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Lamé Parameter Estimation from Static Displacement Field Measurements in the Framework of Nonlinear Inverse Problems

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Abstract

We consider a problem of quantitative static elastography, the estimation of the Lamé parameters from internal displacement field data. This problem is formulated as a nonlinear operator equation. To solve this equation, we investigate the Landweber iteration both analytically and numerically. The main result of this paper is the verification of a nonlinearity condition in an *infinite dimensional* Hilbert space context. This condition guarantees convergence of iterative regularization methods. Furthermore, numerical examples for recovery of the Lamé parameters from displacement data simulating a static elastography experiment are presented.

Keywords: Elastography, Inverse Problems, Nonlinearity Condition, Linearized Elasticity, Lamé Parameters, Parameter Identification, Landweber Iteration

AMS: 65J22, 65J15, 74G75

1 Introduction

Elastography is a common technique for medical diagnosis. Elastography can be implemented based on any imaging technique by recording successive images and evaluating the displacement data (see [34, 42–44], which are some early references on elastographic imaging based on ultrasound imaging). We differ between *standard elastography*, which

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consists in displaying the displacement data, and *quantitative elastography*, which consists in reconstructing elastic material parameters. Again we differ between two kinds of inverse problems related to quantitative elastography: The *all in once approach* attempts to estimate the elastic material parameters from direct measurements of the underlying imaging system (typically recorded outside of the object of interest), while the *two-step approach* consists in successive tomographic imaging, displacement computation and quantitative reconstruction of the elastic parameters from *internal data*, which is computed from reconstructions of a tomographic imaging process. The fundamental difference between these approaches can be seen by a dimensionality analysis: Assuming that the material parameter is isotropic, it is a scalar locally varying parameter in three space dimensions. Therefore, three dimensional measurements of the imaging system should be sufficient to reconstruct the material parameter. On the other hand, the displacement data are a three-dimensional vector field, which requires "three times as much information". The second approach is more intuitive, but less data economic, since it builds up on the well-established reconstruction process taking into account the image formation process, and it can be implemented successfully if appropriate prior information can be used, such as smoothness assumptions or significant speckle for accurate tracking. In this paper we follow the second approach.

In this paper we assume that the model of *linearized elasticity*, describing the relation between forces and displacements, is valid. Then, the inverse problem of *quantitative elastography with internal measurements* consists in estimating the spatially varying *Lamé parameters* λ, μ from displacement field measurements u induced by external forces.

There exist a vast amount of mathematical literature on identifiability of the Lamé parameters, stability, and different reconstruction methods. See for example [6, 8–11, 14, 15, 18, 20, 22, 25, 26, 30, 33, 35, 40, 41, 50] and the references therein. Many of the above papers deal with the time-dependent equations of linearized elasticity, since the resulting inverse problem is arguably more stable because it uses more data. However, in many applications, including the ones we have in mind, no dynamic, i.e., time-dependent displacement field data, are available and hence one has to work with the static elasticity equations.

In this paper we consider the inverse problem of identifying the Lamé parameters from *static* displacement field measurements u. We reformulate this problem as a nonlinear operator equation

$$F(\lambda, \mu) = u \,, \tag{1.1}$$

in an *infinite dimensional* Hilbert space setting, which enables us to solve this equation by gradient based algorithms. In particular, we are studying the convergence of the Landweber iteration, which can be considered a gradient descent algorithm (without line search) in an infinite dimensional function space setting, and reads as follows:

$$(\lambda^{(k+1)}, \mu^{(k+1)}) = (\lambda^{(k)}, \mu^{(k)}) - (F'(\lambda^{(k)}, \mu^{(k)}))^*(F(\lambda^{(k)}, \mu^{(k)}) - u^\delta) \,, \tag{1.2}$$

where k is the iteration index of the Landweber iteration. The iteration is terminated when for the first time $\|F(\lambda^{(k)}, \mu^{(k)}) - u^\delta\| < \tau\delta$, where $\tau > 1$ is a constant and δ is an

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estimate for the amount of noise in the data $u^\delta \approx u$. Denoting the termination index by $k_* := k_*(\delta)$, and assuming a nonlinearity condition on F to hold, guarantees that $(\lambda^{(k_*-1)}, \mu^{(k_*-1)})$ approximates the desired solution of (1.1) (that is, it is convergent in the case of noise free data), and for $\delta \to 0$, $(\lambda^{(k_*(\delta)-1)}, \mu^{(k_*(\delta)-1)})$ is continuously depending on δ (that is, the method is stable [28]). The main ingredient in the analysis is a non-standard nonlinearity condition, called the *tangential cone condition*, in an *infinite dimensional* functional space setting, which is verified in Section 3.4. The *tangential cone condition* has been subject to several studies for particular examples of inverse problems (see for instance [28]). In infinite dimensional function space settings it has only been verified for very simple test case, while after discretization it can be considered a consequence of the inverse function theorem. This condition has been verified for instance for the *discretized* electrical impedance tomography problem [32]. The motivation for studying the Landweber iteration in an infinite dimensional setting is that the convergence is discretization independent, and when actually discretized for numerical purposes, no additional discretization artifacts appear. That means that the outcome of the iterative algorithm after stopping by a discrepancy principle is approximating the desired solution of (1.1) and is also stable with respect to data perturbations in an *infinite dimensional setting*. However, stability estimates, such as [31], cannot be derived from this condition alone, but follow if source conditions, like (3.29), are satisfied (see [46]). For *dynamic* measurement data of the displacement field u, related investigation have been performed in [30, 33].

The outline of this paper is as follows: First, we recall the equations of linear elasticity, describing the forward model (Section 2). Then, we calculate the Frèchet derivative and its adjoint (Sections 3.1 and 3.2), which are needed to implement the Landweber iteration. The main result of this paper is the verification of the *(strong) nonlinearity condition* (Section 3.4) from [21] in an *infinite dimensional* setting, which is the basic assumption guaranteeing convergence of iterative regularization methods. Therefore, together with the general convergence rates results from [21] our paper provides the first successful convergence analysis (guaranteeing convergence to a minimum energy solution) of an iterative method for quantitative elastography in a function space setting. Finally, we present some sample reconstructions with iterative regularization methods from numerically simulated displacement field data (Section 3.5).

2   Mathematical Model of Linearized Elasticity

In this section we introduce the basic notation and recall the basic equation of linearized elasticity:

Notation. Ω denotes a non-empty bounded, open and connected set in $\mathbb{R}^N$, $N = 1, 2, 3$, with a Lipschitz continuous boundary $\partial\Omega$, which has two subsets Γ_D and Γ_T, satisfying $\partial\Omega = \overline{\Gamma_D \cup \Gamma_T}$, $\Gamma_D \cap \Gamma_T = \emptyset$ and meas $(\Gamma_D) > 0$.

Definition 2.1. Given body forces f, displacement g_D, surface traction g_T and Lamé parameters λ and μ, the forward problem of linearized elasticity with displacement-

traction boundary conditions consists in finding $\tilde{u}$ satisfying

$$\begin{aligned} -\operatorname{div}\left(\sigma(\tilde{u})\right) &= f\,, \quad \text{in } \Omega\,, \\ \tilde{u}\,|_{\Gamma_D} &= g_D\,, \\ \sigma(\tilde{u})\vec{n}\,|_{\Gamma_T} &= g_T\,, \end{aligned} \tag{2.1}$$

where $\vec{n}$ is an outward unit normal vector of $\partial\Omega$ and the stress tensor σ defining the stress-strain relation in Ω is defined by

$$\sigma(u) := \lambda \,\operatorname{div}(u)\,I + 2\mu\,\mathcal{E}(u)\,, \qquad \mathcal{E}(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right)\,, \tag{2.2}$$

where I is the identity matrix and $\mathcal{E}$ is called the strain tensor.

It is convenient to homogenize problem (2.1) in the following way: Taking a Φ such that $\Phi|_{\Gamma_D} = g_D$, one then seeks $u := \tilde{u} - \Phi$ such that

$$\begin{aligned} -\operatorname{div}\left(\sigma(u)\right) &= f + \operatorname{div}\left(\sigma(\Phi)\right)\,, \quad \text{in } \Omega\,, \\ u\,|_{\Gamma_D} &= 0\,, \\ \sigma(u)\vec{n}\,|_{\Gamma_T} &= g_T - \sigma(\Phi)\vec{n}\,|_{\Gamma_T}\,. \end{aligned} \tag{2.3}$$

Throughout this paper, we make the following

Assumption 2.1. Let $f \in H^{-1}(\Omega)^N$, $g_D \in H^{\frac{1}{2}}(\Gamma_D)^N$, and $g_T \in H^{-\frac{1}{2}}(\Gamma_T)^N$. Furthermore, let $\Phi \in H^1(\Omega)^N$ be such that $\Phi|_{\Gamma_D} = g_D$.

Since we want to consider weak solutions of (2.3), we make the following

Definition 2.2. Let Assumption 2.1 hold. We define the space

$$V := H^1_{0,\Gamma_D}(\Omega)^N\,, \qquad \text{where} \qquad H^1_{0,\Gamma_D}(\Omega) := \{u \in H^1(\Omega)\,|\,u|_{\Gamma_D} = 0\}\,,$$

the linear form

$$l(v) := \langle\, f, v\,\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \langle\, g_T, v\,\rangle_{H^{-\frac{1}{2}}(\Gamma_T),H^{\frac{1}{2}}(\Gamma_T)}\,, \tag{2.4}$$

and the bilinear form

$$a_{\lambda,\mu}(u,v) := \int_\Omega \left(\lambda \,\operatorname{div}(u)\operatorname{div}(v) + 2\mu\,\mathcal{E}(u) : \mathcal{E}(v)\right)\,dx\,, \tag{2.5}$$

where the expression $\mathcal{E}(u) : \mathcal{E}(v)$ denotes the Frobenius product of the matrices $\mathcal{E}(u)$ and $\mathcal{E}(v)$, which also induces the Frobenius norm $\|\mathcal{E}(u)\|_F := \sqrt{\mathcal{E}(u) : \mathcal{E}(u)}$.

Note that both $a_{\lambda,\mu}(u,v)$ and $l(v)$ are also well defined for $u, v \in H^1(\Omega)^N$.

Definition 2.3. A function $u \in V$ satisfying the variational problem

$$a_{\lambda,\mu}(u,v) = l(v) - a_{\lambda,\mu}(\Phi,v)\,, \qquad \forall\, v \in V\,, \tag{2.6}$$

is called a weak solution of the linearized elasticity problem (2.3).

From now on, we only consider weak solutions of (2.3) in the sense of Definition 2.3.

Definition 2.4. The set $\mathcal{M}(\underline{\mu})$ of admissible Lamé parameters is defined by

$$\mathcal{M}(\underline{\mu}) := \left\{ (\lambda, \mu) \in L^\infty(\Omega)^2 \,\middle|\, \exists\, 0 < \varepsilon \leq \frac{\mu\, c_K^2}{N + 2c_K^2} \;:\; \lambda \geq -\varepsilon\,, \, \mu \geq \underline{\mu} - \varepsilon > 0 \right\}.$$

Concerning existence and uniqueness of weak solutions, by standard arguments of elliptic differential equations we get the following

Theorem 2.1. *Let the Assumption 2.1 hold and assume that the Lamé parameters $(\lambda, \mu) \in \mathcal{M}(\underline{\mu})$ for some $\underline{\mu} > 0$. Then there exists a unique weak solution $u \in V$ of (2.3). Moreover, there exists a constant $c_{LM} > 0$ such that*

$$\|u\|_{H^1(\Omega)} \leq c_{LM} \left(\|f\|_{H^{-1}(\Omega)} + c_T \|g_T\|_{H^{-\frac{1}{2}}(\Gamma_T)} + \left(N \|\lambda\|_{L^\infty(\Omega)} + 2 \|\mu\|_{L^\infty(\Omega)} \right) \|\Phi\|_{H^1(\Omega)} \right),$$

where c_T denotes the constant of the trace inequality (5.1).

Proof. This standard result can for example be found in [49]. For the constant c_{LM} one gets $c_{LM} = (1 + c_F^2)/(\underline{\mu}\, c_K^2)$, where c_F and c_K are the constants of Friedrich's inequality (5.2) and Korn's inequality (5.4), respectively. $\qquad\square$

3   The Inverse Problem

After considering the forward problem of linearized elasticity, we now turn to the inverse problem, which is to estimate the Lamé parameters λ, μ by measurements of the displacement field u. More precisely, we are facing the following inverse problem of quantitative elastography:

Problem. *Let Assumption 2.1 hold and let $u^\delta \in L^2(\Omega)^N$ be a measurement of the true displacement field u satisfying*

$$\left\| u - u^\delta \right\|_{L^2(\Omega)} \leq \delta\,, \tag{3.1}$$

where $\delta \geq 0$ is the noise level. Given the model of linearized elasticity (2.1) in the weak form (2.6), the problem is to find the Lamé parameters λ, μ.

The problem of linearized elastography can be formulated as the solution of the operator equation (1.1) with the operator

$$
\begin{aligned}
F : \mathcal{D}(F) := \left\{ (\lambda, \mu) \in L^\infty(\Omega)^2 \,\middle|\, \lambda \geq 0\,, \, \mu \geq \underline{\mu} > 0 \right\} &\to L^2(\Omega)^N\,, \\
(\lambda, \mu) &\mapsto u(\lambda, \mu)\,,
\end{aligned}
\tag{3.2}
$$

where $u(\lambda, \mu)$ is the solution of (2.6) and hence, we can apply all results from classical inverse problems theory [16], given that the necessary requirements on F hold. For showing them, it is necessary to write F in a different way: We define the space

$$V^* := \left(H^1_{0,\Gamma_D}(\Omega)^N \right)^*\,, \tag{3.3}$$

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which is the dual space of $V = H^1_{0,\Gamma_D}(\Omega)^N$. Next, we introduce the operator $\tilde{A}_{\lambda,\mu}$ connected to the bilinear form $a_{\lambda,\mu}$, defined by

$$
\begin{aligned}
\tilde{A}_{\lambda,\mu} : H^1(\Omega)^N &\to V^* , \\
\tilde{v} &\mapsto (v \mapsto a_{\lambda,\mu}(\tilde{v}, v)) ,
\end{aligned}
\tag{3.4}
$$

and its restriction to V, i.e., $A := \tilde{A}|_V$, namely

$$
\begin{aligned}
A_{\lambda,\mu} : V &\to V^* , \\
v &\mapsto (\bar{v} \mapsto a_{\lambda,\mu}(v, \bar{v})) .
\end{aligned}
\tag{3.5}
$$

Furthermore, for $v \in V$ and $v^* \in V^*$, we define the canonical dual

$$
\langle v^*, v \rangle_{V^*,V} = \langle v, v^* \rangle_{V,V^*} := v^*(v) .
$$

Next, we collect some important properties of $\tilde{A}_{\lambda,\mu}$ and $A_{\lambda,\mu}$. For ease of notation,

$$
\left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty := N \left\| \bar{\lambda} - \lambda \right\|_{L^\infty(\Omega)} + 2 \left\| \bar{\mu} - \mu \right\|_{L^\infty(\Omega)} .
\tag{3.6}
$$

Proposition 3.1. *The operators $\tilde{A}_{\lambda,\mu}$ and $A_{\lambda,\mu}$ defined by (3.4) and (3.5), respectively, are bounded and linear for all $\lambda, \mu \in L^\infty(\Omega)$. In particular, for all $\lambda, \mu, \bar{\lambda}, \bar{\mu} \in L^\infty(\Omega)$*

$$
\left\| A_{\bar{\lambda},\bar{\mu}} - A_{\lambda,\mu} \right\|_{V,V^*} \leq \left\| \tilde{A}_{\bar{\lambda},\bar{\mu}} - \tilde{A}_{\lambda,\mu} \right\|_{H^1(\Omega),V^*} \leq \left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty .
\tag{3.7}
$$

Furthermore, for all $(\lambda, \mu) \in \mathcal{M}(\underline{\mu})$ with $\underline{\mu} > 0$, the operator $A_{\lambda,\mu}$ is bijective and has a continuous inverse $A_{\lambda,\mu}^{-1} : V^* \to V$ satisfying $\left\| A_{\lambda,\mu}^{-1} \right\|_{V^*,V} \leq c_{LM}$, where c_{LM} is the constant of Theorem 2.1. In particular, for all $v^*, \bar{v}^* \in V^*$ and $(\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in \mathcal{M}(\underline{\mu})$

$$
\left\| A_{\bar{\lambda},\bar{\mu}}^{-1} \bar{v}^* - A_{\lambda,\mu}^{-1} v^* \right\|_V \leq c_{LM} \left(\left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty \left\| A_{\lambda,\mu}^{-1} v^* \right\|_V + \left\| \bar{v}^* - v^* \right\|_{V^*} \right) .
\tag{3.8}
$$

Proof. The boundedness and linearity of $A_{\lambda,\mu}$ and $\tilde{A}_{\lambda,\mu}$ for all $\lambda, \mu \in L^\infty(\Omega)$ are immediate consequences of the boundedness and bilinearity of $a_{\lambda,\mu}$ and we have

$$
\begin{aligned}
\left\| \tilde{A}_{\lambda,\mu} - \tilde{A}_{\bar{\lambda},\bar{\mu}} \right\|_{H^1(\Omega),V^*} &= \left\| \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \right\|_{H^1(\Omega),V^*} = \sup_{u \in H^1(\Omega), u \neq 0} \frac{\left\| \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} u \right\|_{V^*}}{\left\| u \right\|_{H^1(\Omega)}} \\
&= \sup_{u \in H^1(\Omega), u \neq 0} \frac{\sup_{v \in V, v \neq 0} \left| a_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}(u, v) \right|}{\left\| u \right\|_{H^1(\Omega)} \left\| v \right\|_V} \leq \left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty ,
\end{aligned}
$$

which also translates to $A_{\lambda,\mu}$, since $V \subset H^1(\Omega)^N$. Moreover, due to the Lax-Milgram Lemma and Theorem 2.1, $A_{\lambda,\mu}$ is bijective for $(\lambda, \mu) \in \mathcal{M}(\underline{\mu})$ with $\underline{\mu} > 0$ and therefore, by the Open Mapping Theorem, $A_{\lambda,\mu}^{-1}$ exists and is linear and continuous. Again by the Lax-Milgram Lemma, there follows $\left\| A_{\lambda,\mu}^{-1} \right\|_{V^*,V} \leq c_{LM}$.

Let $v^*, \bar{v}^* \in V^*$ and $(\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in \mathcal{M}(\underline{\mu})$ with $\underline{\mu} > 0$ be arbitrary but fixed and consider $u := A_{\lambda,\mu}^{-1} v^*$ and $\bar{u} := A_{\bar{\lambda},\bar{\mu}}^{-1} \bar{v}^*$. Subtracting those two equations, we get

$$A_{\lambda,\mu} u - A_{\bar{\lambda},\bar{\mu}} \bar{u} = v^* - \bar{v}^* ,$$

which, by the definition of $A_{\lambda,\mu}$ and $a_{\lambda,\mu}$, can be written as

$$A_{\bar{\lambda},\bar{\mu}} (u - \bar{u}) = A_{\bar{\lambda}-\lambda, \bar{\mu}-\mu} u + v^* - \bar{v}^* .$$

and is equivalent to the variational problem

$$a_{\bar{\lambda},\bar{\mu}} ((u - \bar{u}), v) = a_{\bar{\lambda}-\lambda, \bar{\mu}-\mu}(u, v) + \langle v^* - \bar{v}^*, v \rangle_{V^*, V} , \qquad \forall v \in V . \qquad (3.9)$$

Now since $a_{\lambda,\mu}$ is bounded, the right hand side of (3.9) is bounded by

$$\left(\left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty \|u\|_{H^1(\Omega)} + \|v^* - \bar{v}^*\|_{V^*} \right) \|v\|_V .$$

Hence, due to the Lax-Milgram Lemma the solution of (3.9) is unique and depends continuously on the right hand side, which immediately yields the assertion. $\qquad\square$

Using $A_{\lambda,\mu}$ and $\tilde{A}_{\lambda,\mu}$, the operator F can be written in the alternative form

$$F(\lambda, \mu) = A_{\lambda,\mu}^{-1} \left(l - \tilde{A}_{\lambda,\mu} \Phi \right) , \qquad (3.10)$$

with l defined by (2.4). Now since, due to (3.7),

$$\left\| \left(l - \tilde{A}_{\lambda,\mu} \Phi \right) - \left(l - \tilde{A}_{\bar{\lambda},\bar{\mu}} \Phi \right) \right\|_{V^*} = \left\| \tilde{A}_{\bar{\lambda}-\lambda, \bar{\mu}-\mu} \Phi \right\|_{V^*} \le \left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty \|\Phi\|_{H^1(\Omega)} ,$$

inequality (3.8) implies

$$\left\| F(\bar{\lambda}, \bar{\mu}) - F(\lambda, \mu) \right\|_V \le c_{LM} \left\| (\bar{\lambda}, \bar{\mu}) - (\lambda, \mu) \right\|_\infty \left(\|F(\lambda, \mu)\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega)} \right) , \quad (3.11)$$

showing that F is a continuous operator.

Remark. Note that F can also be considered as an operator from $\mathcal{M}(\underline{\mu})$ to $L^2(\Omega)^N$, in which case Theorem 2.1 and Proposition 3.1 guarantee that it remains well-defined and continuous, which we use later on.

3.1 Calculation of the Fréchet Derivative

In this section, we compute the Fréchet derivative $F'(\lambda, \mu)(h_\lambda, h_\mu)$ of F using the representation (3.10).

Theorem 3.2. *The operator F defined by (3.10) and considered as an operator from $\mathcal{M}(\underline{\mu}) \to L^2(\Omega)^N$ for some $\underline{\mu} > 0$ is Fréchet differentiable for all $(\lambda, \mu) \in \mathcal{D}(F)$ with*

$$F'(\lambda, \mu)(h_\lambda, h_\mu) = -A_{\lambda,\mu}^{-1} \left(A_{h_\lambda, h_\mu} u(\lambda, \mu) + \tilde{A}_{h_\lambda, h_\mu} \Phi \right) . \qquad (3.12)$$

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Proof. We start by defining

$$G_{\lambda,\mu}(h_\lambda, h_\mu) := -A_{\lambda,\mu}^{-1}\left(A_{h_\lambda,h_\mu}u(\lambda,\mu) + \tilde{A}_{h_\lambda,h_\mu}\Phi\right).$$

Due to Proposition 3.1, $G_{\lambda,\mu}$ is a well-defined, bounded linear operator which depends continuously on $(\lambda,\mu) \in D(F)$ with respect to the operator-norm. Hence, if we can prove that $G_{\lambda,\mu}$ is the Gateâux derivative of F it is also the Fréchet derivative of F. For this, we look at

$$
\begin{aligned}
&\frac{F(\lambda + th_\lambda, \mu + th_\mu) - F(\lambda,\mu)}{t} - G_{\lambda,\mu}(h_\lambda, h_\mu) \\
&= \frac{1}{t}\left(A_{\lambda+th_\lambda,\mu+th_\mu}^{-1}(l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi) - A_{\lambda,\mu}^{-1}(l - \tilde{A}_{\lambda,\mu}\Phi)\right) \\
&\quad + A_{\lambda,\mu}^{-1}\left(A_{h_\lambda,h_\mu}u(\lambda,\mu) + \tilde{A}_{h_\lambda,h_\mu}\Phi\right).
\end{aligned}
\tag{3.13}
$$

Note that it can happen that $(\lambda + th_\lambda, \mu + th_\mu) \notin \mathcal{D}(F)$. However, choosing t small enough, one can always guarantee that $(\lambda + th_\lambda, \mu + th_\mu) \in \mathcal{M}(\underline{\mu})$, in which case $F(\lambda + th_\lambda, \mu + th_\mu)$ remains well-defined as noted above. Applying $A_{\lambda,\mu}$ to (3.13) we get

$$
\begin{aligned}
&A_{\lambda,\mu}\left(\frac{F(\lambda + th_\lambda, \mu + th_\mu) - F(\lambda,\mu)}{t} - G_{\lambda,\mu}(h_\lambda, h_\mu)\right) \\
&= \frac{1}{t}\left(A_{\lambda,\mu}A_{\lambda+th_\lambda,\mu+th_\mu}^{-1}(l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi) - (l - \tilde{A}_{\lambda,\mu}\Phi)\right) + \left(A_{h_\lambda,h_\mu}u(\lambda,\mu) + \tilde{A}_{h_\lambda,h_\mu}\Phi\right),
\end{aligned}
$$

which, together with

$$
\begin{aligned}
&A_{\lambda,\mu}A_{\lambda+th_\lambda,\mu+th_\mu}^{-1}(l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi) \\
&= (l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi) - tA_{h_\lambda,h_\mu}A_{\lambda+th_\lambda,\mu+th_\mu}^{-1}(l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi),
\end{aligned}
$$

yields

$$
\begin{aligned}
&A_{\lambda,\mu}\left(\frac{F(\lambda + th_\lambda, \mu + th_\mu) - F(\lambda,\mu)}{t} - G_{\lambda,\mu}(h_\lambda, h_\mu)\right) \\
&= -A_{h_\lambda,h_\mu}A_{\lambda+th_\lambda,\mu+th_\mu}^{-1}(l - \tilde{A}_{\lambda+th_\lambda,\mu+th_\mu}\Phi) + A_{h_\lambda,h_\mu}u(\lambda,\mu) \\
&= -A_{h_\lambda,h_\mu}\left(u(\lambda + th_\lambda, \mu + th_\mu) - u(\lambda,\mu)\right).
\end{aligned}
\tag{3.14}
$$

By the continuity of $A_{\lambda,\mu}$ and $A_{\lambda,\mu}^{-1}$ and due to (3.11) we can deduce that $G_{\lambda,\mu}$ is indeed the Gateâux derivative and, due to the continuous dependence on (λ,μ), also the Fréchet derivative of F, which concludes the proof. $\qquad\square$

Concerning the calculation of $F'(\lambda,\mu)(h_\lambda, h_\mu)$, note that it can be carried out in two distinct steps, requiring the solution of two variational problems involving the same bilinear form $a_{\lambda,\mu}$ (which can be used for efficient implementation) as follows:

1. Calculate $u \in V$ as the solution of the variational problem (2.6).

2. Calculate $F'(\lambda, \mu)(h_\lambda, h_\mu) \in V$ as the solution $\hat{u}$ of the variational problem

$$a_{\lambda,\mu}(\hat{u}, v) = -a_{h_\lambda, h_\mu}(u, v) - a_{h_\lambda, h_\mu}(\Phi, v), \qquad \forall v \in V.$$

Remark. Note that for classical results on iterative regularization methods (see [28]) to be applicable, one needs that both the definition space and the image space are Hilbert spaces. However, the operator F given by (3.2) is defined on $L^\infty(\Omega)^2$. Therefore, one could think of applying Banach space regularization theory to the problem (see for example [29, 47, 48]). Unfortunately, a commonly used assumption is that the involved Banach spaces are reflexive, which excludes $L^\infty(\Omega)^2$. Hence, a commonly used approach is to consider a space which embeds compactly into $L^\infty(\Omega)^2$, for example the Banach space $W^{1,p}(\Omega)^2$ or the Hilbert space $H^s(\Omega)^2$ with p and s large enough, respectively. Although it is preferable to assume as little smoothness as possible for the Lamé parameters, we focus on the $H^s(\Omega)^2$ setting in this paper, since the resulting inverse problem is already difficult enough to treat analytically.

Due to Sobolev's embedding theorem [1], the Sobolev space $H^s(\Omega)$ embeds compactly into $L^\infty(\Omega)$ for $s > N/2$, i.e., there exists a constant $c_E^s > 0$ such that

$$\|v\|_{L^\infty(\Omega)} \leq c_E^s \|v\|_{H^s(\Omega)}, \qquad \forall v \in H^s(\Omega). \tag{3.15}$$

This suggests to consider F as an operator from

$$\mathcal{D}_s(F) := \{(\lambda, \mu) \in H^s(\Omega)^2 \,|\, \lambda \geq 0, \mu \geq \underline{\mu} > 0\} \to L^2(\Omega)^N, \tag{3.16}$$

for some $s > N/2$. Since due to (3.15) there holds $\mathcal{D}_s(F) \subset \mathcal{D}(F)$, our previous results on continuity and Fréchet differentiability still hold in this case. Furthermore, it is now possible to consider the resulting inverse problem $F(\lambda, \mu) = u$ in the classical Hilbert space framework. Hence, in what follows, we always consider F as an operator from $\mathcal{D}_s(F) \to L^2(\Omega)^2$ for some $s > N/2$.

3.2 Calculation of the Adjoint of the Fréchet Derivative

We now turn to the calculation of $F'(\lambda, \mu)^* w$, the adjoint of the Fréchet derivative $F'(\lambda, \mu)$, which is required below for the implementation gradient descent methods. For doing so, note first that for $A_{\lambda,\mu}$ defined by (3.5)

$$\langle A_{\lambda,\mu} v, \bar{v} \rangle_{V^*, V} = \langle A_{\lambda,\mu} \bar{v}, v \rangle_{V^*, V}, \qquad \forall v, \bar{v} \in V. \tag{3.17}$$

This follows immediately from the definition of $A_{\lambda,\mu}$ and the symmetry of the bilinear form $a_{\lambda,\mu}$. Moreover, as an immediate consequence of (3.17), and continuity of $A_{\lambda,\mu}^{-1}$ it follows

$$\langle v^*, A_{\lambda,\mu}^{-1} \bar{v}^* \rangle_{V^*, V} = \langle \bar{v}^*, A_{\lambda,\mu}^{-1} v^* \rangle_{V^*, V}, \qquad \forall v^*, \bar{v}^* \in V^*. \tag{3.18}$$

In order to give an explicit form of $F'(\lambda, \mu)^* w$ we need the following

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Lemma 3.3. *The linear operators* $T : L^2(\Omega)^N \to V^*$, *defined by*

$$Tw := \left(v \mapsto \int_\Omega w \cdot v \, dx \right),$$ (3.19)

and $E_s : L^1(\Omega) \to H^s(\Omega)$,

$$\langle E_s u, v \rangle_{H^s(\Omega)} = \int_\Omega uv \, dx, \qquad \forall v \in H^s(\Omega),$$ (3.20)

respectively, are well-defined and bounded for all $s > N/2$.

Proof. Using the Cauchy-Schwarz inequality it is easy to see that T is bounded with $\|T\|_{L^2(\Omega),V^*} \leq 1$. Furthermore, due to (3.15),

$$\int_\Omega uv \, dx \leq \|u\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} \leq c_E^s \|u\|_{L^1(\Omega)} \|v\|_{H^s(\Omega)}, \qquad \forall v \in H^s(\Omega).$$

Hence, it follows from the Lax-Milgram Lemma that E_s is bounded for $s > N/2$. □

Using this, we can now proof the main result of this section.

Theorem 3.4. *Let* $F : \mathcal{D}_s(F) \to L^2(\Omega)^2$ *with* $\mathcal{D}_s(F)$ *given as in* (3.16) *for some* $s > N/2$. *Then the adjoint of the Fréchet derivative of* F *is given by*

$$F'(\lambda, \mu)^* w = \begin{pmatrix} E_s \left(\operatorname{div} \left(u(\lambda, \mu) + \Phi \right) \operatorname{div} \left(-A_{\lambda,\mu}^{-1} Tw \right) \right) \\ E_s \left(2\,\mathcal{E} \left(u(\lambda, \mu) + \Phi \right) : \mathcal{E} \left(-A_{\lambda,\mu}^{-1} Tw \right) \right) \end{pmatrix}^T,$$ (3.21)

where T *and* E_s *are defined by* (3.19) *and* (3.20), *respectively.*

Proof. Using Theorem 3.2 and (3.19) we get

$$\langle F'(\lambda, \mu)(h_\lambda, h_\mu), w \rangle_{L^2(\Omega)} = \left\langle -A_{\lambda,\mu}^{-1}(A_{h_\lambda, h_\mu} u(\lambda, \mu) + \tilde{A}_{h_\lambda, h_\mu} \Phi), w \right\rangle_{L^2(\Omega)}$$

$$= \left\langle Tw, -A_{\lambda,\mu}^{-1}(A_{h_\lambda, h_\mu} u(\lambda, \mu) + \tilde{A}_{h_\lambda, h_\mu})\Phi) \right\rangle_{V^*, V}$$

Together with (3.18) and the definition of A_{h_λ, h_μ} and a_{h_λ, h_μ} we get

$$\left\langle Tw, -A_{\lambda,\mu}^{-1}(A_{h_\lambda, h_\mu} u(\lambda, \mu) + \tilde{A}_{h_\lambda, h_\mu})\Phi) \right\rangle_{V^*, V} = a_{h_\lambda, h_\mu} \left(u(\lambda, \mu) + \Phi, -A_{\lambda,\mu}^{-1} Tw \right)$$

$$= \int_\Omega h_\lambda \operatorname{div} \left(u(\lambda, \mu) + \Phi \right) \operatorname{div} \left(-A_{\lambda,\mu}^{-1} Tw \right) \, dx + \int_\Omega 2h_\mu \,\mathcal{E} \left(u(\lambda, \mu) + \Phi \right) : \mathcal{E} \left(-A_{\lambda,\mu}^{-1} Tw \right) \, dx.$$

Together with the fact that the product of two $L^2(\Omega)$ functions is in $L^1(\Omega)$, which applies to $\operatorname{div} \left(u(\lambda, \mu) + \Phi \right) \operatorname{div} \left(-A_{\lambda,\mu}^{-1} Tw \right)$ and $\mathcal{E} \left(u(\lambda, \mu) + \Phi \right) : \mathcal{E} \left(-A_{\lambda,\mu}^{-1} Tw \right)$, the statement of the theorem now immediately follows from the definition of E_s (3.20). □

Concerning the calculation of $F'(\lambda,\mu)^*w$, note that it can again be carried out in independent steps, namely:

1. Calculate $u \in V$ as the solution of the variational problem (2.6).

2. Compute $A_{\lambda,\mu}^{-1}Tw$, i.e., find the solution $u(w) \in V$ of the variational problem

$$a_{\lambda,\mu}(u(w),v) = \int_\Omega w \cdot v \, dx \,, \qquad \forall v \in V \,.$$

3. Compute the functions $u_1(w), u_2(w) \in L^1(\Omega)$ given by

$$u_1(w) := \operatorname{div}(u+\Phi)\operatorname{div}(-u(w)) \,,$$
$$u_2(w) := 2\,\mathcal{E}(u+\Phi) : \mathcal{E}(-u(w)) \,.$$

4. Calculate the functions $\hat{\lambda}(w) := E_s\, u_1(w)$ and $\hat{\mu}(w) := E_s\, u_2(w)$ as the solutions of the variational problems

$$\left\langle \hat{\lambda}(w), v \right\rangle_{H^s(\Omega)} = \int_\Omega u_1(w)\,v\,dx\,, \qquad \forall v \in H^s(\Omega)\,,$$

$$\left\langle \hat{\mu}(w), v \right\rangle_{H^s(\Omega)} = \int_\Omega u_2(w)\,v\,dx\,, \qquad \forall v \in H^s(\Omega)\,.$$

5. Combine the results to obtain $F'(\lambda,\mu)^*w = (\hat{\lambda}(w),\hat{\mu}(w))$.

3.3 Reconstruction of compactly supported Lamé parameters

In many cases, the Lamé parameters λ,μ are known in a small neighbourhood of the boundary, for instance when contact materials are used, such as a gel in ultrasound imaging. As a physical problem, we have in mind a test sample consisting of a known material with various inclusions of unknown location and Lamé parameters inside. The resulting inverse problem is better behaved than the original problem and we are even able to prove a nonlinearity condition guaranteeing convergence of iterative solution methods for nonlinear ill-posed problems in this case.

More precisely, assume that we are given a bounded, open, connected Lipschitz domain $\Omega_1 \subset \Omega$ with $\bar{\Omega}_1 \Subset \Omega$ and background functions $0 \le \lambda_b \in H^s(\Omega)$ and $\bar{\mu} \le \mu_b \in H^s(\Omega)$ and assume that the searched for Lamé parameters can be written in the form $(\lambda_b + \lambda, \mu_b + \mu)$, where both $\lambda,\mu \in H^s(\Omega)$ are compactly supported in Ω_1. Hence, after introducing the set

$$\mathcal{D}_s(F_c) := \left\{ (\lambda,\mu) \in H^s(\Omega)^2 \,|\, \lambda \ge -\lambda_b\,,\ \mu \ge \underline{\mu} - \mu_b > 0\,,\ \mathrm{supp}((\lambda,\mu)) \subset \Omega_1 \right\}\,,$$

we define the operator

$$F_c : \mathcal{D}_s(F_c) \to L^2(\Omega)^N\,, \quad (\lambda,\mu) \mapsto F_c(\lambda,\mu) := F(\lambda_b + \lambda, \mu_b + \mu)\,, \tag{3.22}$$

which is well-defined for $s > N/2$. Hence, the sought for Lamé parameters can be reconstructed by solving the problem $F_c(\lambda, \mu) = u$ and taking $(\lambda_b + \lambda, \mu_b + \mu)$.

Continuity and Fréchet differentiability of F also transfer to F_c. For example,

$$F_c'(\lambda, \mu)(h_\lambda, h_\mu) = -A_{(\lambda_b+\lambda,\mu_b+\mu)}^{-1} \left(A_{h_\lambda,h_\mu} u(\lambda, \mu) + \tilde{A}_{h_\lambda,h_\mu} \Phi \right). \qquad (3.23)$$

Furthermore, a similar expression as for the adjoint of the Fréchet derivative of F also holds for F_c. Consequently, the computation and implementation of F_c, its derivative and the adjoint can be carried out in the same way as for the operator F and hence, the two require roughly the same amount of computational work. However, as we see in the next section, for the operator F_c it is possible to prove a nonlinearity condition.

3.4   Strong Nonlinearity Condition

The so-called *(strong) tangential cone condition* or *(strong) nonlinearity condition* is the basis of the convergence analysis of iterative regularization methods for nonlinear ill-posed problems [28]. The nonlinearity condition is a non-standard condition in the field of differential equations, because it requires a stability estimate in the image domain of the operator F. In the theorem below we show a version of this nonlinearity condition, which is sufficient to prove convergence of iterative algorithms for solving (1.1).

Theorem 3.5. *Let* $F : \mathcal{D}_s(F) \to L^2(\Omega)^2$ *for some* $s > N/2 + 1$ *and let* $\Omega_1 \subset \Omega$ *be a bounded, open, connected Lipschitz domain with* $\bar{\Omega}_1 \Subset \Omega$. *Then for each* $(\lambda, \mu) \in \mathcal{D}_s(F)$ *there exists a constant* $c_{NL} = c_{NL}(\lambda, \mu, \Omega_1, \Omega) > 0$ *such that for all* $(\bar{\lambda}, \bar{\mu}) \in \mathcal{D}_s(F)$ *satisfying* $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$ *on* $\Omega \setminus \Omega_1$ *and* $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$ *on* $\partial\Omega_1$ *there holds*

$$\left\| F(\lambda, \mu) - F(\bar{\lambda}, \bar{\mu}) - F'(\lambda, \mu)((\lambda, \mu) - (\bar{\lambda}, \bar{\mu})) \right\|_{L^2(\Omega)}$$
$$\leq c_{NL} \left\| (\bar{\lambda} - \lambda, \bar{\mu} - \mu) \right\|_{W^{1,\infty}(\Omega_1)} \left\| F(\lambda, \mu) - F(\bar{\lambda}, \bar{\mu}) \right\|_{L^2(\Omega)}. \qquad (3.24)$$

Proof. Let $(\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in \mathcal{D}_s(F)$ with $s > N/2 + 1$ such that $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$ on $\Omega \setminus \Omega_1$ and $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$ on $\partial\Omega_1$. For the purpose of this proof, set $u = F(\lambda, \mu)$ and $\bar{u} = F(\bar{\lambda}, \bar{\mu})$. By definition, we have

$$\Big\langle F(\lambda, \mu) - F(\bar{\lambda}, \bar{\mu}) - F'(\lambda, \mu)((\lambda, \mu) - (\bar{\lambda}, \bar{\mu})), w \Big\rangle_{L^2(\Omega)}$$
$$= \Big\langle (u - \bar{u}) - A_{\lambda,\mu}^{-1} \left(A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} u + \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \Phi \right), w \Big\rangle_{L^2(\Omega)}.$$

Together with (3.19) and (3.18), we get

$$\Big\langle (u - \bar{u}) - A_{\lambda,\mu}^{-1} \left(A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} u + \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \Phi \right), w \Big\rangle_{L^2(\Omega)}$$
$$= \Big\langle A_{\lambda,\mu}(u - \bar{u}) - \left(A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} u + \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \Phi \right), A_{\lambda,\mu}^{-1} T w \Big\rangle_{V^*,V},$$

which can be written as

$$\Big\langle A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} (\bar{u} - u), A_{\lambda,\mu}^{-1} T w \Big\rangle_{V^*,V} + \Big\langle A_{\lambda,\mu}(u - \bar{u}) - A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \bar{u} - \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu} \Phi, A_{\lambda,\mu}^{-1} T w \Big\rangle_{V^*,V}.$$

Now since

$$A_{\lambda,\mu}(u - \bar{u}) - A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}\bar{u} - \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}\Phi$$
$$= l - \tilde{A}_{\lambda,\mu}\Phi - A_{\lambda,\mu}\bar{u} - A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}\bar{u} - \tilde{A}_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}\Phi = 0\,,$$

it follows together with (3.17) that

$$\big\langle\, F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu})),\, w \,\big\rangle_{L^2(\Omega)}$$
$$= \big\langle\, A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}\,(\bar{u} - u)\,,\, A_{\lambda,\mu}^{-1}Tw \,\big\rangle_{V^*,V} = \big\langle\, A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}A_{\lambda,\mu}^{-1}Tw,\, \bar{u} - u \,\big\rangle_{V^*,V}\,.$$

Introducing the abbreviation $z := A_{\lambda,\mu}^{-1}Tw$, and using the definition of $A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}$

$$\big\langle\, A_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}z,\, \bar{u} - u \,\big\rangle_{V^*,V} = a_{\bar{\lambda}-\lambda,\bar{\mu}-\mu}(z, \bar{u} - u)$$
$$= \int_{\Omega_1} \big((\bar{\lambda} - \lambda)\,\mathrm{div}\,(z)\,\mathrm{div}\,(\bar{u} - u) + 2(\bar{\mu} - \mu)\,\mathcal{E}\,(z) : \mathcal{E}\,(\bar{u} - u)\big)\,dx\,,$$

where we have used that $(\bar{\lambda} - \lambda, \bar{\mu} - \mu) = 0$ on $\Omega\backslash\Omega_1$. Since we also have $(\bar{\lambda} - \lambda, \bar{\mu} - \mu) = 0$ on $\partial\Omega_1$, partial integration together with the regularity result Lemma 5.1 yields

$$\int_{\Omega_1} \big((\bar{\lambda} - \lambda)\,\mathrm{div}\,(z)\,\mathrm{div}\,(\bar{u} - u) + 2(\bar{\mu} - \mu)\,\mathcal{E}\,(z) : \mathcal{E}\,(\bar{u} - u)\big)\,dx$$
$$= -\int_{\Omega_1} \mathrm{div}\,\big((\bar{\lambda} - \lambda)\,\mathrm{div}\,(z)\,I + 2(\bar{\mu} - \mu)\,\mathcal{E}\,(z)\big)\cdot(\bar{u} - u)\,dx \qquad (3.25)$$
$$\leq \big\|\mathrm{div}\,\big((\bar{\lambda} - \lambda)\,\mathrm{div}\,(z)\,I + 2(\bar{\mu} - \mu)\,\mathcal{E}\,(z)\big)\big\|_{L^2(\Omega_1)}\,\|\bar{u} - u\|_{L^2(\Omega_1)}\,.$$

Now, since there exists a constant $c_G = c_G(N)$ such that for all $v \in H^2(\Omega_1)^N$

$$\|\mathrm{div}\,(\lambda\,\mathrm{div}\,(v)\,I + 2\mu\,\mathcal{E}\,(v))\|_{L^2(\Omega_1)} \leq c_G \max\{\|\lambda\|_{W^{1,\infty}(\Omega_1)}, \|\mu\|_{W^{1,\infty}(\Omega_1)}\}\,\|v\|_{H^2(\Omega_1)}\,.$$

Now since

$$\big\|F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu}))\big\|_{L^2(\Omega)}$$
$$= \sup_{\|w\|_{L^2(\Omega)}=1} \big\langle\, F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu})),\, w \,\big\rangle_{L^2(\Omega)}\,,$$

combining the above results we get

$$\sup_{\|w\|_{L^2(\Omega)}=1} \big\langle\, F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu})),\, w \,\big\rangle_{L^2(\Omega)}$$
$$\leq \sup_{\|w\|_{L^2(\Omega)}=1} c_G \big\|(\bar{\lambda} - \lambda, \bar{\mu} - \mu)\big\|_{W^{1,\infty}(\Omega_1)}\,\|z\|_{H^2(\Omega_1)}\,\|\bar{u} - u\|_{L^2(\Omega_1)}\,.$$

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Together with Lemma 5.1, which implies that there exists a constant $c_R > 0$ such that $\|z\|_{H^2(\Omega_1)} \le c_R \|w\|_{L^2(\Omega_1)}$, we get

$$\left\|F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu}))\right\|_{L^2(\Omega)}$$
$$\le c_G\, c_R \left\|(\bar{\lambda} - \lambda, \bar{\mu} - \mu)\right\|_{W^{1,\infty}(\Omega_1)} \|\bar{u} - u\|_{L^2(\Omega_1)}$$
$$\le c_G\, c_R \left\|(\bar{\lambda} - \lambda, \bar{\mu} - \mu)\right\|_{W^{1,\infty}(\Omega_1)} \|\bar{u} - u\|_{L^2(\Omega)} \, ,$$

which immediately yields the assertion with $c_{NL} := c_G\, c_R$. $\qquad \square$

We get the following useful corollary

Corollary 3.6. *Let F_c be defined as in (3.22) for some $s > N/2 + 1$. Then for each $(\lambda, \mu) \in \mathcal{D}_s(F_c)$ there exists a constant $c_{NL} = c_{NL}(\lambda, \mu, \Omega_1, \Omega) > 0$ such that for all $(\bar{\lambda}, \bar{\mu}) \in \mathcal{D}_s(F_c)$ there holds*

$$\left\|F_c(\lambda,\mu) - F_c(\bar{\lambda},\bar{\mu}) - F_c'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu}))\right\|_{L^2(\Omega)}$$
$$\le c_{NL} \left\|(\bar{\lambda} - \lambda, \bar{\mu} - \mu)\right\|_{W^{1,\infty}(\Omega_1)} \left\|F_c(\lambda,\mu) - F_c(\bar{\lambda},\bar{\mu})\right\|_{L^2(\Omega)} \, . \tag{3.26}$$

Proof. This follows from the definition of F_c and (the proof of) Theorem 3.5. $\qquad \square$

In the following theorem, we establish a similar result as in Corollary 3.6 now for $F : \mathcal{D}_s(F) \to L^2(\Omega)^2$ in case that $\Gamma_T = \emptyset$, i.e., $\Gamma_D = \partial\Omega$ and that $\partial\Omega$ is smooth enough.

Theorem 3.7. *Let $F : \mathcal{D}_s(F) \to L^2(\Omega)^2$ for some $s > N/2 + 1$ and let $\partial\Omega = \Gamma_D \in C^{1,1}$ and $\Gamma_T = \emptyset$. Then for each $(\lambda, \mu) \in \mathcal{D}_s(F)$ there exists a constant $c_{NL} = c_{NL}(\lambda, \mu, \Omega) > 0$ such that for all $(\bar{\lambda}, \bar{\mu}) \in \mathcal{D}_s(F)$, there holds*

$$\left\|F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu}) - F'(\lambda,\mu)((\lambda,\mu) - (\bar{\lambda},\bar{\mu}))\right\|_{L^2(\Omega)}$$
$$\le c_{NL} \left\|(\bar{\lambda} - \lambda, \bar{\mu} - \mu)\right\|_{W^{1,\infty}(\Omega)} \left\|F(\lambda,\mu) - F(\bar{\lambda},\bar{\mu})\right\|_{L^2(\Omega)} \, . \tag{3.27}$$

Proof. The prove of this theorem is analogous to the one of Theorem 3.5, noting that for this choice of boundary condition, the regularity results of Lemma 5.1 also hold on the entire domain, i.e., for $\Omega_1 = \Omega$, which follows for example from [36, Theorem 4.16 and Theorem 4.18]. Furthermore, the boundary integral appearing in the partial integration step in (3.25) also vanishes in this case, since $\bar{u} = u = 0$ on $\partial\Omega$ due to the assumption that $\partial\Omega = \Gamma_D$. $\qquad \square$

As can be found for example in [2, 13, 19, 37], $H^2(\Omega)$ regularity and hence the above theorem can also be proven under weaker smoothness assumptions on the domain Ω. For example, it suffices that Ω is a convex Lipschitz domain.

Remark. Note that (3.26) is already strong enough to prove convergence of the Landweber iteration for the operator F_c to a solution $(\lambda^\dagger, \mu^\dagger)$ given that the initial guess (λ_0, μ_0) is chosen close enough to $(\lambda^\dagger, \mu^\dagger)$ [21, 28]. Furthermore, if there is a $\bar{\rho} > 0$ such that

$$\sup_{(\lambda,\mu) \in \mathcal{B}_{\bar{\rho}}(\lambda^\dagger, \mu^\dagger) \cap \mathcal{D}_s(F_c)} c_R(\lambda, \mu, \Omega_1, \Omega) < \infty \, , \tag{3.28}$$

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then for each $\eta > 0$ there exists a $\rho > 0$ such that

$$\left\| F_c(\lambda, \mu) - F_c(\bar{\lambda}, \bar{\mu}) - F_c'(\lambda, \mu)((\lambda, \mu) - (\bar{\lambda}, \bar{\mu})) \right\|_{L^2(\Omega)} \leq \eta \left\| F_c(\lambda, \mu) - F_c(\bar{\lambda}, \bar{\mu}) \right\|_{L^2(\Omega)},$$
$$\forall (\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in \mathcal{B}_{2\rho}(\lambda_0, \mu_0),$$

which is the original, well-known nonlinearity condition [21]. Obviously, the same statements also hold analogously for the $F : \mathcal{D}_s(F) \to L^2(\Omega)$ under the assumptions of Theorem 3.7. Note further that condition (3.28) follows directly from the proofs of [36, Theorem 4.16 and Theorem 4.18].

3.5   An Informal Discussion of Source Conditions

For general inverse problems of the form $F(x) = y$, source conditions of the form

$$x^\dagger - x_0 \in \mathcal{R}(F'(x^\dagger)^*), \tag{3.29}$$

where $x^\dagger$ and x_0 denote a solution of $F(x) = y$ and an initial guess, respectively, are important for showing convergence rates or even proving convergence of certain gradient-type methods for nonlinear ill-posed problems [28]. In this section, we make an investigation of the source condition for $F : \mathcal{D}_s(F) \to L^2(\Omega)^N$ and $N = 2, 3$.

Lemma 3.8. *Let $F : \mathcal{D}_s(F) \to L^2(\Omega)^N$ with $s > N/2 + 1$. Then (3.29) is equivalent to the existence of a $w \in L^2(\Omega)^N$ such that*

$$\begin{pmatrix} \lambda^\dagger - \lambda_0 \\ \mu^\dagger - \mu_0 \end{pmatrix} = \begin{pmatrix} E_s \left(\mathrm{div} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) \mathrm{div} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right) \right) \\ E_s \left(2\, \mathcal{E} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) : \mathcal{E} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right) \right) \end{pmatrix}. \tag{3.30}$$

Proof. This follows immediately from (3.4). $\qquad\square$

Hence, one has to have that $\lambda^\dagger - \lambda_0 \in \mathcal{R}(E_s)$ and $\mu^\dagger - \mu_0 \in \mathcal{R}(E_s)$ and

$$\begin{pmatrix} E_s^{-1}(\lambda^\dagger - \lambda_0) \\ E_s^{-1}(\mu^\dagger - \mu_0) \end{pmatrix} = \begin{pmatrix} \mathrm{div} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) \mathrm{div} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right) \\ 2\, \mathcal{E} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) : \mathcal{E} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right) \end{pmatrix}. \tag{3.31}$$

If $\mathrm{div} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) \mathrm{div} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right)$ and $2\, \mathcal{E} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right) : \mathcal{E} \left(-A_{\lambda^\dagger, \mu^\dagger}^{-1} Tw \right)$ are in $L^2(\Omega)$, which is for example the case if w as well as f, Φ, g_D and g_T satisfy additional $L^p(\Omega)$ regularity [13], then E_s coincides with i^*, where i is given as the embedding operator from $H^s(\Omega) \to L^2(\Omega)$. In this case, $\lambda^\dagger - \lambda_0 \in \mathcal{R}(E_s)$ and $\mu^\dagger - \mu_0 \in \mathcal{R}(E_s)$ imply a certain differentiability and boundary conditions on $\lambda^\dagger - \lambda_0$ and $\mu^\dagger - \mu_0$. Now, if

$$\frac{E_s^{-1}(\lambda^\dagger - \lambda_0)}{\mathrm{div} \left(u(\lambda^\dagger, \mu^\dagger) + \Phi \right)} \in L^2(\Omega),$$

then (3.31) can be rewritten as

$$\begin{pmatrix} E_s^{-1}(\lambda^\dagger - \lambda_0)/\operatorname{div}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right) \\ E_s^{-1}(\mu^\dagger - \mu_0) \end{pmatrix} = \begin{pmatrix} \operatorname{div}\left(-A_{\lambda^\dagger,\mu^\dagger}^{-1}Tw\right) \\ 2\,\mathcal{E}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right) : \mathcal{E}\left(-A_{\lambda^\dagger,\mu^\dagger}^{-1}Tw\right) \end{pmatrix}.$$

(3.32)

Since $A_{\lambda^\dagger,\mu^\dagger}^{-1}Tw \in V \subset H^1(\Omega)^N$, by the Helmholtz decomposition there exists a function $\phi = \phi(w) \in H^2(\Omega)$ and a vector field $\psi = \psi(w) \in H^2(\Omega)^N$ such that

$$-A_{\lambda^\dagger,\mu^\dagger}^{-1}Tw = \nabla\phi(w) + \nabla \times \psi(w),$$
$$\left(\nabla\phi(w) + \nabla \times \psi(w)\right)|_{\Gamma_D} = 0.$$

Hence, (3.32) is equivalent to

$$\Delta\phi(w) = E_s^{-1}(\lambda^\dagger - \lambda_0)/\operatorname{div}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right),$$
$$E_s^{-1}(\mu^\dagger - \mu_0) = 2\,\mathcal{E}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right) : \mathcal{E}\left(\nabla\phi(w) + \nabla \times \psi(w)\right), \qquad (3.33)$$
$$\left(\nabla\phi(w) + \nabla \times \psi(w)\right)|_{\Gamma_D} = 0.$$

Note that once ϕ and ψ are known such that $-A_{\lambda^\dagger,\mu^\dagger}^{-1}Tw = \nabla\phi + \nabla \times \psi$ holds, w can be uniquely recovered in the following way. Due to the Lax-Milgram Lemma, there exists an element $z(\phi,\psi) \in V$ such that

$$-\left\langle A_{\lambda^\dagger,\mu^\dagger}\left(\nabla\phi + \nabla \times \psi\right), v \right\rangle_{V^*,V} = \left\langle z(\phi,\psi), v \right\rangle_V, \qquad \forall v \in V.$$

However, since

$$-\left\langle A_{\lambda^\dagger,\mu^\dagger}\left(\nabla\phi(w) + \nabla \times \psi(w)\right), v \right\rangle_{V^*,V}$$
$$= \left\langle Tw, v \right\rangle_{V^*,V} = \left\langle w, v \right\rangle_{L^2(\Omega)} = \left\langle i_V^* w, v \right\rangle_V,$$

where i_V denotes the embedding from V to $L^2(\Omega)^N$, there follows $z(\phi,\psi) \in \mathcal{R}(i_V^*)$ and w can be recovered by $w = (i_V^*)^{-1}z(\phi,\psi)$.

Remark. Hence, we derive that the source condition (3.30) holds for the solution $(\lambda^\dagger,\mu^\dagger)$ and the initial guess (λ_0,μ_0) under the following assumptions:

- $\lambda^\dagger - \lambda_0 \in \mathcal{R}(E_s)$ and $\mu^\dagger - \mu_0 \in \mathcal{R}(E_s)$,

- there holds
$$\frac{E_s^{-1}(\lambda^\dagger - \lambda_0)}{\operatorname{div}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right)} \in L^2(\Omega), \tag{3.34}$$

- there exist functions $\phi \in H^2(\Omega)$ and $\psi \in H^2(\Omega)^N$ such that

$$\Delta\phi = E_s^{-1}(\lambda^\dagger - \lambda_0)/\operatorname{div}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right),$$
$$2\,\mathcal{E}\left(u(\lambda^\dagger,\mu^\dagger) + \Phi\right) : \mathcal{E}\left(\nabla\phi + \nabla \times \psi\right) = E_s^{-1}(\mu^\dagger - \mu_0),$$
$$\left(\nabla\phi + \nabla \times \psi\right)|_{\Gamma_D} = 0,$$

- the unique weak solution $z(\phi, \psi) \in V$ of the variational problem

$$-\left\langle A_{\lambda^\dagger, \mu^\dagger} \left(\nabla\phi + \nabla \times \psi\right), v \right\rangle_{V^*, V} = \left\langle z(\phi, \psi), v \right\rangle_V , \qquad \forall\, v \in V ,$$

satisfies $z(\phi, \psi) \in \mathcal{R}(i_V^*)$.

The above assumptions are restrictive, which is as usual [28]. However, without these assumptions one cannot expect convergence rates.

Remark. Note that since $u(\lambda^\dagger, \mu^\dagger) + \Phi$ is the weak solution of the non-homogenized problem (2.1), condition (3.5) implies that in areas of a divergence free displacement field, one has to know the true Lamé parameter $\lambda^\dagger$. This should be compared to similar conditions in [7–9, 50].

Remark. Note that if the source condition is satisfied, then it is known that the iteratively regularized Landweber and Gauss-Newton iterations converge, even if the nonlinearity condition is not satisfied [4, 5, 45].

4 Numerical Examples

In this section, we present some numerical examples demonstrating the reconstructions of Lamé parameters from given noisy displacement field measurements u^δ using both the operators $F|_{\mathcal{D}_s(F)}$ and F_c considered above. The sample problem, described in detail in Section 4.2 is chosen in such a way that it closely mimics a possible real-world setting described below. Furthermore, results are presented showing the reconstruction quality for both smooth and non-smooth Lamé parameters.

4.1 Regularization Approach - Landweber Iteration

For reconstructing the Lamé parameters, we use a Two-Point Gradient (TPG) method [24] based on Landweber's iteration and on Nesterov's acceleration scheme [38] which, using the abbreviation $x_k^\delta = \left(\lambda_k^\delta, \mu_k^\delta\right)$, read as follows,

$$\begin{aligned} z_k^\delta &= x_k^\delta + \alpha_k^\delta \left(x_k^\delta - x_{k-1}^\delta\right) , \\ x_{k+1}^\delta &= z_k^\delta + \omega_k^\delta \left(z_k^\delta\right) s_k^\delta \left(z_k^\delta\right) , \quad s_k^\delta (x) := F'(x)^* \left(u^\delta - F(x)\right) . \end{aligned} \tag{4.1}$$

For linear ill-posed problems, a constant stepsize ω_k^δ and $\alpha_k^\delta = (k-1)/(k+\alpha-1)$, this method was analysed in [39]. For nonlinear problems, convergence of (4.1) under the tangential cone condition was shown in [24] when the discrepancy principle is used as a stopping rule, i.e., the iteration is stopped after k_* steps, with k_* satisfying

$$\left\|u^\delta - F\left(x_{k_*}^\delta\right)\right\| \le \tau\delta \le \left\|u^\delta - F\left(x_k^\delta\right)\right\| , \qquad 0 \le k \le k_* , \tag{4.2}$$

where the parameter τ should be chosen such that

$$\tau > 2\frac{1+\eta}{1-2\eta},$$

although the choices $\tau = 2$ or τ close to 1 suggested by the linear case are also very popular. For the stepsize ω_k^δ we use the steepest descent stepsize [45] and for α_k^δ we use the well-known Nesterov choice, i.e.,

$$\omega_k^\delta(x) := \frac{\left\| s_k^\delta(x) \right\|^2}{\left\| F'(x) s_k^\delta(x) \right\|^2}, \qquad \text{and} \qquad \alpha_k^\delta = \frac{k-1}{k+2}. \tag{4.3}$$

The method (4.1) is known to work well for both linear and nonlinear inverse problems [23, 27] and also serves as the basis of the well-known FISTA algorithm [12] for solving linear ill-posed problems with sparsity constraints.

4.2   Problem Setting, Discretization, and Computation

A possible real-world problem the authors have in mind is a cylinder shaped object made out of agar with a symmetric, ball shaped inclusion of a different type of agar with different material properties and hence, different Lamé parameters. The object is placed on a surface and a constant downward displacement is applied from the top while the outer boundary of the object is allowed to move freely. Due to a marker substance being injected into the object beforehand, the resulting displacement field can be measured inside using a combination of different imaging modalities. Since the object is rotationally symmetric, this also holds for the displacement field, which allows for a relatively high resolution 2D image.

Motivated by this, we consider the following setup for our numerical example problem: For the domain Ω, we choose a rectangle in 2D, i.e., $N = 2$. We split the boundary $\partial\Omega$ of our domain into a part Γ_D consisting of the top and the bottom edge of the rectangle and into a part Γ_T consisting of the remaining two edges. Since the object is free to move on the sides, we set a zero traction condition on Γ_T, i.e., $g_T = 0$. Analogously for Γ_D, since the object is fixed to the surface and a constant displacement is being applied from above, we set $g_D = 0$ and $g_D = c_P = \text{const}$ on the parts of Γ_D corresponding to the bottom and the top edge of the domain.

If, for simplicity, we set $\Omega = (0,1)^2$, then the underlying non-homogenized forward problem (2.1) simplifies to

$$\begin{aligned}
- \operatorname{div}\left(\sigma(\tilde{u}(x))\right) &= 0, & x &\in (0,1)^2, \\
\tilde{u}(x) &= 0, & x &\in [0,1] \times \{0\}, \\
\tilde{u}(x) &= c_P, & x &\in [0,1] \times \{1\}, \\
\sigma(\tilde{u}(x))\vec{n}(x) &= 0, & x &\in \{0,1\} \times [0,1].
\end{aligned} \tag{4.4}$$

The homogenization function Φ can be chosen as $\Phi(x_1, x_2) := c_P\, x_2$ in this case.

In order to define the exact Lamé parameters $(\lambda^\dagger, \mu^\dagger)$, we first need to introduce the following family $\mathcal{B}_{r_1,r_2}^{h_1,h_2}$ of symmetric 2D bump functions with a circular plateau

$$\mathcal{B}_{r_1,r_2}^{h_1,h_2}(x,y) := \begin{cases} h_1, & \sqrt{x^2+y^2} \leq r_1, \\ h_2, & \sqrt{x^2+y^2} \geq r_2, \\ \mathcal{S}_{r_1,r_2}^{h_1,h_2}\left(\sqrt{x^2+y^2}\right), & r_1 < \sqrt{x^2+y^2} < r_2, \end{cases}$$

where $\mathcal{S}_{r_1,r_2}^{h_1,h_2}$ is a 5th order polynomial chosen such that the resulting function $\mathcal{B}_{r_1,r_2}^{h_1,h_2}$ is twice continuously differentiable. The exact Lamé parameters $(\lambda^\dagger, \mu^\dagger)$ are then created by shifting the function $\mathcal{B}_{r_1,r_2}^{h_1,h_2}$ and using different values of r_1, r_2, h_1, h_2; see Figure 4.1.
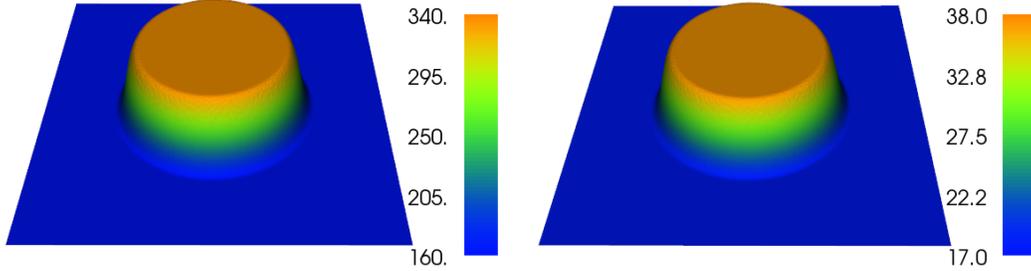

Figure 4.1: Exact Lamé parameters $(\lambda^\dagger, \mu^\dagger)$, in kPa.

As we have seen, a certain smoothness in the exact Lamé parameters is required for reconstruction with the operators $F|_{\mathcal{D}_s(F)}$ and F_c. Although this might be an unnatural assumption in some cases as different materials next to each other may have Lamé parameters of high contrast, it can be justified in the case of the combined agar sample, since when combining the different agar samples into one, the transition from one type of agar into the other can be assumed to be continuous, leading to a smooth behaviour of the Lamé parameters in the transition area.

However, since we also want to see the behaviour of the reconstruction algorithm in case of non-smooth Lamé parameters $(\lambda^\dagger, \mu^\dagger)$, we also look at $(\lambda^\dagger, \mu^\dagger)$ depicted in Figure 4.2, which were created using $\mathcal{B}_{r_1,r_2}^{h_1,h_2}$ with $r_1 \approx r_2$ and which, although being twice continuously differentiable in theory, behave like discontinuous functions after discretization.
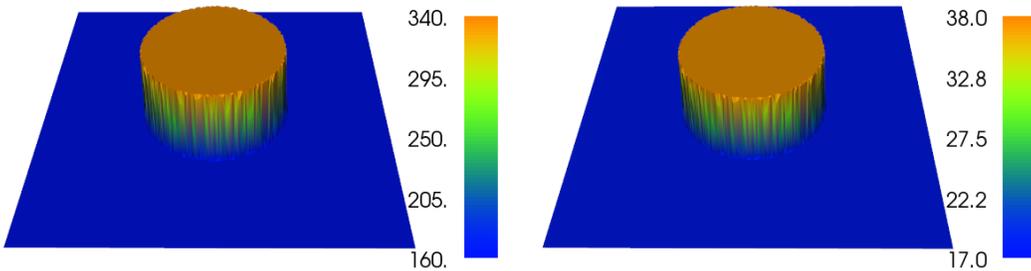

Figure 4.2: Exact Lamé parameters $(\lambda^\dagger, \mu^\dagger)$ created from $\mathcal{B}_{r_1,r_2}^{h_1,h_2}$ with $r_1 \approx r_2$, in kPa.

The discretization, implementation and computation of the involved variational problems was done using Python and the library FEniCS [3]. For the solution of the inverse problem a triangulation with 4691 vertices was introduced for discretizing

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the Lamé parameters. The data u was created by applying the forward model (4.4) to $(\lambda^\dagger, \mu^\dagger)$ using a finer discretization with 28414 vertices in order to avoid an inverse crime. For the constant c_P in (4.4) the choice $c_P = -10^{-4}$ is used. The resulting displacement field for the smooth Lamé parameters $(\lambda^\dagger, \mu^\dagger)$ is depicted in Figure 4.3. Afterwards, a random noise vector with a relative noise level of 0.5% is added to u to arrive at the noisy data u^δ. This leads to the absolute noise level $\delta = \left\| u - u^\delta \right\|_{L^2(\Omega)} \approx 3.1 * 10^{-7}$. Note that while with a smaller noise level more accurate reconstructions can be obtained, the required computational time then drastically increases due to the discrepancy principle. Furthermore, a very small noise level is unrealistic in practice.
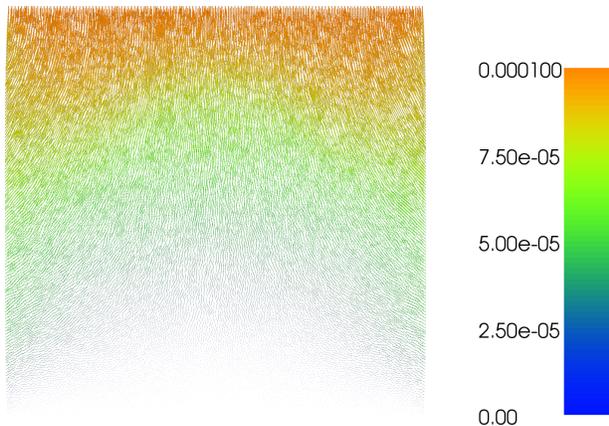
Figure 4.3: Displacement field u corresponding to the Lamé parameters $(\lambda^\dagger, \mu^\dagger)$ depicted in Figure 4.1.

4.3 Numerical Results

In this section we present various reconstruction results for different combinations of operators, Lamé parameters and boundary conditions. Since the domain Ω is two-dimensional, i.e., $N = 2$, the operators $F|_{\mathcal{D}_s(F)}$ and F_c are well-defined for any $s > 1$. By our analysis above, we know that the nonlinearity condition holds for the operator F_c if $s > N/2 + 1$ which suggests to use $s > 2$. However, since numerically there is hardly any difference between using $s = 2$ and $s = 2 + \varepsilon$ for ε small enough, we choose $s = 2$ for ease of implementation in the following examples. When using the operator F_c we chose a slightly smaller square than Ω for the domain Ω_1, which is visible in the reconstructions. Unless noted otherwise, the accelerated Landweber type method (4.1) was used together with the steepest descent stepsize (4.3) and the iteration was terminated using the discrepancy principle (4.2) together with $\tau = 1$. Concerning the initial guess, when using the operator $F|_{\mathcal{D}_s(F)}$ the choice $(\lambda_0, \mu_0) = (2, 0.3)$ was made while when using the operator F_c a zero initial guess was used. For all presented examples, the computation times lay between 15 minutes and 1 hour on a Lenovo ThinkPad W540 with Intel(R) Core(TM) i7-4810MQ CPU @ 2.80GHz, 4 cores.

Example 4.1. As a first test we look at the reconstruction of the smooth Lamé parameters (Figure 4.1), using the operator F_c. The iteration terminated after 642 iterations yields the reconstructions depicted in Figure 4.4. The parameter $\mu^\dagger$ is well reconstructed both qualitatively and quantitatively, with some obvious small artefacts around the border of the inner domain Ω_1. The parameter $\lambda^\dagger$ is less well reconstructed, which is a common theme throughout this section and is due to the smaller sensitivity of the problem to changes of λ. However, the location and also quantitative information of the inclusion is obtained.
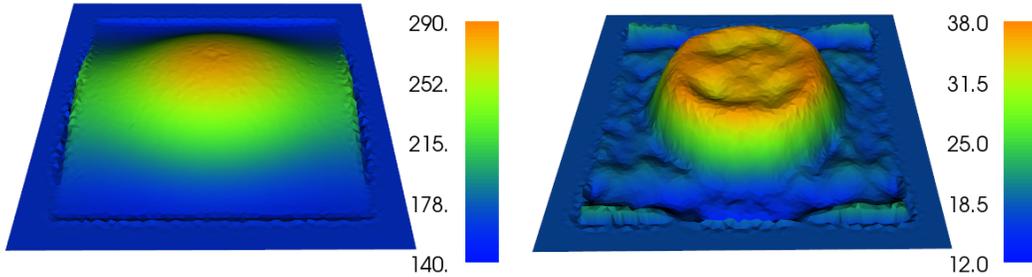
Figure 4.4: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.1. Smooth Lamé parameters (Figure 4.1) - Displacement-Traction boundary conditions - operator F_c.

Example 4.2. Using the same setup as before, but this time with the operator $F|_{\mathcal{D}_s(F)}$ instead of F_c leads to the reconstructions depicted in Figure 4.5, the discrepancy principle being satisfied after 422 iterations in this case. Even though information about the Lamé parameters can be obtained also here, the reconstructions are worse than in the previous case. Note that in the case of mixed boundary conditions the nonlinearity condition has not been verified for the operator $F|_{\mathcal{D}_s(F)}$, and there is no proven convergence result.
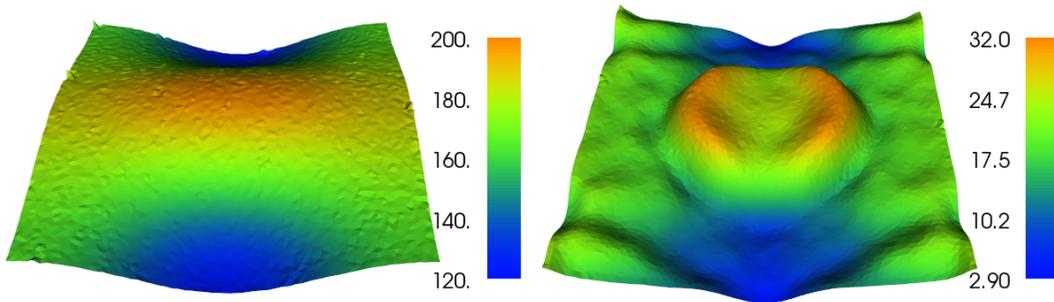

Figure 4.5: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.2. Smooth Lamé parameters (Figure 4.1) - Displacement-Traction boundary conditions - operator $F|_{\mathcal{D}_s(F)}$.

Example 4.3. Going back to the operator F_c but now using the non-smooth Lamé parameters (Figure 4.2), we obtain the reconstructions depicted in Figure 4.6 after 635 iterations. We get similar results as for the first test with the main difference that the reconstructed values of the inclusion now fit less well than before, which is due to the non-smoothness of the used Lamé parameters.
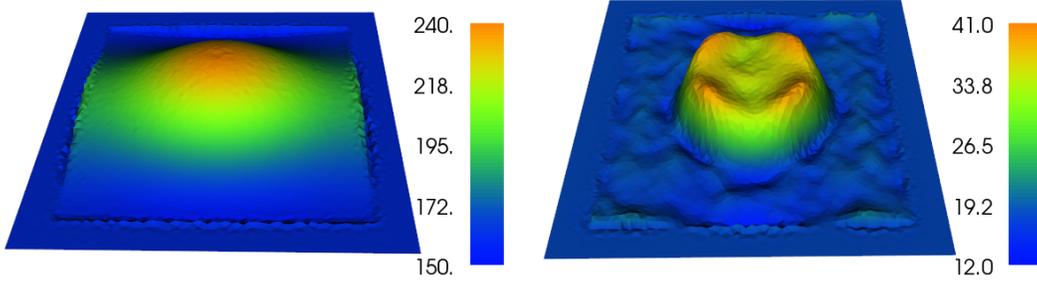

Figure 4.6: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.3. Non-smooth Lamé parameters (Figure 4.2) - Displacement-Traction boundary conditions - operator F_c.

Example 4.4. For the following tests, we want to see what happens if, instead of mixed displacement-traction boundary conditions, only pure displacement conditions are used. For this, we replace the traction boundary condition in (4.4) by a zero displacement condition while leaving everything else the same. The resulting reconstructions using the operator F_c for both smooth and non-smooth Lamé parameters are depicted in Figures 4.7 and 4.8. The discrepancy principle stopped after 177 and 194 iterations, respectively. Compared to the previous tests, it is obvious that the parameter $\lambda^\dagger$ is now much better reconstructed than before in both cases. Also the parameter $\mu^\dagger$ is well reconstructed, although not as good as in the case of mixed boundary conditions. The influence of the non-smooth Lamé parameters in Figure 4.8 can best be seen in the volcano like appearance of the reconstruction of $\mu^\dagger$.
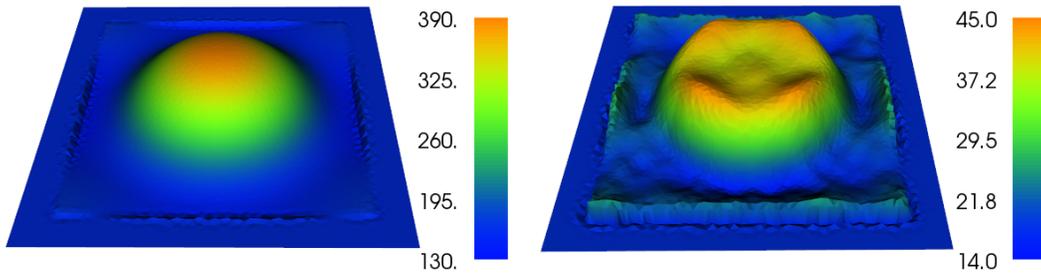

Figure 4.7: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.4. Smooth Lamé parameters (Figure 4.1) - Pure displacement boundary conditions - operator F_c.
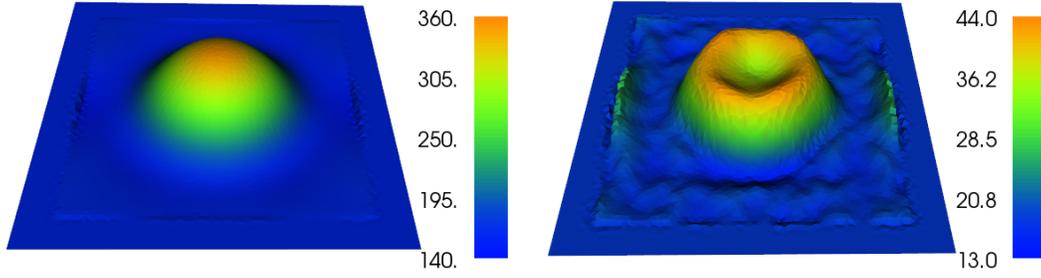
Figure 4.8: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.4. Non-smooth Lamé parameters (Figure 4.2) - Pure displacement boundary conditions - operator F_c.

Example 4.5. Next, we take a look at the reconstruction of the smooth Lamé parameters using $F|_{\mathcal{D}_s(F)}$ and as before the pure displacement boundary conditions. Interestingly, Nesterov acceleration does not work well in this case and so the Landweber iteration with the steepest descent stepsize was used to obtain the reconstructions depicted in Figure 4.9, the discrepancy principle being satisfied after 937 iterations. As with the reconstructions obtained in case of mixed boundary conditions, this case is worse than when using F_c, for the same reasons mentioned above. Note however that in comparison with Figure 4.5, the inclusion in $\lambda^\dagger$ is much better resolved now than in the other case, which is due to the use of pure displacement boundary conditions.
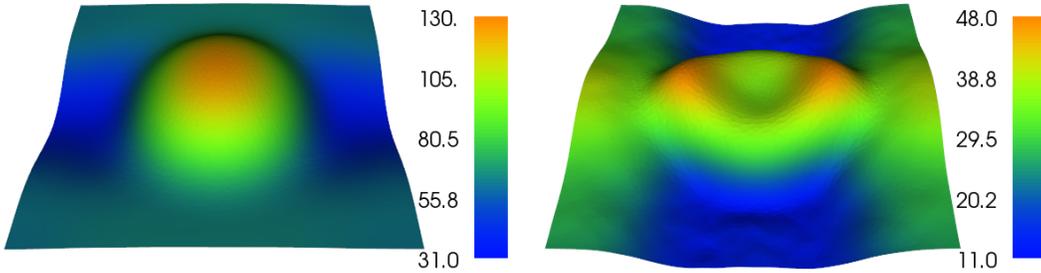

Figure 4.9: Reconstructions of $(\lambda^\dagger, \mu^\dagger)$, in kPa, Example 4.5. Smooth Lamé parameters (Figure 4.1) - Pure displacement boundary conditions - operator $F|_{\mathcal{D}_s(F)}$.

Example 4.6. For the last test we return to the same setting as in Example 4.1, i.e., we again use the operator F_c and mixed displacement-traction boundary conditions. However, this time we consider different exact Lamé parameters modelling a material sample with three inclusions of varying elastic behaviour. The exact parameters and the resulting reconstructions, obtained after 921 iterations, are depicted in Figure 4.10. As expected, the Lamé parameter $\mu^\dagger$ is well reconstructed in shape, value and location of the inclusions. Moreover, even though the reconstruction of $\lambda^\dagger$ does not exhibit the

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same shape as the exact parameter, information about the value and the location of the inclusions was obtained.
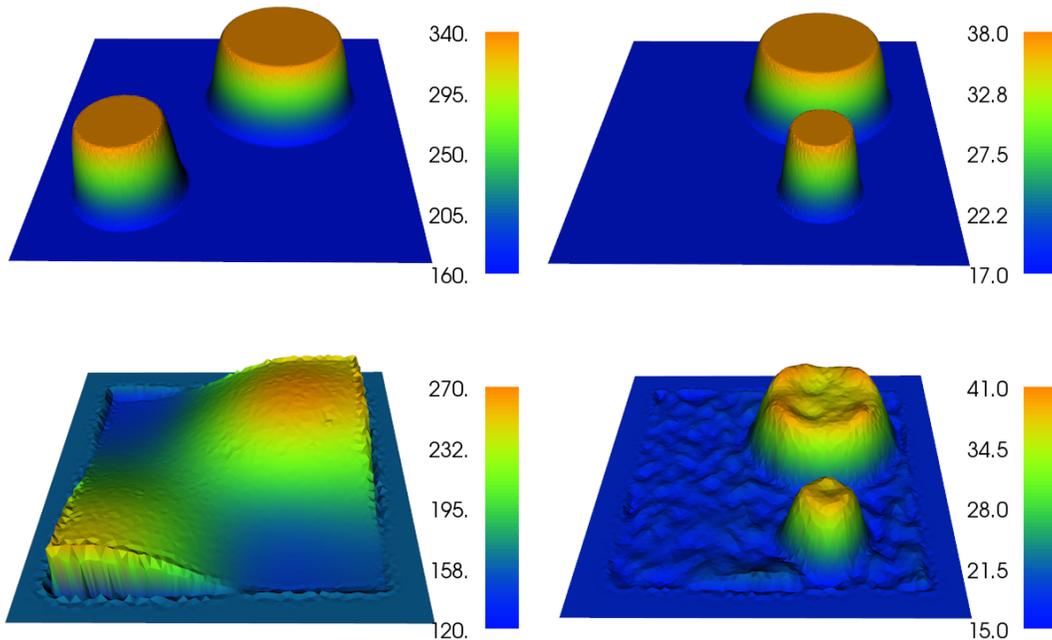

Figure 4.10: Exact Lamé parameters $(\lambda^\dagger, \mu^\dagger)$ (top) and their reconstructions (bottom), in kPa, Example 4.6 - Displacement-Traction boundary conditions - operator F_c.

5   Support and Acknowledgements

Appendix. Important results from PDE theory

Here we collect important results in the theory of partial differential used throughout this paper. Two basic results are the trace inequality [1], which states that there exists a constant $c_T = c_T(\Omega) > 0$ such that

$$\|v\|_{H^{\frac{1}{2}}(\Gamma_T)} \le c_T \|v\|_{H^1(\Omega)} , \quad \forall\, v \in V , \tag{5.1}$$

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and Friedrich's inequality [17], i.e., there exists a constant $c_F = c_F(\Omega) > 0$ such that

$$\|v\|_{L^2(\Omega)} \leq c_F \|\nabla v\|_{L^2(\Omega)} , \quad \forall\, v \in V , \tag{5.2}$$

from which we can deduce

$$\|v\|_{H^1(\Omega)}^2 \leq (1 + c_F^2) \|\nabla v\|_{L^2(\Omega)}^2 , \quad \forall\, v \in V . \tag{5.3}$$

Korn's inequality [49] states that there exists a constant $c_K = c_K(\Omega) > 0$ such that

$$\int_\Omega \|\mathcal{E}(v)\|_F^2 \, dx \geq c_K^2 \|\nabla v\|_{L^2(\Omega)}^2 , \qquad \forall\, v \in V . \tag{5.4}$$

Furthermore, we need the following regularity result

Lemma 5.1. *Let $(\lambda, \mu) \in \mathcal{D}_s(F)$ with $s > N/2 + 1$ and $w \in L^2(\Omega)^N$. Then there exists a unique weak solution u of the elliptic boundary value problem*

$$\begin{aligned} -\operatorname{div}(\sigma(u)) &= w , \quad in\ \Omega , \\ u\,|_{\Gamma_D} &= 0 , \\ \sigma(u)\vec{n}\,|_{\Gamma_T} &= 0 , \end{aligned} \tag{5.5}$$

and for every bounded, open, connected Lipschitz domain $\Omega_1 \subset \Omega$ with $\bar{\Omega}_1 \Subset \Omega$ there holds $u|_{\Omega_1} \in H^2(\Omega_1)^N$ and $-\operatorname{div}(\sigma(u)) = w$ pointwise almost everywhere in Ω_1. Furthermore, there is a constant $c_R = c_R(\lambda, \mu, \Omega_1, \Omega)$ such that

$$\|u\|_{H^2(\Omega_1)} \leq c_R \|w\|_{L^2(\Omega_1)} . \tag{5.6}$$

Proof. This follows immediately from [36, Theorem 4.16]. $\qquad\square$

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