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A note on a formula of Krattenthaler

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A note on a formula of Krattenthaler

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Abstract

In this note, we find a connection between an identity of C. Krattenthaler and some Hankel determinants related to the Hahn polynomials. We also consider some limiting cases related to the Meixner and Charlier polynomials.

1 Introduction

If $\{\mu_n\}$ is a sequence of complex numbers and $L : \mathbb{C}[x] \to \mathbb{C}$ is the linear functional defined by

$$L[x^n] = \mu_n, \quad n = 0, 1, \ldots,$$

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then L is called the *moment functional* [8] determined by the formal moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n.

Suppose that $\{P_n\}$ is a family of monic polynomials, with $\deg(P_n) = n$. If the polynomials $P_n(x)$ satisfy

$$L[P_n P_m] = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots, \tag{1}$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta, then $\{P_n\}$ is called an *orthogonal polynomial sequence* with respect to L.

If we write

$$x^i = \sum_{k=0}^{i} a_{i,k} P_k(x), \quad a_{i,i} = 1,$$

then, we clearly have

$$x^{i+j} = \sum_{k=0}^{i} \sum_{l=0}^{j} a_{i,k} a_{j,l} P_k(x) P_l(x). \tag{2}$$

Applying L to (2) and using (1), we get

$$\begin{aligned} \mu_{i+j} = L\left[x^{i+j}\right] &= \sum_{k=0}^{i} \sum_{l=0}^{j} a_{i,k} a_{j,l} L[P_k P_l] \\ &= \sum_{k=0}^{i} \sum_{l=0}^{j} a_{i,k} a_{j,l} h_k \delta_{k,l} = \sum_{k=0}^{i} a_{i,k} a_{j,k} h_k. \end{aligned} \tag{3}$$

If we define the lower triangular matrix A_n by

$$(A_n)_{i,j} = \begin{cases} a_{i,j} & i \geq j \\ 0 & i < j \end{cases}, \quad 0 \leq i, j \leq n-1,$$

the diagonal matrix D_n by

$$(D_n)_{i,j} = h_i \delta_{i,j}, \quad 0 \leq i, j \leq n-1,$$

and the Hankel matrix H_n by

$$(H_n)_{i,j} = \mu_{i+j}, \quad 0 \leq i, j \leq n-1, \tag{4}$$

then we see from (3) that we have the LDL decomposition

$$H_n = A_n D_n A_n^T. \tag{5}$$

We define the Hankel determinants by $\Delta_0 = 1$ and

$$\Delta_n = \det\left(H_n\right), \quad n = 1, 2, \ldots.$$

Using (5), we see that

$$\Delta_n = \prod_{j=0}^{n-1} h_j. \tag{6}$$

Since

$$L\left[xP_nP_k\right] = 0, \quad k \notin \{n-1, n, n+1\},$$

the monic orthogonal polynomials $P_n(x)$ satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

where

$$\beta_n = \frac{1}{h_n} L\left[xP_n^2\right], \quad \gamma_n = \frac{1}{h_{n-1}} L\left[xP_nP_{n-1}\right].$$

Because

$$L\left[xP_nP_{n-1}\right] = L\left[P_n^2\right],$$

we have

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad n = 1, 2, \ldots, \tag{7}$$

and we define $\gamma_0 = 0$. It follows from (7) that

$$h_n = h_0 \prod_{i=1}^{n} \gamma_i. \tag{8}$$

Using (8) in (6), we get

$$\Delta_n = \prod_{j=0}^{n-1} h_0 \prod_{i=1}^{n} \gamma_i = (h_0)^n \prod_{k=1}^{n-1} (\gamma_k)^{n-k}. \tag{9}$$

The identity (9) is sometimes called "Heilermann formula" [33], since Johannes Bernhard Hermann Heilermann considered the J-fraction expansion

$$\sum_{n=0}^{\infty} \frac{\mu_n}{w^{n+1}} = \cfrac{\mu_0}{w - \beta_1 - \cfrac{\gamma_1}{w - \beta_2 - \cfrac{\gamma_2}{w - \beta_3 - \cdots}}}, \tag{10}$$

in his 1845 Ph.D. thesis "De transformatione serierum in fractiones continuas" [6, 5.2].

Determinants have a long history and an extensive literature, see [3], [7], [41], [46], [52], [53], [54], [56], and the impressive monographs [33] and [35].

The theories of Hankel determinants and orthogonal polynomials are deeply connected, see [8], [13], [17], [23], [27], [30], [32] [36], and [50].

For some applications of Hankel determinants to combinatorial problems, see [1], [9] [10], [19], [24], [26], [29], [48], [49], and [55].

In [34, 3.5], C. Krattenthaler showed (among many other results), the following identity

$$\sum_{0\leq k_1<k_2<\cdots<k_n\leq N} \prod_{i=1}^{n} \left[\frac{(x)_{k_i}\,(y)_{N-k_i}}{(k_i)!\,(N-k_i)!} \right] \prod_{1\leq i<j\leq n} (k_j-k_i)^2 \tag{11}$$
$$= \prod_{k=0}^{n-1} \left[\frac{k!}{(N-k)!} (x)_k\,(y)_k\,(x+y+k+n-1)_{N-n+1} \right],$$

as a limiting case of the $q-$analog formula

$$\sum_{0\leq k_1<k_2<\cdots<k_n\leq N} \prod_{i=1}^{n} \left[\frac{(x;q)_{k_i}\,(y;q)_{N-k_i}}{(q;q)_{k_i}\,(q;q)_{N-k_i}} y^{k_i} \right] \prod_{1\leq i<j\leq n} \left(q^{k_j}-q^{k_i}\right)^2 \tag{12}$$
$$= \prod_{k=0}^{n-1} \left[y^k q^{k(k-1)} \frac{(q;q)_k}{(q;q)_{N-k}} (x;q)_k\,(y;q)_k \left(xyq^{k+n-1};q\right)_{N-n+1} \right].$$

He gave two different proofs of (12) using: 1) a Schur function identity from [38] and 2) a q-integral evaluation from [20], [31].

The purpose of this note is to give a different proof of (11) related to the theory of orthogonal polynomials. We also study some limiting cases, and consider some possible generalizations.

2 Main result

Suppose that the linear functional L has the form

$$L[p] = \sum_{k=0}^{N} c_k p(k), \quad p(x) \in \mathbb{C}[x], \tag{13}$$

for some sequence $\{c_k\}$. Then, the moments μ_l are given by

$$\mu_l = \sum_{k=0}^{N} k^l c_k, \quad l = 0, 1, \ldots,$$

and the Hankel matrix (4) elements are

$$(H_n)_{i,j} = \mu_{i+j} = \sum_{k=0}^{N} k^{i+j} c_k, \quad 0 \leq i, j \leq n-1. \tag{14}$$

We can obtain a representation for the determinants of H_n.

Proposition 1 *The Hankel determinants Δ_n are given by*

$$\Delta_n = \sum_{0 \leq k_1 < k_2 < \cdots < k_n \leq N} \left(\prod_{i=1}^{n} c_{k_i} \right) V_n(k_1, \cdots, k_n), \tag{15}$$

where $V_n(k_1, \cdots, k_n)$ denotes the polynomial

$$V_n(k_1, \cdots, k_n) = \prod_{1 \leq i < j \leq n} (k_j - k_i)^2.$$

Proof. If we rewrite (14) as

$$(H_n)_{i,j} = \sum_{k=0}^{N} \sum_{l=0}^{N} k^i c_k \delta_{k,l} l^j,$$

we see that H_n has the form

$$H_n = V^T C V, \tag{16}$$

where V is the $(N+1) \times n$ Vandermonde matrix

$$(V)_{i,j} = i^j, \quad 0 \leq i \leq N, \quad 0 \leq j \leq n-1,$$

and C is the $(N+1) \times (N+1)$ diagonal matrix

$$(C)_{i,j} = c_i \delta_{i,j}, \quad 0 \leq i, j \leq N.$$

Using the Cauchy–Binet formula [22] in (16), we have

$$\det(H_n) = \sum_{0\le k_1<k_2<\cdots<k_n\le N} \det(c_i \delta_{i,k_i}) \left[\det\left(k_i^j\right)_{0\le j\le n-1}\right]^2,$$

and since

$$\left[\det_{1\le i,j\le n}\left(k_i^{j-1}\right)\right]^2 = \prod_{1\le i<j\le n}(k_j-k_i)^2,$$

the result follows. ■

Remark 2 *The expression (15) is the discrete case of Heine's formula [28]*

$$\det_{1\le i,j\le n}(\mu_{i+j}) = \frac{1}{n!}\int_a^b\int_a^b\cdots\int_a^b V_n(x_1,\cdots,x_n)\, d\alpha(x_1)\, d\alpha(x_2)\cdots d\alpha(x_n),$$

where

$$\mu_i = \int_a^b x^i d\alpha(x), \quad i=0,1,\ldots.$$

Heine's formula is used extensively in random matrix theory [4], [39], [11].

2.1 Hahn polynomials

The monic Hahn polynomials are defined by [43, 18.20.5]

$$Q_n(x) = \frac{(\alpha+1)_n(-N)_n}{(n+\alpha+\beta+1)_n}\, {}_3F_2\left(\begin{matrix} -n,-x,n+\alpha+\beta+1 \\ \alpha+1,-N\end{matrix};1\right),$$

where

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix};z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}$$

denotes the generalized hypergeometric function [43, Chapter 16] and $(u)_k$ is the Pochhammer symbol (or rising factorial) [43, 5.2.4] defined by $(u)_0 = 1$ and

$$(u)_k = u(u+1)\cdots(u+k-1), \quad k=1,2,\ldots.$$

We also have [43, 5.2.5]

$$(u)_z = \frac{\Gamma(u+z)}{\Gamma(u)}, \tag{17}$$

where $\Gamma(z)$ is the Gamma function.

When $\alpha, \beta \in \mathbb{R} \setminus [-N,-1]$, the monic Hahn polynomials satisfy the orthogonality relation [14]

$$\sum_{k=0}^{N} Q_n(k)\, Q_m(k) \binom{N}{k} (\alpha+1)_k (\beta+1)_{N-k} = h_n \delta_{n,m}, \tag{18}$$

where

$$h_n = (n!)^2 \binom{N}{n} (\alpha+1)_n (\beta+1)_n \frac{(\alpha+\beta+2+2n)_{N-n}}{(\alpha+\beta+1+n)_n}. \tag{19}$$

In order to prove our main theorem, we will need the following lemmas.

Lemma 3 *Let $f(x)$ be a function. Then,*

$$\prod_{k=0}^{n} f(x+2k)\, f(x+2k+1) = \prod_{k=0}^{n} f(x+k)\, f(x+k+n+1), \quad n = 0,1,2,\ldots. \tag{20}$$

Proof. Immediate, since

$$\begin{aligned}
\prod_{k=0}^{n} f(x+2k)\, f(x+2k+1) &= \prod_{k=0}^{2n+1} f(x+k) \\
&= \prod_{k=0}^{n} f(x+k) \prod_{k=n+1}^{2n+1} f(x+k) \\
&= \prod_{k=0}^{n} f(x+k) \prod_{k=0}^{n} f(x+k+n+1) \\
&= \prod_{k=0}^{n} f(x+k)\, f(x+k+n+1).
\end{aligned}$$

■

Lemma 4 *For all $0 \leq n \leq N$, we have*

$$\prod_{k=0}^{n} \frac{(x+2k+1)_{N-k}}{(x+k)_k} = \prod_{k=0}^{n} (x+k+n+1)_{N-n}. \tag{21}$$

Proof. Using (20) with $f(z) = \Gamma(z)$, we have

$$\prod_{k=0}^{n} \Gamma(x+k)\,\Gamma(x+k+n+1) = \prod_{k=0}^{n} \Gamma(x+2k)\,\Gamma(x+2k+1),$$

and therefore

$$\prod_{k=0}^{n} (x+2k+1)_{n-k} = \prod_{k=0}^{n} \frac{\Gamma(x+k+n+1)}{\Gamma(x+2k+1)}$$
$$= \prod_{k=0}^{n} \frac{\Gamma(x+2k)}{\Gamma(x+k)} = \prod_{k=0}^{n} (x+k)_k.$$

Multiplying both sides of the previous equation by

$$\prod_{k=0}^{n} (x+k+1)_k$$

we get

$$\prod_{k=0}^{n} (x+k+1)_k\,(x+2k+1)_{n-k} = \prod_{k=0}^{n} (x+k+1)_k\,(x+k)_k.$$

Using the identity [42, 18:5:12]

$$(x)_{s+t} = (x)_s\,(x+s)_t\,, \tag{22}$$

we have

$$\prod_{k=0}^{n} (x+k+1)_n = \prod_{k=0}^{n} (x+k+1)_k\,(x+k)_k,$$

or

$$\prod_{k=0}^{n} \frac{1}{(x+k+1)_k\,(x+k)_k} = \prod_{k=0}^{n} \frac{1}{(x+k+1)_n}.$$

Multiplying both sides by

$$\prod_{k=0}^{n} (x+k+1)_N\,,$$

we obtain

$$\prod_{k=0}^{n} \frac{(x+k+1)_N}{(x+k+1)_k (x+k)_k} = \prod_{k=0}^{n} \frac{(x+k+1)_N}{(x+k+1)_n}.$$

Finally, using the identity [42, 18:5:8]

$$\frac{(x)_s}{(x)_t} = (x+t)_{s-t}, \quad s \geq t, \tag{23}$$

we get

$$\prod_{k=0}^{n} \frac{(x+2k+1)_{N-k}}{(x+k)_k} = \prod_{k=0}^{n} (x+k+n+1)_{N-n}.$$

■

We now have all the elements necessary to show our main result.

Theorem 5 *For all* $1 \leq n \leq N$ *and* $\alpha, \beta \in \mathbb{R} \setminus [-N,-1]$, *we have*

$$\sum_{0 \leq k_1 < k_2 < \cdots < k_n \leq N} \prod_{i=1}^{n} \left[\binom{N}{k_i} (\alpha+1)_{k_i} (\beta+1)_{N-k_i} \right] V_n(k_1, \cdots, k_n) \tag{24}$$

$$= \prod_{k=0}^{n-1} \left[(k!)^2 \binom{N}{k} (\alpha+1)_k (\beta+1)_k (\alpha+\beta+1+k+n)_{N-n+1} \right].$$

Proof. Let

$$c_k = \binom{N}{k} (\alpha+1)_k (\beta+1)_{N-k}.$$

From (18), we see that the monic orthogonal polynomials associated with the linear functional L defined by (13) are the Hahn polynomials.

On the other hand, we have from (15)

$$\det_{0 \leq i,j \leq n-1} \left(L\left[x^{i+j}\right] \right) = \sum_{0 \leq k_1 < k_2 < \cdots < k_n \leq N} \left(\prod_{i=1}^{n} c_{k_i} \right) V_n(k_1, \cdots, k_n).$$

Using (6) and (19), we get

$$\sum_{0 \leq k_1 < k_2 < \cdots < k_n \leq N} \prod_{i=1}^{n} \left[\binom{N}{k_i} (\alpha+1)_{k_i} (\beta+1)_{N-k_i} \right] V_n(k_1, \cdots, k_n)$$

$$= \prod_{k=0}^{n-1} \left[(k!)^2 \binom{N}{k} (\alpha+1)_k (\beta+1)_k \frac{(\alpha+\beta+2+2k)_{N-k}}{(\alpha+\beta+1+k)_k} \right].$$

If we use (21), with $x = \alpha + \beta + 1$, we have

$$\prod_{k=0}^{n-1} \frac{(\alpha + \beta + 2 + 2k)_{N-k}}{(\alpha + \beta + 1 + k)_k} = \prod_{k=0}^{n-1} (\alpha + \beta + 1 + k + n)_{N-n+1} \,,$$

and the result follows. ■

Corollary 6 *For all* $1 \leq n \leq N$, *we have*

$$\sum_{0 \leq k_1 < k_2 < \cdots < k_n \leq N} \prod_{i=1}^{n} \left[\frac{(x)_{k_i} (y)_{N-k_i}}{(k_i)! \, (N-k_i)!} \right] V_n \left(k_1, \cdots, k_n \right) \tag{25}$$
$$= \prod_{k=0}^{n-1} \left[\frac{k!}{(N-k)!} (x)_k \, (y)_k \, (x + y + k + n - 1)_{N-n+1} \right].$$

Proof. If we set $\alpha = x - 1$, $\beta = y - 1$ in (24) and divide both sides of by $(N!)^n$, we obtain (25). To remove any restrictions on x, y, we observe that (25) is an identity between polynomials in x and y of degree

$$\sum_{k=0}^{n-1} (k + k + N - n + 1) = Nn.$$

According to Theorem 5, (25) is true for $x, y \notin [-N+1, 0]$, and therefore it is true for all x, y. ■

2.2 Meixner polynomials

Lemma 7 *Let* $0 \leq k \leq n \leq N$ *and* $w > 0$. *Then, as* $N \to \infty$ *we have*

$$\frac{N!}{(N-k)!} \frac{(1 + wN)_{N-k}}{(1 + wN)_N} \sim (w+1)^{-k} \,, \tag{26}$$

and for all $a > 0$

$$\frac{N!}{(N-k)!} \frac{(1 + wN)_k}{(1 + wN)_N} (a + k + n + wN)_{N-n+1} \sim w^{1-a-n} (w+1)^{a+k} N^{2k-n+1}. \tag{27}$$

Proof. From (17), we have

$$\frac{N!}{(N-k)!}\frac{(1+wN)_{N-k}}{(1+wN)_N}=\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+N+1-k)}{\Gamma(Nw+N+1)},$$

and using Stirling's formula [43, 5.11.1]

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln(z)-z+\frac{1}{2}\ln(2\pi)+O\left(z^{-1}\right),\quad z\to\infty, \tag{28}$$

we obtain

$$\ln\left[\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+N+1-k)}{\Gamma(Nw+N+1)}\right]=-k\ln(w+1)+O\left(N^{-1}\right),\quad N\to\infty.$$

Similarly, from (17) we have

$$\begin{aligned}&\frac{N!}{(N-k)!}\frac{(1+wN)_k}{(1+wN)_N}(a+k+n+wN)_{N-n+1}\\&=\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+k+1)}{\Gamma(Nw+N+1)}\frac{\Gamma(Nw+N+k+a+1)}{\Gamma(Nw+n+k+a)},\end{aligned}$$

and using (28), we get

$$\begin{aligned}&\ln\left[\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+k+1)}{\Gamma(Nw+N+1)}\frac{\Gamma(Nw+N+k+a+1)}{\Gamma(Nw+n+k+a)}\right]\\&=(2k-n+1)\ln(N)+(a+k)\ln(w+1)-(a+n-1)\ln(w)+O\left(N^{-1}\right)\end{aligned}$$

as $N\to\infty$. ■

Corollary 8 *Let* $0<z<1$, $a>0$ *and* $n=1,2,\ldots$. *Then,*

$$\begin{aligned}&\sum_{0\le k_1<k_2<\cdots<k_n}\prod_{i=1}^{n}\left[\frac{(a)_{k_i}\ z^{k_i}}{(k_i)!}\right]V_n(k_1,\cdots,k_n)\\&=\prod_{k=0}^{n-1}\left[(k!)^2\frac{(a)_k\ z^k}{k!}\ (1-z)^{-a-2k}\right].\end{aligned} \tag{29}$$

Proof. Multiplying both sides of (25) by

$$\left[\frac{N!}{(y)_N}\right]^n,$$

setting

$$x = a, \quad y = 1 + \frac{1-z}{z} N,$$

and using (26)-(27), we obtain

$$\sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\frac{N!}{(y)_N} \frac{(a)_{k_i}(y)_{N-k_i}}{(k_i)!(N-k_i)!}\right] V_n(k_1, \cdots, k_n)$$

$$\sim \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\frac{(a)_{k_i} \; z^{k_i}}{(k_i)!}\right] V_n(k_1, \cdots, k_n), \quad N \to \infty$$

and

$$\prod_{k=0}^{n-1} \left[\frac{N!}{(y)_N} \frac{k!}{(N-k)!} (a)_k (y)_k (a+y+k+n-1)_{N-n+1}\right]$$

$$\sim \prod_{k=0}^{n-1} \left[k!(a)_k \left(\frac{1-z}{z}\right)^{1-a-n} z^{-a-k} N^{2k-n+1}\right], \quad N \to \infty.$$

But

$$\prod_{k=0}^{n-1} (1-z)^{1-a-n} = (1-z)^{-n(a+n-1)} = \prod_{k=0}^{n-1} (1-z)^{-a-2k},$$

$$\prod_{k=0}^{n-1} z^{n-1-k} = z^{\frac{1}{2}n(n-1)} = \prod_{k=0}^{n-1} z^k,$$

and

$$\prod_{k=0}^{n-1} N^{2k-n+1} = 1.$$

Therefore,

$$\lim_{N \to \infty} \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\frac{(a)_{k_i} \; z^{k_i}}{(k_i)!}\right] V_n(k_1, \cdots, k_n)$$

$$= \prod_{k=0}^{n-1} \left[k!(a)_k \; z^k \; (1-z)^{-a-2k}\right].$$

■

The monic Meixner polynomials are defined by [43, 18.20.7]

$$M_n(x) = (a)_n \left(1 - z^{-1}\right)^{-n} \; {}_2F_1\left[\begin{matrix} -n, \; -x \\ a \end{matrix}; 1 - z^{-1}\right].$$

When $a > 0$ and $0 < z < 1$, the monic Meixner polynomials satisfy the orthogonality relation [14]

$$\sum_{k=0}^{\infty} M_n(k) M_m(k) \frac{(a)_k}{k!} z^k = n! (a)_n \; z^n (1 - z)^{-a-2n} \delta_{n,m}. \tag{30}$$

If we choose

$$c_k = \frac{(a)_k}{k!} z^k, \quad k = 0, 1, \ldots, \tag{31}$$

we see from (30) that the monic orthogonal polynomials associated with the linear functional L defined by

$$L[p] = \lim_{N \to \infty} \sum_{k=0}^{N} c_k p(k) \tag{32}$$

are the Meixner polynomials. But from (15), we have

$$\det_{0 \le i,j \le n-1} \left(L\left[x^{i+j}\right]\right) = \lim_{N \to \infty} \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \left(\prod_{i=1}^{n} c_{k_i}\right) V_n(k_1, \cdots, k_n), \tag{33}$$

and using (6) and (30), we get

$$\lim_{N \to \infty} \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\frac{(a)_{k_i} \; z^{k_i}}{(k_i)!}\right] V_n(k_1, \cdots, k_n)$$
$$= \prod_{k=0}^{n-1} \left[k! (a)_k \; z^k (1 - z)^{-a-2k}\right],$$

in agreement with (29).

Remark 9 *The moments associated with (31) are given by [15]*

$$\mu_n(z) = \sum_{k=0}^{\infty} k^n \; (a)_k \frac{z^k}{k!} = (1 - z)^{-a-n} P_n(z),$$

where $P_n(z)$ are the polynomials

$$P_n(z) = \sum_{k=0}^{n} S(n,k)\,(a)_k \; z^k (1-z)^{n-k}, \tag{34}$$

and $S(n,k)$ denote the Stirling numbers of the second kind defined by [43, 26.8]

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$

Since

$$\Delta_n = (1-z)^{-an-\frac{1}{2}n(n-1)} \det_{0\le i,j\le n-1} (P_{i+j}),$$

we see from (29) that

$$\det_{0\le i,j\le n-1} (P_{i+j}) = z^{\frac{1}{2}n(n-1)} \prod_{k=0}^{n-1} k! \; (a)_k. \tag{35}$$

The polynomials (34) and their Hankel determinants (35) seem not to have been studied before.

2.3 Charlier polynomials

Corollary 10 *Let $z > 0$ and $n = 1, 2, \ldots$. Then,*

$$\sum_{0\le k_1<k_2<\cdots<k_n} \prod_{i=1}^{n} \left[\frac{z^{k_i}}{(k_i)!}\right] V_n(k_1,\cdots,k_n) = \prod_{k=0}^{n-1} \left[(k!)^2 \frac{z^k}{k!} e^z\right]. \tag{36}$$

Proof. If we set

$$z = \frac{z}{z+a}$$

in (29), we have

$$\sum_{0\le k_1<k_2<\cdots<k_n} \prod_{i=1}^{n} \left[\frac{(a)_{k_i} \left(\frac{z}{z+a}\right)^{k_i}}{(k_i)!}\right] V_n(k_1,\cdots,k_n)$$
$$= \prod_{k=0}^{n-1} \left[(k!)^2 \frac{(a)_k \left(\frac{z}{z+a}\right)^k}{k!} \left(1-\frac{z}{z+a}\right)^{-a-2k}\right].$$

The result follows from the limits

$$\lim_{a\to\infty}\frac{(a)_k}{(z+a)^k}=\lim_{a\to\infty}\prod_{j=0}^{k-1}\frac{a+j}{a+z}=1,$$

and

$$\lim_{a\to\infty}\left(1-\frac{z}{z+a}\right)^{-a-2k}\lim_{a\to\infty}\left(1+\frac{z}{a}\right)^{a+2k}=e^z.$$

■

The monic Charlier polynomials are defined by [43, 18.20.7]

$$C_n(x)=(-z)^n\ {}_2F_0\left[\begin{matrix}-n,\ -x\\ -\end{matrix};-z^{-1}\right].$$

When $z>0$, the monic Charlier polynomials satisfy the orthogonality relation [43, 18.19.1]

$$\sum_{k=0}^{\infty}C_n(k)\,C_m(k)\,\frac{z^k}{k!}=n!z^ne^z\delta_{n,m}. \tag{37}$$

If we choose

$$c_k=\frac{z^k}{k!},\quad k=0,1,\ldots, \tag{38}$$

we see from (37) that the monic orthogonal polynomials associated with the linear functional L defined by (32) are the Charlier polynomials. But using (6), (15) and (37), we get

$$\lim_{N\to\infty}\sum_{0\le k_1<k_2<\cdots<k_n\le N}\prod_{i=1}^{n}\left[\frac{z^{k_i}}{(k_i)!}\right]V_n(k_1,\cdots,k_n)^2=\prod_{k=0}^{n-1}\left(k!z^ke^z\right),$$

in agreement with (36).

Remark 11 *The moments associated with (38) are given by [15]*

$$\mu_n(z)=\sum_{k=0}^{\infty}k^n\frac{z^k}{k!}=e^zT_n(z),$$

where $T_n(z)$ are the Touchard (or exponential, or Bell) polynomials

$$T_n(z)=\sum_{k=0}^{n}S(n,k)\,z^k,$$

We clearly have

$$\Delta_n = e^{nz} \det_{0\le i,j\le n-1} \left(T_{i+j}\right).$$

The determinant

$$\det_{0\le i,j\le n-1} \left(T_{i+j}\right) = z^{\frac{1}{2}n(n-1)} \prod_{k=0}^{n-1} k!$$

has been computed by several authors in many different ways, see [5], [18], [21], [30], [40], [44], [45], and [47]. The special case $z = 1$ *(Bell numbers), was considered in [2], [12], [37], [51], and [57].*

3 Conclusions

We have established a connection between C. Krattenthaler's identity (11) and the Hankel determinants of moments of Hahn polynomials. As we mentioned at end of the last section, the corresponding identity for Hankel determinants of Charlier polynomials has appeared in the literature multiple times.

We have not been able to find any other instance of the determinants

$$\det_{0\le i,j\le n-1} \left(\sum_{k=0}^{N} k^{i+j} c_k\right), \quad \det_{0\le i,j\le n-1} \left(\sum_{k=0}^{\infty} k^{i+j} c_k\right),$$

for general c_k, or at least for c_k being a hypergeometric term (we don't claim that they don't exist, but we have not uncovered a single reference). That's why we were so amazed to learn about (11).

The next case of interest will be

$$c_k = \frac{(\alpha_1)_k (\alpha_2)_k (\alpha_3)_k}{(\beta_1)_k (\beta_2)_k} \frac{1}{k!},$$

which is the weight function for the Generalized Hahn polynomials of type II introduced in [16]. Therefore, we end the note with the following.

Problem 12 *Let* $1 \le n \le N$*. Find a "closed form" for the Hankel determinant*

$$\det_{0\le i,j\le n-1} \left[\sum_{k=0}^{N} k^{i+j} \binom{N}{k} \frac{(u)_k (x)_k (y)_{N-k}}{(v)_k}\right].$$

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