymbol{\tau}) = \sigma\partial_t\eta - \operatorname{div}\boldsymbol{\tau} - f \qquad \text{and} \qquad \mathcal{R}_2(\eta,\boldsymbol{\tau}) = \boldsymbol{\tau} - \nu\nabla\eta$$

can be easily computed. Thus,

$$\begin{aligned}\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 =& T\|\operatorname{div}\boldsymbol{\tau}_0^c + f_0^c\|_\Omega^2\\ &+ \frac{T}{2}\sum_{k=1}^{N}\left(\|-k\omega\,\sigma\eta_k^s + \operatorname{div}\boldsymbol{\tau}_k^c + f_k^c\|_\Omega^2 + \|k\omega\,\sigma\eta_k^c + \operatorname{div}\boldsymbol{\tau}_k^s + f_k^s\|_\Omega^2\right)\\ &+ \frac{T}{2}\sum_{k=N+1}^{\infty}\left(\|f_k^c\|_\Omega^2 + \|f_k^s\|_\Omega^2\right)\\ =& T\|\operatorname{div}\boldsymbol{\tau}_0^c + f_0^c\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^{N}\|k\omega\,\sigma\boldsymbol{\eta}_k^\perp + \mathbf{div}\,\boldsymbol{\tau}_k + \boldsymbol{f}_k\|_\Omega^2 + \underbrace{\frac{T}{2}\sum_{k=N+1}^{\infty}\|\boldsymbol{f}_k\|_\Omega^2}_{=:\mathcal{E}_N},\end{aligned}$$

where $\boldsymbol{\eta}_k^\perp = (-\eta_k^s, \eta_k^c)^T$ and $\mathbf{div}\,\boldsymbol{\tau}_k = (\operatorname{div}\boldsymbol{\tau}_k^c, \operatorname{div}\boldsymbol{\tau}_k^s)^T$, and

$$\begin{aligned}\|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2 &= \int_{Q_T} |\boldsymbol{\tau} - \nu\nabla\eta|^2\, d\boldsymbol{x}\, dt\\ &= T\|\boldsymbol{\tau}_0^c - \nu\nabla\eta_0^c\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^{N}\left(\|\boldsymbol{\tau}_k^c - \nu\nabla\eta_k^c\|_\Omega^2 + \|\boldsymbol{\tau}_k^s - \nu\nabla\eta_k^s\|_\Omega^2\right)\\ &= T\|\boldsymbol{\tau}_0^c - \nu\nabla\eta_0^c\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^{N}\|\boldsymbol{\tau}_k - \nu\nabla\boldsymbol{\eta}_k\|_\Omega^2,\end{aligned}$$

where $\boldsymbol{\tau}_k = ((\boldsymbol{\tau}_k^c)^T, (\boldsymbol{\tau}_k^s)^T)^T$.

Remark 3. *We note that the remainder term*

$$\mathcal{E}_N = \frac{T}{2}\sum_{k=N+1}^{\infty}\|\boldsymbol{f}_k\|_\Omega^2 = \frac{T}{2}\sum_{k=N+1}^{\infty}\left(\|f_k^c\|_\Omega^2 + \|f_k^s\|_\Omega^2\right)$$

is always computable, due to the knowledge on the given data f. In some cases, the computation of $\mathcal{E}_N$ is very easy, for example, if f is multiharmonic. However, even in the most complicated cases, in which $f = f(\boldsymbol{x},t)$ and we do not refer to special (e.g., extra regularity) properties, the term $\mathcal{E}_N$ can be precomputed as $\|f - f_N\|$, where f_N is the truncated Fourier series of f.

In fact, the L^2-norms of $\mathcal{R}_1$ and $\mathcal{R}_2$ corresponding to every single mode k are decoupled. Altogether, it follows that

$$\begin{aligned}\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 =& T\|\mathcal{R}_1{}_0^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^{N}\left(\|\mathcal{R}_1{}_k^c(\eta_k^s,\boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_1{}_k^s(\eta_k^c,\boldsymbol{\tau}_k^s)\|_\Omega^2\right)\\ &+ \frac{T}{2}\sum_{k=N+1}^{\infty}\left(\|f_k^c\|_\Omega^2 + \|f_k^s\|_\Omega^2\right)\end{aligned}$$

and

$$\|\mathcal{R}_2(\eta,\tau)\|^2 = T\|\mathcal{R}_{2}{}_0^c(\eta_0^c,\boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^N \left(\|\mathcal{R}_{2}{}_k^c(\eta_k^c,\boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{2}{}_k^s(\eta_k^s,\boldsymbol{\tau}_k^s)\|_\Omega^2\right),$$

where

$$\begin{aligned}
\mathcal{R}_{1}{}_0^c(\boldsymbol{\tau}_0^c) &:= \operatorname{div}\boldsymbol{\tau}_0^c + f_0^c, \\
\mathcal{R}_{1}{}_k^c(\eta_k^s,\boldsymbol{\tau}_k^c) &:= -k\omega\,\sigma\eta_k^s + \operatorname{div}\boldsymbol{\tau}_k^c + f_k^c, \qquad \forall\, k=1,\dots,N, \\
\mathcal{R}_{1}{}_k^s(\eta_k^c,\boldsymbol{\tau}_k^s) &:= k\omega\,\sigma\eta_k^c + \operatorname{div}\boldsymbol{\tau}_k^s + f_k^s, \qquad \forall\, k=1,\dots,N,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\mathcal{R}_{2}{}_0^c(\eta_0^c,\boldsymbol{\tau}_0^c) &:= \boldsymbol{\tau}_0^c - \nu\nabla\eta_0^c, \\
\mathcal{R}_{2}{}_k^j(\eta_k^j,\boldsymbol{\tau}_k^j) &:= \boldsymbol{\tau}_k^j - \nu\nabla\eta_k^j, \qquad \forall\, k=1,\dots,N, \quad j\in\{c,s\}.
\end{aligned} \tag{33}$$

Corollary 1. *The error majorants $\mathcal{M}^{\oplus}_{|\cdot|}(\eta,\boldsymbol{\tau})$ and $\mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau})$ can be presented in the forms*

$$\begin{aligned}
\mathcal{M}^{\oplus}_{|\cdot|}(\eta,\boldsymbol{\tau}) &= \frac{1}{\mu_{1,|\cdot|}}\Big(C_F\,\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|\Big) \\
&= \frac{1}{\mu_{1,|\cdot|}}\Big(C_F\,\big(T\|\mathcal{R}_{1}{}_0^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2}\sum_{k=1}^N\left(\|\mathcal{R}_{1}{}_k^c(\eta_k^s,\boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{1}{}_k^s(\eta_k^c,\boldsymbol{\tau}_k^s)\|_\Omega^2\right) \\
&\qquad + \frac{T}{2}\sum_{k=N+1}^\infty\left(\|f_k^c\|_\Omega^2 + \|f_k^s\|_\Omega^2\right)\big)^{1/2} + \Big(T\|\mathcal{R}_{2}{}_0^c(\eta_0^c,\boldsymbol{\tau}_0^c)\|_\Omega^2 \\
&\qquad + \frac{T}{2}\sum_{k=1}^N\left(\|\mathcal{R}_{2}{}_k^c(\eta_k^c,\boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{2}{}_k^s(\eta_k^s,\boldsymbol{\tau}_k^s)\|_\Omega^2\right)\Big)^{1/2}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau}) &= \frac{1}{\mu_{1,\|\cdot\|}}\left(\|\mathcal{R}_1(\eta,\boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta,\boldsymbol{\tau})\|^2\right)^{1/2} \\
&= \frac{1}{\mu_{1,\|\cdot\|}}\Big(T\big(\|\mathcal{R}_{1}{}_0^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \|\mathcal{R}_{2}{}_0^c(\eta_0^c,\boldsymbol{\tau}_0^c)\|_\Omega^2\big) + \frac{T}{2}\sum_{k=1}^N\big(\|\mathcal{R}_{1}{}_k^c(\eta_k^s,\boldsymbol{\tau}_k^c)\|_\Omega^2 \\
&\qquad + \|\mathcal{R}_{1}{}_k^s(\eta_k^c,\boldsymbol{\tau}_k^s)\|_\Omega^2 + \|\mathcal{R}_{2}{}_k^c(\eta_k^c,\boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{2}{}_k^s(\eta_k^s,\boldsymbol{\tau}_k^s)\|_\Omega^2\big) \\
&\qquad + \frac{T}{2}\sum_{k=N+1}^\infty\left(\|f_k^c\|_\Omega^2 + \|f_k^s\|_\Omega^2\right)\Big)^{1/2},
\end{aligned}$$

where $\mu_{1,|\cdot|} = \min\{\underline{\nu},\underline{\sigma}\}$ and $\mu_{1,\|\cdot\|} = \min\{\frac{\underline{\nu}}{C_F^2+1},\underline{\sigma}\}$.

We see that the majorants consist of computable quantities related to each harmonic. Therefore, they not only evaluate the overall error, but also provide an information on errors associated with a certain harmonic. Moreover, since the respective quantities are integrals over Ω, their integrands serve as indicators of spatial errors. Thus, the majorants contain a rich amount of information to be utilized in various adaptive procedures.

Remark 4. *Let f has a multiharmonic representation, i.e.,*

$$f(\boldsymbol{x},t) = f_0^c(\boldsymbol{x}) + \sum_{k=1}^{N_f}\left(f_k^c(\boldsymbol{x})\cos(k\omega t) + f_k^s(\boldsymbol{x})\sin(k\omega t)\right),$$

where $N_f\in\mathbb{N}$ is defined by f. If $N\geq N_f$, then η is the exact solution of problem (13) and $\boldsymbol{\tau}$ is the exact flux if and only if the error majorants vanish, i.e.,

$$\begin{aligned}
\mathcal{R}_{1}{}_k^c &= 0 \quad \text{and} \quad \mathcal{R}_{2}{}_k^c = 0 \qquad \forall\, k=0,1,\dots,N_f, \\
\mathcal{R}_{1}{}_k^s &= 0 \quad \text{and} \quad \mathcal{R}_{2}{}_k^s = 0 \qquad \forall\, k=1,2,\dots,N_f.
\end{aligned} \tag{34}$$

Indeed, let the error majorants vanish. Then,

$$
\begin{aligned}
&-\,div\,\boldsymbol{\tau}_0^c = f_0^c, && \boldsymbol{\tau}_0^c = \nu\nabla\eta_0^c,\\
&k\omega\,\sigma\eta_k^s - div\,\boldsymbol{\tau}_k^c = f_k^c,\ -k\omega\,\sigma\eta_k^c - div\,\boldsymbol{\tau}_k^s = f_k^s,\ \boldsymbol{\tau}_k^c = \nu\nabla\eta_k^c,\ \boldsymbol{\tau}_k^s = \nu\nabla\eta_k^s, && \forall\, k = 1,\dots,N_f,
\end{aligned}
$$

so that collecting harmonics, we find that

$$
\begin{aligned}
\boldsymbol{\tau}(\boldsymbol{x},t) &= \boldsymbol{\tau}_0^c(\boldsymbol{x}) + \sum_{k=1}^{N_f}\left(\boldsymbol{\tau}_k^c(\boldsymbol{x})\cos(k\omega t) + \boldsymbol{\tau}_k^s(\boldsymbol{x})\sin(k\omega t)\right),\\
\eta(\boldsymbol{x},t) &= \eta_0^c(\boldsymbol{x}) + \sum_{k=1}^{N_f}\left(\eta_k^c(\boldsymbol{x})\cos(k\omega t) + \eta_k^s(\boldsymbol{x})\sin(k\omega t)\right)
\end{aligned}
$$

and

$$
\sigma\partial_t\eta - div\,\boldsymbol{\tau} = f, \qquad \boldsymbol{\tau} = \nu\nabla\eta.
$$

Since η satisfies the boundary conditions and the equation, we conclude that $\eta = u$.

Another approach to derive a majorant is to insert the Fourier series ansatz directly to the bilinear form $a(u-\eta, v)$ and into the functional $\mathcal{F}_\eta(v)$ as defined in (22). Then, we obtain the following integral identities associated with every mode:

$$
\begin{aligned}
&\int_\Omega \big(\nu(\boldsymbol{x})\nabla(\boldsymbol{u}_k(\boldsymbol{x}) - \boldsymbol{\eta}_k(\boldsymbol{x}))\cdot\nabla\boldsymbol{v}_k(\boldsymbol{x}) + k\omega\,\sigma(\boldsymbol{x})(\boldsymbol{u}_k(\boldsymbol{x}) - \boldsymbol{\eta}_k(\boldsymbol{x}))\cdot\boldsymbol{v}_k^\perp(\boldsymbol{x})\big)\,d\boldsymbol{x}\\
&\quad = \int_\Omega \big(\boldsymbol{f}_k(\boldsymbol{x})\cdot\boldsymbol{v}_k(\boldsymbol{x}) - \nu(\boldsymbol{x})\nabla\boldsymbol{\eta}_k(\boldsymbol{x})\cdot\nabla\boldsymbol{v}_k(\boldsymbol{x}) - k\omega\,\sigma(\boldsymbol{x})\boldsymbol{\eta}_k(\boldsymbol{x})\cdot\boldsymbol{v}_k^\perp(\boldsymbol{x})\big)\,d\boldsymbol{x},
\end{aligned}
\tag{35}
$$

which are valid for all $\boldsymbol{v}_k \in (H_0^1(\Omega))^2$. In the case $k=0$, the integral identity

$$
\int_\Omega \nu(\boldsymbol{x})\nabla(u_0^c(\boldsymbol{x}) - \eta_0^c(\boldsymbol{x}))\cdot\nabla v_0^c(\boldsymbol{x})\,d\boldsymbol{x} = \int_\Omega \left(f_0^c(\boldsymbol{x})\,v_0^c(\boldsymbol{x}) - \nu(\boldsymbol{x})\nabla\eta_0^c(\boldsymbol{x})\cdot\nabla v_0^c(\boldsymbol{x})\right)\,d\boldsymbol{x}
\tag{36}
$$

is valid for all $v_0^c \in H_0^1(\Omega)$. We define the left hand sides of (35) and (36) by

$$
a_k(\boldsymbol{u}_k - \boldsymbol{\eta}_k, \boldsymbol{v}_k) \qquad \text{and} \qquad a_0(u_0^c - \eta_0^c, v_0^c),
$$

and the right hand sides by

$$
\mathcal{F}_{\boldsymbol{\eta}_k}(\boldsymbol{v}_k) \qquad \text{and} \qquad \mathcal{F}_{\eta_0^c}(v_0^c),
$$

respectively. Let us start with the case $k = 1,\dots,N$. Hence, an upper bound for the errors $\boldsymbol{e}_k := \boldsymbol{u}_k - \boldsymbol{\eta}_k$ in $(H_0^1(\Omega))^2$ has to be computed. The bilinear form $a_k(\cdot,\cdot)$ meets the inf-sup condition

$$
\sup_{0\neq\boldsymbol{v}_k\in(H_0^1(\Omega))^2} \frac{a_k(\boldsymbol{u}_k - \boldsymbol{\eta}_k, \boldsymbol{v}_k)}{\|\boldsymbol{v}_k\|_{1,\Omega}} \geq \underline{c}^k_{\|\cdot\|}\,\|\boldsymbol{u}_k - \boldsymbol{\eta}_k\|_{1,\Omega}
\tag{37}
$$

with the inf-sup constant $\underline{c}^k_{\|\cdot\|} = \min\{\underline{\nu}, k\omega\,\underline{\sigma}\}$. By the same method as before, we reform the error functionals and obtain estimates for

$$
\sup_{0\neq\boldsymbol{v}_k\in(H_0^1(\Omega))^2} \frac{\mathcal{F}_{\boldsymbol{\eta}_k}(\boldsymbol{v}_k)}{\|\boldsymbol{v}_k\|_{1,\Omega}}.
$$

We introduce a collection of vector-valued functions

$$
\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T, \qquad \boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\mathrm{div},\Omega) := \{\boldsymbol{\tau}\in[L^2(\Omega)]^d : \mathrm{div}\,\boldsymbol{\tau}\in L^2(\Omega)\}
$$

and use the integral relations

$$
\int_\Omega \mathrm{div}\,\boldsymbol{\tau}\,v\,d\boldsymbol{x} = -\int_\Omega \boldsymbol{\tau}\cdot\nabla v\,d\boldsymbol{x} \qquad \forall\, v\in H_0^1(\Omega).
$$

It is easy to see that

$$
\begin{aligned}
\mathcal{F}_{\boldsymbol{\eta}_k}(\boldsymbol{v}_k) &= \int_\Omega \big(\boldsymbol{f}_k \cdot \boldsymbol{v}_k - k\omega\,\sigma(\boldsymbol{x})\boldsymbol{\eta}_k \cdot \boldsymbol{v}_k^\perp + \mathbf{div}\,\boldsymbol{\tau}_k \cdot \boldsymbol{v}_k \\
&\qquad + \boldsymbol{\tau}_k \cdot \nabla \boldsymbol{v}_k - \nu(\boldsymbol{x})\nabla\boldsymbol{\eta}_k \cdot \nabla \boldsymbol{v}_k\big) d\boldsymbol{x} \\
&= \int_\Omega \big(\boldsymbol{f}_k \cdot \boldsymbol{v}_k + k\omega\,\sigma(\boldsymbol{x})\boldsymbol{\eta}_k^\perp \cdot \boldsymbol{v}_k + \mathbf{div}\,\boldsymbol{\tau}_k \cdot \boldsymbol{v}_k + (\boldsymbol{\tau}_k - \nu(\boldsymbol{x})\nabla\boldsymbol{\eta}_k) \cdot \nabla \boldsymbol{v}_k\big) d\boldsymbol{x} \\
&\le \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\boldsymbol{v}_k\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\nabla \boldsymbol{v}_k\|_\Omega \\
&= \left(\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2\right)^{1/2} \|\boldsymbol{v}_k\|_{1,\Omega},
\end{aligned} \tag{38}
$$

where

$$
\begin{aligned}
\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k) = k\omega\,\sigma\boldsymbol{\eta}_k^\perp + \mathbf{div}\,\boldsymbol{\tau}_k + \boldsymbol{f}_k &= (-k\omega\,\sigma\eta_k^s + \operatorname{div}\boldsymbol{\tau}_k^c + f_k^c, k\omega\,\sigma\eta_k^c + \operatorname{div}\boldsymbol{\tau}_k^s + f_k^s)^T \\
&= (\mathcal{R}_{1k}^{\,c}(\eta_k^s, \boldsymbol{\tau}_k^c), \mathcal{R}_{1k}^{\,s}(\eta_k^c, \boldsymbol{\tau}_k^s))^T
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k) = \boldsymbol{\tau}_k - \nu\nabla\boldsymbol{\eta}_k &= (\boldsymbol{\tau}_k^c - \nu\nabla\eta_k^c, \boldsymbol{\tau}_k^s - \nu\nabla\eta_k^s)^T \\
&= (\mathcal{R}_{2k}^{\,c}(\eta_k^c, \boldsymbol{\tau}_k^c), \mathcal{R}_{2k}^{\,s}(\eta_k^s, \boldsymbol{\tau}_k^s))^T.
\end{aligned}
$$

Hence, the same results as in (32) and (33) for every mode $k = 1, \ldots, N$ have been derived. Using the estimate (38) together with the inf-sup condition (37), we finally arrive at the following upper bounds for every single mode $k = 1, \ldots, N$:

Theorem 3. *Let $\boldsymbol{\eta}_k \in (H_0^1(\Omega))^2$ and the bilinear form $a_k(\cdot,\cdot)$ satisfy (37). Then,*

$$
\|\boldsymbol{u}_k - \boldsymbol{\eta}_k\|_{1,\Omega} \le \frac{1}{\underline{c}_{\|\cdot\|}^k} \left(\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2\right)^{1/2} =: \mathcal{M}_{\|\cdot\|}^{\oplus_k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k), \tag{39}
$$

where $\underline{c}_{\|\cdot\|}^k = \min\{\underline{\nu}, k\omega\,\underline{\sigma}\}$ and $\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T$ with $\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\mathrm{div}, \Omega)$.

Using the inf-sup condition

$$
\begin{aligned}
\sup_{0 \ne \boldsymbol{v}_k \in (H_0^1(\Omega))^2} \frac{a_k(\boldsymbol{u}_k, \boldsymbol{v}_k)}{|\boldsymbol{v}_k|_{1,\Omega}} &= \sup_{0 \ne \boldsymbol{v}_k \in (H_0^1(\Omega))^2} \frac{(\nu\nabla\boldsymbol{u}_k, \nabla\boldsymbol{v}_k)_\Omega + k\omega\left(\sigma\boldsymbol{u}_k, \boldsymbol{v}_k^\perp\right)_\Omega}{|\boldsymbol{v}_k|_{1,\Omega}} \\
&\ge \frac{\left(\nu\nabla\boldsymbol{u}_k, \nabla(\boldsymbol{u}_k - \boldsymbol{u}_k^\perp)\right)_\Omega + k\omega\left(\sigma\boldsymbol{u}_k, (\boldsymbol{u}_k - \boldsymbol{u}_k^\perp)^\perp\right)_\Omega}{|\boldsymbol{u}_k - \boldsymbol{u}_k^\perp|_{1,\Omega}} \\
&= \frac{(\nu\nabla\boldsymbol{u}_k, \nabla\boldsymbol{u}_k)_\Omega + k\omega\left(\sigma\boldsymbol{u}_k, \boldsymbol{u}_k\right)_\Omega}{\sqrt{2}|\boldsymbol{u}_k|_{1,\Omega}} \ge \frac{\underline{\nu}\|\nabla\boldsymbol{u}_k\|_\Omega^2 + k\omega\underline{\sigma}\|\boldsymbol{u}_k\|_\Omega^2}{\sqrt{2}|\boldsymbol{u}_k|_{1,\Omega}} \\
&\ge \frac{\underline{\nu}\|\nabla\boldsymbol{u}_k\|_\Omega^2 + \frac{k\omega\underline{\sigma}}{C_F^2+1}\|\nabla\boldsymbol{u}_k\|_\Omega^2}{\sqrt{2}|\boldsymbol{u}_k|_{1,\Omega}} \ge \frac{\min\{\underline{\nu}, \frac{k\omega\underline{\sigma}}{C_F^2+1}\}}{\sqrt{2}} |\boldsymbol{u}_k|_{1,\Omega}
\end{aligned} \tag{40}
$$

together with the estimate

$$
\begin{aligned}
\mathcal{F}_{\boldsymbol{\eta}_k}(\boldsymbol{v}_k) &\le \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\boldsymbol{v}_k\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\nabla \boldsymbol{v}_k\|_\Omega \\
&\le \left(C_F\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega\right) |\boldsymbol{v}_k|_{1,\Omega}
\end{aligned}
$$

yields the following error majorant for $|\cdot|_{1,\Omega}$:

Theorem 4. *Let $\boldsymbol{\eta}_k \in (H_0^1(\Omega))^2$ and the bilinear form $a_k(\cdot,\cdot)$ satisfy (40). Then,*

$$
|\boldsymbol{u}_k - \boldsymbol{\eta}_k|_{1,\Omega} \le \frac{1}{\underline{c}_{|\cdot|}^k} \left(C_F\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega\right) =: \mathcal{M}_{|\cdot|}^{\oplus_k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k), \tag{41}
$$

where $\underline{c}_{|\cdot|}^k = \min\{\underline{\nu}, k\omega\underline{\sigma}/(C_F^2+1)\}/\sqrt{2}$ and $\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T$ with $\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\mathrm{div}, \Omega)$.

Now, we consider the case $k = 0$. Here, an upper bound for the error $e_0^c := u_0^c - \eta_0^c$ in $H_0^1(\Omega)$ has to be computed. The inf-sup condition

$$\sup_{0 \neq v_0^c \in H_0^1(\Omega)} \frac{a_0(u_0^c - \eta_0^c, v_0^c)}{\|v_0^c\|_{1,\Omega}} \geq \underline{c}^0_{\|\cdot\|} \, \|u_0^c - \eta_0^c\|_{1,\Omega} \tag{42}$$

with the inf-sup constant $\underline{c}^0_{\|\cdot\|} = \underline{\nu}/(C_F^2 + 1)$ can be proved quite analogously to (37). Moreover, one can easily show that

$$\sup_{0 \neq v_0^c \in H_0^1(\Omega)} \frac{a_0(u_0^c - \eta_0^c, v_0^c)}{|v_0^c|_{1,\Omega}} \geq \frac{a_0(u_0^c - \eta_0^c, u_0^c - \eta_0^c)}{|u_0^c - \eta_0^c|_{1,\Omega}} \geq \underline{\nu} \, |u_0^c - \eta_0^c|_{1,\Omega}, \tag{43}$$

since ν satisfies the assumptions (4). By arguments similar to those used above for the modes k, we deduce the following estimates:

$$\|u_0^c - \eta_0^c\|_{1,\Omega} \leq \frac{1}{\underline{c}^0_{\|\cdot\|}} \left(\|\mathcal{R}_1{}_0^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \|\mathcal{R}_2{}_0^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 \right)^{1/2} =: \mathcal{M}^{\oplus_0}_{\|\cdot\|}(\eta_0^c, \boldsymbol{\tau}_0^c) \tag{44}$$

and

$$|u_0^c - \eta_0^c|_{1,\Omega} \leq \frac{1}{\underline{\nu}} \left(C_F \|\mathcal{R}_1{}_0^c(\boldsymbol{\tau}_0^c)\|_\Omega + \|\mathcal{R}_2{}_0^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega \right) =: \mathcal{M}^{\oplus_0}_{|\cdot|}(\eta_0^c, \boldsymbol{\tau}_0^c), \tag{45}$$

where $\boldsymbol{\tau}_0^c \in H(\mathrm{div}, \Omega)$,

$$\mathcal{R}_1{}_0^c(\boldsymbol{\tau}_0^c) = f_0^c + \mathrm{div}\, \boldsymbol{\tau}_0^c \qquad \text{and} \qquad \mathcal{R}_2{}_0^c(\eta_0^c, \boldsymbol{\tau}_0^c) = \boldsymbol{\tau}_0^c - \nu \nabla \eta_0^c.$$

4.2. **Error majorant of the second type.** In this section, we deduce another upper bound of the error $e := u - \eta$, which is valid for approximations that are less regular with respect to the time, i.e., $\eta \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$. In fact, we will choose a multiharmonic finite element approximation u_{Nh} as η, which is, of course, more regular in time, but the abstract functional a posteriori error estimates, which are obtained, can be used in a more general setting. Let us again consider the functional

$$\mathcal{F}_\eta(v) = \int_{Q_T} \Big(f\, v - \sigma(\boldsymbol{x}) \partial_t^{1/2} \eta \, \partial_t^{1/2} v^\perp - \nu(\boldsymbol{x}) \nabla \eta \cdot \nabla v \Big) \, d\boldsymbol{x} \, dt$$

defined for all $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$. In addition to the vector-valued function $\boldsymbol{\tau} \in H(\mathrm{div}, Q_T)$, we introduce the function $\kappa \in H^{0,\frac{1}{2}}_{per}(Q_T)$. We rearrange the functional $\mathcal{F}_\eta(v)$ and write it as

$$\begin{aligned}
\mathcal{F}_\eta(v) &= \int_{Q_T} \Big(f\, v - \sigma \partial_t^{1/2} \eta \, \partial_t^{1/2} v^\perp - \nu \nabla \eta \cdot \nabla v \Big) \, d\boldsymbol{x} \, dt \\
&= \int_{Q_T} \Big(f\, v - \sigma \partial_t^{1/2} \eta \, \partial_t^{1/2} v^\perp + \sigma\, \kappa \, \partial_t^{1/2} v^\perp + \sigma \partial_t^{1/2} \kappa^\perp \, v + \mathrm{div}\, \boldsymbol{\tau}\, v \\
&\qquad\qquad + \boldsymbol{\tau} \cdot \nabla v - \nu \nabla \eta \cdot \nabla v \Big) \, d\boldsymbol{x} \, dt \\
&= \int_{Q_T} \Big(\Big(f + \mathrm{div}\, \boldsymbol{\tau} + \sigma \partial_t^{1/2} \kappa^\perp \Big) \, v + \Big(\sigma(-\partial_t^{1/2} \eta + \kappa) \Big) \, \partial_t^{1/2} v^\perp + (\boldsymbol{\tau} - \nu \nabla \eta) \cdot \nabla v \Big) \, d\boldsymbol{x} \, dt
\end{aligned}$$

for all $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$.

Remark 5. *The function $\boldsymbol{\tau}$ can be interpreted as "an image" of $\nu \nabla u$ and the function κ as "an image" of $\partial_t^{1/2} u$.*

Let

$$\begin{aligned}
\mathcal{R}_1(\boldsymbol{\tau}, \kappa) &:= f + \mathrm{div}\, \boldsymbol{\tau} + \sigma \partial_t^{1/2} \kappa^\perp, \\
\mathcal{R}_2(\boldsymbol{\tau}, \eta) &:= \boldsymbol{\tau} - \nu \nabla \eta, \\
\mathcal{R}_3(\kappa, \eta) &:= \sigma(\kappa - \partial_t^{1/2} \eta).
\end{aligned}$$

Then, the functional $\mathcal{F}_\eta(v)$ can be estimated from above as follows

$$\begin{aligned}\mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|\|v\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\|\nabla v\| + \|\mathcal{R}_3(\kappa,\eta)\|\|\partial_t^{1/2} v\| \\ &\leq \left(\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|^2 + \|\mathcal{R}_3(\kappa,\eta)\|^2\right)^{1/2} \|v\|_{H^{1,\frac{1}{2}}(Q_T)},\end{aligned}$$

using the Cauchy-Schwarz inequality and $\|\partial_t^{1/2} v^\perp\| = \|\partial_t^{1/2} v\|$, since

$$\left\|\partial_t^{1/2} v^\perp\right\|^2 = \frac{T}{2}\sum_{k=1}^{\infty} k\omega \|\boldsymbol{v}_k^\perp\|_\Omega^2 = \frac{T}{2}\sum_{k=1}^{\infty} k\omega\|\boldsymbol{v}_k\|_\Omega^2 = \left\|\partial_t^{1/2} v\right\|^2.$$

Altogether, we obtain the upper bound

$$\sup_{0\neq v\in H_{0,per}^{1,\frac{1}{2}}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} \leq \left(\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|^2 + \|\mathcal{R}_3(\kappa,\eta)\|^2\right)^{1/2}. \tag{46}$$

From (20) follows that

$$\|u-\eta\|_{H^{1,\frac{1}{2}}(Q_T)} \leq \frac{1}{\mu_1} \sup_{0\neq v\in H_{0,per}^{1,\frac{1}{2}}(Q_T)} \frac{a(u-\eta,v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}} = \frac{1}{\mu_1} \sup_{0\neq v\in H_{0,per}^{1,\frac{1}{2}}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,\frac{1}{2}}(Q_T)}},$$

which leads together with (46) to the following result:

Theorem 5. *Let $\eta \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ defined in (14) satisfy (20). Then,*

$$\begin{aligned}\|u-\eta\|_{H^{1,\frac{1}{2}}(Q_T)} &\leq \frac{1}{\mu_1}\left(\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|^2 + \|\mathcal{R}_3(\kappa,\eta)\|^2\right)^{1/2} \\ &=: \mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau},\kappa),\end{aligned} \tag{47}$$

where $\boldsymbol{\tau} \in H(\mathrm{div}, Q_T)$, $\kappa \in H_{per}^{0,\frac{1}{2}}(Q_T)$ and $\mu_1 = \min\{\frac{\nu}{C_F^2+1}, \underline{\sigma}\}$.

Remark 6. *If $\mathcal{R}_1(\boldsymbol{\tau},\kappa) = 0$, $\mathcal{R}_2(\boldsymbol{\tau},\eta) = 0$ and $\mathcal{R}_3(\kappa,\eta) = 0$, then*

$$-\sigma\partial_t^{1/2}\kappa^\perp - div\,\boldsymbol{\tau} = f, \qquad \boldsymbol{\tau} = \nu\nabla\eta, \qquad \kappa = \partial_t^{1/2}\eta.$$

Since η satisfies the Dirichlet condition on Σ_T, η is the solution. In other words, $\mathcal{M}^{\oplus}_{\|\cdot\|}(\eta,\boldsymbol{\tau},\kappa)$ vanishes if and only if η is the exact solution, $\boldsymbol{\tau}$ is the exact flux and κ is the exact half time derivative of the solution. Moreover, if $\eta \in H_{0,per}^{1,1}(Q_T)$, one derives the original equation (1) in the weak sense, due to

$$-\left(\sigma\partial_t^{1/2}(\partial_t^{1/2}\eta)^\perp, v\right) = \left(\sigma\partial_t^{1/2}\eta, \partial_t^{1/2}v^\perp\right) = \left(\sigma\partial_t\eta, v\right)$$

using the σ-weighted counterparts of the identities (11) and (9), cf. (10) as well.

It is obvious that similar results to the ones obtained in Theorem 1 for $|\cdot|_{H^{1,\frac{1}{2}}(Q_T)}$ can be shown here together with the estimate

$$\begin{aligned}\mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|\|v\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\|\nabla v\| + \|\mathcal{R}_3(\kappa,\eta)\|\|\partial_t^{1/2}v\| \\ &\leq C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|\|\nabla v\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\|\nabla v\| + \|\mathcal{R}_3(\kappa,\eta)\|\|\partial_t^{1/2}v\| \\ &= \left(C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\right)\|\nabla v\| + \|\mathcal{R}_3(\kappa,\eta)\|\|\partial_t^{1/2}v\| \\ &\leq \Big(\left(C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\right)^2 + \|\mathcal{R}_3(\kappa,\eta)\|^2\Big)^{1/2}|v|_{H^{1,\frac{1}{2}}(Q_T)}.\end{aligned} \tag{48}$$

From (21) and (48), we deduce the following a posteriori estimate analogously proven as the one of Theorem 5:

Theorem 6. *Let $\eta \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ defined by (14) satisfy (21). Then,*

$$\begin{aligned}|u-\eta|_{H^{1,\frac{1}{2}}(Q_T)} &\leq \frac{1}{\mu_1}\Big(\left(C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\right)^2 + \|\mathcal{R}_3(\kappa,\eta)\|^2\Big)^{1/2} \\ &=: \mathcal{M}^{\oplus}_{|\cdot|}(\eta,\boldsymbol{\tau},\kappa),\end{aligned} \tag{49}$$

where $\boldsymbol{\tau} \in H(\operatorname{div}, Q_T)$, $\kappa \in H^{0,\frac{1}{2}}(Q_T)$ *and* $\mu_1 = \min\{\underline{\nu}, \underline{\sigma}\}$.

Remark 7. *If* $\eta \in H^{1,1}_{0,per}(Q_T)$, *then* $\kappa = \partial_t^{1/2}\eta$ *and the term* $\mathcal{R}_3$ *vanishes. Hence, we arrive at the estimate (29), which can be now viewed as a particular case of (49).*

Similarly, one can prove a posteriori error estimates using the following weighted norm:

$$\begin{aligned} |u|^2_{V_0} &= (\nu \nabla u, \nabla u) + \big(\sigma \partial_t^{1/2} u, \partial_t^{1/2} u\big) \\ &= T\,(\nu \nabla u_0^c, \nabla u_0^c)_\Omega + \frac{T}{2}\sum_{k=1}^{\infty} \left((\nu \nabla \boldsymbol{u}_k, \nabla \boldsymbol{u}_k)_\Omega + k\omega\,(\sigma \boldsymbol{u}_k, \boldsymbol{u}_k)_\Omega\right). \end{aligned} \tag{50}$$

First, we need to obtain the corresponding inf-sup and sup-sup inequalities.

Lemma 4. *The space-time bilinear form* $a(\cdot,\cdot)$ *defined by (14) meets the following inf-sup and sup-sup conditions:*

$$\mu_1 |u|_{V_0} \le \sup_{0 \ne v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{|v|_{V_0}} \le \mu_2 |u|_{V_0} \tag{51}$$

for all $u \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ *with constants* $\mu_1 = 1/\sqrt{2}$ *and* $\mu_2 = 1$.

Proof. Let us start with the proof of the sup-sup condition. Using the triangle inequality and the σ- and ν-weighted counterparts of the Cauchy-Schwarz inequalities in the Fourier space, e.g.,

$$\begin{aligned} \left|(\sigma \partial_t^{1/2} u, \partial_t^{1/2} v)\right| &= \left|\frac{T}{2}\sum_{k=1}^{\infty} k\omega (\sigma \boldsymbol{u}_k, \boldsymbol{v}_k)_\Omega\right| \le \frac{T}{2}\sum_{k=1}^{\infty} k\omega\,(\sigma \boldsymbol{u}_k, \boldsymbol{u}_k)_\Omega^{1/2} (\sigma \boldsymbol{v}_k, \boldsymbol{v}_k)_\Omega^{1/2} \\ &\le \left(\frac{T}{2}\sum_{k=1}^{\infty} k\omega (\sigma \boldsymbol{u}_k, \boldsymbol{u}_k)_\Omega\right)^{1/2} \left(\frac{T}{2}\sum_{k=1}^{\infty} k\omega (\sigma \boldsymbol{v}_k, \boldsymbol{v}_k)_\Omega\right)^{1/2} \\ &= (\sigma \partial_t^{1/2} u, \partial_t^{1/2} u)^{1/2} (\sigma \partial_t^{1/2} v, \partial_t^{1/2} v)^{1/2} \qquad \forall\, u, v \in H^{0,\frac{1}{2}}_{per}(Q_T), \end{aligned}$$

leads to the estimate

$$\begin{aligned} |a(u,v)| &= \left|\int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2} u\, \partial_t^{1/2} v^\perp + \nu(\boldsymbol{x})\nabla u \cdot \nabla v\Big)\, d\boldsymbol{x}\, dt\right| \\ &\le \left|\int_{Q_T} \sigma(\boldsymbol{x})\partial_t^{1/2} u\, \partial_t^{1/2} v^\perp\, d\boldsymbol{x}\, dt\right| + \left|\int_{Q_T} \nu(\boldsymbol{x})\nabla u \cdot \nabla v\, d\boldsymbol{x}\, dt\right| \\ &\le \big(\sigma \partial_t^{1/2} u, \partial_t^{1/2} u\big)^{1/2} \big(\sigma \partial_t^{1/2} v^\perp, \partial_t^{1/2} v^\perp\big)^{1/2} + (\nu \nabla u, \nabla u)^{1/2} (\nu \nabla v, \nabla v)^{1/2} \\ &= \big(\sigma \partial_t^{1/2} u, \partial_t^{1/2} u\big)^{1/2} \big(\sigma \partial_t^{1/2} v, \partial_t^{1/2} v\big)^{1/2} + (\nu \nabla u, \nabla u) (\nu \nabla v, \nabla v)^{1/2}, \end{aligned}$$

since

$$\big(\sigma \partial_t^{1/2} v^\perp, \partial_t^{1/2} v^\perp\big) = \frac{T}{2}\sum_{k=1}^{\infty} k\omega (\sigma \boldsymbol{v}_k^\perp, \boldsymbol{v}_k^\perp)_\Omega = \frac{T}{2}\sum_{k=1}^{\infty} k\omega (\sigma \boldsymbol{v}_k, \boldsymbol{v}_k)_\Omega = \big(\sigma \partial_t^{1/2} v, \partial_t^{1/2} v\big).$$

Finally, the sup-sup condition is proven by

$$\begin{aligned} |a(u,v)| &\le \big(\sigma \partial_t^{1/2} u, \partial_t^{1/2} u\big)^{1/2} \big(\sigma \partial_t^{1/2} v, \partial_t^{1/2} v\big)^{1/2} + (\nu \nabla u, \nabla u)^{1/2} (\nu \nabla v, \nabla v)^{1/2} \\ &\le \Big(\big(\sigma \partial_t^{1/2} u, \partial_t^{1/2} u\big) + (\nu \nabla u, \nabla u)\Big)^{1/2} \Big(\big(\sigma \partial_t^{1/2} v, \partial_t^{1/2} v\big) + (\nu \nabla v, \nabla v)\Big)^{1/2} \\ &= \mu_2 |u|_{V_0} |v|_{V_0} \end{aligned}$$

with the constant $\mu_2 = 1$.

Let us now prove the inf-sup condition by choosing the test function $\pi_u = u - u^\perp$ and using the σ- and ν-weighted orthogonality relations (10). The choice $\pi_u = u - u^\perp$ yields

$$\begin{aligned} a(u,u) &= \int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2} u\, \partial_t^{1/2} u^\perp + \nu(\boldsymbol{x})\nabla u \cdot \nabla u\Big)\, d\boldsymbol{x}\, dt = (\nu\nabla u, \nabla u), \\ a(u,-u^\perp) &= \int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2} u\, \partial_t^{1/2} u + \nu(\boldsymbol{x})\nabla u \cdot \nabla u^\perp\Big)\, d\boldsymbol{x}\, dt = \big(\sigma\partial_t^{1/2} u, \partial_t^{1/2} u\big), \\ a(u, u-u^\perp) &= |u|_{V_0}^2. \end{aligned}$$

Using the σ- and ν-weighted orthogonalities (10) again leads to $|\pi_u|_{V_0} = \sqrt{2}\,|u|_{V_0}$, i.e.,

$$\begin{aligned} |\pi_u|_{V_0}^2 &= |u-u^\perp|_{V_0}^2 = \big(\nu\nabla(u-u^\perp), \nabla(u-u^\perp)\big) + \big(\sigma\partial_t^{1/2}(u-u^\perp), \partial_t^{1/2}(u-u^\perp)\big) \\ &= (\nu\nabla u, \nabla u) - \big(\nu\nabla u^\perp, \nabla u\big) - \big(\nu\nabla u, \nabla u^\perp\big) + \big(\nu\nabla u^\perp, \nabla u^\perp\big) \\ &\quad + \big(\sigma\partial_t^{1/2} u, \partial_t^{1/2} u\big) - \big(\sigma\partial_t^{1/2} u^\perp, \partial_t^{1/2} u\big) - \big(\sigma\partial_t^{1/2} u, \partial_t^{1/2} u^\perp\big) + \big(\sigma\partial_t^{1/2} u^\perp, \partial_t^{1/2} u^\perp\big) \\ &= (\nu\nabla u, \nabla u) + (\nu\nabla u, \nabla u) + \big(\sigma\partial_t^{1/2} u, \partial_t^{1/2} u\big) + \big(\sigma\partial_t^{1/2} u, \partial_t^{1/2} u\big) \\ &= 2\,|u|_{V_0}^2. \end{aligned}$$

Altogether, we arrive at the following estimate of the supremum from below:

$$\sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{|v|_{V_0}} \geq \frac{a(u,u-u^\perp)}{|u-u^\perp|_{V_0}} = \frac{|u|_{V_0}^2}{\sqrt{2}|u|_{V_0}} = \frac{1}{\sqrt{2}}|u|_{V_0},$$

which finally yields the inf-sup constant $\mu_1 = 1/\sqrt{2}$. □

In order to deduce an upper bound of the quantity

$$\sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{V_0}},$$

we rearrange the functional $\mathcal{F}_\eta(v)$ as follows

$$\begin{aligned} \mathcal{F}_\eta(v) &= \int_{Q_T} \Big(f\,v - \sigma\partial_t^{1/2}\eta\,\partial_t^{1/2} v^\perp - \nu\nabla\eta\cdot\nabla v\Big)\, d\boldsymbol{x}\, dt \\ &= \int_{Q_T} \Big(f\,v - \sigma\partial_t^{1/2}\eta\,\partial_t^{1/2} v^\perp + \sigma\,\kappa\,\partial_t^{1/2} v^\perp + \sigma\partial_t^{1/2}\kappa^\perp\, v \\ &\qquad + \operatorname{div}\boldsymbol{\tau}\, v + \boldsymbol{\tau}\cdot\nabla v - \nu\nabla\eta\cdot\nabla v\Big)\, d\boldsymbol{x}\, dt \\ &= \int_{Q_T} \Big(\Big(f + \operatorname{div}\boldsymbol{\tau} + \sigma\partial_t^{1/2}\kappa^\perp\Big)\, v + (\boldsymbol{\tau} - \nu\nabla\eta)\cdot\nabla v + \sigma(-\partial_t^{1/2}\eta + \kappa)\partial_t^{1/2} v^\perp\Big)\, d\boldsymbol{x}\, dt \end{aligned}$$

and then obtain the estimate

$$\begin{aligned} \mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|\|v\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\|\nabla v\| + (\sigma\,\mathcal{R}_3(\kappa,\eta), \mathcal{R}_3(\kappa,\eta))^{1/2}(\sigma\partial_t^{1/2} v, \partial_t^{1/2} v)^{1/2} \\ &\leq C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\|\|\nabla v\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\|\nabla v\| + (\sigma\,\mathcal{R}_3(\kappa,\eta), \mathcal{R}_3(\kappa,\eta))^{1/2}(\sigma\partial_t^{1/2} v, \partial_t^{1/2} v)^{1/2} \\ &= \big(C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\big)\|\nabla v\| + (\sigma\,\mathcal{R}_3(\kappa,\eta), \mathcal{R}_3(\kappa,\eta))^{1/2}(\sigma\partial_t^{1/2} v, \partial_t^{1/2} v)^{1/2} \end{aligned}$$

for all $v \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$, where

$$\mathcal{R}_1(\boldsymbol{\tau},\kappa) := f + \operatorname{div}\boldsymbol{\tau} + \sigma\partial_t^{1/2}\kappa^\perp, \qquad \mathcal{R}_2(\boldsymbol{\tau},\eta) := \boldsymbol{\tau} - \nu\nabla\eta, \qquad \mathcal{R}_3(\kappa,\eta) := \kappa - \partial_t^{1/2}\eta.$$

Hence, it follows an a posteriori error estimate for $|\cdot|_{V_0}$.

Theorem 7. *Let $\eta \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ satisfy (51). Then,*

$$
\begin{aligned}
|u-\eta|_{V_0} &\le \frac{1}{\mu_1^\nu}\Big(\big(C_F\,\|\mathcal{R}_1(\boldsymbol{\tau},\kappa)\| + \|\mathcal{R}_2(\boldsymbol{\tau},\eta)\|\big)^2 + (\sigma\,\mathcal{R}_3(\kappa,\eta),\mathcal{R}_3(\kappa,\eta))\Big)^{1/2} \\
&=: \mathcal{M}^{\oplus}_{|\cdot|_{V_0}}(\eta,\boldsymbol{\tau},\kappa),
\end{aligned} \tag{52}
$$

where $\boldsymbol{\tau} \in H(\mathrm{div},Q_T)$, $\kappa \in H^{0,\frac{1}{2}}_{per}(Q_T)$ and $\mu_1^\nu = \min\{\sqrt{\underline{\nu}},1\}/\sqrt{2}$.

Proof. Using the left inequality of (51) yields the upper bound

$$
|u-\eta|_{V_0} \le \sqrt{2} \sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u-\eta,v)}{|v|_{V_0}} = \sqrt{2} \sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{V_0}},
$$

which immediately leads to (52) with the estimation

$$
\begin{aligned}
\sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{V_0}} &\le \frac{\Big(\big(C_F\,\|\mathcal{R}_1\| + \|\mathcal{R}_2\|\big)^2 + (\sigma\,\mathcal{R}_3,\mathcal{R}_3)\Big)^{1/2}\Big(\|\nabla v\|^2 + (\sigma\partial_t^{1/2}v,\partial_t^{1/2}v)\Big)^{1/2}}{\Big((\nu\nabla v,\nabla v) + \big(\sigma\partial_t^{1/2}v,\partial_t^{1/2}v\big)\Big)^{1/2}} \\
&\le \frac{\Big(\big(C_F\,\|\mathcal{R}_1\| + \|\mathcal{R}_2\|\big)^2 + (\sigma\,\mathcal{R}_3,\mathcal{R}_3)\Big)^{1/2}}{\min\{\sqrt{\underline{\nu}},1\}}.
\end{aligned}
$$

□

In order to derive a posteriori estimates for the full weighted $H^{1,\frac{1}{2}}$-norm defined as

$$
\|v\|^2_{V_0} = \|v\|^2 + (\nu\nabla v,\nabla v) + \big(\sigma\partial_t^{1/2}v,\partial_t^{1/2}v\big),
$$

the functional $\mathcal{F}_\eta(v)$ has to be rearranged again, i.e.,

$$
\begin{aligned}
\mathcal{F}_\eta(v) &= \int_{Q_T}\Big(f\,v - \sigma\partial_t^{1/2}\eta\,\partial_t^{1/2}v^\perp - \nu\nabla\eta\cdot\nabla v\Big)\,d\boldsymbol{x}\,dt \\
&= \int_{Q_T}\Big(f\,v - \sigma\partial_t^{1/2}\eta\,\partial_t^{1/2}v^\perp + \sigma\,\kappa\,\partial_t^{1/2}v^\perp + \sigma\partial_t^{1/2}\kappa^\perp\,v \\
&\qquad + \mathrm{div}\,(\nu\,\tilde{\boldsymbol{\tau}})\,v + (\nu\,\tilde{\boldsymbol{\tau}})\cdot\nabla v - \nu\nabla\eta\cdot\nabla v\Big)\,d\boldsymbol{x}\,dt \\
&= \int_{Q_T}\Big(\Big(f + \mathrm{div}\,(\nu\,\tilde{\boldsymbol{\tau}}) + \sigma\partial_t^{1/2}\kappa^\perp\Big)\,v + \sigma\big(-\partial_t^{1/2}\eta + \kappa\big)\,\partial_t^{1/2}v^\perp \\
&\qquad + \nu\big(\tilde{\boldsymbol{\tau}} - \nabla\eta\big)\cdot\nabla v\Big)\,d\boldsymbol{x}\,dt \qquad\qquad \forall\, v \in H^{1,\frac{1}{2}}_{0,per}(Q_T).
\end{aligned}
$$

Here, a vector-valued function $\tilde{\boldsymbol{\tau}}$ has been introduced, which satisfies the identity

$$
\int_\Omega \mathrm{div}\,(\nu\,\tilde{\boldsymbol{\tau}})\,v\,d\boldsymbol{x} = -\int_\Omega (\nu\,\tilde{\boldsymbol{\tau}})\cdot\nabla v\,d\boldsymbol{x} \qquad \forall\, v \in H^1_0(\Omega).
$$

Let now

$$
\mathcal{R}_1(\tilde{\boldsymbol{\tau}},\kappa) = f + \mathrm{div}\,(\nu\,\tilde{\boldsymbol{\tau}}) + \sigma\partial_t^{1/2}\kappa^\perp, \qquad \mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta) := \tilde{\boldsymbol{\tau}} - \nabla\eta.
$$

Then, the functional $\mathcal{F}_\eta(v)$ can be estimated from above as follows

$$
\begin{aligned}
\mathcal{F}_\eta(v) &\le \|\mathcal{R}_1(\tilde{\boldsymbol{\tau}},\kappa)\|\|v\| + (\nu\,\mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta),\mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta))^{1/2}(\nu\nabla v,\nabla v)^{1/2} \\
&\qquad + (\sigma\,\mathcal{R}_3(\kappa,\eta),\mathcal{R}_3(\kappa,\eta))^{1/2}(\sigma\partial_t^{1/2}v,\partial_t^{1/2}v)^{1/2} \\
&\le \Big(\|\mathcal{R}_1(\tilde{\boldsymbol{\tau}},\kappa)\|^2 + (\nu\,\mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta),\mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta)) + (\sigma\,\mathcal{R}_3(\kappa,\eta),\mathcal{R}_3(\kappa,\eta))\Big)^{1/2}\|v\|_{V_0}.
\end{aligned} \tag{53}
$$

Moreover, the following inf-sup conditions can be proven:

Lemma 5. *The space-time bilinear form $a(\cdot,\cdot)$ defined by (14) satisfies the following inf-sup and sup-sup conditions:*

$$\mu_1 \|u\|_{V_0} \leq \sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{\|v\|_{V_0}} \leq \mu_2 \|u\|_{V_0} \tag{54}$$

for all $u \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ with constants $\mu_1 = \min\{1, \underline{\nu}/C_F^2\}/\sqrt{5}$ and $\mu_2 = 1$.

Proof. The sup-sup condition is analogously proven as it is done in Lemma 4 with the final result

$$|a(u,v)| \leq |u|_{V_0}|v|_{V_0} \leq \mu_2 \|u\|_{V_0}\|v\|_{V_0},$$

where $\mu_2 = 1$. The inf-sup condition is proven by choosing the test function $\pi_u = u + u - u^\perp$ and using the σ- and ν-weighted orthogonality relations (10) as well as the Friedrichs inequality (27) in the Fourier space. With the choice $\pi_u = 2u - u^\perp$, we obtain

$$\begin{aligned}
a(u,2u) &= \int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2}u\,\partial_t^{1/2}(2u)^\perp + \nu(\boldsymbol{x})\nabla u\cdot\nabla(2u)\Big)\, d\boldsymbol{x}\, dt = 2(\nu\nabla u, \nabla u),\\
a(u,-u^\perp) &= \int_{Q_T} \Big(\sigma(\boldsymbol{x})\partial_t^{1/2}u\,\partial_t^{1/2}u + \nu(\boldsymbol{x})\nabla u\cdot\nabla u^\perp\Big)\, d\boldsymbol{x}\, dt = \big(\sigma\partial_t^{1/2}u, \partial_t^{1/2}u\big),\\
a(u,2u-u^\perp) &= |u|_{V_0}^2 + (\nu\nabla u,\nabla u) \geq |u|_{V_0}^2 + \underline{\nu}\|\nabla u\|^2\\
&\geq |u|_{V_0}^2 + \frac{\underline{\nu}}{C_F^2}\|u\|^2 \geq \min\{1, \frac{\underline{\nu}}{C_F^2}\}\|u\|_{V_0}^2.
\end{aligned}$$

Since

$$\begin{aligned}
\|\pi_u\|_{V_0}^2 &= \|2u-u^\perp\|_{V_0}^2 = \|2u-u^\perp\|^2 + \big(\nu\nabla(2u-u^\perp), \nabla(2u-u^\perp)\big)\\
&\quad + \big(\sigma\partial_t^{1/2}(2u-u^\perp), \partial_t^{1/2}(2u-u^\perp)\big)\\
&= \|2u\|^2 + \|u^\perp\|^2 + (\nu\nabla(2u),\nabla(2u)) + \big(\nu\nabla u^\perp, \nabla u^\perp\big)\\
&\quad + \big(\sigma\partial_t^{1/2}(2u), \partial_t^{1/2}(2u)\big) + \big(\sigma\partial_t^{1/2}u^\perp, \partial_t^{1/2}u^\perp\big)\\
&= 5\|u\|^2 + 5\,(\nu\nabla u,\nabla u) + 5\big(\sigma\partial_t^{1/2}u, \partial_t^{1/2}u\big) = 5\,\|u\|_{V_0}^2
\end{aligned}$$

we use (10) and find that $\|\pi_u\|_{V_0} = \sqrt{5}\,\|u\|_{V_0}$. Hence, we arrive at the following estimate of the supremum from below:

$$\sup_{0\neq v\in H^{1,\frac{1}{2}}_{0,per}(Q_T)} \frac{a(u,v)}{\|v\|_{V_0}} \geq \frac{a(u,2u-u^\perp)}{\|2u-u^\perp\|_{V_0}} \geq \frac{\min\{1,\frac{\underline{\nu}}{C_F^2}\}\|u\|_{V_0}^2}{\sqrt{5}\|u\|_{V_0}} = \mu_1\|u\|_{V_0},$$

which finally yields the inf-sup constant $\mu_1 = \min\{1, \underline{\nu}/C_F^2\}/\sqrt{5}$. □

From (53) and (54), we deduce the following a posteriori estimate:

Theorem 8. *Let $\eta \in H^{1,\frac{1}{2}}_{0,per}(Q_T)$ and the bilinear form $a(\cdot,\cdot)$ satisfy (54). Then,*

$$\begin{aligned}
\|u-\eta\|_{V_0} &\leq \frac{1}{\mu_1}\Big(\|\mathcal{R}_1(\tilde{\boldsymbol{\tau}},\kappa)\|^2 + (\nu\,\mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta), \mathcal{R}_2(\tilde{\boldsymbol{\tau}},\eta)) + (\sigma\,\mathcal{R}_3(\kappa,\eta), \mathcal{R}_3(\kappa,\eta))\Big)^{1/2}\\
&=: \mathcal{M}^{\oplus}_{\|\cdot\|_{V_0}}(\eta,\tilde{\boldsymbol{\tau}},\kappa),
\end{aligned} \tag{55}$$

where $(\nu\,\tilde{\boldsymbol{\tau}}) \in H(\mathrm{div}, Q_T)$, $\kappa \in H^{0,\frac{1}{2}}_{per}(Q_T)$ and $\mu_1 = \min\{1, \underline{\nu}/C_F^2\}/\sqrt{5}$.

Finally, we briefly discuss a posteriori estimates for Fourier modes in the context of V_0-norms. In main, they are derived by the same arguments as before. Therefore, we present only the results and comments related to specific features of this case. We introduce

$$|\boldsymbol{u}_k|_{V_0}^2 = (\nu\nabla\boldsymbol{u}_k, \nabla\boldsymbol{u}_k)_\Omega + k\omega\,(\sigma\boldsymbol{u}_k, \boldsymbol{u}_k)_\Omega,$$

where the following inf-sup condition holds:

$$\sup_{0\neq \boldsymbol{v}_k\in (H^1_0(\Omega))^2} \frac{a_k(\boldsymbol{u}_k-\boldsymbol{\eta}_k,\boldsymbol{v}_k)}{|\boldsymbol{v}_k|_{V_0}} \geq \underline{c}_{|\cdot|_{V_0}}\, |\boldsymbol{u}_k-\boldsymbol{\eta}_k|_{V_0} \tag{56}$$

with the parameter-independent constant $\underline{c}_{|\cdot|_{V_0}} = 1/\sqrt{2}$. In addition to $\boldsymbol{\tau}_k = (\boldsymbol{\tau}^c_k, \boldsymbol{\tau}^s_k)^T$ with $\boldsymbol{\tau}^c_k, \boldsymbol{\tau}^s_k \in H(\mathrm{div},\Omega)$, we introduce the functions $\boldsymbol{\kappa}_k = (\kappa^c_k, \kappa^s_k)^T \in (L^2(\Omega))^2$ fulfilling the simple orthogonality relation

$$\int_\Omega k\omega\,\sigma(\boldsymbol{x})\boldsymbol{\kappa}_k\cdot \boldsymbol{v}^\perp\,d\boldsymbol{x} = -\int_\Omega k\omega\,\sigma(\boldsymbol{x})\boldsymbol{\kappa}^\perp_k\cdot \boldsymbol{v}\,d\boldsymbol{x} \qquad \forall\, \boldsymbol{v}\in (L^2(\Omega))^2.$$

Let

$$\begin{aligned}
\mathcal{R}_{1k}(\boldsymbol{\kappa}_k,\boldsymbol{\tau}_k) &:= k\omega\,\sigma\boldsymbol{\kappa}^\perp_k + \mathbf{div}\,\boldsymbol{\tau}_k + \boldsymbol{f}_k = (-k\omega\,\sigma\kappa^s_k + \mathrm{div}\,\boldsymbol{\tau}^c_k + f^c_k, k\omega\,\sigma\kappa^c_k + \mathrm{div}\,\boldsymbol{\tau}^s_k + f^s_k)^T\\
&= (\mathcal{R}_{1}{}^c_k(\kappa^s_k,\boldsymbol{\tau}^c_k), \mathcal{R}_{1}{}^s_k(\kappa^c_k,\boldsymbol{\tau}^s_k))^T,\\
\mathcal{R}_{2k}(\boldsymbol{\eta}_k,\boldsymbol{\tau}_k) &:= \boldsymbol{\tau}_k - \nu\nabla\boldsymbol{\eta}_k = (\boldsymbol{\tau}^c_k - \nu\nabla\eta^c_k, \boldsymbol{\tau}^s_k - \nu\nabla\eta^s_k)^T = (\mathcal{R}_{2}{}^c_k(\eta^c_k,\boldsymbol{\tau}^c_k), \mathcal{R}_{2}{}^s_k(\eta^s_k,\boldsymbol{\tau}^s_k))^T,\\
\mathcal{R}_{3k}(\boldsymbol{\eta}_k,\boldsymbol{\kappa}_k) &:= \boldsymbol{\kappa}_k - \boldsymbol{\eta}_k = (\kappa^c_k - \eta^c_k, \kappa^s_k - \eta^s_k)^T = (\mathcal{R}_{3}{}^c_k(\eta^c_k,\kappa^c_k), \mathcal{R}_{2}{}^s_k(\eta^s_k,\kappa^s_k))^T.
\end{aligned}$$

By arguments similar to those used for proving Theorems 3 and 4, we deduce the following upper bounds for every single mode k with $|\cdot|_{V_0}$:

Theorem 9. *Let $\boldsymbol{\eta}_k \in (H^1_0(\Omega))^2$ and the bilinear form $a_k(\cdot,\cdot)$ satisfy (56). Then, it follows the estimate*

$$\begin{aligned}
|\boldsymbol{u}_k-\boldsymbol{\eta}_k|_{V_0} &\leq \frac{1}{\underline{c}^\nu_{V_0}}\Big(\big(C_F\,\|\mathcal{R}_{1k}(\boldsymbol{\kappa}_k,\boldsymbol{\tau}_k)\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k,\boldsymbol{\tau}_k)\|_\Omega\big)^2\\
&\qquad\qquad + (\sigma\mathcal{R}_{3k}(\boldsymbol{\eta}_k,\boldsymbol{\kappa}_k), \mathcal{R}_{3k}(\boldsymbol{\eta}_k,\boldsymbol{\kappa}_k))_\Omega\Big)^{1/2}\\
&=: \mathcal{M}^{\oplus_k}_{|\cdot|_{V_0}}(\boldsymbol{\eta}_k,\boldsymbol{\tau}_k,\boldsymbol{\kappa}_k),
\end{aligned} \tag{57}$$

where $\boldsymbol{\tau}_k = (\boldsymbol{\tau}^c_k,\boldsymbol{\tau}^s_k)^T \in (H(\mathrm{div},\Omega))^2$, $\boldsymbol{\kappa}_k = (\kappa^c_k,\kappa^s_k)^T \in (L^2(\Omega))^2$ and the constant $\underline{c}^\nu_{|\cdot|_{V_0}} = \min\{\sqrt{\underline{\nu}},1\}/\sqrt{2}$. In the case $k=0$, we derive the estimate (45).

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