Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

JOHANNES KEPLER UNIVERSITY LINZ

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DK-Report No. 2018-08 06 2018

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by
Austrian Science Fund (FWF) Upper Austria

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June 11, 2018

Abstract

In this article, we find a power series expansion for the Hankel determinant whose entries are the moments of a linear functional related to discrete semiclassical orthogonal polynomials. We provide explicit formulas that allow the computation of coefficients to arbitrary order, and give examples for the first few terms in the series.

We reinterpret these results in the context of the theory of Young tableaux and Schur functions, and find closed-form expressions for the cases corresponding to the Charlier and Meixner polynomials.

We also discuss further research directions.

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Keywords: Hankel determinants, orthogonal polynomials, integer partitions, Young tableaux, Schur polynomials.

Subject Classification Codes: 15A15 (primary), 33C45, 05A17, 05E05 (secondary).

1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0$ the set of nonnegative integers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

If $\{\mu_n\}$ is a sequence of complex numbers and $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ is the linear functional defined by

$$\mathcal{L}[x^n] = \mu_n, \quad n \in \mathbb{N}_0,$$

then $\mathcal{L}$ is called the *moment functional* [4] determined by the formal moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n.

Suppose that $\{P_n\}$ is a family of monic polynomials, with $\deg(P_n) = n$. If the polynomials $P_n(x)$ satisfy

$$\mathcal{L}[P_n P_m] = h_n \delta_{n,m}, \quad n, m \in \mathbb{N}_0, \tag{1}$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta, then $\{P_n\}$ is called an *orthogonal polynomial sequence* with respect to $\mathcal{L}$.

If we write

$$x^n = \sum_{k=0}^{n} a_{n,k} P_k(x), \quad n \in \mathbb{N}_0,$$

we can define a lower triangular matrix A_n by

$$(A_n)_{i,j} = a_{i,j}, \quad 0 \leq i, j \leq n-1.$$

If we define the diagonal matrix D_n by

$$(D_n)_{i,j} = h_i \delta_{i,j}, \quad 0 \leq i, j \leq n-1,$$

and the Hankel matrix H_n by

$$(H_n)_{i,j} = \mu_{i+j}, \quad 0 \leq i, j \leq n-1, \tag{2}$$

then we have the *LDL decomposition* [12, 4.1.2]

$$H_n = A_n D_n A_n^T. \tag{3}$$

We define the *Hankel determinants* Δ_n by $\Delta_0 = 1$ and

$$\Delta_n = \det(H_n), \quad n \in \mathbb{N}. \tag{4}$$

Determinants have a long history and an extensive literature, see [1], [3], [21], [25], [29], [30], [31], [32], and the impressive monographs [17] and [18].

The theories of Hankel determinants and orthogonal polynomials are deeply connected, see for example [4], [6], [9], [10], [14], [15], [16] [19], and [28].

In [8], we studied the discrete semiclassical orthogonal polynomials of class 1, and considered linear functionals $\mathcal{L}$ that have the form

$$\mathcal{L}[p] = \sum_{k=0}^{\infty} p(k)\omega(k) z^k, \quad p(x) \in \mathbb{C}[x], \tag{5}$$

for $z \in \mathbb{C}$ and some function ω with

$$\omega(k) \neq 0, \quad k \in \mathbb{N}_0. \tag{6}$$

In this case, the moments $\mu_i(z)$ are given by

$$\mu_i(z) = \sum_{k=0}^{\infty} k^i \omega(k) z^k, \quad i \in \mathbb{N}_0, \tag{7}$$

and the entries of the Hankel matrix (2) are

$$(H_n)_{i,j} = \mu_{i+j}(z) = \sum_{k=0}^{\infty} k^{i+j} \omega(k) z^k, \quad 0 \leq i, j \leq n-1. \tag{8}$$

We note from (7) that all the moments $\mu_i(z)$ can be obtained from the first one $\mu_0(z)$, since

$$\mu_i(z) = \sum_{k=0}^{\infty} k^i \omega(k) z^k = \sum_{k=0}^{\infty} \omega(k) \vartheta^i\left(z^k\right) = \vartheta^i \mu_0(z),$$

where the differential operator ϑ is defined by [23, Chapter 6]

$$\vartheta = z\frac{d}{dz}.$$

It follows that if the first moment $\mu_0(z)$ is analytic in some disk $|z_0| < r$, the same will be true for all the other moments. In fact, we showed in [7] that for all families of discrete semiclassical orthogonal polynomials of class s, the moments have the form

$$\mu_i(z) = (\lambda + \tau z)^{-i} \sum_{k=0}^{s} p_k(z) \mu_k(z),$$

where the constants $\lambda, \tau \in \{-1, 0, 1\}$ and the polynomials $p_k(z)$ depend on the given family.

Since the Hankel determinants $\Delta_n(z)$ are analytic functions of z (in the same domain as μ_0), it is natural to consider the Taylor series

$$\Delta_n(z) = \sum_{m=0}^{\infty} d_m(n) z^m, \quad |z| < r, \tag{9}$$

and try to determine the coefficients $d_m(n)$. Surprisingly enough, we haven't been able to find many references on this topic. In [24], Rusk considered at the $n^{th}-$derivative of a general determinant, whose elements are functions of a variable t, and found some connection with a symbolic version of the multinomial theorem. In [5], Christiano and Hall used the general Leibniz rule and obtained a formula for the $n^{th}-$derivative in terms of determinants.

In [13], Hochstadt considered the derivative of a Wronskian determinant

$$W(t) = \det X(t),$$

where $X(t)$ satisfies the matrix equation

$$\frac{d}{dt}X(t) = A(t) X(t),$$

and obtained

$$\frac{d}{dt}W(t) = \operatorname{tr}(A) W(t),$$

where $\operatorname{tr}(A)$ denotes the trace of A. In [11], Golberg generalized this result and proved Jacobi's formula

$$\frac{d}{dt}\det A(t) = \operatorname{tr}\left(\operatorname{adj} A \frac{d}{dt} A\right),$$

where $\operatorname{adj}(A)$ is the adjugate of A defined by

$$\operatorname{adj}(A)\, A = \det(A)\, I.$$

Lastly, in [33] Withers and Nadarajah studied the Taylor series of a determinant with general entries, and obtained a formula involving traces of powers of the matrix and the complete exponential Bell polynomials.

In this article, we obtain explicit expressions for the coefficients $d_m(n)$ in the Taylor series (9) when the entries of the Hankel matrix are given by (8). The paper is organized as follows: in Section 2, we derive a formula for $d_m(n)$ when $\omega(k)$ is a general function. We give some examples for special cases of $\omega(k)$.

In Section 3 we relate the results from Section 2 to the theory of Schur polynomials. We obtain exact evaluations of $d_m(n)$ for the Charlier and Meixner polynomials. Finally, in Section 4 we give a summary of results and point out some future directions.

2 Main result

2.1 Vandermonde polynomials

Definition 1 *The* Vandermonde determinant $V(x_1, x_2, \ldots, x_n)$ *is defined by*

$$V(x_1, x_2, \ldots, x_n) = \det_{1\leq i,j\leq n}\left(x_i^{j-1}\right) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{j-1},$$

where $\operatorname{sgn}(\sigma)$ *denotes the sign of the permutation* σ.

Remark 2 *It is well known that the Vandermonde determinant* $V(x_1, x_2, \ldots, x_n)$ *is an alternating multivariate polynomial given by [31, 4.1.2]*

$$V(x_1, x_2, \ldots, x_n) = \prod_{1\leq i<j\leq n} (x_j - x_i), \quad n > 1. \tag{10}$$

Definition 3 *If $a, m \in \mathbb{Z}$ and $m \leq 0$, we define*

$$V(x_a, x_{a+1}, \ldots, x_{a+m}) = 1. \tag{11}$$

Next, we derive some basic results about Vandermonde polynomials.

Lemma 4 *If $a \in \mathbb{Z}$ and $m \in \mathbb{N}$, then*

$$V(x_a, x_{a+1}, \ldots, x_{a+m}) = \prod_{j=0}^{m}\prod_{i=0}^{j-1}(x_{j+a} - x_{i+a}), \tag{12}$$

Proof. Using (10), we have

$$\begin{aligned} V(x_a, x_{a+1}, \ldots, x_{a+m}) &= \prod_{a \leq i < j \leq a+m}(x_j - x_i) = \prod_{j=a}^{a+m}\prod_{i=a}^{j-1}(x_j - x_i) \\ &= \prod_{j=0}^{m}\prod_{i=a}^{j+a-1}(x_{j+a} - x_i) = \prod_{j=0}^{m}\prod_{i=0}^{j-1}(x_{j+a} - x_{i+a}). \end{aligned}$$

■

Corollary 5 *If $m \in \mathbb{N}_0$, then*

$$V(0, 1, 2, \ldots, m) = \mathcal{G}(m+1), \tag{13}$$

where the function $\mathcal{G}(m)$ is defined by

$$\mathcal{G}(0) = 1$$

and

$$\mathcal{G}(m) = \prod_{i=0}^{m-1}(i!), \quad m \in \mathbb{N}. \tag{14}$$

Note that $\mathcal{G}(m)$ satisfies the recurrence

$$\mathcal{G}(m+1) = m!\,\mathcal{G}(m).$$

Proof. If we set $a = 0$ and

$$x_i = i, \quad 0 \le i \le m$$

in (12), we get

$$\begin{aligned} V\left(0,1,2,\ldots,m\right) &= \prod_{j=0}^{m}\prod_{i=0}^{j-1}\left(j-i\right) = \prod_{j=0}^{m}\prod_{k=1}^{j}k \qquad (k = j-i) \\ &= \prod_{j=0}^{m} j! = \mathcal{G}\left(m+1\right). \end{aligned}$$

■

Remark 6 *The function* $\mathcal{G}\left(m\right)$ *can be written as*

$$\mathcal{G}\left(m\right) = G(m+1),$$

where $G(m)$ *is Barnes' G-Function [22, 5.17], defined by* $G(1) = 1$ *and*

$$G\left(z+1\right) = \Gamma\left(z\right) G(z), \quad z \in \mathbb{C}.$$

Proposition 7 *For all* $n \in \mathbb{N}$ *and* $r \in \mathbb{N}_0$*, we have*

$$\begin{aligned} V\left(x_1,x_2,\ldots,x_n\right) &= V\left(x_1,x_2,\ldots,x_{n-r}\right) V\left(x_{n-r+1},\ldots,x_n\right) \\ &\times \prod_{j=n-r+1}^{n}\prod_{i=1}^{n-r}\left(x_j - x_i\right), \end{aligned} \tag{15}$$

where empty products are assumed to be equal to 1.

Proof. Setting $a = 1$ and $m = n-r-1$ in (12), we get

$$V\left(x_1,x_2,\ldots,x_{n-r}\right) = \prod_{j=0}^{n-r-1}\prod_{i=0}^{j-1}\left(x_{j+1} - x_{i+1}\right).$$

Similarly, if we set $a = n-r+1$ and $m = r-1$ in (12), we have

$$V\left(x_{n-r+1},\ldots,x_n\right) = \prod_{j=0}^{r-1}\prod_{i=0}^{j-1}\left(x_{j+n-r+1} - x_{i+n-r+1}\right).$$

Thus,

$$\frac{V(x_1, x_2, \ldots, x_n)}{V(x_1, x_2, \ldots, x_{n-r})\, V(x_{n-r+1}, \ldots, x_n)}$$

$$= \frac{\prod_{j=0}^{n-1}\prod_{i=0}^{j-1}(x_{j+1} - x_{i+1})}{\left[\prod_{j=0}^{n-r-1}\prod_{i=0}^{j-1}(x_{j+1} - x_{i+1})\right]\left[\prod_{j=0}^{r-1}\prod_{i=0}^{j-1}(x_{j+n-r+1} - x_{i+n-r+1})\right]}$$

$$= \frac{\prod_{j=n-r}^{n-1}\prod_{i=0}^{j-1}(x_{j+1} - x_{i+1})}{\prod_{j=0}^{r-1}\prod_{i=0}^{j-1}(x_{j+n-r+1} - x_{i+n-r+1})}.$$

But

$$\begin{aligned}\prod_{j=0}^{r-1}\prod_{i=0}^{j-1}(x_{j+n-r+1} - x_{i+n-r+1}) &= \prod_{j=n-r}^{n-1}\prod_{i=0}^{j-n-r-1}(x_{j+1} - x_{i+n-r+1}) \\ &= \prod_{j=n-r}^{n-1}\prod_{i=n-r}^{j-1}(x_{j+1} - x_{i+1}).\end{aligned}$$

Hence,

$$\frac{V(x_1, x_2, \ldots, x_n)}{V(x_1, x_2, \ldots, x_{n-r})\, V(x_{n-r+1}, \ldots, x_n)}$$

$$= \prod_{j=n-r}^{n-1}\frac{\prod_{i=0}^{j-1}(x_{j+1} - x_{i+1})}{\prod_{i=n-r}^{j-1}(x_{j+1} - x_{i+1})} = \prod_{j=n-r}^{n-1}\prod_{i=0}^{n-r-1}(x_{j+1} - x_{i+1}).$$

We conclude that

$$\frac{V(x_1, x_2, \ldots, x_n)}{V(x_1, x_2, \ldots, x_{n-r})\, V(x_{n-r+1}, \ldots, x_n)} = \prod_{j=n-r+1}^{n}\prod_{i=1}^{n-r}(x_j - x_i).$$

■

We now obtain a representation for $\Delta_n(z)$ in terms of Vandermonde polynomials.

Theorem 8 *Let the moments $\mu_i(z)$ be defined by*

$$\mu_i(z) = \sum_{k=0}^{\infty} k^i \omega(k) z^k,$$

and let $\Delta_n(z)$ denote the Hankel determinant

$$\Delta_n(z) = \det_{1 \le i,j \le n} (\mu_{i+j-2}).$$

Then,

$$\Delta_n(z) = \sum_{0 \le k_1 < k_2 < \cdots < k_n} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] V^2(k_1, k_2, \ldots, k_n). \tag{16}$$

Proof. This result is essentially contained in the proof of the Theorem in Section 6.10.4 of [31]. We reproduce the main steps for the purpose of completion.

Let $n \le N$ and

$$\Delta_{n,N}(z) = \det_{1 \le i,j \le n} \left(\sum_{k_i=0}^{N} k_i^{i+j-2} \omega(k_i) z^{k_i} \right).$$

Using the linearity of the determinant, we have

$$\begin{aligned} \Delta_{n,N}(z) &= \sum_{k_1,\ldots,k_n=0}^{N} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] \left[\prod_{i=1}^{n} k_i^{i-1} \right] \det_{1 \le i,j \le n} \left(k_i^{j-1} \right) \\ &= \sum_{k_1,\ldots,k_n=0}^{N} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] \left[\prod_{i=1}^{n} k_i^{i-1} \right] V(k_1, k_2, \ldots, k_n). \end{aligned}$$

Since

$$\prod_{i=1}^{n} \omega(k_i) z^{k_i}$$

is a symmetric function of $(k_1, k_2, \ldots, k_n)$, we get

$$\Delta_{n,N}(z) = \frac{1}{n!} \sum_{k_1,\ldots,k_n=0}^{N} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} k_{\sigma(i)}^{i-1} \right) V\left(k_{\sigma(1)}, \ldots, k_{\sigma(n)}\right).$$

Using the identity [31, 4.1.9]

$$\sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} k_{\sigma(i)}^{i-1} \right) V\left(k_{\sigma(1)}, \ldots, k_{\sigma(n)}\right) = V^2(k_1, k_2, \ldots, k_n),$$

we obtain

$$\Delta_{n,N}(z) = \frac{1}{n!} \sum_{k_1,\ldots,k_n=0}^{N} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] V^2(k_1, k_2, \ldots, k_n).$$

Finally, since

$$V^2(k_1, k_2, \ldots, k_n)$$

is a symmetric function of $(k_1, k_2, \ldots, k_n)$ that vanishes if $k_i = k_j$, we can rewrite $\Delta_{n,N}(z)$ as

$$\Delta_{n,N}(z) = \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \left[\prod_{i=1}^{n} \omega(k_i) z^{k_i} \right] V^2(k_1, k_2, \ldots, k_n).$$

Taking the limit as $N \to \infty$, the result follows. ■

2.2 Integer partitions

In this section, we will collect powers of z in (16) to find a power series for $\Delta_n(z)$. First, we define the concept of integer partitions.

Definition 9 *A* partition Λ *is any (finite or infinite) weakly decreasing sequence*

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \tag{17}$$

of non-negative integers

$$\lambda_i \ge \lambda_{i+1} \ge 0, \quad i \in \mathbb{N},$$

containing only finitely many non-zero terms. We do not distinguish between two partitions which differ only by a string of zeros at the end. For example,

$$(2,1) = (2,1,0) = (2,1,0,0).$$

The non-zero elements λ_i *in (17) are called the* parts *of* Λ*. The number of parts is the* length *of* Λ*, denoted by* $l(\Lambda)$

$$l(\Lambda) = \operatorname{card}\{\lambda_i \in \Lambda \mid \lambda_i \neq 0\} < \infty.$$

The sum of the parts is the weight *of* Λ*, denoted by* $\|\Lambda\|$*,*

$$\|\Lambda\| = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{l(\Lambda)} \lambda_i.$$

If $\|\Lambda\| = m$*, we say that* Λ *is a partition of* m*. The set of all partitions of* m *is denoted by* $\mathcal{P}(m)$*. In particular,* $\mathcal{P}(0)$ *consists of a single element, the unique partition of zero, which we denote by* $\Lambda = (0)$*.*

We will denote by $\mathcal{P}_n(m)$ *the set of all partitions of* m *into at most* n *parts,*

$$\mathcal{P}_n(m) = \{\Lambda \mid \|\Lambda\| = m \quad \text{and} \quad l(\Lambda) \leq n\}.$$

We now establish a connection between the sum in (16) and those indexed by partitions.

Lemma 10 *Let* $n \in \mathbb{N}$ *and* f *be an arbitrary function. Then,*

$$\sum_{0 \leq k_1 < k_2 < \cdots < k_n} f(k_1, k_2, \ldots, k_n) = \sum_{l(\Lambda) \leq n} f(\lambda_n, \lambda_{n-1} + 1, \ldots, \lambda_1 + n - 1). \quad (18)$$

Proof. We define the bijection

$$\varphi : \{\Lambda \mid l(\Lambda) \leq n\} \to \{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \mid 0 \leq k_1 < k_2 < \cdots < k_n\}$$

by

$$\varphi = T \circ \sigma,$$

where $\sigma \in S_n$ is given by

$$\sigma(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$$

and the affine transformation T is defined by

$$T(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) + (0, 1, \ldots, n-1).$$

Note that

$$\sigma : \{\Lambda \mid l(\Lambda) \leq n\} \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n\}$$

and

$$T : \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n\}$$
$$\to \{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \mid 0 \leq k_1 < k_2 < \cdots < k_n\}$$

since $k_1 = x_1 \geq 0$ and

$$k_i = x_i + i - 1 \leq x_{i+1} + i - 1 < x_{i+1} + i = k_{i+1}, \quad 1 \leq i \leq n-1.$$

■

We can now write (16) as a sum over partitions.

Proposition 11 *Let $n \in \mathbb{N}$. Then,*

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}} = \sum_{m=0}^{\infty} z^m \sum_{\Lambda \in \mathcal{P}_n(m)} \left[\prod_{i=1}^{n} \omega(\lambda_{n-i+1} + i - 1) \right] \times V^2(\lambda_n, \lambda_{n-1} + 1, \ldots, \lambda_1 + n - 1). \tag{19}$$

Proof. Using (18) in (16), we have

$$\Delta_n(z) = \sum_{l(\Lambda) \leq n} \left[\prod_{i=1}^{n} \omega(\lambda_{n-i+1} + i - 1) z^{\lambda_{n-i+1} + i - 1} \right] \times V^2(\lambda_n, \lambda_{n-1} + 1, \ldots, \lambda_1 + n - 1).$$

But

$$\sum_{i=1}^{n} (\lambda_{n-i+1} + i - 1) = \sum_{i=1}^{n} \lambda_i + \frac{n(n-1)}{2} = m + \binom{n}{2},$$

and therefore

$$\sum_{l(\Lambda) \leq n} \left[\prod_{i=1}^{n} \omega(\lambda_{n-i+1} + i - 1) z^{\lambda_{n-i+1} + i - 1} \right]$$
$$= z^{\binom{n}{2}} \sum_{m=0}^{\infty} z^m \sum_{\Lambda \in \mathcal{P}_n(m)} \left[\prod_{i=1}^{n} \omega(\lambda_{n-i+1} + i - 1) \right].$$

■

We have now all the elements to prove our main theorem.

Theorem 12 *If* $\Lambda \in \mathcal{P}_n(m)$, *then*

$$V(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1) = \mathcal{G}(n)\, L_\Lambda \Psi_\Lambda(n), \tag{20}$$

where

$$L_\Lambda = \prod_{i=1}^{l(\Lambda)} \frac{1}{(\lambda_i)!} \prod_{j=1}^{i-1} \left(1 - \frac{\lambda_i}{\lambda_j + i - j}\right), \tag{21}$$

and

$$\Psi_\Lambda(n) = \prod_{i=1}^{l(\Lambda)} (n-i+1)_{\lambda_i}. \tag{22}$$

Proof. Let $l(\Lambda) = r$. (i) If $r = 0$, then $\Lambda = (0)$ and using (13), we have

$$\begin{aligned} V(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1) &= V(0,1,2,\ldots,n-1) \\ &= \mathcal{G}(n) = \mathcal{G}(n)\, L_{(0)} \Psi_{(0)}(n), \end{aligned}$$

since

$$L_{(0)} = \Psi_{(0)}(n) = 1. \tag{23}$$

(ii) If $r = 1$, then $\Lambda = (m)$. Using (13) and (15), we get

$$\begin{aligned} V(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1) &= V(0,1,2,\ldots,n-2,m+n-1) \\ &= V(0,1,2,\ldots,n-2) \prod_{i=1}^{n-1} (m+n-i) = \mathcal{G}(n-1) \frac{\Gamma(m+n)}{\Gamma(m+1)}. \end{aligned}$$

But

$$\frac{\Gamma(m+n)}{\Gamma(m+1)} = \frac{\Gamma(n)}{\Gamma(m+1)} \frac{\Gamma(m+n)}{\Gamma(n)} = \frac{(n-1)!}{m!} (n)_m,$$

and therefore

$$\begin{aligned} V(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1) &= \mathcal{G}(n-1) \frac{(n-1)!}{m!} (n)_m \\ &= \mathcal{G}(n)\, L_{(m)} \Psi_{(m)}(n), \end{aligned}$$

since

$$\mathcal{G}(n) = (n-1)!\,\mathcal{G}(n-1), \quad L_{(m)} = \frac{1}{m!}, \quad \Psi_{(m)}(n) = (n)_m.$$

(iii) If $r \geq 2$, then using (13) and (15), we obtain

$$V\left(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1\right) = V\left(0,1,2,\ldots,n-r-1\right)$$
$$\times V\left(\lambda_r+n-r, \ldots, \lambda_1+n-1\right) \prod_{j=n-r+1}^{n} \prod_{i=1}^{n-r} \left(\lambda_{n-j+1}+j-i\right)$$
$$= \mathcal{G}\left(n-r-2\right) V\left(\lambda_r+n-r, \ldots, \lambda_1+n-1\right) \prod_{j=n-r+1}^{n} \prod_{i=1}^{n-r} \left(\lambda_{n-j+1}+j-i\right).$$

(iv) We have

$$\prod_{j=n-r+1}^{n} \prod_{i=1}^{n-r} \left(\lambda_{n-j+1}+j-i\right) = \prod_{l=1}^{r} \prod_{i=1}^{n-r} \left(\lambda_l+n-l+1-i\right) \qquad (l=n-j+1)$$
$$= \prod_{l=1}^{r} \prod_{s=0}^{n-r-1} \left(\lambda_l-l+r+1+s\right) \qquad (s=n-r-i)$$
$$= \prod_{l=1}^{r} \left(\lambda_l-l+r+1\right)_{n-r}.$$

Setting $x=\lambda_l+1$, $a=r-l$ and $b=n-r$ in the identity

$$(x)_{a+b} = (x)_a \, (x+a)_b \, ,$$

we get

$$\left(\lambda_l-l+r+1\right)_{n-r} = \frac{\left(\lambda_l+1\right)_{n-l}}{\left(\lambda_l+1\right)_{r-l}}.$$

Hence,

$$\prod_{l=1}^{r} \left(\lambda_l-l+r+1\right)_{n-r} = \prod_{l=1}^{r} \frac{\left(\lambda_l+1\right)_{n-l}}{\left(\lambda_l+1\right)_{r-l}} = \frac{\prod_{l=1}^{r} \left(\lambda_l+1\right)_{n-l}}{\prod_{l=1}^{r-1} \left(\lambda_l+1\right)_{r-l}},$$

since $(x)_0 = 1$. We conclude that

$$\prod_{j=n-r+1}^{n} \prod_{i=1}^{n-r} (\lambda_{n-j+1} + j - i) = \frac{\prod_{l=1}^{r} (\lambda_l + 1)_{n-l}}{\prod_{l=1}^{r-1} \prod_{s=0}^{r-l-1} (\lambda_l + 1 + s)} = \frac{\prod_{l=1}^{r} (\lambda_l + 1)_{n-l}}{\prod_{l=1}^{r-1} \prod_{s=l+1}^{r} (\lambda_l + s - l)}.$$

(v) Using (12), we get

$$\begin{aligned} V(\lambda_r + n - r, \ldots, \lambda_1 + n - 1) &= \prod_{j=0}^{r-2} \prod_{i=0}^{j} (\lambda_{r-j-1} - \lambda_{r-i} + j - i + 1) \\ &= \prod_{l=1}^{r-1} \prod_{i=0}^{r-1-l} (\lambda_l - \lambda_{r-i} + r - l - i) \qquad (l = r - j - 1) \\ &= \prod_{l=1}^{r-1} \prod_{s=l+1}^{r} (\lambda_l - \lambda_s + s - l) \qquad (s = r - i). \end{aligned}$$

(vi) From the previous results, we have

$$\begin{aligned} & V(\lambda_n, \lambda_{n-1} + 1, \ldots, \lambda_1 + n - 1) \\ &= \left(\prod_{l=1}^{n-r-1} l! \right) \left[\frac{\prod_{l=1}^{r} (\lambda_l + 1)_{n-l}}{\prod_{l=1}^{r-1} \prod_{s=l+1}^{r} (\lambda_l + s - l)} \right] \left[\prod_{l=1}^{r-1} \prod_{s=l+1}^{r} (\lambda_l - \lambda_s + s - l) \right] \\ &= \left[\prod_{l=1}^{r} (\lambda_l + 1)_{n-l} \right] \prod_{l=1}^{r-1} \prod_{s=l+1}^{r} \left(\frac{\lambda_l + s - l - \lambda_s}{\lambda_l + s - l} \right). \end{aligned}$$

Setting $x = 1$, $a = \lambda_l$ and $b = n - l$ in the identity

$$\frac{(x + a)_b}{(x)_b} = \frac{(x + b)_a}{(x)_a},$$

we obtain

$$(\lambda_l + 1)_{n-l} = \frac{(1)_{n-l} (n - l + 1)_{\lambda_l}}{(1)_{\lambda_l}}.$$

Thus,

$$\prod_{l=1}^{r} (\lambda_l + 1)_{n-l} = \prod_{l=1}^{r} \frac{(1)_{n-l} (n-l+1)_{\lambda_l}}{(1)_{\lambda_l}} = \left[\prod_{l=1}^{r} (n-l)!\right] \left[\prod_{l=1}^{r} \frac{(n-l+1)_{\lambda_l}}{(\lambda_i)!}\right]$$
$$= \left(\prod_{i=n-r}^{n-1} i!\right) \left[\prod_{l=1}^{r} \frac{(n-l+1)_{\lambda_l}}{(\lambda_i)!}\right] \qquad (i = n-l).$$

We conclude that

$$V(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1)$$
$$= \left[\prod_{i=1}^{n-1} i!\right] \left[\prod_{l=1}^{r} \frac{(n-l+1)_{\lambda_l}}{(\lambda_i)!}\right] \prod_{l=1}^{r-1} \prod_{s=l+1}^{r} \left(\frac{\lambda_l + s - l - \lambda_s}{\lambda_l + s - l}\right).$$

(vii) Finally,

$$\prod_{l=1}^{r-1} \prod_{s=l+1}^{r} \left(\frac{\lambda_l + s - l - \lambda_s}{\lambda_l + s - l}\right) = \prod_{l=1}^{r-1} \prod_{s=l+1}^{r} \left(1 - \frac{\lambda_s}{\lambda_l + s - l}\right)$$
$$= \prod_{s=2}^{r} \prod_{l=1}^{s-1} \left(1 - \frac{\lambda_s}{\lambda_l + s - l}\right),$$

since

$$2 \leq l+1 \leq s \leq r.$$

But for $s = 1$ we get

$$\prod_{l=1}^{0} \left(1 - \frac{\lambda_1}{\lambda_l + 1 - l}\right) = 1$$

and therefore we can write

$$\prod_{l=1}^{r-1} \prod_{s=l+1}^{r} \frac{\lambda_l + s - l - \lambda_s}{\lambda_l + s - l} = \prod_{s=2}^{r} \prod_{l=1}^{s-1} \left(1 - \frac{\lambda_s}{\lambda_l + s - l}\right).$$

■

With the help of the previous result, we obtain a new expression for $\Delta_n(z)$ as a sum over partitions.

Corollary 13 *Let $n \in \mathbb{N}$. Then,*

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}\mathcal{G}^2(n)\left[\prod_{i=0}^{n-1}\omega(i)\right]} = \sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} \left[\prod_{i=1}^{l(\Lambda)} \frac{\omega(\lambda_i+n-i)}{\omega(n-i)}\right] [L_\Lambda \Psi_\Lambda(n)]^2 . \tag{24}$$

Proof. Let $\Lambda \in \mathcal{P}_n(m)$. Since

$$\lambda_s = 0, \quad l(\Lambda) < s \le n,$$

we have

$$\prod_{i=1}^{n}\omega(\lambda_{n-i+1}+i-1) = \prod_{s=1}^{n}\omega(\lambda_s+n-s) \qquad (i = n-s+1)$$
$$= \left[\prod_{s=1}^{n}\frac{\omega(\lambda_s+n-s)}{\omega(n-s)}\right]\prod_{s=0}^{n-1}\omega(s) = \left[\prod_{s=1}^{l(\Lambda)}\frac{\omega(\lambda_s+n-s)}{\omega(n-s)}\right]\prod_{s=0}^{n-1}\omega(s),$$

where we recall from (6) that $\omega(s) \neq 0$.

If $l(\Lambda) \le n$, then the formula follows after using (20) in (19). If $l(\Lambda) > n$, then

$$\Psi_\Lambda(n) = \prod_{i=1}^{n}(n-i+1)_{\lambda_i}\prod_{i=n+1}^{l(\Lambda)}(n-i+1)_{\lambda_i} = 0,$$

and therefore

$$\sum_{\Lambda\in\mathcal{P}_n(m)}\left[\prod_{i=1}^{l(\Lambda)}\frac{\omega(\lambda_i+n-i)}{\omega(n-i)}\right][L_\Lambda\Psi_\Lambda(n)]^2$$
$$= \sum_{\Lambda\in\mathcal{P}(m)}\left[\prod_{i=1}^{l(\Lambda)}\frac{\omega(\lambda_i+n-i)}{\omega(n-i)}\right][L_\Lambda\Psi_\Lambda(n)]^2 .$$

■

We now compute some particular values of $L_\Lambda\Psi_\Lambda(n)$.

Proposition 14 *We have*

$$L_{(0)}\Psi_{(0)}(n) = 1, \tag{25}$$

$$L_{(m-2,2)}\Psi_{(m-2,2)}(n) = \frac{m-3}{2} n \binom{n+m-3}{m-1}, \tag{26}$$

and for $0 \leq a \leq m-1$,

$$L_{(m-a,1,\ldots,1)}\Psi_{(m-a,1,\ldots,1)}(n) = \binom{m-1}{a}\binom{n+m-1-a}{m}. \tag{27}$$

In particular, we have

$$L_{(m)}\Psi_{(m)}(n) = \binom{n-1+m}{m}, \tag{28}$$

and

$$L_{(1,\ldots,1)}\Psi_{(1,\ldots,1)}(n) = \binom{n}{m}. \tag{29}$$

Proof. Using (21) and (22), we get

$$L_{(m-2,2)} = \frac{1}{(m-2)!}\frac{1}{2!}\left(1 - \frac{2}{m-2+1}\right) = \frac{1}{2}\frac{m-3}{(m-1)!}$$

and

$$\Psi_{(m-2,2)}(n) = (n)_{m-2}(n-1)_2 = n\frac{(n+m-3)!}{(n-2)!}. \tag{30}$$

Thus,

$$L_{(m-2,2)}\Psi_{(m-2,2)}(n) = (m-3)\,\frac{n}{2}\frac{(n+m-3)!}{(m-1)!\,(n-2)!}.$$

Similarly,

$$\begin{aligned} L_{(m-a,1,\ldots,1)} &= \frac{1}{(m-a)!}\prod_{i=2}^{a+1}\left[\left(1 - \frac{1}{m-a+i-1}\right)\prod_{j=2}^{i-1}\left(1 - \frac{1}{1+i-j}\right)\right] \\ &= \frac{1}{(m-a)!}\prod_{i=2}^{a+1}\left[\left(1 - \frac{1}{m-a+i-1}\right)\frac{1}{i-1}\right] = \frac{1}{(m-a)!}\frac{m-a}{m}\frac{1}{a!} \end{aligned}$$

and

$$\Psi_{(m-a,1,\ldots,1)}(n) = (n)_{m-a}\prod_{i=2}^{a+1}(n-i+1) = \frac{(n+m-a-1)!}{(n-1)!}\frac{(n-1)!}{(n-1-a)!}.$$

Thus,

$$L_{(m-a,1,\dots,1)}\Psi_{(m-a,1,\dots,1)}(n) = \frac{m!}{a!\,(m-a-1)!m}\frac{(n+m-a-1)!}{m!\,(n-1-a)!}.$$

Formula (28) follows from (27) if we set $a=0$, and (28) follows from (27) if we set $a=m-1$. ∎

Using the previous result, we compute $L_\Lambda\Psi_\Lambda(n)$ for all $\Lambda\in\mathcal{P}(m)$, $0\le m\le 5$.

Corollary 15 *Let $n\in\mathbb{N}$. Then, we have:*

(i) For $\Lambda\in\mathcal{P}(0)$,

$$L_{(0)}\Psi_{(0)}(n) = 1.$$

(ii) For $\Lambda\in\mathcal{P}(1)$,

$$L_{(1)}\Psi_{(1)}(n) = \binom{n}{1}.$$

(iii) For $\Lambda\in\mathcal{P}(2)$,

$$L_{(2)}\Psi_{(2)}(n) = \binom{n+1}{2},\quad L_{(1,1)}\Psi_{(1,1)}(n) = \binom{n}{2}.$$

(iv) For $\Lambda\in\mathcal{P}(3)$,

$$L_{(3)}\Psi_{(3)}(n) = \binom{n+2}{3},\quad L_{(2,1)}\Psi_{(2,1)}(n) = 2\binom{n+1}{3},$$
$$L_{(1,1,1)}\Psi_{(1,1,1)}(n) = \binom{n}{3}.$$

(v) For $\Lambda\in\mathcal{P}(4)$,

$$L_{(4)}\Psi_{(4)}(n) = \binom{n+3}{4},\quad L_{(3,1)}\Psi_{(3,1)}(n) = 3\binom{n+2}{4},$$
$$L_{(2,2)}\Psi_{(2,2)}(n) = \frac{n}{2}\binom{n+1}{3},\quad L_{(2,1,1)}\Psi_{(2,1,1)}(n) = 3\binom{n+1}{4},$$
$$L_{(1,1,1,1)}\Psi_{(1,1,1,1)}(n) = \binom{n}{4}.$$

(vi) For $\Lambda \in \mathcal{P}(5)$,

$$
\begin{aligned}
L_{(5)}\Psi_{(5)}(n) &= \binom{n+4}{5}, \quad L_{(4,1)}\Psi_{(4,1)}(n) = 4\binom{n+3}{5} \\
L_{(3,2)}\Psi_{(3,2)}(n) &= n\binom{n+2}{4}, \quad L_{(3,1,1)}\Psi_{(3,1,1)}(n) = 6\binom{n+2}{5} \\
L_{(2,2,1)}\Psi_{(2,2,1)}(n) &= n\binom{n+1}{4}, \quad L_{(2,1,1,1)}\Psi_{(2,1,1,1)}(n) = 4\binom{n+1}{5} \\
L_{(1,1,1,1,1)}\Psi_{(1,1,1,1,1)}(n) &= \binom{n}{5}.
\end{aligned}
\tag{31}
$$

Proof. All the results follow from Proposition 14, except for $L_{(2,2,1)}\Psi_{(2,2,1)}(n)$. In this case, we have

$$L_{(2,2,1)} = \frac{1}{6}$$

and

$$
\begin{aligned}
\Psi_{(2,2,1)}(n) &= (n)_2 (n-1)_2 (n-2)_1 \\
&= \frac{1}{4} n^2 (n-1)(n-2)(n+1) = \frac{4!}{4} n \binom{n+1}{4}.
\end{aligned}
$$

Hence,

$$L_{(2,2,1)}\Psi_{(2,2,1)}(n) = n\binom{n+1}{4}.$$

■

We can now an explicit formula for the first few terms in the power series of $\Delta_n(z)$.

Corollary 16 *Let* $g_n(z)$ *be defined by*

$$g_n(z) = \frac{\Delta_n(z)}{z^{\binom{n}{2}} \mathcal{G}^2(n) \prod_{i=0}^{n-1} \omega(i)}.$$

Then,

$$g_n(z) = 1 + \sum_{m=1}^{\infty} r_m(n) z^m, \tag{32}$$

where

$$r_1(n) = n^2 \frac{\omega(n)}{\omega(n-1)} z, \quad r_2(n) = \binom{n+1}{2}^2 \frac{\omega(n+1)}{\omega(n-1)} + \binom{n}{2}^2 \frac{\omega(n)}{\omega(n-2)},$$

$$r_3(n) = \binom{n+2}{3}^2 \frac{\omega(n+2)}{\omega(n-1)} + 4\binom{n+1}{3}^2 \frac{\omega(n+1)}{\omega(n-2)} + \binom{n}{3}^2 \frac{\omega(n)}{\omega(n-3)},$$

$$\begin{aligned} r_4(n) &= \binom{n+3}{4}^2 \frac{\omega(n+3)}{\omega(n-1)} + 9\binom{n+2}{4}^2 \frac{\omega(n+2)}{\omega(n-2)} \\ &+ \frac{n^2}{4}\binom{n+1}{3}^2 \frac{\omega(n)\,\omega(n+1)}{\omega(n-1)\,\omega(n-2)} \\ &+ 9\binom{n+1}{4}^2 \frac{\omega(n+1)}{\omega(n-3)} + \binom{n}{4}^2 \frac{\omega(n)}{\omega(n-4)}, \end{aligned}$$

and

$$\begin{aligned} r_5(n) &= \binom{n+4}{5}^2 \frac{\omega(n+4)}{\omega(n-1)} + 16\binom{n+3}{5}^2 \frac{\omega(n+3)}{\omega(n-2)} \\ &+ n^2\binom{n+2}{4}^2 \frac{\omega(n)\,\omega(n+2)}{\omega(n-1)\,\omega(n-2)} + 36\binom{n+2}{5}^2 \frac{\omega(n+2)}{\omega(n-3)} \\ &+ n^2\binom{n+1}{4}^2 \frac{\omega(n)\,\omega(n+1)}{\omega(n-1)\,\omega(n-3)} \\ &+ 16\binom{n+1}{5}^2 \frac{\omega(n+1)}{\omega(n-4)} + \binom{n}{5}^2 \frac{\omega(n)}{\omega(n-5)}. \end{aligned}$$

2.3 Examples

In this section we apply the previous results to some specific families of orthogonal polynomials.

Example 17 *Charlier polynomials. If*

$$\omega(k) = \frac{1}{k!}, \tag{33}$$

then the polynomials orthogonal with respect to the linear functional (5) are known as Charlier polynomials *[22, 18.19]. The monic Charlier polynomials* *have the hypergeometric representation [22, 18.20.7]*

$$P_n(x) = (-z)^n \ {}_2F_0\left[\begin{array}{c} -n, \ -x \\ - \end{array}; -z^{-1}\right],$$

where

$${}_pF_q\left(\begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

denotes the generalized hypergeometric function *[22, Chapter 16]. When $z > 0$, the monic Charlier polynomials satisfy the orthogonality relation*

$$\sum_{k=0}^{\infty} P_n(k)\, P_m(k)\, \frac{z^k}{k!} = n! z^n e^z \delta_{n,m}. \tag{34}$$

If we use (33) in (32), we obtain

$$g_n(z) = \sum_{k=0}^{5} n^k \frac{z^k}{k!} + O\left(z^6\right).$$

In the next section, we will prove that

$$g_n(z) = e^{nz}.$$

Example 18 *Meixner polynomials. If*

$$\omega(n) = \frac{(a)_n}{n!}, \tag{35}$$

then the polynomials orthogonal with respect to the linear functional (5) are known as Meixner polynomials *[22, 18.19]. The monic Meixner polynomials can be represented by [22, 18.20.7]*

$$M_n(x) = (a)_n \left(1 - z^{-1}\right)^{-n} \ {}_2F_1\left[\begin{array}{c} -n, \ -x \\ a \end{array}; 1 - z^{-1}\right],$$

where $a > 0$ and $0 < z < 1$. They satisfy the orthogonality relation

$$\sum_{k=0}^{\infty} M_n(k)\, M_m(k)\, \frac{(a)_k}{k!} z^k = n!\, (a)_n \ z^n (1 - z)^{-a-2n} \delta_{n,m}. \tag{36}$$

If we use (35) in (32), we obtain

$$g_n(z) = \sum_{k=0}^{5} (n(n+a-1))_k \frac{z^k}{k!} + O\left(z^6\right).$$

In the next section, we will prove that

$$g_n(z) = (1-z)^{-n(a+n-1)}.$$

Example 19 *Generalized Charlier polynomials. If*

$$\omega(k) = \frac{1}{(b+1)_k} \frac{1}{k!}, \quad b > -1, \tag{37}$$

then the polynomials orthogonal with respect to the linear functional (5) are known as Generalized Charlier polynomials *[27].*

If we use (37) in (32), we obtain

$$r_1(n) = \frac{1}{n+b}, \quad r_2(n) = \frac{n^2+bn-1}{(n+b-1)_3},$$

$$r_3(n) = \frac{n^4+2bn^3+\left(b^2-5\right)n^2-3bn+4}{(n+b-2)_5},$$

and

$$r_4(n) = \frac{1}{(n+b)(n+b-3)_7}\left[n^7+4bn^6+\left(6b^2-14\right)n^5+2b\left(2b^2-17\right)n^4 + \left(b^4-26b^2+49\right)n^3-2b\left(3b^2-31\right)n^2+\left(19b^2-36\right)n-30b\right].$$

It is of course possible to compute higher terms, but we haven't found a convenient way to represent them, and the expressions become increasingly cumbersome.

Example 20 *In general, let*

$$\omega(k) = \frac{\xi(k)}{k!}. \tag{38}$$

We have

$$\frac{\omega(n-i+\lambda_i)}{\omega(n-i)} = \frac{\xi(n-i+\lambda_i)}{\xi(n-i)} \frac{(n-i)!}{(n-i+\lambda_i)!} = \frac{\xi(n-i+\lambda_i)}{\xi(n-i)} \frac{1}{(n-i+1)_{\lambda_i}}.$$

Thus,

$$\prod_{i=1}^{l(\Lambda)}\frac{\omega\left(n-i+\lambda_i\right)}{\omega\left(n-i\right)}=\frac{1}{\Psi_\Lambda\left(n\right)}\prod_{i=1}^{l(\Lambda)}\frac{\xi\left(n-i+\lambda_i\right)}{\xi\left(n-i\right)}.$$

From (14), we get

$$\mathcal{G}^2\left(n\right)\prod_{i=0}^{n-1}\omega\left(i\right)=\mathcal{G}\left(n\right)\prod_{i=0}^{n-1}\xi\left(i\right).$$

Hence, replacing in (24) we obtain

$$\frac{\Delta_n\left(z\right)}{z^{\binom{n}{2}}\mathcal{G}\left(n\right)\prod_{i=0}^{n-1}\xi\left(i\right)}=\sum_{m=0}^{\infty}z^m\sum_{\Lambda\in\mathcal{P}(m)}\left[\prod_{i=1}^{l(\Lambda)}\frac{\xi\left(n-i+\lambda_i\right)}{\xi\left(n-i\right)}\right]L_\Lambda^2\Psi_\Lambda\left(n\right). \tag{39}$$

All families of discrete semiclassical orthogonal polynomials have weight functions like (38) [8]. They include:

Polynomials	$\xi\left(k\right)$
Charlier	1
Meixner	$\left(a\right)_k, \quad a>0$
Generalized Charlier	$\frac{1}{\left(b+1\right)_k}, \quad b>-1$
Generalized Meixner	$\frac{\left(a\right)_k}{\left(b+1\right)_k}, \quad a>0, b>-1$
Generalized Krawtchouk	$\left(a\right)_k\left(-N\right)_k, \quad a>0, N\in\mathbb{N}$
Generalized Hahn	$\frac{\left(a_1\right)_k\left(a_2\right)_k}{\left(b+1\right)_k}, \quad a_1,a_2>0, b>-1.$

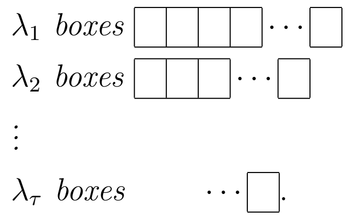
3 Combinatorial interpretation

3.1 Young tableaux

In this section, we will relate our previous results to some combinatorial objects. We begin with some definitions and examples, to aid those reader who are not experts in these topics.

Definition 21 *The* (Ferrers or Young) diagram $\mathfrak{D}_\Lambda$ *of the partition* Λ *with*

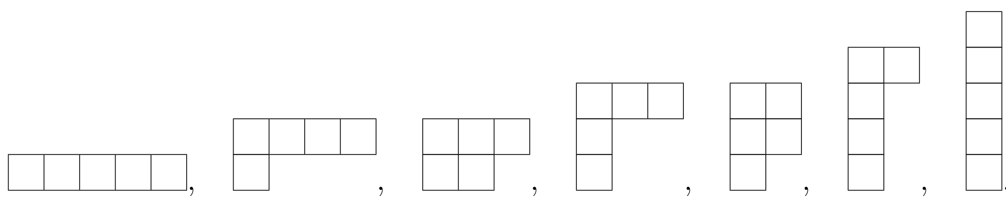
$l(\Lambda) = \tau$ is the set of boxes



In other words (with coordinates representing boxes)

$$\mathfrak{D}_\Lambda = \{(i,j) : 1 \le i \le l(\Lambda), \quad 1 \le j \le \lambda_i\}.$$

Example 22 *The diagrams for the partitions $\Lambda \in \mathcal{P}(5)$ are*



Definition 23 *A* Young tableau *$T(\Lambda)$ (of shape Λ) is a Young diagram $\mathfrak{D}_\Lambda$ for a partition $\Lambda \in \mathcal{P}(n)$ and a bijective assignment of the numbers $(1, ..., n)$ to the n boxes of the diagram $\mathfrak{D}_\Lambda$.*

A Young tableau $T(\Lambda)$ is called standard *if and only if the numbers increase in each row and each column.*

Example 24 *The standard Young tableaux for $\Lambda = (3,2)$ are*

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}\,.$$

Definition 25 *Let Λ be a partition. We define*

$$f(\Lambda) = \text{number of standard Young tableaux } T(\Lambda). \tag{40}$$

Example 26 *For $m = 5$, we have*

$$\begin{aligned} f(1,1,1,1,1) = 1, \quad f(2,1,1,1) = 4, \quad f(2,2,1) = 5, \\ f(3,1,1) = 6, \quad f(3,2) = 5, \quad f(4,1) = 4, \quad f(5) = 1. \end{aligned} \tag{41}$$

The next result shows that the values of $f(\Lambda)$ for $\Lambda \in \mathcal{P}(m)$ also give the dimensions of the irreducible representations of the symmetric group S_m [26, Chapter V].

Theorem 27 *We have*

$$\sum_{\Lambda \in \mathcal{P}(m)} f^2(\Lambda) = m!.$$

Proof. See [26, VI.2.2.]. ∎

Next, we relate the function L_Λ defined in the previous section with $f(\Lambda)$.

Theorem 28 *Let $\Lambda \in \mathcal{P}(m)$ with $l(\Lambda) = \tau$ and*

$$t_i = \lambda_i + \tau - i, \quad 1 \le i \le \tau.$$

Then,

$$f(\Lambda) = m! \frac{\prod\limits_{1 \le i < k \le \tau} (t_i - t_k)}{\prod\limits_{i=1}^{r} (t_i)!}. \tag{42}$$

Proof. See [26, Vl.2.3.]. ∎

Corollary 29 *Let Λ be a partition. The function L_Λ defined in (21) can be written as*

$$L_\Lambda = \frac{f(\Lambda)}{(\|\Lambda\|)!}.$$

Proof. Let Λ be a partition with $l(\Lambda) = \tau$. From (42), we get

$$\frac{f(\Lambda)}{m!} = \frac{\prod\limits_{1 \le i < k \le \tau} (\lambda_i - \lambda_k + k - i)}{\prod\limits_{i=1}^{\tau} (\lambda_i + \tau - i)!} = \prod_{i=1}^{\tau} \frac{\prod\limits_{k=i+1}^{\tau} (\lambda_i - \lambda_k + k - i)}{(\lambda_i + \tau - i)!}.$$

We have

$$(\lambda_i + \tau - i)! = \prod_{j=1}^{\lambda_i + \tau - i} j = (\lambda_i)! \prod_{j=\lambda_i+1}^{\lambda_i + \tau - i} j = (\lambda_i)! \prod_{j=1}^{\tau - i} (\lambda_i + j)$$
$$= (\lambda_i)! \prod_{k=i+1}^{\tau} (\lambda_i + k - i) \qquad (j = k - i).$$

Thus,

$$\frac{f(\Lambda)}{m!} = \prod_{i=1}^{\tau} \frac{1}{(\lambda_i)!} \prod_{k=i+1}^{\tau} \frac{(\lambda_i - \lambda_k + k - i)}{(\lambda_i + k - i)} = \prod_{i=1}^{\tau} \frac{1}{(\lambda_i)!} \prod_{k=i+1}^{\tau} \left(1 - \frac{\lambda_k}{\lambda_i + k - i}\right).$$

But

$$\prod_{i=1}^{\tau} \prod_{k=i+1}^{\tau} \left(1 - \frac{\lambda_k}{\lambda_i + k - i}\right) = \prod_{k=1}^{\tau} \prod_{i=1}^{k-1} \left(1 - \frac{\lambda_k}{\lambda_i + k - i}\right),$$

and therefore

$$\frac{f(\Lambda)}{m!} = \prod_{k=1}^{\tau} \frac{1}{(\lambda_k)!} \prod_{i=1}^{k-1} \left(1 - \frac{\lambda_k}{\lambda_i + k - i}\right) = L_\Lambda.$$

■

We now find a connection between the function $\Psi_\Lambda(n)$ defined in the previous section, and some properties of the diagram $\mathfrak{D}_\Lambda$.

Definition 30 *Let* Λ *be a partition with diagram* $\mathfrak{D}_\Lambda$.

1) We define the hook length $h_{i,j}(\Lambda)$ *of* Λ *at* $(i,j) \in \mathfrak{D}_\Lambda$ *as the number of boxes directly to the right or directly below the box located at* (i,j), *counting* (i,j) *itself once.*

2) We define the content $c_{i,j}(\Lambda)$ *of* Λ *at* $(i,j) \in \mathfrak{D}_\Lambda$ *by*

$$c_{i,j}(\Lambda) = j - i. \tag{43}$$

Example 31 *The hook lengths of* $\Lambda = (3,2)$ *are*

4	3	1
2	1	

. (44)

The contents of $\Lambda = (3,2)$ *are*

0	1	2
−1	0	

. (45)

Remark 32 *Let* Λ *be a partition. Then,*

$$\sum_{(i,j)\in\mathfrak{D}_\Lambda} \left[h_{i,j}^2(\Lambda) - c_{i,j}^2(\Lambda)\right] = \|\Lambda\|^2.$$

For a proof see [20, I.1. Example 5.].

Proposition 33 *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ *and*

$$t_i = \lambda_i + r - i, \quad 1 \le i \le r.$$

Then,

$$\prod_{(i,j)\in\mathfrak{D}_\Lambda} h_{i,j}(\Lambda) = \frac{\prod_{i=1}^{r} (t_i)!}{\prod_{1\le i<k\le r} (t_i - t_k)}.$$

Proof. See [20, I.1. Example 1.]. ■

Corollary 34 *Hook length formula. Let* Λ *be a partition. Then,*

$$f(\Lambda) = \frac{(\|\Lambda\|)!}{\prod_{(i,j)\in\mathfrak{D}_\Lambda} h_{i,j}(\Lambda)}. \tag{46}$$

Example 35 *Let* $\Lambda = (3, 2)$. *Using (44) in (46), we obtain*

$$f(3,2) = \frac{5!}{4\times 3\times 1\times 2\times 1} = 5,$$

in agreement with (41).

Definition 36 *Let* Λ *be a partition. We define the* content polynomial *of* Λ *by [20, I.1 Example 11]*

$$\mathcal{C}_\Lambda(x) = \prod_{(i,j)\in\mathfrak{D}_\Lambda} [x + c_{i,j}(\Lambda)]. \tag{47}$$

Example 37 *Let* $\Lambda = (3, 2)$. *Using (45) in (47), we get*

$$\mathcal{C}_{(3,2)}(x) = (x-1)\,x^2\,(x+1)\,(x+2).$$

Note that from (30) we have

$$\Psi_{(3,2)}(n) = n\frac{(n+5-3)!}{(n-2)!} = n^2(n-1)(n+2)(n+1) = C_{(3,2)}(n).$$

Proposition 38 *Let* $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. *Then,*

$$\mathcal{C}_\Lambda(x) = \Psi_\Lambda(x).$$

Proof. Using (43) and (47), we have

$$\prod_{(i,j)\in\mathfrak{D}_\Lambda} [x + c_{i,j}(\Lambda)] = \prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}(x+j-i) = \prod_{i=1}^{r}\prod_{j=0}^{\lambda_i-1}(x-i+1+j)$$
$$= \prod_{i=1}^{r}(x-i+1)_{\lambda_i} = \Psi_\Lambda(x).$$

■

3.2 Schur polynomials

In this section now introduce the concept of Schur polynomials, which will be essential in finding closed-form formulas for the Charlier and Meixner polynomials.

Definition 39 *Let* $n \in \mathbb{N}$ *and* Λ *be a partition. We define the* Schur polynomial s_Λ *by [20, I.3]*

$$s_\Lambda(x_1, \ldots, x_n) = 0, \quad l(\Lambda) > n$$

and

$$s_\Lambda(x_1, \ldots, x_n) = \frac{\det\limits_{1\le i,j\le n}\left(x_i^{n-j+\lambda_j}\right)}{\det\limits_{1\le i,j\le n}\left(x_i^{n-j}\right)}, \quad l(\Lambda) \le n.$$

Example 40 *Let* $n = 3$ *and* $m = 5$. *We have*

$$s_{(1,1,1,1,1)} = 0, \quad s_{(2,1,1,1)} = 0, \quad s_{(2,2,1)} = e_2e_3, \quad s_{(3,1,1)} = \left(e_1^2 - e_2\right)e_3,$$
$$s_{(3,2)} = e_1e_2^2 - \left(e_1^2 + e_2\right)e_3, \quad s_{(4,1)} = (e_3 - e_1e_2)\left(2e_2 - e_1^2\right)$$
$$s_{(5)} = e_1^5 - 4e_1^3e_2 + 3e_3e_1^2 + 3e_1e_2^2 - 2e_3e_2,$$

where e_k *denotes the* elementary symmetric polynomials *defined by*

$$\prod_{k=1}^{n}(t - x_k) = \sum_{k=0}^{n}(-1)^k e_k(x_1, \ldots, x_n)\, t^{n-k}.$$

The following is known as the principal specialization of Schur polynomials.

Theorem 41 *Let $n \in \mathbb{N}$ and Λ be a partition with $l(\Lambda) \leq n$. Then,*

$$s_{\Lambda}\left(1^{n}\right) = \prod_{(i,j) \in \mathfrak{D}_{\Lambda}} \frac{n + c_{i,j}(\Lambda)}{h_{i,j}(\Lambda)},$$

where

$$s_{\Lambda}\left(1^{n}\right) = s_{\Lambda}\left(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}\right).$$

Proof. See [20, I.3. Example 4]. ■

We have now all the elements to represent $L_{\Lambda}\Psi_{\Lambda}(n)$ and $\Delta_n(z)$ in terms of Schur polynomials.

Corollary 42 *Let $n \in \mathbb{N}$ and $\Lambda \in \mathcal{P}_n(m)$. Then,*

$$s_{\Lambda}\left(1^{n}\right) = \frac{f(\Lambda)}{m!}\mathcal{C}_{\Lambda}(n) = L_{\Lambda}\Psi_{\Lambda}(n) \tag{48}$$

and

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}\mathcal{G}(n)\left[\prod_{i=0}^{n-1}\xi(i)\right]} = \sum_{m=0}^{\infty}\frac{z^m}{m!}\sum_{\Lambda \in \mathcal{P}_n(m)}\left[\prod_{i=1}^{n}\frac{\xi(n-i+\lambda_i)}{\xi(n-i)}\right] f(\Lambda)\, s_{\Lambda}\left(1^{n}\right). \tag{49}$$

The next result will be needed in finding the Hankel determinants for the Charlier polynomials.

Proposition 43 *Let $n, m \in \mathbb{N}$. Then,*

$$\sum_{\Lambda \in \mathcal{P}_n(m)} f(\Lambda)\, s_{\Lambda}(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)^m = e_1^m. \tag{50}$$

Proof. See [20, I.4. Example 3]. ■

Example 44 *Let $n = 3$ and $m = 5$. We have*

$$\sum_{\Lambda\in\mathcal{P}_3(5)} f(\Lambda)\, s_\Lambda(x_1,x_2,x_3) = 5e_2e_3 + 6\left(e_1^2 - e_2\right)e_3$$
$$+5\left(e_1e_2^2 - \left(e_1^2+e_2\right)e_3\right) + 4\left(e_3 - e_1e_2\right)\left(2e_2 - e_1^2\right)$$
$$+e_1^5 - 4e_1^3e_2 + 3e_3e_1^2 + 3e_1e_2^2 - 2e_3e_2 = e_1^5.$$

Corollary 45 *Charlier polynomials. Let $\xi(k) = 1$. Then,*

$$\Delta_n(z) = z^{\binom{n}{2}}\mathcal{G}(n)\, e^{nz}.$$

Proof. Setting

$$x_1 = x_2 = \cdots = x_n = 1$$

in (50), we have

$$\sum_{\Lambda\in\mathcal{P}(m)} f(\Lambda)\, s_\Lambda(1^n) = n^m. \tag{51}$$

Thus, from (49) we get

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}\mathcal{G}(n)} = \sum_{m=0}^{\infty}\frac{z^m}{m!}n^m = e^{nz}.$$

■

The next result will be essential in finding the Hankel determinants for the Meixner polynomials.

Theorem 46 *Let $n_1, n_2 \in \mathbb{N}$. Then,*

$$\sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} s_\Lambda(x_1,\ldots,x_{n_1})\, s_\Lambda(y_1,\ldots,y_{n_2}) = \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(1 - x_iy_jz\right)^{-1}. \tag{52}$$

Proof. See [20, I.4.]. ■

Proposition 47 *Let $m \in \mathbb{N}_0$ and x, y be indeterminate variables. Then,*

$$\sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x)\, \Psi_\Lambda(y) = \frac{(xy)_m}{m!}. \tag{53}$$

Proof. Setting

$$x_i = y_j = 1, \quad 1 \le i \le n_1, \quad 1 \le j \le n_2$$

in (52), we get

$$\sum_{m=0}^{\infty} z^m \sum_{\Lambda \in \mathcal{P}(m)} s_\Lambda \left(1^{n_1}\right) s_\Lambda \left(1^{n_2}\right) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1-z)^{-1}$$
$$= (1-z)^{-n_1 n_2} = \sum_{m=0}^{\infty} \frac{(n_1 n_2)_m}{m!} z^m.$$

Hence, from (48) we obtain

$$\sum_{\Lambda \in \mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda (n_1) \Psi_\Lambda (n_2) = \sum_{\Lambda \in \mathcal{P}(m)} s_\Lambda \left(1^{n_1}\right) s_\Lambda \left(1^{n_2}\right) = \frac{(n_1 n_2)_m}{m!}.$$

But this is an identity between polynomials of degree m, and since it's valid for all $n_1, n_2 \in \mathbb{N}$ it must be valid for indeterminate variables x, y. ■

Example 48 *Let $m = 5$. Using (31), we have*

$$(5!)^2 \sum_{\Lambda \in \mathcal{P}(5)} L_\Lambda^2 \Psi_\Lambda (x) \Psi_\Lambda (y) = (x-4)_5 (y-4)_5 + 16 (x-3)_5 (y-3)_5$$
$$+ 25xy (x-2)_4 (y-2)_4 + 36 (x-2)_5 (y-2)_5 + 25xy (x-1)_4 (y-1)_4$$
$$+ 16 (x-1)_5 (y-1)_5 + (x)_5 (y)_5 = 5! (xy)_5 .$$

Corollary 49 *Meixner polynomials. Let $\xi(k) = (a)_k$. Then,*

$$\Delta_n (z) = z^{\binom{n}{2}} \mathcal{G}(n) \left[\prod_{i=0}^{n-1} (a)_i \right] (1-z)^{-n(a+n-1)} .$$

Proof. Replacing

$$\prod_{i=1}^{l(\Lambda)} \frac{\xi(n-i+\lambda_i)}{\xi(n-i)} = \prod_{i=1}^{l(\Lambda)} (a+n-i)_{\lambda_i} = \Psi_\Lambda (a+n-1)$$

in (39), we have

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}\mathcal{G}(n)\prod_{i=0}^{n-1}(a)_i} = \sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(n)\,\Psi_\Lambda(a+n-1),$$

and using (53), we get

$$\frac{\Delta_n(z)}{z^{\binom{n}{2}}\mathcal{G}(n)\prod_{i=0}^{n-1}(a)_i} = \sum_{m=0}^{\infty} z^m \frac{(n(a+n-1))_m}{m!} = (1-z)^{-n(a+n-1)}.$$

■

We can also use the previous results to find a general identity for $\Psi_\Lambda(x)$, corresponding to the Charlier weight.

Remark 50 *From (53), we see that*

$$\sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x)\,\Psi_\Lambda(y) = (1-z)^{-xy} = {}_1F_0\left(\begin{matrix} xy \\ - \end{matrix};z\right).$$

Thus,

$$\begin{aligned}&\lim_{y\to\infty}\sum_{m=0}^{\infty}\left(\frac{z}{y}\right)^m \sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x)\,\Psi_\Lambda(y)\\ &= \lim_{y\to\infty} {}_1F_0\left(\begin{matrix} xy \\ - \end{matrix};\frac{z}{y}\right) = e^{xz}.\end{aligned}$$

But since $\Psi_\Lambda(y)$ is a monic polynomial with

$$\deg \Psi_\Lambda(y) = m, \quad \Lambda\in\mathcal{P}(m),$$

we have

$$\lim_{y\to\infty}\frac{\Psi_\Lambda(y)}{y^m} = 1,$$

and using Tannery's theorem [2], we obtain

$$\sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x) = e^{xz},$$

or

$$\sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x) = \frac{x^m}{m!}$$

which is the extension of (51) for an indeterminate variable.

Finally, for weight functions corresponding to discrete semiclassical orthogonal polynomials, we have the following.

Example 51 *In general, let*

$$\xi(k) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1+1)_k \cdots (b_q+1)_k}.$$

Then,

$$\frac{\xi(n-i+\lambda_i)}{\xi(n-i)} = \frac{(a_1+n-i)_{\lambda_i} \cdots (a_p+n-i)_{\lambda_i}}{(b_1+1+n-i)_{\lambda_i} \cdots (b_q+1+n-i)_{\lambda_i}},$$

and therefore

$$\prod_{i=1}^{l(\Lambda)} \frac{\xi(n-i+\lambda_i)}{\xi(n-i)} = \frac{\Psi_\Lambda(a_1+n-1)\cdots\Psi_\Lambda(a_p+n-1)}{\Psi_\Lambda(b_1+n)\cdots\Psi_\Lambda(b_q+n)}.$$

Replacing in (39), we conclude that

$$\Delta_n(z) = z^{\binom{n}{2}} \mathcal{G}(n) \left[\prod_{k=0}^{n-1} \frac{(a_1)_k \cdots (a_p)_k}{(b_1+1)_k \cdots (b_q+1)_k}\right] \times \sum_{m=0}^{\infty} z^m \sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(n) \left[\frac{\Psi_\Lambda(a_1+n-1)\cdots\Psi_\Lambda(a_p+n-1)}{\Psi_\Lambda(b_1+n)\cdots\Psi_\Lambda(b_q+n)}\right].$$

Unfortunately, we haven't been able to find any identities for Schur polynomials that will allow the computation of

$$\sum_{\Lambda\in\mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(n) \left[\frac{\Psi_\Lambda(a_1+n-1)\cdots\Psi_\Lambda(a_p+n-1)}{\Psi_\Lambda(b_1+n)\cdots\Psi_\Lambda(b_q+n)}\right]$$

for $p > 1$ and $q > 0$.

4 Conclusions

We have obtained a Taylor series

$$\Delta_n(z) = z^{\binom{n}{2}} \mathcal{G}^2(n) \left[\prod_{i=0}^{n-1} \omega(i) \right] \sum_{m=0}^{\infty} r_m(n) z^m,$$

for the Hankel determinant

$$\Delta_n(z) = \det_{0 \le i,j \le n-1} \left(\sum_{k=0}^{\infty} k^{i+j} \omega(k) z^k \right),$$

where $\omega(k)$ is a known nonzero function, $\mathcal{G}(n)$ is defined by

$$\mathcal{G}(n) = \prod_{i=0}^{n-1} (i!),$$

and the coefficients $r_m(n)$ are given by

$$r_m(n) = \sum_{\Lambda \in \mathcal{P}(m)} \left[\prod_{i=1}^{l(\Lambda)} \frac{\omega(\lambda_i + n - i)}{\omega(n-i)} \right] \left[L_\Lambda \Psi_\Lambda(n) \right]^2,$$

with

$$L_\Lambda = \prod_{i=1}^{l(\Lambda)} \frac{1}{(\lambda_i)!} \prod_{j=1}^{i-1} \left(1 - \frac{\lambda_i}{\lambda_j + i - j} \right),$$

and

$$\Psi_\Lambda(x) = \prod_{i=1}^{l(\Lambda)} (x - i + 1)_{\lambda_i}.$$

In particular, when $\omega(k)$ is the weight function associated with discrete semiclassical orthogonal polynomials

$$\omega(k) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1+1)_k \cdots (b_q+1)_k} \frac{1}{k!},$$

we have

$$r_m(n) = \sum_{\Lambda \in \mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(n) \frac{\Psi_\Lambda(a_1+n-1) \cdots \Psi_\Lambda(a_p+n-1)}{\Psi_\Lambda(b_1+n) \cdots \Psi_\Lambda(b_q+n)}.$$

Using Schur polynomials, we showed that

$$\sum_{\Lambda \in \mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x) = \frac{x^m}{m!}$$

and

$$\sum_{\Lambda \in \mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(x) \Psi_\Lambda(y) = \frac{(xy)_m}{m!}.$$

From these identities, we obtained exact evaluations for the Hankel determinants of the Charlier polynomials

$$\det_{0 \le i,j \le n-1} \left(\sum_{k=0}^{\infty} k^{i+j} \frac{z^k}{k!} \right) = z^{\binom{n}{2}} \mathcal{G}(n) e^{nz}$$

and Meixner polynomials

$$\det_{0 \le i,j \le n-1} \left(\sum_{k=0}^{\infty} k^{i+j} (a)_k \frac{z^k}{k!} \right) = z^{\binom{n}{2}} \mathcal{G}(n) \left[\prod_{i=0}^{n-1} (a)_i \right] (1-z)^{-n(a+n-1)}.$$

We plan to extend these results and study the multivariate rational functions

$$R(t, x_1, \ldots, x_p, y_1, \ldots, y_q) = \sum_{\Lambda \in \mathcal{P}(m)} L_\Lambda^2 \Psi_\Lambda(t) \frac{\Psi_\Lambda(a_1 + x_1 - 1) \cdots \Psi_\Lambda(a_p + x_p - 1)}{\Psi_\Lambda(b_1 + y_1) \cdots \Psi_\Lambda(b_q + y_q)},$$

in order to obtain closed-form formulas for the Hankel determinants of other families of discrete semiclassical orthogonal polynomials.

Acknowledgement 52 *This work was done while visiting the Johannes Kepler Universität Linz. It was supported by the Austrian Science Fund (FWF) under grant SFB F50-07 and by the strategic program "Innovatives OÖ– 2010 plus" from the Upper Austrian Government.*

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