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Laguerre-Freud equations for Generalized Hahn polynomials of type I

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Abstract

We derive a system of difference equations satisfied by the three-term recurrence coefficients of some families of discrete orthogonal polynomials.

1 Introduction

Let $\{\mu_n\}$ be a sequence of complex numbers and $L : \mathbb{C}[x] \to \mathbb{C}$ be a linear functional defined by

$$L[x^n] = \mu_n, \quad n = 0, 1, \ldots.$$

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Then, L is called the *moment functional* determined by the formal moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n. A sequence $\{P_n(x)\} \subset \mathbb{C}[x]$, of monic polynomials with $\deg(P_n) = n$ is called an *orthogonal polynomial sequence* with respect to L provided that [4]

$$L[P_n P_m] = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots,$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta.

Since

$$L[x P_n P_k] = 0, \quad k \notin \{n-1, n, n+1\},$$

the monic orthogonal polynomials $P_n(x)$ satisfy the *three-term recurrence relation*

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \tag{1}$$

where

$$\beta_n = \frac{1}{h_n} L\left[x P_n^2\right], \quad \gamma_n = \frac{1}{h_{n-1}} L[x P_n P_{n-1}]. \tag{2}$$

If we define $P_{-1}(x) = 0$, $P_0(x) = 1$, we see that

$$P_1(x) = x - \beta_0, \tag{3}$$

and

$$P_2(x) = (x - \beta_1)(x - \beta_0) - \gamma_1. \tag{4}$$

Because

$$L[x P_n P_{n-1}] = L\left[P_n^2\right],$$

we have

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad n = 1, 2, \ldots, \tag{5}$$

and we define

$$\gamma_0 = 0. \tag{6}$$

Note that from (2) we get

$$\beta_0 = \frac{1}{h_0} L[x] = \frac{\mu_1}{\mu_0}. \tag{7}$$

If the coefficients β_n, γ_n are known, the recurrence (1) can be used to compute the polynomials $P_n(x)$. Stability problems and numerical aspects arising in the calculations have been studied by many authors [12], [14], [34], [46].

If explicit representations of the polynomials $P_n(x)$ are given, symbolic computation techniques can be applied to obtain recurrence relations and, in particular, to find expressions for the coefficients β_n, γ_n (see [5], [20], [37], [38], [47]).

If, alas, the only knowledge we have is the linear functional L, the computation of β_n and γ_n is a real challenge. One possibility is to use the Modified Chebyshev algorithm [13, 2.1.7]. Another is to obtain recurrences for β_n, γ_n of the form [2], [43]

$$\begin{aligned}\gamma_{n+1} &= F_1\left(n, \gamma_n, \gamma_{n-1}, \ldots, \beta_n, \beta_{n-1}, \ldots\right), \\ \beta_{n+1} &= F_2\left(n, \gamma_{n+1}, \gamma_n, \ldots, \beta_n, \beta_{n-1}, \ldots\right),\end{aligned}$$

for some functions F_1, F_2. This system of recurrences is known as the *Laguerre-Freud equations* [11], [23]. The name was coined by Alphonse Magnus as part of his work on Freud's conjecture [24], [25], [26], [27]. In terms of performance, the Modified Chebyshev algorithm requires $O(n^2)$ operations, while the Laguerre-Freud equations require only $O(n)$ operations for the computation of β_n and γ_n [3].

There are several papers on the Laguerre-Freud equations for different types of orthogonal polynomials including continuous [1], [31], [41], discrete [16], [17], [39], [44], D_ω polynomials [10], [30], Laguerre-Hahn [9], and q-polynomials [18].

Most of the known examples belong to the set of semiclassical orthogonal polynomials [28], where the linear functional satisfies an equation of the form

$$L[\phi U(\pi)] = L[\psi \pi], \quad \pi \in \mathbb{C}[x],$$

called the *Pearson equation* [36], where $U : \mathbb{C}[x] \to \mathbb{C}[x]$ is a linear operator and $\phi(x)$, $\psi(x)$ are fixed polynomials. The *class* of the semiclassical orthogonal polynomials is defined by

$$c = \max\left\{\deg(\phi) - 2,\ \deg(\phi - \psi) - 1\right\}.$$

In this paper, we focus our attention on linear functionals defined by

$$L[f] = \sum_{x=0}^{\infty} f(x)\rho(x), \tag{8}$$

where the weight function $\rho(x)$ is of the form

$$\rho(x) = \frac{(a_1)_x (a_2)_x \cdots (a_p)_x}{(b_1+1)_x (b_2+1)_x \cdots (b_q+1)_x} \frac{z^x}{x!}, \tag{9}$$

and $(a)_x$ denotes the Pochhammer symbol (also called shifted or rising factorial) defined by [35, 5.2.4]

$$\begin{aligned} (a)_0 &= 1 \\ (a)_x &= a(a+1)\cdots(a+x-1), \quad x \in \mathbb{N}, \end{aligned}$$

or by

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)},$$

where $\Gamma(z)$ is the Gamma function. Note that we have

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\psi(x)}{\phi(x+1)}, \tag{10}$$

with

$$\begin{aligned} \psi(x) &= z(x+a_1)(x+a_2)\cdots(x+a_p), \\ \phi(x) &= x(x+b_1)(x+b_2)\cdots(x+b_q). \end{aligned} \tag{11}$$

Hence, the weight function $\rho(x)$ satisfies an alternative form of the Pearson equation

$$\Delta_x(\phi\rho) = (\psi - \phi)\rho, \tag{12}$$

where

$$\Delta_x f(x) = f(x+1) - f(x) \tag{13}$$

is the forward difference operator. Using (10) in (8), we get the Pearson equation

$$L[\psi(x)\pi(x)] = L[\phi(x)\pi(x-1)], \quad \pi \in \mathbb{C}[x]. \tag{14}$$

The rest of the paper is organized as follows: in Section 2 we use (14) and obtain two difference equations satisfied by the discrete semiclassical orthogonal polynomials. As an example, we apply the method to obtain the recurrence coefficients of the Meixner polynomials.

In Section 3, we derive the Laguerre-Freud equations for the Generalized Hahn polynomials of type I, introduced in [7] as part of the classification of discrete semiclassical orthogonal polynomials of class one. Specializing one of the parameters in the polynomials, we obtain the recurrence coefficients of the Hahn polynomials.

We finish the paper with some remarks and future directions.

2 Laguerre-Feud equations

As Maroni remarks at the beginning of [29], "the history of finite-type relations is as old as the history of orthogonality since

$$r(x)P_n(x) = \sum_{k=n-t}^{n+t} \psi_{n,k} P_k(x),$$

when $P_n(x)$ is a sequence of orthogonal polynomials and $r(x)$ is a polynomial with $\deg(r) = t$." The three-term recurrence relation (1) is the most used example, with $r(x) = x$.

We now derive difference equations for orthogonal polynomials whose linear functional satisfies (14). We follow an approach similar to the one used in [40] to find the Laguerre-Freud equations for the generalized Charlier polynomials. Another method used in many articles is to use ladder operators [19].

Proposition 1 *Let $\{P_n(x)\}$ be a family of orthogonal polynomials with respect to a linear functional satisfying (14). Then, we have*

$$\psi(x) P_n(x+1) = \sum_{k=-q-1}^{p} A_k(n) P_{n+k}(x) \tag{15}$$

and

$$\phi(x) P_n(x-1) = \sum_{k=-p}^{q+1} B_k(n) P_{n+k}(x), \tag{16}$$

for some coefficients $A_k(n)$, $B_k(n)$.

Proof. Since $\deg \psi(x) P_n(x+1) = n+p$, we can write

$$\psi(x) P_n(x+1) = \sum_{k=-n}^{p} A_k(n) P_{n+k}(x).$$

Using orthogonality and (14), we have

$$\begin{aligned} h_{n+k} A_k(n) &= L[\psi(x) P_n(x+1) P_{n+k}(x)] \\ &= L[\phi(x) P_n(x) P_{n+k}(x-1)] = 0, \quad k < -q-1. \end{aligned}$$

Similarly, writing

$$\phi(x) P_n(x-1) = \sum_{k=-n}^{q+1} B_k(n) P_{n+k}(x),$$

we get

$$\begin{aligned} h_{n+k} B_k(n) &= L[\phi(x) P_n(x-1) P_{n+k}(x)] \\ &= L[\psi(x) P_n(x) P_{n+k}(x+1)] = 0, \quad k < -p. \end{aligned}$$

■

The coefficients $A_k(n)$ and $B_k(n)$ are not independent of each other.

Corollary 2

$$A_k(n) = \frac{h_n}{h_{n+k}} B_{-k}(n+k), \quad -q-1 \leq k \leq p. \tag{17}$$

Proof. If $-q-1 \leq k \leq p$, then

$$\begin{aligned} A_k(n) &= \frac{1}{h_{n+k}} L[\phi(x) P_n(x) P_{n+k}(x-1)] \\ &= \frac{1}{h_{n+k}} L\left[P_n(x) \sum_{j=-p}^{q+1} B_j(n+k) P_{n+k+j}(x)\right] \\ &= \frac{1}{h_{n+k}} \sum_{j=-p}^{q+1} B_j(n+k) L[P_n(x) P_{n+k+j}(x)] \\ &= \frac{h_n}{h_{n+k}} B_{-k}(n+k). \end{aligned}$$

■

We can now state our main result.

Theorem 3 *For $-q-1 \leq k \leq p$, we have*

$$\begin{aligned}&\gamma_{n+k+1} A_{k+1}(n) - \gamma_n A_{k+1}(n-1) + A_{k-1}(n) - A_{k-1}(n+1) \\ &= (\beta_n - \beta_{n+k} - 1) A_k(n),\end{aligned} \tag{18}$$

with

$$A_p(n) = z, \tag{19}$$

$$A_{-q-1}(n) = \gamma_n \gamma_{n-1} \cdots \gamma_{n-q}, \tag{20}$$

and

$$A_{p+1}(n) = 0 = A_{-q-2}(n).$$

Proof. Using (1), we have

$$\begin{aligned}&\psi(x)(x+1) P_n(x+1) = \psi(x) P_{n+1}(x+1) \\ &+\beta_n \psi(x) P_n(x+1) + \gamma_n \psi(x) P_{n-1}(x+1),\end{aligned}$$

and from (15)

$$\psi(x)(x+1) P_n(x+1) = \sum_{k=-q}^{p+1} A_{k-1}(n+1) P_{n+k}(x) \tag{21}$$

$$+\sum_{k=-q-1}^{p} \beta_n A_k(n) P_{n+k}(x) + \sum_{k=-q-2}^{p-1} \gamma_n A_{k+1}(n-1) P_{n+k}(x).$$

On the other hand, if we multiply (15) by x, we get

$$\psi(x) x P_n(x+1) = \sum_{k=-q-1}^{p} A_k(n) x P_{n+k}(x),$$

and using (1) we obtain

$$\psi(x) x P_n(x+1) = \sum_{k=-q}^{p+1} A_{k-1}(n) P_{n+k}(x) \tag{22}$$

$$+\sum_{k=-q-1}^{p} \beta_{n+k} A_k(n) P_{n+k}(x) + \sum_{k=-q-2}^{p-1} \gamma_{n+k+1} A_{k+1}(n) P_{n+k}(x).$$

Using (15), (21) and (22) in the identity

$$\psi(x) P_n(x+1) = (x+1)\psi(x) P_n(x+1) - x\psi(x) P_n(x+1),$$

we have

$$\begin{aligned}\sum_{k=-q-1}^{p} A_k(n) P_{n+k}(x) = &\sum_{k=-q}^{p+1} \left[A_{k-1}(n+1) - A_{k-1}(n)\right] P_{n+k}(x) \\ &+ \sum_{k=-q-1}^{p} (\beta_n - \beta_{n+k}) A_k(n) P_{n+k}(x) \\ &+ \sum_{k=-q-2}^{p-1} \left[\gamma_n A_{k+1}(n-1) - \gamma_{n+k+1} A_{k+1}(n)\right] P_{n+k}(x).\end{aligned}$$

Since the polynomials $P_n(x)$ are linearly independent, we get:

$$k = p+1: \quad A_p(n+1) - A_p(n) = 0, \tag{23}$$

$$k = -q-2: \quad \gamma_n A_{-q-1}(n-1) - \gamma_{n-q-1} A_{-q-1}(n) = 0, \tag{24}$$

and for $-q-1 \leq k \leq p$,

$$\begin{aligned}(1 + \beta_{n+k} - \beta_n) A_k(n) = &A_{k-1}(n+1) - A_{k-1}(n) \\ &+ \gamma_n A_{k+1}(n-1) - \gamma_{n+k+1} A_{k+1}(n).\end{aligned}$$

Comparing leading coefficients in (15) we obtain

$$A_p(n) = z,$$

in agreement with (23).

Rewriting (24) as

$$\frac{A_{-q-1}(n)}{A_{-q-1}(n-1)} = \frac{\gamma_n}{\gamma_{n-q-1}},$$

we see that

$$\frac{A_{-q-1}(n)}{A_{-q-1}(q+1)} = \frac{\gamma_n \gamma_{n-1} \cdots \gamma_{n-q}}{\gamma_1 \gamma_2 \cdots \gamma_{q+1}}.$$

From (17) we have

$$A_{-q-1}(q+1) = \frac{h_{q+1}}{h_0} B_{q+1}(0).$$

Since $\phi(x) P_n(x-1)$ is a monic polynomial, (16) gives

$$B_{q+1}(n) = 1, \tag{25}$$

and using (5) we get

$$\frac{h_{q+1}}{h_0} B_{q+1}(0) = \gamma_1 \gamma_2 \cdots \gamma_{q+1},$$

proving (20). ∎

2.1 Meixner polynomials

To illustrate the use of Theorem 3, we consider the family of Meixner polynomials introduced by Josef Meixner in [32]. These polynomials are orthogonal with respect to the weight function

$$\rho(x) = (a)_x \frac{z^x}{x!},$$

and using (11) we have

$$\psi(x) = z(x+a), \quad \phi(x) = x,$$

and $p = 1, \quad q = 0$.

From (19) and (20) we get

$$A_1(n) = z, \quad A_{-1}(n) = \gamma_n, \tag{26}$$

while (18) gives:

$$k = 1: \quad (1 + \beta_{n+1} - \beta_n) A_1(n) = A_0(n+1) - A_0(n),$$

$$k = 0: \quad A_0(n) = A_{-1}(n+1) - A_{-1}(n) + \gamma_n A_1(n-1) - \gamma_{n+1} A_1(n),$$

and

$$k = -1: \quad (1 + \beta_{n-1} - \beta_n) A_{-1}(n) = \gamma_n A_0(n-1) - \gamma_n A_0(n).$$

Using (26) we obtain

$$z(1 + \beta_{n+1} - \beta_n) = A_0(n+1) - A_0(n), \tag{27}$$

$$A_0(n) = \gamma_{n+1} - \gamma_n + z(\gamma_n - \gamma_{n+1}) = (1-z)(\gamma_{n+1} - \gamma_n), \tag{28}$$

and

$$1 + \beta_{n-1} - \beta_n = A_0(n-1) - A_0(n). \tag{29}$$

Summing (27) from $n = 0$ and (29) from $n = 1$, we get

$$\begin{aligned} z(\beta_n - \beta_0 + n) &= A_0(n) - A_0(0), \\ \beta_n - \beta_0 - n &= A_0(n) - A_0(0). \end{aligned}$$

Using (28) and (6), gives

$$\beta_n - \beta_0 - n = z(\beta_n - \beta_0 + n) = (1-z)(\gamma_{n+1} - \gamma_n - \gamma_1).$$

Therefore,

$$\beta_n = \beta_0 + \frac{1+z}{1-z}n,$$

and

$$\gamma_{n+1} - \gamma_n - \gamma_1 = \frac{2nz}{(1-z)^2}. \tag{30}$$

Summing (30) from $n = 0$, we conclude that

$$\gamma_n = n\gamma_1 + \frac{n(n-1)z}{(1-z)^2}.$$

If we use (26) and (28) in (15), we get

$$\begin{aligned} z(x+a)P_n(x+1) &= \gamma_n P_{n-1}(x) \\ &+ (1-z)(\gamma_{n+1} - \gamma_n)P_n(x) + zP_{n+1}(x), \end{aligned} \tag{31}$$

and using (17),

$$\begin{aligned} B_1(n) &= \frac{h_n}{h_{n+1}} A_{-1}(n+1) = \frac{A_{-1}(n+1)}{\gamma_{n+1}} = 1, \\ B_0(n) &= A_0(n) = (1-z)(\gamma_{n+1} - \gamma_n), \\ B_{-1}(n) &= \frac{h_n}{h_{n-1}} A_1(n-1) = \gamma_n z. \end{aligned}$$

Hence, from (16) we obtain

$$xP_n(x-1) = z\gamma_n P_{n-1}(x) + (1-z)(\gamma_{n+1} - \gamma_n)P_n(x) + P_{n+1}(x). \tag{32}$$

Setting $n = 0$ in (31) and (32) gives

$$z(x+a) = (1-z)\gamma_1 + z(x-\beta_0),$$
$$x = (1-z)\gamma_1 + x - \beta_0,$$

from which we find

$$(1-z)\gamma_1 = \beta_0 = -a + \frac{1-z}{z}\gamma_1,$$

and therefore

$$\beta_0 = \frac{az}{1-z}, \quad \gamma_1 = \frac{az}{(1-z)^2}.$$

Thus, we recover the well known coefficients [35, 18.22.2]

$$\beta_n = \frac{n+(n+a)z}{1-z}, \quad \gamma_n = \frac{n(n+a-1)z}{(1-z)^2}. \tag{33}$$

Using the hypergeometric representation [35, 18.20.7]

$$P_n(x) = (a)_n \left(1-\frac{1}{z}\right)^{-n} {}_2F_1\left[\begin{matrix} -n, \ -x \\ a \end{matrix}; 1-\frac{1}{z}\right],$$

one can easily verify (or re-derive) (33) using (for instance) the Mathematica package HolonomicFunctions [22].

3 Generalized Hahn polynomials of type I

The Generalized Hahn polynomials of type I were introduced in [7]. They are orthogonal with respect to the weight function

$$\rho(x) = \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}, \quad |z| < 1, \quad b \neq -1, -2, \ldots.$$

The first moments are given by

$$\mu_0 = {}_2F_1\left[\begin{matrix} a_1, \ a_2 \\ b+1 \end{matrix}; z\right], \tag{34}$$
$$\mu_1 = z\frac{a_1 a_2}{b+1} {}_2F_1\left[\begin{matrix} a_1+1, \ a_2+1 \\ b+2 \end{matrix}; z\right].$$

Since

$$\frac{\rho(x+1)}{\rho(x)} = \frac{z(x+a_1)(x+a_2)}{(x+1)(x+b+1)},$$

we have

$$\psi(x) = z(x+a_1)(x+a_2), \quad \phi(x) = x(x+b),$$

and $p = 2, \quad q = 1$.

We can now derive the Laguerre-Freud equations for the Generalized Hahn polynomials of type I.

Theorem 4 *The recurrence coefficients of the Generalized Hahn polynomials of type I satisfy the Laguerre-Freud equations*

$$(1-z)\nabla_n(\gamma_{n+1}+\gamma_n) = zv_n\nabla_n(\beta_n+n) - u_n\nabla_n(\beta_n-n), \tag{35}$$

$$\Delta_n\nabla_n[(u_n-zv_n)\gamma_n] = u_n\nabla_n(\beta_n-n) + \nabla_n(\gamma_{n+1}+\gamma_n). \tag{36}$$

with initial conditions $\beta_0 = \frac{\mu_1}{\mu_0}$ *and*

$$\gamma_1 = \frac{(a_1+a_2-b)\beta_0 + a_1a_2}{1-z} - (\beta_0+a_1)(\beta_0+a_2), \tag{37}$$

where

$$u_n = \beta_n + \beta_{n-1} - n + b + 1,$$
$$v_n = \beta_n + \beta_{n-1} + n - 1 + a_1 + a_2,$$

and

$$\nabla_x f(x) = f(x) - f(x-1). \tag{38}$$

Proof. From (19) and (20), we get

$$A_2(n) = z, \quad A_{-2}(n) = \gamma_n\gamma_{n-1}, \tag{39}$$

while (18) gives:

$$k=2: \quad A_1(n+1) - A_1(n) = z(1+\beta_{n+2}-\beta_n), \tag{40}$$

$$\begin{aligned} k=1: &\quad A_0(n+1) - A_0(n) = A_1(n)(1+\beta_{n+1}-\beta_n) + z(\gamma_{n+2}-\gamma_n), \\ k=0: &\quad A_{-1}(n+1) - A_{-1}(n) = A_0(n) + A_1(n)\gamma_{n+1} - A_1(n-1)\gamma_n, \\ k=-1: &\quad A_{-2}(n+1) - A_{-2}(n) \\ &= A_{-1}(n)(1+\beta_{n-1}-\beta_n) + \gamma_n[A_0(n) - A_0(n-1)], \end{aligned} \tag{41}$$

and

$$k=-2: \quad A_{-2}(n)(1+\beta_{n-2}-\beta_n)=A_{-1}(n-1)\gamma_n-A_{-1}(n)\gamma_{n-1}. \tag{42}$$

Solving (40) we get

$$A_1(n)=A_1(0)+z(\beta_{n+1}+\beta_n+n-\beta_0-\beta_1). \tag{43}$$

Setting $n=0$ in (15) we have

$$z(x+a_1)(x+a_2)=A_0(0)+A_1(0)P_1(x)+zP_2(x),$$

and using (3)-(4), we get

$$A_0(0)=z\left[a_1a_2+\gamma_1+(a_1+a_2)\beta_0+\beta_0^2\right], \tag{44}$$

and

$$A_1(0)=z(a_1+a_2+\beta_0+\beta_1). \tag{45}$$

Using (45) in (43), we obtain

$$A_1(n)=z(\beta_{n+1}+\beta_n+n+a_1+a_2). \tag{46}$$

If we use (39) in (42), we get

$$1+\beta_{n-2}-\beta_n=\frac{A_{-1}(n-1)}{\gamma_{n-1}}-\frac{A_{-1}(n)}{\gamma_n},$$

and summing from $n=2$ we see that

$$n-1+\beta_0+\beta_1-\beta_{n-1}-\beta_n=\frac{A_{-1}(1)}{\gamma_1}-\frac{A_{-1}(n)}{\gamma_n}. \tag{47}$$

Setting $n=0$ in (16), we have

$$x(x+b)=(x-\beta_1)(x-\beta_0)-\gamma_1+B_1(0)(x-\beta_0)+B_0(0)$$

and hence

$$B_1(0)=\beta_0+\beta_1+b, \tag{48}$$

$$B_0(0)=\beta_0^2+b\beta_0+\gamma_1. \tag{49}$$

Using (17) with $k=-1$ and (48), we obtain

$$A_{-1}(1)=\gamma_1B_1(0)=\gamma_1(\beta_0+\beta_1+b). \tag{50}$$

Combining (47) and (50), we conclude that

$$A_{-1}(n) = \gamma_n (\beta_n + \beta_{n-1} - n + b + 1). \tag{51}$$

If we introduce the functions

$$\begin{aligned} u_n &= \frac{A_{-1}(n)}{\gamma_n} = \beta_n + \beta_{n-1} - n + b + 1, \\ v_n &= \frac{A_1(n-1)}{z} = \beta_n + \beta_{n-1} + n - 1 + a_1 + a_2, \end{aligned}$$

and use (46),(51) in (41), we get

$$\begin{aligned} \nabla_n A_0 &= z v_n \nabla_n (\beta_n + n) + z \nabla_n (\gamma_{n+1} + \gamma_n), \\ A_0 &= \Delta_n [(u_n - z v_n) \gamma_n], \\ \nabla_n A_0 &= u_n \nabla_n (\beta_n - n) + \nabla_n (\gamma_{n+1} + \gamma_n). \end{aligned} \tag{52}$$

Using (17) with $k = 0$ and (49), we obtain

$$A_0(0) = B_0(0) = \beta_0^2 + b\beta_0 + \gamma_1. \tag{53}$$

From (44) and (53) we have

$$(1 - z) [\gamma_1 + (\beta_0 + a_1)(\beta_0 + a_2)] = (a_1 + a_2 - b) \beta_0 + a_1 a_2. \tag{54}$$

Finally, if we eliminate A_0 from (52), we conclude that

$$z v_n \nabla_n (\beta_n + n) + z \nabla_n (\gamma_{n+1} + \gamma_n) = u_n \nabla_n (\beta_n - n) + \nabla_n (\gamma_{n+1} + \gamma_n)$$

and

$$\begin{aligned} &\Delta_n [(u_n - z v_n) \gamma_n] - \Delta_n [(u_{n-1} - z v_{n-1}) \gamma_{n-1}] \\ &= u_n \nabla_n (\beta_n - n) + \nabla_n (\gamma_{n+1} + \gamma_n) \end{aligned}$$

or

$$\Delta_n \nabla_n [(u_n - z v_n) \gamma_n] = u_n \nabla_n (\beta_n - n) + \nabla_n (\gamma_{n+1} + \gamma_n).$$

■

3.1 Hahn polynomials

We now consider the case $z = 1$. Under the assumptions

$$\operatorname{Re}(b - a_1 - a_2) > 0, \quad b - a_1 - a_2 \neq 1, 2, \ldots,$$

the first two moments (34) are given by [35, 15.4(ii)]

$$\mu_0 = \frac{\Gamma(b+1)\Gamma(b+1-a_1-a_2)}{\Gamma(b+1-a_1)\Gamma(b+1-a_2)},$$
$$\mu_1 = \frac{a_1 a_2}{b+1}\frac{\Gamma(b+2)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)}.$$

Hence,

$$\beta_0 = \frac{\mu_1}{\mu_0} = \frac{a_1 a_2}{b - a_1 - a_2}. \tag{55}$$

Note that we get the same result if we set $z = 1$ in (54).

Taking limits in (37) as $z \to 1^-$, we obtain

$$\gamma_1 = \frac{a_1 a_2 (b-a_1)(b-a_2)}{(b-a_1-a_2)(b-1-a_1-a_2)} - a_1 \frac{b-a_1}{b-a_1-a_2} a_2 \frac{b-a_2}{b-a_1-a_2},$$

or

$$\gamma_1 = \frac{a_1 a_2 (b-a_1)(b-a_2)}{(b-a_1-a_2)^2 (b-a_1-a_2-1)}, \tag{56}$$

where we have used the formula [35, 15.5.1]

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; z\right] = \frac{ab}{c}\,{}_2F_1\left[\begin{matrix} a+1,\ b+1 \\ c+1 \end{matrix}; z\right].$$

When $z = 1$, the Laguerre-Freud equations (35)-(36) decouple, and we get

$$u_n \nabla_n (\beta_n - n) = v_n \nabla_n (\beta_n + n), \tag{57}$$

$$\Delta_n \nabla_n [(b - a_1 - a_2 + 2 - 2n)\gamma_n] - \nabla_n(\gamma_{n+1} + \gamma_n) = u_n \nabla_n (\beta_n - n), \tag{58}$$

since in this case

$$u_n - v_n = b - a_1 - a_2 + 2 - 2n.$$

Solving for β_n in (57), we have

$$\beta_n = \frac{2n + a_1 + a_2 - b - 4}{2n + a_1 + a_2 - b}\beta_{n-1} - \frac{a_1 + a_2 + b}{2n + a_1 + a_2 - b}. \tag{59}$$

As it is well known, the general solution of the initial value problem

$$y_{n+1} = c_n y_n + g_n, \quad y_{n_0} = y_0, \quad n \geq n_0,$$

is [8, 1.2.4]

$$y_n = y_0 \prod_{j=n_0}^{n-1} c_j + \sum_{k=n_0}^{n-1} \left(g_k \prod_{j=k+1}^{n-1} c_j \right).$$

Thus, the solution of (59) is given by

$$\begin{aligned} \beta_n = & \frac{(a_1 + a_2 - b)(a_1 + a_2 - b - 2)}{(2n + a_1 + a_2 - b)(2n + a_1 + a_2 - b - 2)} \beta_0 \\ & - \frac{(a_1 + a_2 + b)(a_1 + a_2 - b + n - 1)}{(2n + a_1 + a_2 - b)(2n + a_1 + a_2 - b - 2)} n, \end{aligned}$$

where we have used the identity

$$\prod_{k=n_0}^{n_1} \frac{2n + K - 2}{2n + K + 2} = \frac{(2n_0 + K)(2n_0 + K - 2)}{(2n_1 + K)(2n_1 + K + 2)}.$$

If we use the initial condition (55), we conclude that

$$\beta_n = \frac{(b + 2 - a_1 - a_2) a_1 a_2 - n (a_1 + a_2 + b)(n + a_1 + a_2 - b - 1)}{(2n + a_1 + a_2 - b)(2n + a_1 + a_2 - b - 2)}.$$

Re-writing (58), we have

$$\begin{aligned} & (b - a_1 - a_2 - 2n - 1) \gamma_{n+1} - 2 (b - a_1 - a_2 - 2n + 2) \gamma_n \\ & + (b - a_1 - a_2 - 2n + 5) \gamma_{n-1} = u_n \nabla_n (\beta_n - n). \end{aligned}$$

Summing from $n = 1$, we get

$$\begin{aligned} & (b - a_1 - a_2 - 2n - 1) \gamma_{n+1} + (a_1 + a_2 - b + 2n - 3) \gamma_n \\ & + (a_1 + a_2 - b + 1) \gamma_1 = -\sum_{k=0}^{n-1} \beta_k + \beta_n^2 - \beta_0^2 \\ & + b (\beta_n - \beta_0 - n) - n \beta_n + \frac{n (n-1)}{2}. \end{aligned}$$

The solution of this difference equation with initial condition (56) is

$$\gamma_n = -n\frac{(n+a_1-1)(n+a_2-1)(n+a_1-b-1)}{(2n+a_1+a_2-b-1)(2n+a_1+a_2-b-3)} \times \frac{(n+a_2-b-1)(n+a_1+a_2-b-2)}{(2n+a_1+a_2-b-2)^2}.$$

We summarize the results in the following proposition.

Proposition 5 *The recurrence coefficients of the Hahn polynomials, orthogonal with respect to the weight function*

$$\rho(x) = \frac{(a_1)_x (a_2)_x}{x! (b+1)_x},$$

with

$$\operatorname{Re}(b-a_1-a_2) > 0, \quad b-a_1-a_2 \neq 1, 2, \ldots,$$

are given by

$$\beta_n = \frac{(b+2-a_1-a_2)a_1a_2 - n(a_1+a_2+b)(n+a_1+a_2-b-1)}{(2n+a_1+a_2-b)(2n+a_1+a_2-b-2)}, \tag{60}$$

and

$$\gamma_n = -n\frac{(n+a_1-1)(n+a_2-1)(n+a_1-b-1)}{(2n+a_1+a_2-b-1)(2n+a_1+a_2-b-3)} \times \frac{(n+a_2-b-1)(n+a_1+a_2-b-2)}{(2n+a_1+a_2-b-2)^2}. \tag{61}$$

This family of orthogonal polynomials was introduced by Hahn in [15]. They have the hypergeometric representation [45]

$$P_n(x) = \frac{(a_1)_n (a_2)_n}{(n+a_1+a_2-b-1)_n} \, {}_3F_2\left[\begin{matrix} -n, \; -x, \; n+a_1+a_2-b-1 \\ a_1, \; a_2 \end{matrix} ;1\right],$$

from which (60) and (61) can be obtained using HolonomicFunctions.

As we observed in [6], the finite family of polynomials that are usually called "Hahn polynomials" in the literature [35, 18.19] correspond to the special case

$$a_1 = \alpha + 1, \quad a_2 = -N, \quad b = -N-1-\beta.$$

4 Conclusions

We have presented a method that allows the computation of the recurrence coefficients of discrete orthogonal polynomials. In some cases, a closed-form expression can be given. We plan to extend the results to include other families of polynomials.

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5 Appendix

In this section we review the theory of orthogonal polynomial which are solutions of the difference equation of hypergeometric type (see [33, Chapter 2]) and also list some of the main properties of the Meixner and Hahn polynomials (see [21, 2.5,2.9] and [35, 18.19-18.23]).

5.1 A.0. Second order hypergeometric difference equation

Let's consider the difference equation

$$\sigma(x)\,\Delta\nabla y + \tau(x)\,\Delta y + \nu y = 0, \tag{62}$$

where

$$\Delta f(x) = f(x+1) - f(x),$$
$$\nabla f(x) = f(x) - f(x-1),$$

and $\sigma(x), \tau(x)$ are polynomials with $\deg(\sigma) \leq 2$, $\deg(\tau) \leq 1$, and ν is a constant.

The higher-order differences

$$d_m(x) = \Delta^m [y(x)],$$

satisfy the equation

$$\sigma(x)\Delta\nabla d_m + \tau_m(x)\Delta d_m + \nu_m d_m = 0, \tag{63}$$

where

$$\tau_m(x) = \tau(x+m) + \sigma(x+m) - \sigma(x), \tag{64}$$

and

$$\nu_m = \nu + m\tau' + \frac{m(m-1)}{2}\sigma''.$$

The solution of (62) is a polynomial $y_n(x)$ of degree n if and only if the function $d_n(x)$ is a constant. From (63) we see that ν_n must be zero and therefore

$$v = -n\tau' - \frac{n(n-1)}{2}\sigma'' = \lambda_n. \tag{65}$$

If we multiply both sides of (62) by a function $\rho(x)$ satisfying the Pearson equation

$$\Delta(\sigma\rho) = \tau\rho,$$

then we can write (62) in the self adjunct form

$$\Delta(\sigma\rho\nabla y_n) + \lambda_n \rho y_n = 0. \tag{66}$$

Similarly, the equation (63) can be written as

$$\Delta(\sigma\rho_m\nabla d_m) + \nu_m\rho_m d_m = 0, \tag{67}$$

where $\rho_m(x)$ satisfies the Pearson equation

$$\Delta(\sigma\rho_m) = \tau_m\rho_m. \tag{68}$$

Solving (68), we obtain

$$\rho_m(x) = \rho(x+m)\prod_{k=1}^{m}\sigma(x+k). \tag{69}$$

Note that

$$\rho_m(x) = \sigma(x+1)\rho_{m-1}(x+1).$$

Considering two solutions y_n, y_m of (66), we see that

$$(\lambda_m - \lambda_n) \, y_n(x) \, y_m(x) \, \rho(x) = \Delta \left[(y_m \nabla y_n - y_n \nabla y_m) \, \sigma(x) \, \rho(x) \right].$$

Hence,

$$(\lambda_m - \lambda_n) \sum_{x=a}^{b-1} y_n(x) \, y_m(x) \, \rho(x) = \left[(y_m \nabla y_n - y_n \nabla y_m) \, \sigma(x) \, \rho(x) \right]_{x=a}^{x=b},$$

where $y_m \nabla y_n - y_n \nabla y_m$ is a polynomial. If we impose the boundary conditions

$$\left[x^i \sigma(x) \, \rho(x) \right]_{x=a}^{x=b} = 0, \quad i = 0, 1, \ldots,$$

then we obtain the orthogonality relation

$$\sum_{x=a}^{b-1} y_n(x) \, y_m(x) \, \rho(x) = 0, \quad n \neq m.$$

From (67) we can derive the Rodrigues formula

$$y_n(x) = \frac{C_n}{\rho(x)} \nabla^n \left[\rho_n(x) \right], \tag{70}$$

where C_n is a normalizing constant. Writing

$$y_n(x) = \kappa_n x^n + \cdots,$$

we find that

$$\kappa_n = C_n \prod_{k=0}^{n-1} \left(\tau' + \frac{n+k-1}{2} \sigma'' \right). \tag{71}$$

Using (70), we also get the backward difference

$$\sigma(x) \, \nabla y_n = \frac{\lambda_n}{n \tau_n'} \left(\tau_n y_n - \frac{C_n}{C_{n+1}} y_{n+1} \right). \tag{72}$$

Finally, using the formula

$$\nabla^n \left[f(x) \right] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k)$$

we can rewrite (70) as

$$y_n(x) = C_n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\rho_n(x-k)}{\rho(x)},$$

or

$$y_n(x) = (-1)^n C_n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{\rho_n(x-n+k)}{\rho(x)}. \tag{73}$$

But from (68), it follows that

$$\frac{\rho_n(x-n+k+1)}{\rho_n(x-n+k)} = \frac{\sigma(x-n+k) + \tau_n(x-n+k)}{\sigma(x-n+k+1)}$$

is a rational function of k, and therefore $y_n(x)$ admits a representation as a hypergeometric function.

5.2 A.1. Meixner polynomials

From (33) we see that the monic Meixner polynomials satisfy the three-term recurrence relation

$$xM_n = M_{n+1} + \frac{n+(n+a)z}{1-z} M_n + \frac{n(n+a-1)z}{(1-z)^2} M_{n-1}. \tag{74}$$

From (31) and (74) we obtain the forward difference

$$z(x+a) M_n(x+1) = (z-1) M_{n+1}(x) + (x-n) M_n(x), \tag{75}$$

and from (32) and (74) we get the backward difference

$$xM_n(x-1) = (1-z) M_{n+1}(x) + z(x+a+n) M_n(x). \tag{76}$$

Combining (75) and (76), we have the difference equation

$$z(x+a) M_n(x+1) + [n-x-z(x+a+n)] M_n(x) + xM_n(x-1) = 0,$$

which can be written in the hypergeometric form

$$x\Delta\nabla M_n + (zx - x + az) \Delta M_n + n(1-z) M_n = 0. \tag{77}$$

Comparing (77) with (62), we see that

$$\sigma(x) = x, \quad \tau(x) = (z-1)x + az, \quad \lambda_n = n(1-z). \tag{78}$$

Note that from (64) and (78) we have

$$\tau_n(x) = (z-1)(x+n) + az + n, \tag{79}$$

and that

$$\lambda_n = n(1-z) = -n(z-1) = -n\tau' - \frac{n(n-1)}{2}\sigma'',$$

in agreement with (65).

Introducing the weight function

$$\rho(x) = (a)_x \frac{z^x}{x!},$$

we can write (77) in self adjunct form. Note that $\rho(x)$ satisfies the Pearson equation

$$\Delta(x\rho) = (zx - x + az)\rho$$

and that

$$\mu_0 = \sum_{x=0}^{\infty} \rho(x) = \sum_{x=0}^{\infty} (a)_x \frac{z^x}{x!} = (1-z)^{-a}, \tag{80}$$

as long as $|z| < 1$. In order to have

$$\rho(x) > 0, \quad x = 0, 1, \ldots,$$

we need $a > 0$ and $0 < z < 1$.

Using (69), we get

$$\rho_n(x) = \rho(x+n) \prod_{k=1}^{n} (x+k) = \rho(x+n)(x+1)_n. \tag{81}$$

Since we are considering monic polynomials, we set $\kappa_n = 1$ in (71) and obtain

$$C_n = \prod_{k=0}^{n-1} \frac{1}{\tau'} = \prod_{k=0}^{n-1} \frac{1}{z-1} = (z-1)^{-n}. \tag{82}$$

Using (81) and (82) in (70), we have the Rodrigues formula

$$M_n(x) = \frac{(z-1)^{-n}}{\rho(x)} \nabla^n \left[\frac{(a)_{x+n} (x+1)_n}{(1)_{x+n}} z^{x+n} \right],$$

which we can rewrite as

$$M_n(x) = \frac{(z-1)^{-n}}{\rho(x)} \nabla^n \left[\frac{(a)_n (a+n)_x}{(1)_x} z^{x+n} \right],$$

using the identity

$$(x)_{n+m} = (x)_n (x+n)_m .$$

Note that using (78), (79) and (82) in (72), we obtain

$$x \nabla M_n = (z-1) M_{n+1} - (-x + az + nz + xz) M_n,$$

in agreement with (76).

Using (5) and (33), we get

$$h_n = h_0 \prod_{k=1}^{n} \gamma_k = n! (a)_n (1-z)^{-2n-a} z^n,$$

since from (80)

$$h_0 = \mu_0 = (1-z)^{-a} .$$

Using (73), we have

$$\begin{aligned} M_n(x) &= (1-z)^{-n} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{\rho(x+k)}{\rho(x)} (x-n+k+1)_n \\ &= (x+1-n)_n (1-z)^{-n} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(x+a)_k}{(x+1-n)_k} z^k. \end{aligned}$$

Thus, we obtain the hypergeometric representation

$$M_n(x) = (x+1-n)_n (1-z)^{-n} \ {}_2F_1\left(\begin{matrix} -n, x+a \\ x+1-n \end{matrix} ; z \right). \tag{83}$$

Using the linear transformation [35, 15.8.7]

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z \right) = \frac{(c-b)_n}{(c)_n} z^n \ {}_2F_1\left(\begin{matrix} -n, 1-c-n \\ 1+b-c-n \end{matrix} ; 1 - \frac{1}{z} \right),$$

we can rewrite (83) as

$$M_n(x) = (a)_n \left(1-\frac{1}{z}\right)^{-n} {}_2F_1\left(\begin{matrix} -n,-x \\ a \end{matrix};1-\frac{1}{z}\right). \tag{84}$$

Using the expansion [35, 16.10.2]

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(a_1)_n}{n!}\, {}_{q+1}F_q\left(\begin{matrix} -n,a_2,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix};\zeta\right)\omega^n \\ &= (1-\omega)^{-a_1}\, {}_{q+1}F_q\left(\begin{matrix} a_1,a_2,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix};\frac{\omega\zeta}{\omega-1}\right),\end{aligned} \tag{85}$$

valid for

$$|1-\zeta|<1, \quad \mathrm{Re}\left(\frac{\omega}{\omega-1}\right)<\frac{1}{2},$$

and (84) we obtain the generating function

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{(a)_n}M_n(x)\frac{t^n}{n!} = \left(1-\frac{zt}{z-1}\right)^{-\xi} {}_2F_1\left(\begin{matrix} \xi,-x \\ a \end{matrix};\frac{(z-1)\,t}{1+(t-1)\,z}\right), \tag{86}$$

where we chose

$$\begin{aligned}&q=1, \quad a_1=\xi, \quad a_2=-x, \quad b_1=a, \\ &\zeta = 1-\frac{1}{z}, \quad \omega=\frac{zt}{z-1}.\end{aligned}$$

If we set $\xi = a$ in (86), we get

$$\begin{aligned}\sum_{n=0}^{\infty}M_n(x)\frac{t^n}{n!} &= \left(1-\frac{zt}{z-1}\right)^{-a} {}_1F_0\left(\begin{matrix} -x \\ - \end{matrix};\frac{(z-1)\,t}{1+(t-1)\,z}\right) \\ &= \left(1-\frac{zt}{z-1}\right)^{-a}\left[1-\frac{(z-1)\,t}{1+(t-1)\,z}\right]^{x} \\ &= \left(1+\frac{t}{1-z}\right)^{x}\left(1-\frac{zt}{z-1}\right)^{-x-a}.\end{aligned}$$

Finally, if we set $\xi = \theta\xi$, $t=\frac{t}{\xi\theta}$ in (86), and let $\theta \to \infty$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{M_n(x)}{(a)_n}\frac{t^n}{n!} &= \lim_{\theta\to\infty}\left(1-\frac{z}{z-1}\frac{t}{\xi\theta}\right)^{-\theta\xi} {}_2F_1\left(\begin{matrix} \theta\xi,-x \\ a \end{matrix};\frac{(z-1)\,t}{tz+\xi\theta\,(1-z)}\right) \\ &= \exp\left(\frac{zt}{z-1}\right) {}_1F_1\left(\begin{matrix} -x \\ a \end{matrix};-t\right),\end{aligned}$$

where we have used the identity

$$\lim_{\theta\to\infty}\frac{(\xi\theta)_k}{\theta^k}=\xi^k,$$

and Tannery's theorem [42].

5.3 A.2. Hahn polynomials

In this section we consider the monic Hahn polynomials with the special choice of parameters

$$a_1=\alpha+1,\quad a_2=-N,\quad b=-N-1-\beta,\quad N\in\mathbb{N},$$

denoted $Q_n(x)$. From (60) and (61) we see that they satisfy the three-term recurrence relation

$$xQ_n=Q_{n+1}+\beta_n Q_n+\gamma_n Q_{n-1}, \tag{87}$$

with

$$\beta_n=\frac{(2N-\alpha+\beta)\,n^2+(\alpha+\beta+1)\,(2N-\alpha+\beta)\,n+(\alpha+1)\,(\alpha+\beta)\,N}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)},$$

and

$$\gamma_n=\frac{n\,(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)\,(N-n+1)\,(N+n+\alpha+\beta+1)}{(2n+\alpha+\beta)^2\,(2n+\alpha+\beta-1)\,(2n+\alpha+\beta+1)}. \tag{88}$$

From (15) and (87), we obtain the forward difference

$$\begin{aligned}(x+\alpha+1)\,(N-x)\,\Delta Q_n&=-\,(2n+\alpha+\beta+1)\,Q_{n+1}\\+\,(n+\alpha+\beta+1)&\left[x-\frac{(n+\alpha+1)\,N+n\,(n+\beta+1)}{2n+\alpha+\beta+2}\right]Q_n,\end{aligned} \tag{89}$$

and from (16) and (87), we get the backward difference

$$\begin{aligned}x\,(N+\beta+1-x)\,\nabla Q_n&=-\,(2n+\alpha+\beta+1)\,Q_{n+1}\\+\,(n+\alpha+\beta+1)&\left[x-\frac{(n+\alpha+1)\,(N-n)}{2n+\alpha+\beta+2}\right]Q_n.\end{aligned} \tag{90}$$

Combining (89) and (90), we have the hypergeometric difference equation

$$x\left(N+\beta+1-x\right)\Delta\nabla Q_n+\left[\left(\alpha+1\right)N-\left(\alpha+\beta+2\right)x\right]\Delta Q_n \\ +n\left(n+\alpha+\beta+1\right)Q_n=0. \tag{91}$$

Comparing (91) with (62), we see that

$$\sigma\left(x\right)=x\left(N+\beta+1-x\right),\quad \tau\left(x\right)=\left(\alpha+1\right)N-\left(\alpha+\beta+2\right)x, \\ \lambda_n=n\left(n+\alpha+\beta+1\right). \tag{92}$$

Note that from (64) and (92), we have

$$\tau_n\left(x\right)=\left(n+\alpha+1\right)\left(N-n\right)-\left(2n+\alpha+\beta+2\right)x. \tag{93}$$

Introducing the weight function

$$\rho\left(x\right)=\frac{\left(\alpha+1\right)_x\left(-N\right)_x}{x!\left(-N-\beta\right)_x}, \tag{94}$$

we can write (91) in self adjunct form. Note that $\rho\left(x\right)$ satisfies the Pearson equation

$$\Delta\left[x\left(N+\beta+1-x\right)\rho\right]=\left[\left(\alpha+1\right)N-\left(\alpha+\beta+2\right)x\right]\rho,$$

and that

$$\mu_0=\sum_{x=0}^{\infty}\rho\left(x\right)=\sum_{x=0}^{\infty}\frac{\left(\alpha+1\right)_x\left(-N\right)_x}{x!\left(-N-\beta\right)_x}=\frac{\left(\alpha+\beta+2\right)_N}{\left(\beta+1\right)_N}, \tag{95}$$

as long as $\beta\notin\left[-N,-1\right]$. In order to have

$$\rho\left(x\right)>0,\quad x=0,1,\ldots,N,$$

we need $\alpha,\beta<-N$ or $\alpha,\beta>-1$.

Using (94), we get

$$\rho_n\left(x\right)=\rho\left(x+n\right)\prod_{k=1}^{n}\left(x+k\right)\left(N+\beta+1-x-k\right) \\ =\left(-1\right)^n\rho\left(x+n\right)\left(x+1\right)_n\left(x-N-\beta\right)_n. \tag{96}$$

Since we are considering monic polynomials, we set $\kappa_n = 1$ in (71) and obtain

$$C_n = \prod_{k=0}^{n-1} \frac{-1}{k+n+\alpha+\beta+1} = \frac{(-1)^n}{(n+\alpha+\beta+1)_n}. \tag{97}$$

Using (96) and (97) in (70), we have the Rodrigues formula

$$Q_n(x) = \frac{(-1)^n}{(n+\alpha+\beta+1)_n} \frac{1}{\rho(x)} \nabla^n \left[(-1)^n \rho(x+n)(x+1)_n (x-N-\beta)_n\right],$$

which we can rewrite as

$$Q_n(x) = \frac{1}{(n+\alpha+\beta+1)_n \ \rho(x)} \nabla^n \left[\frac{(\alpha+1)_{x+n}(-N)_{x+n}}{(1)_x(-N-\beta)_x}\right].$$

Using (5) and (88), we get

$$h_n = h_0 \prod_{k=1}^{n} \gamma_k = (-1)^n n! \frac{(-N)_n (\alpha+1)_n (\alpha+\beta+2+n)_N}{(\alpha+\beta+2)_n (\alpha+\beta+1+n)_n},$$

since from (95)

$$h_0 = \mu_0 = \frac{(\alpha+\beta+2)_N}{(\beta+1)_N}.$$

Using (73), we have

$$\begin{aligned}
Q_n(x) &= \sum_{k=0}^{n} \frac{(-1)^n (-n)_k}{k!} \frac{\rho(x+k)}{\rho(x)} \frac{(x-n+k+1)_n (x-n+k-N-\beta)_n}{(n+\alpha+\beta+1)_n} \\
&= \frac{(-1)^n (x+1-n)_n (x-n-N-\beta)_n}{(n+\alpha+\beta+1)_n} \\
&\times \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(x-N)_k (x+\alpha+1)_k}{(x-n-N-\beta)_k (x-n+1)_k}.
\end{aligned}$$

Thus, we obtain the hypergeometric representation

$$\begin{aligned}
Q_n(x) = {} & \frac{(-1)^n (x-N-\beta-n)_n (x+1-n)_n}{(n+\alpha+\beta+1)_n} \\
& \times \ {}_3F_2\left(\begin{array}{c} -n, x-N, x+\alpha+1 \\ x-N-\beta-n, x+1-n \end{array}; 1\right).
\end{aligned} \tag{98}$$

The linear transformation

$$
{}_3F_2\left(\begin{matrix} -n, a, b \\ c, d \end{matrix}; 1\right) = (-1)^n \frac{(d-a)_n (d-b)_n}{(c)_n (d)_n} \times {}_3F_2\left(\begin{matrix} -n, 1-d-n, 1+a+b-c-d-n \\ 1+b-d-n, 1+a-d-n \end{matrix}; 1\right) \tag{99}
$$

can be proved symbolically using HolonomicFunctions (or in other ways by hand). Using (99), we can rewrite (98) as

$$
Q_n(x) = \frac{(\alpha+1)_n (-N)_n}{(n+\alpha+\beta+1)_n} \, {}_3F_2\left[\begin{matrix} -n, -x, \ n+\alpha+\beta+1 \\ \alpha+1, \ -N \end{matrix}; 1\right]. \tag{100}
$$

Finally, the polynomials $Q_n(x)$ have the generating functions [35, 18.23.1]

$$
\sum_{n=0}^{N} \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n (\beta+1)_n} Q_n(x) \frac{t^n}{n!} = {}_1F_1\left[\begin{matrix} -x \\ \alpha+1 \end{matrix}; -t\right] {}_1F_1\left[\begin{matrix} x-N \\ \beta+1 \end{matrix}; t\right],
$$

and [35, 18.23.2]

$$
\begin{aligned}
&\sum_{n=0}^{N} (n+\alpha+\beta+1)_n \ Q_n(x) \frac{t^n}{n!} \\
&= {}_2F_0\left[\begin{matrix} -x, -x+\beta+N+1 \\ - \end{matrix}; -t\right] {}_2F_0\left[\begin{matrix} x-N, x+\alpha+1 \\ - \end{matrix}; t\right],
\end{aligned}
$$

valid for $x = 0, 1, \ldots, N$.

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