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Computation of Dimension in Filtered Free Modules by Gröbner Reduction

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ABSTRACT

We present an axiomatic approach to Gröbner basis techniques in free multi-filtered modules over a not necessarily commutative multi-filtered ring. It is shown that classical Gröbner basis concepts can be viewed as models of our axioms. Within this theory it is possible to prove a general theorem about the dimension of filter spaces in multi-filtered modules. We use these ideas for computing the Hilbert function of finitely generated multi-filtered modules over difference-differential rings. Thus the presented method allows to compute a multivariate generalization of the univariate and the bivariate dimension polynomial considered in the papers of Winkler and Zhou.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms – *Algebraic algorithms*

General Terms

Algorithms

Keywords

Filtered free modules, difference and differential operators, Gröbner bases

1. INTRODUCTION

Gröbner bases, as introduced in [1], are a well established algorithmic concept for solving problems occurring in polynomial ideal theory, that is, performing computations in finitely generated modules over $K[x_1, \ldots, x_n]$. As the theory and its applications evolved, increasing interest came up in generalizing the notion of Gröbner bases to modules over more general rings.

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In [10] Winkler and Zhou introduced the concept of Gröbner bases in difference-differential modules.

Already in 1964 Kolchin formulated a fundamental theorem on univariate differential dimension polynomials [3] and ([4], Sect. II.12. Thm. 6). In 2007, by using serveral term orders, Levin was able to extend the computation of univariate and bivariate dimension polynomials to multivariate dimension polynomials [7]. In 2008, Winkler and Zhou extended their 2006-approach to the notion of relative Gröbner bases and applied it to the computation of difference-differential dimension polynomials [12]. Splitting the set of derivations and the set of automorphisms, they provided algorithms for the univariate and the bivariate case [11, 12].

In his 2013 paper, C. Dönch pointed out that the algorithm which generates a relative Gröbner basis out of a finite set of generators, as formulated in [12], might not terminate [2].

Different viewpoints on the computation and applications of dimension polynomials are presented in [5].

The backbone of Gröbner basis techniques in a module is the existence of monomials. In case that the set of monomials is appropriately contained in some monoid $\mathbb{N}^n$ we always may find an admissible linear extension of the product order in $\mathbb{N}^n$ (a monomial order). Then the usual Gröbner basis algorithms terminate, i.e., the classical concepts apply. There are but situations where this is not the case. For example, in rings of difference-differential operators, the set of monomials is isomorphic to $\mathbb{N}^m \times \mathbb{Z}^n$, and it is not obvious how to design a reduction process in a way that unique normal forms are produced. Several methods to overcome this problem have been developed, each with its own facet of technical difficulties, e.g. in [12] the set of monomials is covered by finitely many subsets in each of which reduction terminates, while in [6] characteristic sets are used to come to a solution.

In any case there is some filtration present derived naturally from the respective type of monomials and such that the reduction process is compatible with it.

Carefully inspecting the procedures provided by the papers mentioned above we gained increasing evidence that the interplay of filtrations and Gröbner bases must have a key role. Thus we tried to set up a general theory of reduction in a free module that takes into account a given filtration inherited naturally from the basic ring. The resulting computational tool is applicable to general filtered rings including polynomial rings $K[x_1,\dots,x_n]$ as well as modules of difference-differential operators as special cases.

2. FILTERED MODULES

$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of non-negative integers, integers and rational numbers respectively. Throughout, the letter K will denote a field, and R will be an associative ring with 1, such that $K \subseteq R$. All modules over R are assumed to be left modules without further mention. The field K is not assumed to be central, so R is not necessarily an algebra in the classical sense. In addition we always assume given a distinguished basis Λ of the K-vector space R whose members are called monomials. We will indicate this by writing $R = K^{(\Lambda)}$ when necessary. Thus, elements $a \in R$ admit a unique representation $a = \sum_{\lambda\in\Lambda} a_\lambda \lambda$ as a K-linear combination of monomials.

The basis Λ extends naturally to a basis of free modules: let $F = Re_1 \oplus \cdots \oplus Re_q$ be the free R-module on the set $E = \{e_1,\dots,e_q\}$. Then the set $\Lambda E = \{\lambda e\colon \lambda \in \Lambda \wedge e \in E\}$ is a K-basis of F. Again we will call its members monomials, and elements $f \in F$ are represented uniquely as K-linear combinations of monomials.

As we will work over fields exclusively, we do not distinguish formally between monomials and terms. So we write $\mathrm{T}(f)$ for the set of terms of f, i.e., the set of all monomials which appear with a non-zero coefficient in a standard representation of f

$$\mathrm{T}\big(\sum_{t\in\Lambda E} f_t t\big) = \{t \in \Lambda E\colon f_t \neq 0\}.$$

This applies in particular to elements of the ring R. Also we will write $\mathrm{lt}(f)$ and $\mathrm{lc}(f)$ for the leading term and leading coefficient in contexts where these notions apply, so that, in such a situation, each $f \neq 0$ has a representation

$$f = \mathrm{lc}(f)\cdot \mathrm{lt}(f) + \text{lower order terms}.$$

Obviously we have that $\mathrm{T}(f \pm g) \subseteq \mathrm{T}(f) \cup \mathrm{T}(g)$.

One object of particular interest is the ring D of difference-differential operators, defined over a field K, and its finitely generated (left) modules. On K there are assumed two distinguished finite sets Δ, Σ where $\Delta = \{\delta_1,\dots,\delta_m\}$ consists of derivations and $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ contains automorphisms of K, all commuting with one another (a difference-differential field, cf. [12]). The ring D is then constructed as the free K-vector space on the set of formal expressions

$$\delta_1^{k_1}\cdots\delta_m^{k_m}\sigma_1^{l_1}\cdots\sigma_n^{l_n}\ (k_i \in \mathbb{N},\ l_j \in \mathbb{Z}) \tag{1}$$

and a product that reflects the properties of derivations and automorphisms, that is

$$\delta_i \cdot a = a\delta_i + \delta_i(a) \text{ and } \sigma_j \cdot a = \sigma_j(a)\sigma_j\ (a \in K). \tag{2}$$

In the ring D the natural K-basis is the set Λ of all expressions (1). Note that the elements $\lambda \in \Lambda$ involve negative exponents in the automorphisms σ_j, and from (2) one derives that Λ is a multiplicative monoid that is isomorphic to $\mathbb{N}^m \times \mathbb{Z}^n$. The elements of D, called `difference-differential operators`, are thus finite K-linear combinations

$$\sum_{(k,l)\in\mathbb{N}^m\times\mathbb{Z}^n} a_{k,l}\delta^k\sigma^l.$$

A left module over D is also called a `difference-differential module` (over K) or a $\Delta - \Sigma$ module.

In the sequel, the letter D will be reserved for the ring of difference-differential operators, whereas R may denote an arbitrary ring of the type mentioned above.

For $r, s \in \mathbb{N}^p$ set $r \leq_\pi s \iff r_i \leq s_i$ $(1 \leq i \leq p)$. By a (p-fold) filtration of R we mean a family of additive subgroups $R_r \subseteq R$, indexed by $\mathbb{N}^p$, such that

- $R_r \cdot R_s \subseteq R_{r+s} \quad (r, s \in \mathbb{N}^p)$;
- $R_r \subseteq R_s \quad (r \leq_\pi s \in \mathbb{N}^p)$;
- $R = \bigcup_{r\in\mathbb{N}^p} R_r$;
- $1 \in R_0$.

R together with such a filtration will be called a `multi-filtered ring`. In a filtered ring R, R_0 is a subring and each R_r is a left and a right R_0-module.

Definition 1. *A filtration of R is called* `monomial` *iff*

$$R_0 = K \text{ and } \forall r \in \mathbb{N}^p\, \forall f\, (f \in R_r \Rightarrow \mathrm{T}(f) \subseteq R_r).$$

Example 1. *For a monomial $\lambda = \delta^k\sigma^l$ in D we set*

$$|\lambda|_1 := k_1 + \cdots + k_m \text{ and } |\lambda|_2 := |l_1| + \cdots + |l_n|.$$

For a general operator $a = \sum_{\lambda\in\Lambda} a_\lambda\lambda$ in D we define the order functions $|a|_\nu := \max\{|\lambda|_\nu\colon a_\lambda \neq 0\}$ $(\nu = 1, 2)$. Then, for $r, s \in \mathbb{N}$, the sets

$$D_{r,s} := \{a \in D\colon |a|_1 \leq r \wedge |a|_2 \leq s\}$$

define a (bivariate) monomial filtration. We call it the `standard filtration` *of D (see [12]).*

Let M be a left R-module. A (p-fold) filtration of M w.r.t. the (p-fold) filtered ring R is a family $(M_r)_{r\in\mathbb{N}^p}$ of additive subgroups $M_r \subseteq M$ with the properties

- $R_r \cdot M_s \subseteq M_{r+s} \quad (r, s \in \mathbb{N}^p)$;
- $M_r \subseteq M_s \quad (r \leq_\pi s \in \mathbb{N}^p)$;
- $M = \bigcup_{r\in\mathbb{N}^p} M_r$.

M together with such a filtration is called a `filtered module` (over the filtered ring R). Plainly, each M_r is an R_0-module. If in addition we have $M_r = R_rM_0\ \forall r$, the filtration is called `standard`. Note that the filtration on the ring D is standard.

Notation 1. *If X is an arbitrary subset of a filtered module $M = \bigcup_{r\in\mathbb{N}^p} M_r$ we set $X_r = X \cap M_r$.*

A (p-fold) filtration of R extends naturally to a (p-fold) filtration of free modules: Let R be a filtered ring, and $F = Re_1 \oplus \cdots \oplus Re_q$ the free R-module on the set $E = \{e_1,\dots,e_q\}$. Then

$$F_r := R_re_1 \oplus \cdots \oplus R_re_q \quad (r \in \mathbb{N}^p)$$

defines a filtration on F. If the filtration of R is monomial (w.r.t. the basis Λ) then so is the extended filtration of F (w.r.t. ΛE), meaning that always $f \in F_r \Rightarrow \mathrm{T}(f) \subseteq F_r$.

Example 2. *We extend the order functions of the difference-differential ring D to the free module $F = D^q$: For $\lambda e \in \Lambda E$ and $\nu = 1, 2$ let $|\lambda e|_\nu := |\lambda|_\nu$ and for a module element $f = \sum_{t \in \Lambda E} f_t t \in F$ let $|f|_\nu := \max\{|t|_\nu : t \in \mathrm{T}(f)\}$. This gives the extended filtration on F - for $r, s \in \mathbb{N}$*

$$F_{r,s} = D_{r,s}e_1 \oplus \cdots \oplus D_{r,s}e_q = \{f \in F : |f|_1 \leq r \wedge |f|_2 \leq s\}.$$

From $|e_j|_\nu = 0$ it is clear that $E \subseteq F_{0,0}$ whence $(F_{r,s})$ is a standard filtration. We will call it `the standard filtration of` *F. Since the ring filtration is monomial, the extended filtration is so too. Obviously*

$$f \in F_{r,s} \iff \forall t \in \mathrm{T}(f): |t|_1 \leq r \wedge |t|_2 \leq s \iff \mathrm{T}(f) \subseteq F_{r,s}.$$

Let the ring R be a filtered ring, and M, N filtered R-modules. An R-homomorphism $\varphi\colon M \longrightarrow N$ is called a `morphism` if it respects the filter structure, that is, if

$$\varphi(M_r) \subseteq N_r, \ \forall r \in \mathbb{N}^p.$$

A morphism induces R_0-linear maps $M_r \longrightarrow N_r \ \forall r \in \mathbb{N}^p$.

Lemma 1. *Let R be a filtered ring and $\varphi\colon M \longrightarrow N$ a homomorphism of R-modules.*

1. *If M is filtered over R then $\mathrm{im}(\varphi)$ is filtered by setting $\mathrm{im}(\varphi)_r = \varphi(M_r)$. φ is then a morphism $M \longrightarrow \mathrm{im}(\varphi)$.*
2. *If N is filtered over R then M is filtered by setting $M_r = \varphi^{-1}(N_r)$. φ is then a morphism $M \longrightarrow N$.*

Thus, each finitely generated R-module $M = Rh_1 + \cdots + Rh_q$ inherits a filtration by first extending the family R_r to the free module $F \cong R^q$ and then pushing down with a map

$$\pi\colon F \longrightarrow M, \ e_i \mapsto h_i. \tag{3}$$

By specializing Lemma 1 to inclusion $N \hookrightarrow M$ any submodule $N \subseteq M$ naturally inherits a filtration from M via

$$N_r = N \cap M_r.$$

3. REDUCTION RELATIONS

Let X be a set and $\rho \subseteq X \times X$ a binary relation. We write $f \longrightarrow h$ to indicate that $(f, h) \in \rho$, and $f \longrightarrow^\star h$ when there is a chain of finite length

$$f = f_0 \longrightarrow f_1 \longrightarrow \cdots \longrightarrow f_k = h \quad (k \in \mathbb{N})$$

from f to h, that is

$$f \longrightarrow^\star h \iff (f, h) \in \rho^\star = \bigcup_{k \in \mathbb{N}} \rho^k.$$

With I we denote the set of ρ-irreducible elements, that is

$$I = \{x \in X \mid \nexists y \in X \text{ such that } x \longrightarrow y\}.$$

A subset $Y \subseteq X$ is called `ρ-stable` if $y \in Y$ and $y \longrightarrow z$ implies that $z \in Y$.

If $\rho \subseteq M \times M$ is a relation on a `module` M then, for $k \in \mathbb{N}$ we set

$$Z_k = \{f \mid (f, 0) \in \rho^k\}, \ Z_{\leq k} = \bigcup_{l \leq k} Z_l \text{ and } Z = \bigcup_{k=0}^{\infty} Z_k.$$

It is plain that $Z = \bigcup_{k=0}^{\infty} Z_{\leq k} = \{f \in M : f \longrightarrow^\star 0\}$.

We consider a list of axioms which make a relation appropriate for reducing module elements to normal forms.

Definition 2. *Let M be a module, $N \subseteq M$ a submodule and ρ a binary relation on M. ρ is called a* `(weak) reduction` *for N provided that*

1. *ρ is noetherian, i.e., every sequence*
$$f_1 \longrightarrow f_2 \longrightarrow \cdots$$
terminates;
2. *I is a monomial K-linear subspace of M, that is, I is a vector space and*
$$\forall f \in M \, (f \in I \Rightarrow \mathrm{T}(f) \subseteq I);$$
3. *$f \longrightarrow h \Rightarrow f \equiv h \mod N$;*

ρ is a `strong reduction` *for N if it satisfies in addition*

4. *$I \cap N = 0$ that is, every non-zero element in N is reducible.*

We will refer to these items as axioms 1 to 4. A relation satisfying Axiom 2 is used only when M is a free module, so that the passage 'monomial' does make sense.

Lemma 2. *Let $N \subseteq M$ be a submodule, and the relation $\rho \subseteq M \times M$ be such that it satisfies axioms 1. and 3. Then we have*

1. *$M = N + I$;*
2. *$I \cap N \subseteq 0 \iff Z = N$.*

Consequently, if F is a free module and ρ is a strong reduction for $N \subseteq F$ then

$$F = N \oplus I \text{ and } Z = N.$$

Proof. By axioms 1. and 3., $Z \subseteq N$. Assume $I \cap N \subseteq 0$. Let $n \in N$. Then there is an irreducible element $r \in N$ with $n \longrightarrow^\star r$. Thus $r \in I \cap N \subseteq 0$ and so $n \longrightarrow^\star 0$, i.e., $n \in Z$.

Conversely, assume that $Z = N$. Then, for $x \in I \cap N$, $x \longrightarrow^\star 0$ and x is irreducible. Therefore $x = 0$. Consequently $I \cap N \subseteq 0$. □

Note that a relation satisfying axioms 1 - 4 is noetherian and confluent. If F is a free module we will write $\mathrm{NF}(f)$ for the unique normal form of $f \in F$. Thus we always have $f \longrightarrow^\star \mathrm{NF}(f)$.

Theorem 1. *Let $M = Rm_1 + \cdots + Rm_q$ be a finitely generated R-module with free presentation*

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

where $F = R^q$. Assume given a strong reduction for N with set of irreducibles I. Let $V \subseteq F$ be a monomial K-linear subspace that is ρ-stable and let U be the set of irreducible monomials in V. Then $\pi(U)$ is a K-vector space basis for $\pi(V)$. In particular we obtain that

$$\dim_K \pi(V) = |\pi(U)| = |U|.$$

Proof. Let $f, h \in I$. Then $\pi(f) = \pi(h)$ implies that $f - h \in N \cap I = 0$ whence $\pi|I$ is injective. Since $U = I \cap \Lambda E \cap V \subseteq I$ it is plain that $\pi|U$ is injective, whence $|\pi(U)| = |U|$. Let

$$\sum_j c_j \pi(\mu_j) = 0 \ (c_j \in K, \ \mu_j \in I \cap \Lambda E).$$

Then $\sum_j c_j \mu_j \in N \cap I = 0$. Therefore $c_j = 0 \; \forall j$. This demonstrates that $\pi(I \cap \Lambda E)$ is K-linearly independent. Thus $\pi(U) \subseteq \pi(I \cap \Lambda E)$ is linearly independent. Now we may reduce elements $f \in F$ until an irreducible r is reached. Doing this for elements $f \in V$ and taking into account that the reduction stays inside V we obtain an irreducible $r \in V$. Thus

$$\forall f \in V \, \exists r \in I \cap V \text{ with } \pi(r) = \pi(f).$$

Now take $m \in \pi(V)$. $\exists f \in V$ with $m = \pi(f)$. Choose $r \in I \cap V$ with $\pi(r) = \pi(f)$,

$$r = \sum_j c_j \mu_j \; (c_j \in K, \mu_j \in \Lambda E).$$

Since V is monomial, all μ_j are in V and because $r \in I$, all terms of r must be in I. Therefore

$$\mu_j \in V \cap \Lambda E \cap I = U \; \forall j.$$

Consequently

$$m = \pi(r) = \sum_j c_j \pi(\mu_j) \in K \cdot \pi(U).$$

So $\langle \pi(U) \rangle_K = \pi(V)$ and $\pi(U)$ is a K-basis. $\square$

4. GRÖBNER REDUCTION

We return to a monomially filtered ring $R = \bigcup_{r \in \mathbb{N}^p} R_r$ and a finitely generated free R-module F with extended filtration.

Definition 3. *Let $N \subseteq F$ be a submodule. A strong reduction $\rho \subseteq F \times F$ for N is called a* `Gröbner reduction for` *N if it satisfies the axiom*

5. F_r is ρ-stable $\forall r \in \mathbb{N}^p$.

Proposition 1. *Let $N \subseteq F$ be a submodule, $\rho \subseteq F \times F$ be a relation satisfying axioms 1, 3, 5. Then*

$$F_r = N_r + I_r \; \forall r \in \mathbb{N}^p. \tag{4}$$

Consequently, if ρ is a Gröbner reduction for N then

$$F = N \oplus I \text{ and } \forall r \in \mathbb{N}^p \, F_r = N_r \oplus I_r \tag{5}$$

Proof. Let $f \in F_r$. Reduce f to normal form $f \longrightarrow^\star z$. $f \equiv z \mod N$ whence $f - z = n \in N$. By axiom 5, $z \in F_r$. Thus $z \in I \cap F_r = I_r$. As both f and z are in F_r, so is n. Therefore $f = n + z \in N_r + I_r$. $\square$

Equation (4) of Proposition 1 corresponds to 'division with remainder' in the classical theory. Similar, equation (5) describes 'uniqueness of normal forms' in Gröbner basis computations.

For classical monomials it is easy to deal with monomial submodules:

Proposition 2. *Assume that the set Λ of monomials in R satisfies $\Lambda\Lambda \subseteq \Lambda$. Let $N \subseteq F$ be a monomial submodule. Choose a monomial K-linear complement I of N in F (e.g., $I = KS$ where $S = \{t \in \Lambda E \colon t \notin N\}$). Let p_I denote projection $N \oplus I \longrightarrow I$ and let $\rho \subset F \times F$ be the relation*

$$\rho = p_I|_{F \setminus I}.$$

Then, with arbitrary monomial filtration, ρ is a Gröbner reduction for N.

Proof. Let N be generated by $X \subseteq \Lambda E$. The general element of N is $n = \sum_{x \in X} a_x x$. The elements $a_x \in R$ are

$$a_x = \sum_{\lambda \in \Lambda} a_x^\lambda \lambda \; (a_x^\lambda \in K) \text{ whence } n = \sum_{x \in X} \sum_{\lambda \in \Lambda} a_x^\lambda \lambda x. \tag{6}$$

Since $\Lambda\Lambda \subseteq \Lambda$, the expressions λx are monomials in ΛE. After (possibly) some cancellations, equation (6) results in the unique representation of n as K-linear combination of ΛE. Since each surviving term is a (monomial) multiple of a generator monomial of N, it is in N, this means, N is a monomial module.

Let $S = \{t \in \Lambda E \colon t \notin N\}$, and let $I = KS$, the vector space generated by elements from S. By construction, I is a `monomial` subspace of F.

Evidently $N \cap I = 0$.

Write $f \in F$ as K-linear combination of elements of ΛE. We may split this expression as

$$f = \sum_{t \in S} f_t t + \sum_{t \notin S} f_t t \quad (f_t \in K)$$

which shows that $f \in I + N$. Consequently $F = N \oplus I$. The relation ρ results in

$$\rho \colon f \longrightarrow h \iff f \in F \setminus I \wedge h = p_I(f)$$

Thus, with exception of elements in I, every $f \in F$ reduces to normal form in 1 step. If $f \longrightarrow h$ then $f \in F \setminus I$ and $h = p_I(f) = p_I(n + r) = r$; thus, $f - h = n \in N$, i.e., $f \equiv h \mod N$. Consequently ρ is a strong reduction for N.

Let $f \in F_r$ and $f \longrightarrow h$. By monomiality of F_r, $T(f) \subseteq F_r$. Because $f = n + h$ is a direct decomposition, it follows that $T(h) \subseteq F_r$. Consequently $h \in F_r$ and ρ is a Gröbner reduction. $\square$

The following example is less artificial.

Example 3. *$R = K[x_1, \ldots, x_n]$, $N \trianglelefteq R$ an ideal, $R_s = \{f \in R \mid \deg f \leq s\}$. Then R is monomially filtered ($p = 1$). The reduction relation coming from a Gröbner basis of N w.r.t a degree lexicographic order obeys axioms 1. to 5. Consequently such a Gröbner basis induces a Gröbner reduction.*

This filtration is not appropriate for arbitrary term orders. For instance in $R = K[x, y]$ with lexicographic order $x \succ y$ and ideal $N = \langle x - y^2 \rangle \lhd R$, the polynomial x reduces by means of the Gröbner basis $\{x - y^2\}$ to y^2, thereby leaving the filter space R_1. For arbitrary term orders we have the following.

Proposition 3. *If $N \trianglelefteq R = K[x_1, \ldots, x_n]$ is an ideal and G a Gröbner basis of N w.r.t. any term order $\prec$, then for $r \in \mathbb{N}^n$*

$$R_r := \{f \in K[x_1, \ldots, x_n] \colon \forall m \in \mathrm{T}(f) \colon m \preceq x_1^{r_1} \cdots x_n^{r_n}\}$$

defines a monomial filtration with the additional property

$$m \in R_r \wedge n \preceq m \Rightarrow n \in R_r.$$

Consequently $\longrightarrow_G$ is a Gröbner-reduction w.r.t. $(R_r)_{r \in \mathbb{N}^n}$.

Notation 2. *If $\prec$ is a (partial) order on ΛE (of any kind whatsoever), $\lambda \in \Lambda$ and $f \in F$, we will write $f \prec \lambda$ to indicate that $t \prec \lambda \; \forall t \in \mathrm{T}(f)$. In particular, if $\prec$ is a term order, we have $f \preceq \lambda \iff \mathrm{lt}_\preceq(f) \preceq \lambda$.*

5. RELATIVE REDUCTION OVER DIFFERENCE DIFFERENTIAL FIELDS

For details within this section we refer to the paper [12]. As before, we treat the ring D of difference-differential operators on the field K with given derivations $\delta_1, \dots, \delta_m$ and automorphisms $\sigma_1, \dots, \sigma_n$, and the finitely generated free D-module F on the set $E = \{e_1, \dots, e_q\}$. In the paper [12], the troubles caused by negative exponents in reduction relations are solved by introducing the notion of `orthant decomposition`. This is a finite family of monoid homomorphisms $\phi_u \colon \mathbb{N}^n \longrightarrow \mathbb{Z}^n$ each of whose images generate the group $\mathbb{Z}^n$ and being such that

$$\bigcup_u \operatorname{im}(\phi_u) = \mathbb{Z}^n.$$

The decomposition extends naturally to the set

$$\Lambda E \cong \mathbb{N}^m \times \mathbb{Z}^n \times E.$$

Consequently the set of monomials ΛE of F is covered by finitely many isomorphic copies of $\mathbb{N}^m \times \mathbb{N}^n \times E$ in which term orders are well founded and reduction is supposed to behave well. Remark that only the $\mathbb{Z}$-part contributes to the orthant of a monomial $t = \delta^k \sigma^l e_i$, i.e., the position of $l = (l_1, \dots, l_n)$ in $\bigcup_u \operatorname{im}(\phi_u)$ determines the orthant of t. The orthant decomposition concept provides the basis for a special type of order:

Definition 4. *Given an orthant decomposition on ΛE. A* `generalized term order` *is a total order $\prec$ on ΛE such that*

1. *e_i is the smallest element in Λe_i $(1 \le i \le q)$;*
2. *if $\lambda e_i \prec \mu e_j$ and $\nu \in \Lambda$ is in the same orthant as μ then $\nu\lambda e_i \prec \nu\mu e_j$.*

In [10] it is proved that a generalized term order is always a well order.

In [12] the following orders $\prec$ and $\prec'$ on ΛE are considered:

For monomials $t = \delta^k \sigma^l e_i$ in ΛE, $\prec$ is given lexicographically by $(|t|_2, |t|_1, e_i, k, |l|, l)$ and $\prec'$ by $(|t|_1, |t|_2, e_i, k, |l|, l)$. Precisely, for $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_n^{l_n}$, $\mu = \delta_1^{r_1} \dots \delta_m^{r_m} \sigma_1^{s_1} \dots \sigma_n^{s_n}$

$$\begin{aligned} \lambda e_i \prec \mu e_j \quad &:\Longleftrightarrow \\ &(|\lambda|_2, |\lambda|_1, e_i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n) \\ &<_{\mathrm{lex}} \\ &(|\mu|_2, |\mu|_1, e_j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n) \end{aligned}$$

respectively

$$\begin{aligned} \lambda e_i \prec' \mu e_j \quad &:\Longleftrightarrow \\ &(|\lambda|_1, |\lambda|_2, e_i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n) \\ &<_{\mathrm{lex}} \\ &(|\mu|_1, |\mu|_2, e_j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n) \end{aligned}$$

where the set E of basis elements is assumed ordered by $e_i < e_j \iff i < j$. Both, $\prec$ and $\prec'$ are generalized term orders w.r.t. the canonical orthant decomposition, ($\mathbb{Z}^n$ covered by several arrangements of cartesian products of $\mathbb{N}$ and $-\mathbb{N}$).

Relative reduction, invented in [12] and called $\prec$`-reduction relative to` $\prec'$ amounts to the following.

Let $f, g, h \in F$. Then $f \xrightarrow{\text{rel}}_g h$ iff $\exists \lambda \in \Lambda$ such that

$$\operatorname{lt}_\prec(\lambda g) = \operatorname{lt}_\prec(f) \wedge \operatorname{lt}_{\prec'}(\lambda g) \preceq' \operatorname{lt}_{\prec'}(f) \wedge h = f - \frac{\operatorname{lc}_\prec(f)}{\operatorname{lc}_\prec(\lambda g)} \lambda g.$$

Therefore, writing $\longrightarrow_g$ for ordinary leading term reduction w.r.t. $\prec$ by g, we obtain

$$f \xrightarrow{\text{rel}}_g h \iff f \longrightarrow_g h \wedge \operatorname{lt}_{\prec'}(\lambda g) \preceq' \operatorname{lt}_{\prec'}(f).$$

For a set $G \subseteq F$ relative reduction is defined as

$$f \xrightarrow{\text{rel}}_G h \iff \exists g \in G \text{ with } f \xrightarrow{\text{rel}}_g h.$$

Proposition 4. *Let $F_{r,s}$ denote standard filtration of the free D-module F. Then*

$$f \xrightarrow{\text{rel}}_g h \text{ and } f \in F_{r,s} \;\Rightarrow\; h \in F_{r,s}$$

that is, $\prec$-reduction relative to $\prec'$ is a reduction compatible with the filtration $(F_{r,s})$. Consequently $\xrightarrow{\text{rel}}_G$ gives rise to a Gröbner reduction.

Proof. Assume $f \xrightarrow{\text{rel}}_g h$ and $f \in F_{r,s}$. Thus $|f|_1 \le r$ and $|f|_2 \le s$. We set

$$u := \operatorname{lt}_\prec(f) = \operatorname{lt}_\prec(\lambda g), u' := \operatorname{lt}_{\prec'}(f), c = \operatorname{lc}_\prec(\lambda g).$$

Thus we may write

$$\begin{aligned} f &= f_u u + \varphi = f_{u'} u' + \varphi' \\ \lambda g &= cu + \psi. \end{aligned}$$

From the assumption we obtain that $\lambda g \preceq' u'$ and

$$h = f - \frac{\operatorname{lc}_\prec(\mathrm{f})}{\operatorname{lc}_\prec(\lambda g)} \lambda g = f_u u + \varphi - \frac{f_u}{c}(cu + \psi) = \varphi - \frac{f_u}{c}\psi.$$

Therefore

$$\mathrm{T}(h) \subseteq \mathrm{T}(\varphi) \cup \mathrm{T}(\psi) = \big(\mathrm{T}(f) \cup \mathrm{T}(\lambda g)\big) \setminus \{u\}.$$

Take $\mu \in \mathrm{T}(h)$. If $\mu \in \mathrm{T}(f)$ then $|\mu|_1 \le r \wedge |\mu|_2 \le s$. If $\mu \in \mathrm{T}(\lambda g)$ then, since $\lambda g \preceq' u'$, we obtain $\mu \preceq' u'$ and therefore $|\mu|_1 \le |u'|_1 \le r$. Because $u = \operatorname{lt}_\prec(\lambda g)$ we obtain $\mu \prec u$ and thus $|\mu|_2 \le |u|_2 \le s$. So in any case we obtain $|\mu|_1 \le r \wedge |\mu|_2 \le s$, that is, $|h|_1 \le r \wedge |h|_2 \le s$. Therefore $h \in F_{r,s}$. Obviously $f \xrightarrow{\text{rel}}_g h$ implies that $\operatorname{lt}_\prec(h) \prec \operatorname{lt}_\prec(f)$. Consequently $\xrightarrow{\text{rel}}_G$ is a noetherian reduction compatible with the filtration. $\square$

6. EXISTENCE OF GRÖBNER REDUCTION AND ALGORITHMIC ASPECTS

We return to the general setting of a free module F over a ring R of the type introduced in Section 2.

Proposition 5. *Let N be an arbitrary submodule of F. Then there is a strong reduction for N.*

Proof. Assume that $N \subset F$ whence $\Lambda E \not\subseteq N$. Choose a set S being maximal in the non-empty inductively ordered set $\{T \subseteq \Lambda E \mid KT \cap N = 0\}$. Put $C = KS$. Obviously $F = K^{(\Lambda E)} = N \oplus C$, so consider projection $p_C \colon N \oplus C \longrightarrow C$ and define a reduction relation

$$\rho = \{(f, h) \in F \times F \colon f \notin C \wedge h = p_C(f)\}.$$

It is clear that ρ terminates. The set I of ρ-irreducible elements is C which is a monomial K-linear space. If $f \longrightarrow h$ then $f = n + c \in N \oplus C$ and $h = c$ which shows that $f \equiv h$ mod N. Finally, $I \cap N = C \cap N = 0$, and thus ρ is a strong reduction for N. $\square$

Whereas the situation in Proposition 2 is decidable as long as we know which monomials are in N, the present construction is totally non-constructive. Comparing this with the reduction relation induced by an ordinary Gröbner basis computation we see that, in order to be algorithmically applicable, a Gröbner reduction $\tau \subseteq F \times F$ for N has to be an extension of ρ (i.e. such that $\rho^\star \subseteq \tau^\star$) being strong enough to be decidable but weak enough to terminate. It depends on the nature of the ring R how to design such a reduction for algorithmic purposes. The same remark applies to the choice of a filtration. In our examples they have been selected with the aim to weaken usual Gröbner basis reduction w.r.t. a term order.

Now assume we are concerned with two rings and modules joined by a homomorphism, precisely, consider a ring $R = K^{(\Lambda)}$, a free module $F = Re_1 \oplus \cdots \oplus Re_q$ and a submodule $N \subseteq F$. Let $S = K^{(\Omega)}$ be another such ring and let $\varphi \colon S \longrightarrow R$ denote a surjective homomorphism of rings such that $\varphi(K) = K$ and $\varphi(\Omega) = \Lambda$. Further let $G = Se_1 \oplus \cdots \oplus Se_q$ be the free S-module (with $\operatorname{rank} S = \operatorname{rank} R$). We extend the map φ to a homomorphism of S-modules denoted by the same symbol

$$\varphi \colon G \longrightarrow F,\ \sum_{i=1}^{q} r_i e_i \mapsto \sum_{i=1}^{q} \varphi(r_i) e_i.$$

Proposition 6. *If $\sigma \subseteq G \times G$ is a strong reduction for $\varphi^{-1}(N)$ then there is a strong reduction $\rho \subseteq F \times F$ such that*

$$\varphi(\mathrm{NF}(g)) = \mathrm{NF}(\varphi(g)).$$

Further, if σ is a Gröbner reduction for $\varphi^{-1}(N)$ w.r.t. a monomial filtration $S = \bigcup_{r\in\mathbb{N}^p} S_r$ then ρ is a Gröbner reduction for N w.r.t. filtration $R = \bigcup_{r\in\mathbb{N}^p} \varphi(S_r)$.

Proof. Let $I = \{g \in G : \nexists z \text{ with } g \longrightarrow z\}$ denote the monomial subspace of irreducibles in G. By Proposition 1 we have that $G = \varphi^{-1}(N) \oplus I$. Then $F = N \oplus \varphi(I)$. Let $\pi \colon F \longrightarrow \varphi(I)$ denote projection. We define the relation $\rho \subseteq F \times F$ by

$$f \longrightarrow_\rho h \iff f \notin \varphi(I) \wedge h = \pi(f).$$

It is clear that ρ is noetherian. $\varphi(I)$ is the K-space of ρ-irreducibles. $\varphi(I)$ is monomial. Indeed, if $f = \varphi(i) \in \varphi(I)$ with $i = \sum_{t\in\Omega E} i_t t \in I$ then

$$f = \sum_{t\in\Omega E} \varphi(i_t)\varphi(t). \tag{7}$$

By monomiality of I we know that all monomials t occuring in this sum are in I and so the corresponding $\varphi(t)$ are in $\varphi(I)$. Since $\varphi(\Omega E) = \Lambda E$, i.e., φ maps monomials in G onto monomials in F, by collecting terms in (7) we see, that $\mathrm{T}(f) \subseteq \varphi(I)$ demonstrating Axiom 2. Axiom 3 and 4 are obvious.

Take $g \in G$ and let $i = \mathrm{NF}(g)$ w.r.t. σ. Then $g \longrightarrow_\sigma^\star i$ and $g - i = \nu \in \varphi^{-1}(N)$ (according to Ax 3 for σ). $\varphi(g) = \varphi(\nu) + \varphi(i) \in N \oplus \varphi(I)$. If $\nu \in \ker\varphi$ then $\varphi(g) = \varphi(i)$ equals its own normal form. If $\nu \notin \ker\varphi$ then $\varphi(g) \longrightarrow_\rho \varphi(i)$. In both cases we derive $\varphi(i) = \mathrm{NF}(\varphi(g))$ w.r.t. ρ.

Now assume that $S = \bigcup_{r\in\mathbb{N}^p} S_r$ is a filtration and that σ is a Gröbner reduction w.r.t. the extended filtration $G_r = S_r e_1 \oplus \cdots \oplus S_r e_q$. Then Proposition 1 assures that $G_r = \varphi^{-1}(N)_r \oplus I_r\ \forall r \in \mathbb{N}^p$. By Lemma 1, $R_r = \varphi(S_r)$ is a filtration on R and $F_r = \varphi(G_r)$ yields the extended filtration $F = \bigcup_{r\in\mathbb{N}^p} F_r$. Let $f \longrightarrow_\rho h$ and $f \in F_r$. There is a $g \in G_r$ with $\varphi(g) = f$. Let $i = \mathrm{NF}(g)$. Then $i \in G_r$ and so $\varphi(i) \in \varphi(G_r) = F_r$. But $\varphi(i) = \mathrm{NF}(f)$ and therefore we see that $h \in F_r$. Consequently ρ is a Groebner reduction. ☐

Applying the last proposition to the ring D provides an alternative method for constructing a Gröbner reduction in free D-modules.

Corollary 1. *Consider the ring D (as mentioned on page 2) and let S be the ring constructed from the same data as D but using positive exponents exclusively. More precisely, set $\tau_j = \sigma_j^{-1}$ and let S be the free K-vector space on the set of expressions*

$$\delta_1^{k_1} \cdots \delta_m^{k_m} \sigma^{l_1} \cdots \sigma_n^{l_n} \tau_1^{p_1} \cdots \tau_n^{p_n} \quad (k_i, l_j, p_j \in \mathbb{N})$$

with product being formally the same as the one in D (cf. Equation (2)). Let φ be the K-linear map $S \longrightarrow D$ defined on basis elements

$$\varphi(\delta^k \sigma^l \tau^p) = \delta^k \sigma^{l-p} \quad (k \in \mathbb{N}^m, l, p \in \mathbb{N}^n).$$

Then φ is a surjective homomorphism of rings and Proposition 6 applies. That is, given a submodule N of a finitely generated free D-module F, we may derive a Gröbner reduction for N by means of one constructed in a corresponding free module over S.

Of course the purpose is to represent the ring D as the quotient of S by $\ker\varphi$

$$D \cong S/\langle \sigma_1\tau_1 - 1, \ldots, \sigma_n\tau_n - 1\rangle$$

This construction can be used to execute computational tasks over D omitting negative exponents.

7. DIMENSION OF FILTER SPACES AND THE HILBERT POLYNOMIAL

In the general situation consider a finitely generated module M over an arbitrary monomially filtered ring R

$$R = \bigcup_{r\in\mathbb{N}^p} R_r \quad M = Rm_1 + \cdots + Rm_q.$$

Choose a free presentation

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

with $F = R^q$. We get the following corollary.

Corollary 2. *Let F be equipped with extended filtration from R and consider M with the filtration $M_r = \pi(F_r)$. For $r \in \mathbb{N}^p$ let U_r be the set of irreducible monomials in the filter space F_r. Assume given a Gröbner reduction for N. Then the sets $\pi(U_r)$ provide K-vector space bases for the spaces M_r. In particular*

$$\dim_K M_r = |\pi(U_r)| = |U_r| \quad (r \in \mathbb{N}^p).$$

Proof. Apply Theorem 1 with $V = F_r$. ☐

Combining this corollary with Proposition 2 gives:

Corollary 3. *Assume that the monomials in R satisfy $\Lambda\Lambda \subseteq \Lambda$ and let $N \subseteq F$ be a monomial submodule. Let $S = \{t \in \Lambda E \colon t \notin N\}$. Then, for arbitrary monomial filtration $R = \bigcup_{r\in\mathbb{N}^p} R_r$ and extended filtration $F = \bigcup_{r\in\mathbb{N}^p} F_r$, we have*

$$\dim_K \left(F/N\right)_r = |S_r|.$$

PROOF. Let $I = KS$. Then $I \cap \Lambda E = S$ und thus $I \cap \Lambda E \cap F_r = S \cap F_r = S_r$. Using Corollary 2 proves the assertion. $\square$

We may now determine the Hilbert function for filtered modules. This is most simple for monomial modules.

EXAMPLE 4. *Let $R = K[x, y]$ and N the ideal*

$$N = \langle x^4y^3,\ x^2y^5,\ 2x^5y^2 - 4x^3y^5\rangle.$$

It is easy to see that N is generated by the set

$$G = \{x^4y^3, x^2y^5, x^5y^2\}.$$

Thus, N is a monomial ideal and G is a Gröbner basis for N (w.r.t. arbitrary term-order).

Thus, Corollary 3 is applicable. For example when

$$R_k = \{f \in D\colon \deg(f) \le k\} \text{ and}$$

$$R_{r,s} = \{f \in D\colon \deg_x(f) \le r \wedge \deg_y(f) \le s\}$$

counting irreducible monomials that are not multiples of elements in G produces the dimensions in R/N. Let

$$\begin{aligned} p_1(k) &= \#\ \textit{irred. monomials in } R_k \\ p_2(r,s) &= \#\ \textit{irred. monomials in } R_{r,s} \end{aligned}$$

k	7	8	9	10	11	12
$p_1(k)$	33	37	41	45	49	53

For example, there are $\binom{8+2}{2} = 45$ monomials in 2 variables of total degree ≤ 8. From this 45 monomials, 37 monomials are irreducible, leaving 8 reducible elements w.r.t. graded lexicographic order. They are given by:

$$R_8 \setminus I = \{x^5y^2, x^6y^2, x^4y^3, y^3x^5, x^4y^4, x^2y^5, x^3y^5, x^2y^6\}.$$

From the 8 elements in $R_8 \setminus I$ there are 3 elements of degree 7, hence, in two variables, there are in total $\binom{7+2}{2} = 36$ monomials of degree 7, 3 of them reducible modulo G, giving us the value 33.

Since the degree of the Hilbert polynomial is bounded by the number of variables, interpolation gives

$$\begin{aligned} p_1(k) &= 4k + 5\ (k \ge 7) \\ p_2(r,s) &= 2r + 2s + 7\ ((r,s) \ge_\pi (4,4)). \end{aligned}$$

From this we see that the growth of elements is linear *by increasing the degree in one direction. Moreover p_2 is symmetric ($p_2(r,s) = p_2(s,r)$) which shows that the growth of dimension is the same in x and y direction.*

An obvious relation between $R_{r,s}$ and R_{r+s} is

$$\forall\, (r,s) \in \mathbb{N}^2\colon R_{r,s} \subseteq R_{r+s}.$$

A less obvious relation is the following. We have that

$$\begin{aligned} \#\ \textit{irred. elements in } R_{k,k} &= p_2(k,k) = 4k + 7 \\ \#\ \textit{irred. elements in } R_{k+k} &= p_1(k+k) = 8k + 5. \end{aligned}$$

Consequently $p_2(k,k) \le p_1(k+k)$ for $k \ge 1$.

8. COMPUTATION OF MULTIVARIATE DIFFERENCE DIFFERENTIAL DIMENSION POLYNOMIALS

In the general case, Corollary 2 applies. We will generalize the dimension polynomial computed in [12].

For example, using Corollary 2 to a finitely generated module over the ring D using relative reduction, the resulting sets U_r coincide with those computed in [12]. We will now set up a refined filtration of the ring D, controlled by a partition of the basic operators in the difference-differential field K. After designing a Gröbner reduction for a submodule, Corollary 2 will give us an improved picture of the filter spaces in the quotient.

Consider the sets $\Delta = \{\delta_1, \ldots, \delta_m\}$, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of the difference-differential field K.

We divide Δ and Σ into p respectively q pairwise disjoint subsets

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_p \text{ and } \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_q \tag{8}$$

where

$$\begin{aligned} \Delta_1 &= \{\delta_1, \ldots, \delta_{m_1}\} \\ \Delta_k &= \{\delta_{m_1+\cdots+m_{k-1}+1}, \ldots, \delta_{m_1+\cdots+m_k}\}, \quad (2 \le k \le p) \end{aligned}$$

and $m_1 + \cdots + m_p = m$. Similar for Σ

$$\begin{aligned} \Sigma_1 &= \{\sigma_1, \ldots, \sigma_{n_1}\} \\ \Sigma_k &= \{\sigma_{n_1+\cdots+n_{k-1}+1}, \ldots, \sigma_{n_1+\cdots+n_k}\}, \quad (2 \le k \le q) \end{aligned}$$

where $n_1 + \cdots + n_q = n$.

DEFINITION 5. *For a monomial*

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in \Lambda$$

we define

$$\begin{aligned} |\lambda|_{\Delta_j} &= \sum_{\delta_i \in \Delta_j} k_i \quad (1 \le j \le p) \\ |\lambda|_{\Sigma_j} &= \sum_{\sigma_i \in \Sigma_j} |l_i| \quad (1 \le j \le q) \end{aligned}$$

For a general difference-differential operator

$$a = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$$

we set

$$|a|_\Phi := \max\{|\lambda|_\Phi \colon \lambda \in T(a)\},$$

with $\Phi \in \{\Delta_1, \ldots, \Delta_p, \Sigma_1, \ldots, \Sigma_q\}$.

The following device defines a $p+q$-variate filtration on D. For $r \in \mathbb{N}^{p+q}$ set

$$D_r = \{u \in D\colon \forall_{1 \le i \le p}\ |u|_{\Delta_i} \le r_i \wedge \forall_{1 \le j \le q}\ |u|_{\Sigma_j} \le r_{p+j}\}. \tag{9}$$

DEFINITION 6. *Let M be a Δ-Σ module over a Δ-Σ field with m derivations and n automorphisms, partitioned as given in (8) and set $s = p + q$. The numerical polynomial $p(t_1, \ldots, t_s)$ is called difference-differential dimension polynomial associated to M, if*

1. $\deg(p) \le m + n$

2. $p(r_1, \dots, r_s) = \dim_K M_{r_1,\dots,r_s}$ *for all* $(r_1, \dots, r_s) \in \mathbb{N}^s$ *large enough.*

By a change of the vector space basis of polynomials of degree less than or equal to s to the Newton basis p admits a canonical representation of the form

$$\sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_s=0}^{n_s} a_{i_1,i_2,\dots,i_s} \binom{t_1+i_1}{i_1} \binom{t_2+i_2}{i_2} \cdots \binom{t_s+i_s}{i_s},$$

s.t. $a_{i_1,i_2,\dots,i_s} \in \mathbb{Z}$*, the dimension polynomial mentioned in [8, 9].*

Theorem 2. *Let* K *be a* $\Delta-\Sigma$ *field and* M *a finitely generated difference-differential module. Produce a partition of the sets* Δ, Σ *as described in (8) and equip the operator ring* D *with the filtration described in (9). Extend the filtration to the finite free presentation*

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

where F *has* K*-basis* E*, and let* $\prec$ *be a generalized term order on* ΛE*. If* G *is a Gröbner basis of* N *then the cardinality of the sets*

$$\begin{aligned} U_r = \{ & t \in \Lambda E \cap F_r : \forall_{g\in G} \forall_{\lambda\in\Lambda} \\ & (t = \mathrm{lt}_\prec(\lambda g) \Rightarrow \exists_i\, |\lambda g|_{\Delta_i} > r_i \vee \exists_j\, |\lambda g|_{\Sigma_j} > r_{p+j})\} \end{aligned}$$

provide the values of the Hilbert function of M*, i.e.,*

$$\dim_K M_r = |U_r| \quad \forall r \in \mathbb{N}^{p+q}.$$

Proof. The relation

$$\begin{aligned} f \longrightarrow h \iff & \exists_{g\in G}\, \exists_{\lambda\in\Lambda}\, \mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(f) \wedge \\ & \forall_{1\le i\le p}\, |\lambda g|_{\Delta_i} \le |f|_{\Delta_i} \wedge \forall_{1\le j\le q}\, |\lambda g|_{\Sigma_j} \le |f|_{\Sigma_j} \wedge \\ & h = f - \frac{\mathrm{lc}_\prec(f)}{\mathrm{lc}_\prec(\lambda g)} \lambda g \end{aligned}$$

defines a Gröbner reduction for N and Corollary 2 is applicable. □

9. CONCLUSIONS

We have set up a theory of reduction intended to extend Gröbner basis computations to modules over (possibly) non-commutative rings which contain a field as a subring. This applies in particular to rings of difference-differential operators. Assuming a multivariate filtration in the ground ring compatible with a given vector space basis, we have formulated natural axioms such a reduction should obey. It was possible to demonstrate that relative reduction as introduced in [12] as well as the reduction relations defined by classical Gröbner bases in polynomial rings or free modules over them can be viewed as an instance of our axioms. Further we have proved a general theorem on the dimension of filter spaces in finitely generated modules over such rings. The concepts have been demonstrated to be applicable to the computation of the Hilbert polynomial of multivariate difference-differential modules.

So far we have not given a general algorithm for computing such a reduction relation in nontrivial instances. In a continuing paper we plan to refine our approach and by giving additional features to the data of the ground ring, to formulate Buchberger criteria for such reductions. In doing so we try to make our approach suitable for actual computations in such general rings.

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