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Abstract

We present some elements of the theory of orthogonal polynomials based on matrix decompositions. We focus on discrete linear functionals, and use the Meixner polynomials as a concrete example.

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1 Introduction

Let $\{\phi_n\}$, $\{\psi_n\}$, and $\{P_n\}$ be three bases of $\mathbb{C}[x]$, related by

$$\phi_i(x) = \sum_{k=0}^{i} a_{i,k} P_k(x),$$

and

$$\psi_j(x) = \sum_{m=0}^{j} b_{j,m} P_m(x).$$

Then, we clearly have

$$\phi_i(x)\psi_j(x) = \sum_{k=0}^{i}\sum_{m=0}^{j} a_{i,k} b_{j,m} P_k(x) P_m(x). \tag{1}$$

If $\{\mu_n\}$ is a sequence of complex numbers and $\mathfrak{L} : \mathbb{C}[x] \to \mathbb{C}$ is the linear functional defined by

$$\mathfrak{L}[x^n] = \mu_n, \quad n = 0, 1, \ldots,$$

then $\mathfrak{L}$ is called the **moment functional** determined by the formal moment sequence $\{\mu_n\}$ [31]. The number μ_n is called the **moment** of order n.

If the polynomials $P_n(x)$ satisfy

$$\mathfrak{L}[P_n P_m] = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots, \tag{2}$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta

$$\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases},$$

then $\{P_n\}$ is called an **orthogonal polynomial sequence** with respect to $\mathfrak{L}$.

Applying $\mathfrak{L}$ to (1) and using (2), we get

$$\mathfrak{L}[\phi_i\psi_j] = \sum_{k,m} a_{i,k} b_{j,m} \mathfrak{L}[P_k P_m] = \sum_{k,m} a_{i,k} b_{j,m} h_k \delta_{k,m} = \sum_{k} a_{i,k} b_{j,k} h_k. \tag{3}$$

If we define the matrices A_n, B_n, D_n, and H_n by

$$(A_n)_{i,j} = \begin{cases} a_{i,j} & i \geq j \\ 0 & i < j \end{cases}, \quad 0 \leq i, j \leq n-1,$$

$$(B_n)_{i,j} = \begin{cases} b_{i,j} & i \geq j \\ 0 & i < j \end{cases}, \quad 0 \leq i, j \leq n-1,$$

$$(D_n)_{i,j} = \begin{cases} h_i & i = j \\ 0 & i \neq j \end{cases}, \quad 0 \leq i, j \leq n-1,$$

and

$$(H_n)_{i,j} = \mathfrak{L}\left[\phi_i \psi_j\right], \quad 0 \leq i, j \leq n-1,$$

then we see from (3) that

$$H_n = A_n D_n B_n^T. \tag{4}$$

In particular, if we choose $\phi_n(x) = \psi_n(x) = x^n$ and the polynomials $P_n(x)$ are monic, H_n becomes a **Hankel matrix**

$$(H_n)_{i,j} = \mu_{i+j-2}, \quad 1 \leq i, j \leq n,$$

and we define the **Hankel determinants** by $\Delta_0 = 1$ and

$$\Delta_n = \det(H_n), \quad n = 1, 2, \ldots.$$

It is well known that (2) is equivalent to the condition

$$\Delta_n \neq 0, \quad n = 1, 2, \ldots.$$

The theory of orthogonal polynomials has a long history, from the first work of Adrien-Marie Legendre [51] on the gravitational potential in spherical coordinates, to the present day. See [18], [22], [28], [31], [44], [47], [49], [50], [52], [55], [59], [63], [66], [67].

The standard approach is to use the monomial basis and apply the theory of Hankel determinants. However, one could consider more general bases and work instead with matrix factorizations of the form (4). In particular, if the bases $\phi_n(x)$, $\psi_n(x)$ are the same, one would have an LDL decomposition [72], [46].

The links between the theory of (infinite, semi-infinite) matrices and orthogonal polynomials have been studied by multiple authors. The bibliography on this subject has grown exponentially in the last years, and any attempt to review all the references will be almost impossible. Moreover, there are now connections with several fields of mathematics, physics, and statistics, including:

1. Random matrices [7], [10], [12], [27], [32], [57].
2. Toda lattices [3], [4], [5], [11], [14], [53], [54], [68].
3. Matrix orthogonal polynomials [17], [37].
4. Multiple orthogonal polynomials [15], [19], [38].
5. Multivariate orthogonal polynomials [21], [36].
6. Orthogonal polynomials on the unit circle [16], [20], [64], [65].
7. Skew orthogonal polynomials [1], [2], [45], [58].
8. Darboux transformations [8], [9].
9. Christoffel transformations [13].
10. Geronimus transformations [33], [41].
11. Riemann–Hilbert Problems [6], [25], [26].

    and

12. Painlevé equations [23], [29], [30], [39], [40].

The objective of this article is to present some basic elements of the theory of orthogonal polynomials based on matrix factorizations. We apply these ideas to one specific family of discrete semiclassical orthogonal polynomials (the Meixner polynomials), and outline the general case.

The paper is structured as follows: in Section 2, we introduce all the elements necessary for the theory. We show how to compute the coefficients in the three-term recurrence relation satisfied by the orthogonal polynomials using the Modified Chebyshev algorithm.

In Section 3, we consider some special classes of orthogonal polynomials, called semiclassical. We focus our attention on the linear functionals that satisfy a Pearson equation with respect to the shift operator. We derive a system of partial difference equations that can be used to compute the coefficients in the three-term recurrence relation.

Finally, in Section 4 we summarize the results and discuss possible extensions.

2 Main theory

2.1 Definitions

We begin with a few definitions.

Definition 1 *Let* $\mathbb{N}_0$ *denote the set*

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = 0, 1, 2, \ldots .$$

A ***semi-infinite matrix*** $M \in \mathbb{C}^{\infty\times\infty}$ *is a function* $M : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$. *We write*

$$M(i, j) = M_{i,j}.$$

(i) We say that U *is an* ***upper triangular matrix*** *if*

$$U_{i,j} = 0, \quad i > j.$$

We say that U *is a* ***unit upper triangular*** *(**UUT**) matrix if* U *is upper triangular and*

$$U_{i,i} = 1, \quad i \in \mathbb{N}_0.$$

(ii) We say that L *is a* ***lower triangular matrix*** *if*

$$L_{i,j} = 0, \quad i < j.$$

We say that L *is a* ***unit lower triangular*** *(**ULT**) matrix if* L *is lower triangular and*

$$L_{i,i} = 1, \quad i \in \mathbb{N}_0.$$

Remark 2 *For material on semi-infinite matrices and their connections with orthogonal polynomials, see [70].*

Definition 3 *We say that* $\overrightarrow{q} \in \mathbb{C}[x]^{\infty\times 1}$ *is a* ***basis*** *of* $\mathbb{C}[x]$ *if* $q_n(x) \in \mathbb{C}[x]$ *and* $\deg(q_n) = n$.

We say that $\overrightarrow{q}$ *is a* ***monic basis*** *if* $q_n(x)$ *is a monic polynomial for all* $n \in \mathbb{N}_0$.

The basis that we will use in our examples is constructed with the falling factorials.

Example 4 *The basis of **falling factorial** (or binomial) polynomials is defined by* $\phi_0(x) = 1$ *and*

$$\phi_n(x) = \prod_{j=0}^{n-1} (x - j), \quad n \in \mathbb{N}. \tag{5}$$

Remark 5 *Using the definition (5), we immediately obtain the recurrence relation*

$$\phi_{n+1}(x) = (x - n)\, \phi_n(x). \tag{6}$$

Definition 6 *We define the **Pochhammer** (or rising factorial) polynomials by* $(x)_0 = 1$ *and*

$$(x)_n = \prod_{k=0}^{n-1} (x + k), \quad n \in \mathbb{N}. \tag{7}$$

Remark 7 *The Pochhammer polynomials can be generalized to complex values of* n *using the formula [61, 5.2.5]*

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad -(x+n) \notin \mathbb{N}_0, \tag{8}$$

where $\Gamma(z)$ *is the Gamma function.*

The Pochhammer polynomials satisfy many identities, including the recurrence [60, 18:5:12]

$$(x)_{n+m} = (x)_n\, (x+n)_m, \quad n, m \in \mathbb{N}_0, \tag{9}$$

the change of sign identity

$$(-x)_n = (-1)^n\, (x - n + 1)_n, \tag{10}$$

and the ratio formulas [60, 18:5:10]

$$\frac{(x-m)_n}{(x)_n} = \frac{(x-m)_m}{(x-m+n)_m} = \frac{(1-x)_m}{(1-x-n)_m}, \quad m \in \mathbb{N}_0. \tag{11}$$

We see from the definitions (5) and (7) that the polynomials $\phi_n(x)$ and $(x)_n$ are related by [61, 5.2.6]

$$\phi_n(x) = \prod_{j=0}^{n-1} (x - j) = (-1)^n \prod_{j=0}^{n-1} (-x + j) = (-1)^n\, (-x)_n.$$

Using (10), we get

$$\phi_n(x) = (-1)^n (-x)_n = (x-n+1)_n. \tag{12}$$

Remark 8 *Note that from (8) and (12) we get*

$$\phi_n(x) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = n!\binom{x}{n}. \tag{13}$$

In particular, using the recurrence relation for the binomial coefficients [61, 26.3.5], we obtain

$$\frac{\phi_n(x+1)}{n!} = \binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1} = \frac{\phi_n(x)}{n!} + \frac{\phi_{n-1}(x)}{(n-1)!}.$$

Therefore,

$$\phi_n(x+1) = \phi_n(x) + n\phi_{n-1}(x). \tag{14}$$

Using the **forward difference operator (**acting on the variable x)

$$\Delta f(x) = f(x+1) - f(x),$$

we can write (14) as $\Delta\phi_n = n\phi_{n-1}$. For higher powers of Δ, we have the following lemma.

Lemma 9 *For all $i, j \in \mathbb{N}_0$, we have*

$$\Delta^i \phi_j(x) = \phi_i(j)\, \phi_{j-i}(x). \tag{15}$$

Note that from (5) we see that

$$\phi_i(j) = 0, \quad i > j.$$

Proof. We use induction on i. The case $i = 0$ is an identity. Assuming the result to be true for $i \geq 0$, we have

$$\begin{aligned}\Delta^{i+1}\phi_j(x) &= \Delta[\phi_i(j)\,\phi_{j-i}(x)] = \phi_i(j)\,\Delta\phi_{j-i}(x) \\ &= \phi_i(j)\,(j-i)\,\phi_{j-i-1}(x),\end{aligned}$$

where we have used (14). The result now follows from (6), since

$$(j-i)\,\phi_i(j) = \phi_{i+1}(j).$$

■

Using the Pochhammer polynomials we can construct the generalized hypergeometric function.

Definition 10 *The* ***generalized hypergeometric function*** ${}_pF_q$ *is defined by [61, 16.2]*

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{16} $$

Remark 11 *The convergence of the series (16) depends on the values of p and q. We have three different cases to consider:*

1. If $p < q+1$, ${}_pF_q$ is an entire function of z.
2. If $p = q+1$, ${}_pF_q$ is analytic inside the unit circle, $|z| < 1$.
3. If $p > q+1$, ${}_pF_q$ diverges for $z \neq 0$, unless one or more of the top parameters a_i is a negative integer. If we take $a_1 = -N$, with $N \in \mathbb{N}_0$, then ${}_pF_q$ becomes a polynomial of degree N.

For example, we write the exponential generating function for the Pochhammer polynomials as a ${}_1F_0$ function.

Example 12 *Using the binomial theorem and (13), we have*

$$ (1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \frac{\phi_n(x)}{n!} z^n. $$

From (12), we get

$$ {}_1F_0\left(\begin{array}{c} x \\ - \end{array};z\right) = \sum_{n=0}^{\infty} (x)_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \phi_n(-x) \frac{z^n}{n!} = (1-z)^{-x}, \quad |z| < 1. \tag{17} $$

In the next section, we will need the following result.

Proposition 13 *The polynomials $\phi_n(x)$ satisfy the connection formula*

$$ \phi_n(x)\,\phi_m(x) = \sum_{k=0}^{\infty} \binom{n}{k}\binom{m}{k} k!\phi_{n+m-k}(x). \tag{18} $$

Proof. Using (13), we can write

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=\sum_{k=0}^{\infty}\frac{\phi_k(n)\,\phi_k(m)\,\phi_{n+m-k}(x)}{k!},$$

or, using (12),

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=\sum_{k=0}^{\infty}(-n)_k\,(-m)_k\,(x+1+k-n-m)_{n+m-k}\,\frac{1}{k!}.$$

But from (9), we have

$$(x+1-n-m)_k\,(x+1-n-m+k)_{n+m-k}=(x+1-n-m)_{n+m}$$

and therefore

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=(x+1-n-m)_{n+m}\sum_{k=0}^{\infty}\frac{(-n)_k\,(-m)_k}{(x+1-n-m)_k}\frac{1}{k!}.$$

Using (16), we get

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=(x+1-n-m)_{n+m}\;{}_2F_1\left(\begin{matrix}-n,-m\\ x+1-n-m\end{matrix};1\right).$$

If we use the Chu–Vandermonde identity [61, 15.4.24]

$${}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};1\right)=\frac{(c-b)_n}{(c)_n},\quad n\in\mathbb{N}_0,$$

we obtain

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=(x+1-n-m)_{n+m}\,\frac{(x+1-n)_n}{(x+1-n-m)_n},$$

and (9) gives

$$\frac{(x+1-n-m)_{n+m}}{(x+1-n-m)_n}=(x+1-m)_m\,.$$

Thus, using (12),

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{k}k!\phi_{n+m-k}(x)=(x+1-n)_n\,(x+1-m)_m=\phi_n(x)\,\phi_m(x)\,.$$

■

2.2 Linear functionals

In this section we consider linear functional acting on the space of polynomials $\mathbb{C}[x]$, i.e., belonging to the dual vector space $\mathbb{C}^*[x]$.

Definition 14 *Let $\mathfrak{L} : \mathbb{C}[x] \to \mathbb{C}$ be a linear functional and $\overrightarrow{q} \in \mathbb{C}[x]^{\infty \times 1}$ be a monic basis.*

(i) The numbers

$$\nu_n = \mathfrak{L}[q_n], \quad n \in \mathbb{N}_0,$$

are called the ***(generalized) moments*** *of $\mathfrak{L}$. We write*

$$\overrightarrow{\nu} = \mathfrak{L}[\overrightarrow{q}] \in \mathbb{C}[x]^{\infty \times 1}.$$

(ii) We define the ***Gram matrix*** *G by*

$$G = \mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{q}^T\right] \in \mathbb{C}^{\infty \times \infty}.$$

As an example, we consider the basis of falling factorial polynomials.

Example 15 *Let $\mathfrak{L} : \mathbb{C}[x] \to \mathbb{C}$ be defined by*

$$\mathfrak{L}[q] = \sum_{x=0}^{\infty} q(x)\,(a)_x \frac{z^x}{x!}, \quad q \in \mathbb{C}[x]. \tag{19}$$

The moments of $\mathfrak{L}$ on the falling factorial basis are given by

$$\nu_n(z) = \mathfrak{L}[\phi_n] = \sum_{x=0}^{\infty} \phi_n(x)\,(a)_x \frac{z^x}{x!}.$$

From (13), we get

$$\nu_n(z) = \sum_{x=n}^{\infty} \frac{x!}{(x-n)!}(a)_x \frac{z^x}{x!} = \sum_{x=0}^{\infty} \frac{(a)_{x+n}}{x!} z^{x+n},$$

or using (9) and (17)

$$\nu_n(z) = z^n (a)_n \sum_{x=0}^{\infty} (a+n)_x \frac{z^x}{x!} = z^n (a)_n (1-z)^{-a-n}, \quad |z| < 1. \tag{20}$$

Using (20) and (18), we get

$$G_{i,j} = \mathfrak{L}[\phi_i, \phi_j] = \sum_{k=0}^{\infty} \binom{i}{k} \binom{j}{k} k! z^{i+j-k} (a)_{i+j-k} (1-z)^{-a-i-j+k}$$
$$= (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} \sum_{k=0}^{\infty} \frac{(-i)_k (-j)_k}{k!} (a)_{i+j-k} \left(\frac{1-z}{z} \right)^k .$$

From (9) and (10), we have

$$\frac{(a)_{i+j}}{(a)_{i+j-k}} = (a+i+j-k)_k = (-1)^k (1-i-j-a)_k .$$

Therefore,

$$G_{i,j} = (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_{i+j} \sum_{k=0}^{\infty} \frac{(-i)_k (-j)_k}{k!} \frac{(-1)^k}{(1-i-j-a)_k} \left(\frac{1-z}{z} \right)^k$$
$$= (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_{i+j} \; {}_2F_1 \left(\begin{matrix} -i, -j \\ 1-i-j-a \end{matrix} ; \frac{z-1}{z} \right).$$

Using the identity [61, 15.8.7]

$${}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix} ; z \right) = \frac{(c-b)_n}{(c)_n} \; {}_2F_1 \left(\begin{matrix} -n, b \\ 1+b-c-n \end{matrix} ; 1-z \right), \quad n \in \mathbb{N}_0,$$

we obtain

$$G_{i,j} = (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_{i+j} \; \frac{(1-i-a)_i}{(1-i-j-a)_i} \; {}_2F_1 \left(\begin{matrix} -i, -j \\ a \end{matrix} ; \frac{1}{z} \right).$$

But using (11) and (9),

$$\frac{(1-i-a)_i}{(1-i-a-j)_i} = \frac{(a)_i}{(a+j)_i} = (a)_i \frac{(a)_j}{(a)_{i+j}}.$$

Thus, we conclude that

$$G_{i,j} = (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_i \; (a)_j \; {}_2F_1 \left(\begin{matrix} -i, -j \\ a \end{matrix} ; \frac{1}{z} \right), \tag{21}$$

where $-a \notin \mathbb{N}_0$, and we choose the branch $z \notin [1, \infty)$.

Remark 16 *Note that the matrix* G *defined by (21) is symmetric, and all the entries are finite sums, since the hypergeometric series terminates for all* $i, j \in \mathbb{N}_0$.

Example 17 *Also,* $z = 0$ *is not a singularity of* $G_{i,j}$*, since the power* z^{i+j} *cancels the powers of* z^{-1}.

Definition 18 *We say that* $\mathfrak{L}$ *is a* ***quasi-definite functional*** *with respect to a monic basis* $\overrightarrow{q} \in \mathbb{C}[x]^{\infty\times 1}$ *if the matrix* $\mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{q}^T\right]$ *admits the LDL decomposition [46, 4.12]*

$$\mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{q}^T\right] = G = CHC^T, \tag{22}$$

where $C \in \mathbb{C}^{\infty\times\infty}$ *is a ULT matrix and* $H \in \mathbb{C}^{\infty\times\infty}$ *is a nonsingular diagonal matrix*

$$H_{i,j} = h_i \delta_{i,j}, \quad h_i \neq 0, \quad i, j \in \mathbb{N}_0.$$

If $h_i > 0$ *for all* $i \in \mathbb{N}_0$*, we say that* $\mathfrak{L}$ *is a* ***positive-definite functional****.*

Proposition 19 *If* $\mathfrak{L}$ *is a quasi-definite functional with respect to* $\overrightarrow{q}$*, then we can compute the entries of* C *and* H *in (22) by the following iterative formula:*

$$h_0 = G_{0,0}, \quad C_{i,0} = \frac{G_{i,0}}{h_0}, \quad C_{i,i} = 1, \quad i \in \mathbb{N}_0,$$
$$C_{i,j} = 0, \quad i < j,$$

and for $i \in \mathbb{N}$,

$$h_i = G_{i,i} - \sum_{k=0}^{i-1} (C_{i,k})^2 h_k,$$
$$C_{i,j} = \frac{1}{h_j}\left(G_{i,j} - \sum_{k=0}^{j-1} C_{i,k} C_{j,k} h_k\right), \quad j = 1, \ldots, i-1.$$

Proof. Let $i \geq j$. Then,

$$G_{i,j} = \left(CHC^T\right)_{i,j} = \sum_{k=0}^{\infty} C_{i,k} h_k C_{j,k}$$
$$= \sum_{k=0}^{j} C_{i,k} h_k C_{j,k} = C_{i,j} h_j + \sum_{k=0}^{j-1} C_{i,k} h_k C_{j,k}.$$

Solving for $C_{i,j}$, we get

$$C_{i,j} = \frac{1}{h_j} \left(G_{i,j} - \sum_{k=0}^{j-1} C_{i,k} h_k C_{j,k} \right).$$

In particular, when $i = j$

$$1 = C_{i,i} = \frac{1}{h_i} \left[G_{i,i} - \sum_{k=0}^{i-1} \left(C_{i,k} \right)^2 h_k \right].$$

■

Example 20 *Let the matrix G be defined by (21). Since*

$$\begin{aligned}
\sum_{k=0}^{\infty} C_{i,k} h_k C_{j,k} &= \sum_{k=0}^{\infty} \binom{i}{k} \frac{(a)_i}{(a)_k} \left(\frac{z}{1-z} \right)^{i-k} \frac{(a)_k \, k! z^k}{(1-z)^{2k+a}} \binom{j}{k} \frac{(a)_j}{(a)_k} \left(\frac{z}{1-z} \right)^{j-k} \\
&= (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_i \, (a)_j \sum_{k=0}^{\infty} \binom{i}{k} \binom{j}{k} \frac{k!}{(a)_k} z^{-k} \\
&= (1-z)^{-a} \left(\frac{z}{1-z} \right)^{i+j} (a)_i \; (a)_j \; {}_2F_1 \left(\begin{matrix} -i, -j \\ a \end{matrix} ; \frac{1}{z} \right),
\end{aligned}$$

we see that the matrices C and H in the LDL decomposition (22) have entries

$$C_{i,j} = \binom{i}{j} \frac{(a)_i}{(a)_j} \left(\frac{z}{1-z} \right)^{i-j}, \quad i, j \in \mathbb{N}_0, \tag{23}$$

and $H_{i,j} = h_i \delta_{i,j}$, with

$$h_i = \frac{(a)_i \; i! \; z^i}{(1-z)^{2i+a}}, \quad i \in \mathbb{N}_0. \tag{24}$$

We conclude that $\mathfrak{L}$ is a quasi-definite functional if $-a \notin \mathbb{N}_0$, $z \neq 0$, and $z \notin [1, \infty)$. The functional $\mathfrak{L}$ will be positive definite if $a > 0$ and $0 < z < 1$.

2.3 Orthogonal polynomials

In this section, we introduce sequences of polynomials orthogonal with respect to linear functionals.

Definition 21 *If $\mathfrak{L}$ is a quasi-definite functional with respect to $\overrightarrow{q}$, we define the sequence of* ***monic orthogonal polynomials*** *(**MOPS**) with respect to $\mathfrak{L}$ by*

$$\overrightarrow{p} = C^{-1}\overrightarrow{q} \in \mathbb{C}\left[x\right]^{\infty\times 1}. \tag{25}$$

Example 22 *Let the matrix C be defined by (23). We have*

$$\begin{aligned}\sum_{k=0}^{\infty}\left(-1\right)^{i-k}C_{i,k}C_{k,j} &= \sum_{k=0}^{\infty}\left(-1\right)^{i-k}\binom{i}{k}\frac{(a)_i}{(a)_k}\left(\frac{z}{1-z}\right)^{i-k}\binom{k}{j}\frac{(a)_k}{(a)_j}\left(\frac{z}{1-z}\right)^{k-j}\\ &= \frac{(a)_i}{(a)_j}\left(\frac{z}{1-z}\right)^{i-j}\sum_{k=0}^{\infty}\left(-1\right)^{i-k}\binom{i}{k}\binom{k}{j}.\end{aligned}$$

If we use the formula for higher order differences [62, 6.1]

$$\Delta^{p}f\left(x\right) = \sum_{j=0}^{p}\binom{p}{j}\left(-1\right)^{p-j}f\left(x+j\right), \tag{26}$$

we see that

$$\sum_{k=0}^{\infty}\left(-1\right)^{i-k}\binom{i}{k}\binom{k}{j} = \sum_{k=0}^{i}\binom{i}{k}\left(-1\right)^{i-k}\binom{k}{j} = \left[\Delta^{i}\binom{x}{j}\right]_{x=0}.$$

From (13) and (15), we have

$$\left[\Delta^{i}\binom{x}{j}\right]_{x=0} = \frac{1}{j!}\left[\Delta^{i}\phi_j\left(x\right)\right]_{x=0} = \frac{\phi_i\left(j\right)}{j!}\phi_{j-i}\left(0\right) = \delta_{i,j},$$

since $\phi_i\left(i\right) = i!$.

Thus, we conclude that

$$\sum_{k=0}^{\infty}\left(-1\right)^{i-k}C_{i,k}C_{k,j} = \frac{(a)_i}{(a)_j}\left(\frac{z}{1-z}\right)^{i-j}\delta_{i,j} = \delta_{i,j},$$

and therefore

$$\left(-1\right)^{i-k}C_{i,k} = \left(C^{-1}\right)_{i,k}. \tag{27}$$

The polynomials $\overrightarrow{p} = C^{-1}\overrightarrow{\phi}$ are known as (monic) **Meixner polynomials** [47, 6.1]. Using (23) and (27), we get

$$p_n(x) = \sum_{j=0}^{\infty} \left(C^{-1}\right)_{n,j} \phi_j(x) = \sum_{j=0}^{\infty} (-1)^{n-j} \binom{n}{j} \frac{(a)_n}{(a)_j} \left(\frac{z}{1-z}\right)^{n-j} (-1)^j (-x)_j .$$

From (13) and (12), we have

$$\binom{n}{j} = \frac{\phi_j(n)}{j!} = \frac{(-1)^j (-n)_j}{j!}.$$

Therefore, we obtain the hypergeometric representation [34]

$$p_n(x) = (a)_n \left(\frac{z}{z-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ a \end{matrix}; 1 - z^{-1}\right).$$

Theorem 23 *Let $\mathfrak{L}$ be a quasi-definite functional with respect to $\overrightarrow{q}$ and $\overrightarrow{p}$ be the corresponding MOPS. Then,*

Proposition 24 *(i) The polynomials $p_n(x)$ satisfy the* ***orthogonality relation***

$$\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = H. \tag{28}$$

(ii) We have

$$\mathfrak{L}\left[\overrightarrow{p}\right] = h_0 \overrightarrow{e_0}, \tag{29}$$

where

$$(\overrightarrow{e_k})_j = \delta_{k,j}.$$

(iii) If $\overrightarrow{\psi}$ is a monic basis of $\mathbb{C}[x]$, then

$$\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{\psi}^T\right] = HU,$$

where U is a UUT matrix. In other words, for all $i, j \in \mathbb{N}_0$

$$\mathfrak{L}[p_i \psi_j] = \begin{cases} h_i, & i = j \\ 0, & i > j \end{cases} . \tag{30}$$

Proof. (i) Using (25), we have

$$\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[C^{-1}\overrightarrow{q}\ \overrightarrow{q}^T C^{-T}\right] = C^{-1}GC^{-T} = H,$$

where

$$C^{-T} = \left(C^T\right)^{-1} = \left(C^{-1}\right)^T.$$

(ii) Using (28), we have

$$\left(\mathfrak{L}\left[\overrightarrow{p}\right]\right)_j = \mathfrak{L}\left[p_j\right] = \mathfrak{L}\left[p_j p_0\right] = h_0 \delta_{j,0}.$$

(iii) If $\overrightarrow{\psi}$ is a monic basis of $\mathbb{C}[x]$, then there exists a ULT matrix L such that

$$\overrightarrow{\psi} = L\ \overrightarrow{q}.$$

Using (25), we get

$$\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{\psi}^T\right] = \mathfrak{L}\left[C^{-1}\overrightarrow{q}\ \overrightarrow{q}^T L^T\right] = C^{-1}GL^T = HC^T L^T.$$

Since C and L are ULT matrices, the matrix $C^T L^T$ is UUT. ■

Example 25 *Meixner polynomials. Using (19), (24) and (28), we obtain the orthogonality relation for the (monic) Meixner polynomials [34]*

$$\sum_{x=0}^{\infty} p_n(x)\, p_m(x)\, (a)_x \frac{z^x}{x!} = \frac{n!\ z^n\ (a)_n}{(1-z)^{a+2n}} \delta_{n,m}, \quad n, m \in \mathbb{N}_0.$$

Definition 26 *Let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional $\mathfrak{L}$. We define the* ***Jacobi matrix*** *$J \in \mathbb{C}^{\infty\times\infty}$ by*

$$J = \mathfrak{L}\left[x\overrightarrow{p}\ \overrightarrow{p}^T\right] H^{-1}. \tag{31}$$

Theorem 27 *(i) The Jacobi matrix J defined by (31) is a tridiagonal matrix with entries*

$$J_{i,j} = \delta_{i+1,j} + \beta_i \delta_{i,j} + \gamma_i \delta_{i-1,j}, \tag{32}$$

where the coefficients β_i, γ_i are given by

$$\beta_i = \frac{\mathfrak{L}\left[x p_i^2\right]}{h_i}, \quad i \in \mathbb{N}_0,$$

$\gamma_0 = 0$ and

$$\gamma_i = \frac{\mathfrak{L}\left[x p_i p_{i-1}\right]}{h_{i-1}} = \frac{h_i}{h_{i-1}} \neq 0, \quad i \in \mathbb{N}. \tag{33}$$

Proposition 28 *(ii) The polynomials* $\overrightarrow{p}$ *satisfy the eigenvalue equation*

$$J\ \overrightarrow{p} = x\overrightarrow{p}. \tag{34}$$

By linearity, this extends to

$$q(x)\overrightarrow{p} = q(J)\overrightarrow{p}, \quad q \in \mathbb{C}[x]. \tag{35}$$

(iii) Let $q \in \mathbb{C}[x]$ *. Then,* $q(J)H$ *is a symmetric matrix.*
(iv) Let $q \in \mathbb{C}[x]$ *be given by*

$$q(x) = \ \overrightarrow{p}^T\ \overrightarrow{\omega}, \quad \overrightarrow{\omega} \in \mathbb{C}[x]^{\infty\times 1}. \tag{36}$$

Then,

$$\omega_k = \frac{h_0}{h_k}[q(J)]_{k,0}. \tag{37}$$

Proof. (i) Using (30) in two different ways, we have

$$\mathfrak{L}[p_i\ xp_j] = \begin{cases} h_i, & i = j+1 \\ 0, & i > j+1 \end{cases},$$

and

$$\mathfrak{L}[p_j\ xp_i] = \begin{cases} h_j, & j = i+1 \\ 0, & j > i+1 \end{cases}.$$

Thus, from (31) we obtain

$$(JH)_{i,j} = 0, \quad j \notin \{i-1, i, i+1\}.$$

The three nonzero entries are given by

$$J_{i,i-1}h_{i-1} = \mathfrak{L}[xp_ip_{i-1}] = h_i,$$

$$J_{i,i}h_i = \mathfrak{L}\left[xp_i^2\right] = h_i\beta_i,$$

and

$$J_{i,i+1}h_{i+1} = \mathfrak{L}[xp_ip_{i+1}] = h_{i+1}.$$

(ii) Representing $x\overrightarrow{p}$ with respect to the basis $\overrightarrow{p}$, we have

$$x\overrightarrow{p} = M\overrightarrow{p},$$

for some matrix M. Multiplying by $\overrightarrow{p}^T$ and applying $\mathfrak{L}$ on both sides of the equation, we get

$$JH = \mathfrak{L}\left[x\overrightarrow{p}\ \overrightarrow{p}^T\right] = M\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = MH,$$

where we have used (28) and (31). Since H is nonsingular, $M = J$.

(iii) Using (35), we have

$$\mathfrak{L}\left[q\overrightarrow{p}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[q(J)\overrightarrow{p}\ \overrightarrow{p}^T\right] = q(J)\,\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = q(J)\,H.$$

But on the other hand,

$$\mathfrak{L}\left[q\overrightarrow{p}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T q\right] = \mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T q\left(J^T\right)\right] = Hq\left(J^T\right).$$

Therefore,

$$[q(J)\,H]^T = H^T[q(J)]^T = Hq\left(J^T\right) = q(J)\,H. \tag{38}$$

(iv) From (36), we have

$$\mathfrak{L}\left[\overrightarrow{p}q\right] = \mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\ \overrightarrow{\omega}\right] = H\ \overrightarrow{\omega}.$$

Using (35),

$$\mathfrak{L}\left[\overrightarrow{p}q\right] = \mathfrak{L}\left[q\overrightarrow{p}\right] = \mathfrak{L}\left[q(J)\,\overrightarrow{p}\right] = q(J)\,\mathfrak{L}\left[\overrightarrow{p}\right].$$

Finally, from (29)

$$q(J)\,\mathfrak{L}\left[\overrightarrow{p}\right] = q(J)\,h_0\overrightarrow{e_0}.$$

Thus, we conclude that

$$h_j\omega_j = \left(H\ \overrightarrow{\omega}\right)_j = \sum_k [q(J)]_{j,k}\,h_0\delta_{k,0} = h_0\,[q(J)]_{j,0}\,.$$

■

Corollary 29 *Let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional $\mathfrak{L}$. Then, the polynomials $\overrightarrow{p}$ satisfy the* ***three-term recurrence relation***

$$xp_n = p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \quad n \in \mathbb{N}_0, \tag{39}$$

with initial conditions

$$p_{-1} = 0, \quad p_0 = 1.$$

The following result is known as the Modified Chebyshev algorithm [43, 2.1.7].

Proposition 30 *Let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional $\mathfrak{L}$ and $\overrightarrow{q}$ be a monic basis of $\mathbb{C}[x]$ satisfying*

$$x\overrightarrow{q} = T\ \overrightarrow{q}, \tag{40}$$

where T is a tridiagonal matrix with entries

$$T_{i,j} = \delta_{i+1,j} + \eta_i \delta_{i,j} + \xi_i \delta_{i-1,j}. \tag{41}$$

Let the "modified moments" be defined by

$$R = \mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{p}^T\right].$$

Then, the entries of R satisfy the recurrence

$$R_{i,j+1} = R_{i+1,j} + (\eta_i - \beta_j) R_{i,j} + \xi_i R_{i-1,j} - \gamma_j R_{i,j-1},$$

with initial values

$$R_{i,-1} = 0, \quad R_{i,0} = \mathfrak{L}[q_i] = \nu_i, \quad i \in \mathbb{N}_0.$$

Moreover, the coefficients in the three-term recurrence relation (39) are given by

$$\beta_i = \eta_i + \frac{R_{i+1,i}}{R_{i,i}} - \frac{R_{i,i-1}}{R_{i-1,i-1}}, \tag{42}$$

and

$$\gamma_i = \frac{R_{i,i}}{R_{i-1,i-1}}. \tag{43}$$

Proof. Let L be the ULT matrix satisfying

$$\overrightarrow{q} = L\ \overrightarrow{p}.$$

Then,

$$R = \mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[L\ \overrightarrow{p}\ \overrightarrow{p}^T\right] = LH. \tag{44}$$

Hence, R is a lower triangular matrix and

$$R_{i,i} = h_i. \tag{45}$$

Using (34) and (40), we have

$$T \overrightarrow{q} \ \overrightarrow{p}^T = x\overrightarrow{q} \ \overrightarrow{p}^T = \overrightarrow{q} \ x\overrightarrow{p}^T = \overrightarrow{q} \ \overrightarrow{p}^T J^T,$$

and therefore

$$TR = \mathfrak{L}\left[T \overrightarrow{q} \ \overrightarrow{p}^T\right] = \mathfrak{L}\left[\overrightarrow{q} \ \overrightarrow{p}^T J^T\right] = R \ J^T.$$

Using (32) and (41), we get

$$R_{i+1,j} + \eta_i R_{i,j} + \xi_i R_{i-1,j} = R_{i,j+1} + \beta_j R_{i,j} + \gamma_j R_{i,j-1}. \tag{46}$$

Since R is a lower triangular matrix, we have

$$R_{i,j} = 0, \quad i < j, \tag{47}$$

and setting $i = j - 1$ in (46), we obtain

$$\gamma_j = \frac{R_{j,j}}{R_{j-1,j-1}}. \tag{48}$$

Note that from (45) and (48) we have

$$\gamma_j = \frac{h_j}{h_{j-1}},$$

in agreement with (33).

If we set $i = j$ in (46) and use (48) and (47), we obtain

$$\beta_j = \eta_j + \frac{R_{j+1,j} - \gamma_j R_{j,j-1}}{R_{j,j}} = \eta_j + \frac{R_{j+1,j}}{R_{j,j}} - \frac{R_{j,j-1}}{R_{j-1,j-1}}.$$

Finally, solving for $R_{i,j+1}$ in (46), we get

$$R_{i,j+1} = R_{i+1,j} + (\eta_i - \beta_j) R_{i,j} + \xi_i R_{i-1,j} - \gamma_j R_{i,j-1}.$$

■

Example 31 *Meixner polynomials. The falling factorial polynomials satisfy the 3-term recurrence relation (6). Comparing with (41), we see that*

$$\eta_n = n, \quad \xi_n = 0,$$

and therefore

$$T_{i,j} = \delta_{i+1,j} + i\delta_{i,j}.$$

Using (44), we get

$$R_{i,j} = \sum_{k=0}^{\infty} C_{i,k} H_{k,j} = C_{i,j} h_j = \binom{i}{j} \frac{(a)_i}{(a)_j} \left(\frac{z}{1-z} \right)^{i-j} \frac{(a)_j \; j! \; z^j}{(1-z)^{2j+a}}$$
$$= j! \binom{i}{j} (a)_i \frac{z^i}{(1-z)^{i+j+a}}.$$

Finally, using (42) and (43) we obtain [34]

$$\beta_n = n + \frac{R_{n+1,n}}{R_{n,n}} - \frac{R_{n,n-1}}{R_{n-1,n-1}} = \frac{n + (n+a) z}{1-z} \tag{49}$$

and

$$\gamma_n = \frac{R_{n,n}}{R_{n-1,n-1}} = \frac{n (n-1+a) z}{(1-z)^2}. \tag{50}$$

In the next section, we will consider a class of orthogonal polynomials that includes the Meixner family as a particular case.

3 Semiclassical orthogonal polynomials

Let $\Upsilon : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator and let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional $\mathfrak{L}$. We say that $\overrightarrow{p}$ is **semiclassical** (with respect to Υ) if there exist fixed polynomials $\lambda(x), \tau(x)$ such that the functional $\mathfrak{L}$ satisfies the **Pearson equation**

$$\mathfrak{L}[\lambda \Upsilon q] = \mathfrak{L}[\tau q], \quad q \in \mathbb{C}[x]. \tag{51}$$

We define the class of $\overrightarrow{p}$ to be the number

$$s = \max\{\deg(\lambda) - 2,\ \deg(\lambda - \tau) - 1\}. \tag{52}$$

The polynomials of class $s = 0$ are called **classical**.

In particular, let's suppose that the operator Υ is the shift operator

$$\Upsilon q(x) = q(x+1), \quad q \in \mathbb{C}[x], \tag{53}$$

and the functional $\mathfrak{L}$ has the form

$$\mathfrak{L}[q] = \sum_{x=0}^{\infty} q(x)\,\rho(x), \quad q \in \mathbb{C}[x], \tag{54}$$

for some weight function $\rho : \mathbb{N}_0 \to \mathbb{C}$ with

$$\rho(-1) = 0. \tag{55}$$

Using (54) in (51), we have

$$\sum_{x=0}^{\infty} \lambda(x-1)\,q(x)\,\rho(x-1) = \sum_{x=-1}^{\infty} \lambda(x)\,q(x+1)\,\rho(x) = \sum_{x=0}^{\infty} \tau(x)\,q(x)\,\rho(x),$$

where we have used (55).

We conclude that the weight function $\rho(x)$ must satisfy

$$\lambda(x-1)\,\rho(x-1) = \tau(x)\,\rho(x),$$

or

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\lambda(x)}{\tau(x+1)}. \tag{56}$$

If the polynomials $\lambda(x), \tau(x)$ are given by

$$\begin{aligned} \lambda(x) &= z(x+a_1)(x+a_2)\cdots(x+a_l), \\ \tau(x) &= x(x+b_1)(x+b_2)\cdots(x+b_{t-1}), \end{aligned} \tag{57}$$

then solving (56) we see that

$$\rho(x) = \frac{(a_1)_x (a_2)_x \cdots (a_l)_x}{(b_1+1)_x (b_2+1)_x \cdots (b_{t-1}+1)_x} \frac{z^x}{x!}. \tag{58}$$

Note that (55) is satisfied, since $\frac{1}{x!} = 0$ at $x = -1$. In [35], we classified the weight functions satisfying (56), with $\deg(\lambda - \tau) = 2$ and $1 \leq \deg(\tau) \leq 3$. For a recent book on discrete semiclassical polynomials, see [69].

Suppose that the shift operator Υ is represented by the matrix S on the basis $\overrightarrow{p}$,

$$\Upsilon \overrightarrow{p} = S \overrightarrow{p}. \tag{59}$$

Since $\overrightarrow{p}$ is a monic basis, it follows that S is a ULT matrix. It's relation to the Jacobi matrix is given in the following result.

Proposition 32 *Let J be the Jacobi matrix defined in (31) and S be defined by (59). Then,*

$$[J,S]=S, \tag{60}$$

*where the **commutator** $[A,B]$ is defined by*

$$[A,B]=AB-BA.$$

Proof. Using (34), we have

$$(x+1)\overrightarrow{p}(x+1)=J\overrightarrow{p}(x+1)=J\Upsilon\overrightarrow{p}=JS\overrightarrow{p}.$$

On the other hand, from (59) we get

$$(x+1)\overrightarrow{p}(x+1)=(x+1)\Upsilon(\overrightarrow{p})=(x+1)S\overrightarrow{p}=Sx\overrightarrow{p}+S\overrightarrow{p}=SJ\overrightarrow{p}+S\overrightarrow{p}.$$

Thus,

$$SJ+S=JS.$$

■

Remark 33 *If we use (32), we can write the commutator equation (60) in extended form*

$$\begin{aligned}([J,S])_{i,j}&=\sum_k\left(\delta_{i+1,k}+\beta_i\delta_{i,k}+\gamma_i\delta_{i-1,k}\right)S_{k,j}-\sum_k S_{i,k}\left(\delta_{k+1,j}+\beta_k\delta_{k,j}+\gamma_k\delta_{k-1,j}\right)\\&=S_{i+1,j}+\beta_iS_{i,j}+\gamma_iS_{i-1,j}-S_{i,j-1}-\beta_jS_{i,j}-\gamma_{j+1}S_{i,j+1}=S_{i,j}.\end{aligned}$$

Hence,

$$S_{i+1,j}-S_{i,j-1}+(\beta_i-\beta_j-1)S_{i,j}+\gamma_iS_{i-1,j}-\gamma_{j+1}S_{i,j+1}=0. \tag{61}$$

Since S is a ULT matrix, the equations (61) are automatically true for $i<j-1$. For $i=j-1$, we have

$$\begin{aligned}&S_{j,j}-S_{j-1,j-1}+(\beta_{j-1}-\beta_j-1)S_{j-1,j}+\gamma_{j-1}S_{j-2,j}-\gamma_{j+1}S_{j-1,j+1}\\&=1-1+0+0-0=0.\end{aligned}$$

Finally, for $i=j$, we obtain

$$\begin{aligned}&S_{j+1,j}-S_{j,j-1}+(\beta_j-\beta_j-1)S_{j,j}+\gamma_jS_{j-1,j}-\gamma_{j+1}S_{j,j+1}\\&=S_{j+1,j}-S_{j,j-1}-1=0.\end{aligned}$$

Summing the last equation from $j = 0$ to $n - 1$, we get

$$S_{n,n-1} - S_{0,-1} = n,$$

and since $S_{0,-1} = 0$, we conclude that

$$S_{n,n-1} = n. \tag{62}$$

The same result can be obtained by using the fact that

$$p_n(x+1) - p_n(x) = (x+1)^n - x^n + \cdots = nx^{n-1} + \cdots = np_{n-1}(x) + \cdots,$$

where $\cdots$ denotes lower order terms.

Example 34 *Meixner polynomials. We claim that for these polynomials,*

$$S_{i,j} = \left(\frac{z}{z-1}\right)^{i-j-1+\delta_{i,j}} \frac{i!}{j!}\chi(i \geq j), \tag{63}$$

where $\chi(i \geq j)$ denotes the characteristic function

$$\chi(i \geq j) = \begin{cases} 1, & i \geq j \\ 0, & i < j \end{cases}.$$

Clearly $S_{i,i} = 1$ and

$$S_{i,i-1} = \frac{i!}{(i-1)!} = i,$$

in agreement with (62). Hence, we only need to verify (61) for $i > j$.

If $i = j + 1$,

$$\begin{aligned} &S_{j+2,j} - S_{j+1,j-1} + (\beta_{j+1} - \beta_j - 1) S_{j+1,j} + \gamma_{j+1} S_{j,j} - \gamma_{j+1} S_{j+1,j+1} \\ &= S_{j+2,j} - S_{j+1,j-1} + (\beta_{j+1} - \beta_j - 1) S_{j+1,j}. \end{aligned}$$

Using (63), we have

$$S_{j+2,j} - S_{j+1,j-1} = \frac{2z}{z-1}(j+1), \quad S_{j+1,j} = j+1,$$

and from (49)

$$\beta_{j+1} - \beta_j - 1 = -\frac{2z}{z-1}.$$

Hence,

$$S_{j+2,j} - S_{j+1,j-1} + (\beta_{j+1} - \beta_j - 1)\, S_{j+1,j} = 0.$$

Finally, for $i > j+1$

$$S_{i+1,j} - S_{i,j-1} = \left(\frac{z}{z-1}\right)^{i-j} (i-j+1)\,\frac{i!}{j!},$$

$$(\beta_i - \beta_j - 1)\, S_{i,j} = -\left(\frac{z}{z-1}\right)^{i-j} \frac{(i+1-j)\,z + i - 1 - j}{z}\,\frac{i!}{j!},$$

and from (50)

$$\gamma_i S_{i-1,j} - \gamma_{j+1} S_{i,j+1} = \left(\frac{z}{z-1}\right)^{i-j} \frac{(i-j-1)}{z}\,\frac{i!}{j!}.$$

Therefore,

$$S_{i+1,j} - S_{i,j-1} + (\beta_i - \beta_j - 1)\, S_{i,j} + \gamma_i S_{i-1,j} - \gamma_{j+1} S_{i,j+1}$$
$$= \left(\frac{z}{z-1}\right)^{i-j} \frac{i!}{j!}\left[i - j + 1 - \frac{(i+1-j)\,z + i - 1 - j}{z} + \frac{(i-j-1)}{z}\right] = 0.$$

The inverse of the shift operator is given by

$$\Upsilon^{-1} q(x) = q(x-1), \quad q \in \mathbb{C}[x],$$

and is represented by the matrix S^{-1}, since

$$\overrightarrow{p}(x) = \Upsilon^{-1}\left[\Upsilon \overrightarrow{p}(x)\right] = \Upsilon^{-1} S \overrightarrow{p}(x) = S \Upsilon^{-1} \overrightarrow{p}(x).$$

Example 35 *Meixner polynomials. The inverse of the matrix* S *defined by (63) is given by*

$$\left(S^{-1}\right)_{i,j} = \frac{(-1)^{1+\delta_{i,j}}}{(z-1)^{i-j-1+\delta_{i,j}}}\,\frac{i!}{j!}\chi(i \geq j). \tag{64}$$

To see this, consider

$$U_{i,j} = \sum_k \left(\frac{z}{z-1}\right)^{i-k-1+\delta(i,k)} \frac{i!}{k!}\chi(i \geq k)\,\frac{(-1)^{1+\delta(k,j)}}{(z-1)^{k-j-1+\delta(k,j)}}\,\frac{k!}{j!}\chi(k \geq j).$$

Clearly, $U_{i,j} = 0$ *for* $i < j$*. For* $i \geq j$*, we have*

$$U_{i,j} = \frac{i!}{j!} \sum_{k=j}^{i} (-1)^{1+\delta(k,j)} \frac{z^{i-k-1+\delta(i,k)}}{(z-1)^{i-2+\delta(i,k)-j+\delta(k,j)}},$$

and we see that $U_{i,i} = 1$*.*

When $i > j$*,*

$$\begin{aligned} U_{i,j} &= \frac{i!}{j!} \left[\left(\frac{z}{z-1} \right)^{i-j-1} - \frac{1}{(z-1)^{i-j-1}} - \sum_{k=j+1}^{i-1} \frac{z^{i-k-1}}{(z-1)^{i-2-j}} \right] \\ &= \frac{i!}{j!} \left[\left(\frac{z}{z-1} \right)^{i-j-1} - \frac{1}{(z-1)^{i-j-1}} - \frac{z^{i-j-1} - 1}{(z-1)^{i-j-1}} \right] = 0. \end{aligned}$$

Therefore, $U_{i,j} = \delta_{i,j}$*.*

3.1 Laguerre-Freud equations

If $\overrightarrow{p}$ is a family of semiclassical polynomials, then the coefficients in the 3-term recurrence relation (39) satisfy a (in general) nonlinear system of equations, known as "Laguerre-Freud equations" [24], [56]. In this section, we derive a system of matrix equations that leads to the Laguerre-Freud equations. We presented some of these ideas at the meeting "Challenges in 21st Century Experimental Mathematical Computation" held at ICERM, Brown University, Providence, RI, on July 21-25, 2014.

We begin with a matrix analogue of the Pearson equation.

Theorem 36 *Let* J *be the Jacobi matrix defined in (31) and* S *be defined by (59). If the linear functional* $\mathfrak{L}$ *satisfies the Pearson equation (51), then*

$$S\lambda(J) HS^T = H\tau\left(J^T\right). \tag{65}$$

Proof. Since the shift operator Υ is multiplicative, we can use (59) and obtain

$$\Upsilon\left(\overrightarrow{p}\ \overrightarrow{p}^T\right) = \left(\Upsilon\overrightarrow{p}\right)\left(\Upsilon\overrightarrow{p}\right)^T = S\ \overrightarrow{p}\ \overrightarrow{p}^T S^T.$$

Thus, from (51), we get

$$\begin{aligned} \tau(J) H &= \mathfrak{L}\left[\tau(J)\overrightarrow{p}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[\tau\overrightarrow{p}\ \overrightarrow{p}^T\right] = \mathfrak{L}\left[\lambda\Upsilon\left(\overrightarrow{p}\ \overrightarrow{p}^T\right)\right] \\ &= \mathfrak{L}\left[\lambda S\ \overrightarrow{p}\ \overrightarrow{p}^T S^T\right] = S\mathfrak{L}\left[\lambda\ \overrightarrow{p}\ \overrightarrow{p}^T\right] S^T \\ &= S\mathfrak{L}\left[\lambda(J)\ \overrightarrow{p}\ \overrightarrow{p}^T\right] S^T = S\lambda(J) HS^T. \end{aligned}$$

Finally, from (38), we have

$$S\lambda(J)\,HS^{T}=\tau(J)\,H=H\tau\left(J^{T}\right).$$

■

Remark 37 *If we eliminate S from the system*

$$[J,S]=S,\quad S\lambda(J)\,HS^{T}=H\tau\left(J^{T}\right),$$

we obtain nonlinear relations among the entries of J, i.e., between the coefficients in the 3-term recurrence relation β_n and γ_n. For a different approach, see [48].

In [34], we developed a method for obtaining Laguerre-Freud equations from (60) and (65). Our approach was to introduce the matrices

$$A=S\lambda(J)\,,\quad B=S^{-1}\tau(J)\,. \tag{66}$$

Then, it follows from (60) that

$$[J,A]=JS\lambda(J)-S\lambda(J)\,J=JS\lambda(J)-SJ\lambda(J)=[J,S]\,\lambda(J)=S\lambda(J)=A, \tag{67}$$

where we have used the fact that

$$q(J)\,J=Jq(J)\,,\quad q\in\mathbb{C}[x]\,.$$

From (65), we obtain

$$B^{T}=\tau\left(J^{T}\right)S^{-T}=H^{-1}S\lambda(J)\,H=H^{-1}AH. \tag{68}$$

Next, we need the concept of banded matrices [46, 4.3].

Definition 38 *Let $A\in\mathbb{C}^{\infty\times\infty}$. We say that A is a $(k_1,k_2)-$**banded matrix** if*

$$A_{i,j}=0,\quad j>i+k_1 \ \text{ or } \ j<i-k_2,$$

*where $k_1,k_2\in\mathbb{N}_0$. The quantities k_1 and k_2 are called the **upper and lower bandwidth**, respectively.*

*The **bandwidth** of the matrix is defined by $k=\max\{k_1,k_2\}$. Note that*

$$A_{i,j}=0,\quad |i-j|>k.$$

The advantage of using A, B is that they are banded matrices.

Theorem 39 *Let*

$$\lambda(x)\overrightarrow{p}(x+1) = A\overrightarrow{p}(x) \tag{69}$$

and

$$\tau(x)\overrightarrow{p}(x-1) = B\overrightarrow{p}(x). \tag{70}$$

Then, A is a (l,t) −banded matrix and B is a (t,l) −banded matrix.

Proof. From (69), we have

$$\lambda(x) p_n(x+1) = \sum_k A_{n,k} p_k(x).$$

Since $\deg(\lambda) = l$, we get $A_{n,k} = 0, \quad k > n+l$. Similarly, from (70)

$$\tau(x) p_n(x-1) = \sum_k B_{n,k} p_k(x),$$

and since $\deg(\tau) = t$, we obtain $B_{n,k} = 0, \quad k > n+t$.

But from (68), we see that

$$\frac{h_k}{h_n} A_{n,k} = B_{k,n} = 0, \quad n > k+t,$$

and

$$B_{n,k} = \frac{h_n}{h_k} A_{k,n} = 0, \quad n > k+l.$$

■

Remark 40 *In [42], the authors study a characterization of semiclassical polynomials with respect to the derivative operator using banded matrices.*

Next, we introduce a sequence of functions that can be used to find the entries of the matrix A.

Theorem 41 *Let $\alpha_k(n)$ be defined by*

$$\lambda(x) p_n(x+1) = \sum_{k=-t}^{l} \alpha_k(n) p_{n+k}(x). \tag{71}$$

Then, $\alpha_k(n)$ satisfies the system of partial difference equations

$$\alpha_{k-1}(n+1) - \alpha_{k-1}(n) = (1 + \beta_{n+k} - \beta_n)\,\alpha_k(n) \qquad (72)$$
$$+\gamma_{n+k+1}\alpha_{k+1}(n) - \gamma_n\alpha_{k+1}(n-1)\,, \quad -t \le k \le l,$$

with boundary conditions

$$\alpha_k(n) = 0, \quad k \notin [-t, l]\,, \quad \alpha_l(n) = z, \quad \alpha_{-t}(n) = \frac{h_n}{h_{n-t}}, \qquad (73)$$

and initial conditions

$$\alpha_k(0) = \frac{h_0}{h_k}\,[\lambda(J)]_{k,0}\,, \quad \alpha_{-k}(k) = [\tau(J)]_{k,0}\,. \qquad (74)$$

Proof. Comparing (69) with (71), we see that

$$\alpha_{k-n}(n) = A_{n,k}. \qquad (75)$$

From (67), we know that $A = [J, A]$, and using (61) with S replaced by A, we get

$$A_{i+1,j} - A_{i,j-1} + (\beta_i - \beta_j - 1)\,A_{i,j} + \gamma_i A_{i-1,j} - \gamma_{j+1}A_{i,j+1} = 0.$$

Thus, using (75) we obtain

$$\alpha_{j-i-1}(i+1) - \alpha_{j-i-1}(i) + (\beta_i - \beta_j - 1)\,\alpha_{j-i}(i) + \gamma_i\alpha_{j-i+1}(i-1) - \gamma_{j+1}\alpha_{j-i+1}(i) = 0,$$

from which (72) follows after setting $i \to n$, $j - i \to k$.

From (68) and (75), we have

$$B_{n,k} = \frac{h_n}{h_k}A_{k,n} = \frac{h_n}{h_k}\alpha_{n-k}(k)\,.$$

Therefore, we can rewrite (70) as

$$\tau(x)\,p_n(x-1) = \sum_{k=n-l}^{n+t}\frac{h_n}{h_k}\alpha_{n-k}(k)\,p_k(x) = \sum_{k=-l}^{t}\frac{h_n}{h_{n+k}}\alpha_{-k}(n+k)\,p_{n+k}(x)\,. \qquad (76)$$

Comparing coefficients in (71) and using (57), we get

$$\alpha_l(n) = z,$$

while from (76) we obtain

$$\frac{h_n}{h_{n+t}}\alpha_{-t}(n+t)=1,$$

from which (73) follows.

Finally, setting $n=0$ in (71) we have

$$\lambda(x)=\sum_{k=0}^{l}\alpha_k(0)\,p_k(x)\,.$$

Hence, using (37)

$$\alpha_k(0)=\frac{h_0}{h_k}\left[\lambda(J)\right]_{k,0}.$$

Similarly, setting $n=0$ in (76), we get

$$\tau(x)=\sum_{k=0}^{t}\frac{h_0}{h_k}\alpha_{-k}(k)\,p_k(x)\,,$$

and (37) gives

$$\frac{h_0}{h_k}\alpha_{-k}(k)=\frac{h_0}{h_k}\left[\tau(J)\right]_{k,0}.$$

■

Example 42 *Meixner polynomials. From (19), we have*

$$\rho(x)=(a)_x\frac{z^x}{x!},$$

and using (9) we get

$$\frac{\rho(x+1)}{\rho(x)}=\frac{z(x+a)}{x+1}.$$

Comparing with (56), we conclude that

$$\lambda(x)=z(x+a)\,,\quad \tau(x)=x. \tag{77}$$

Hence, $l=t=1$, and it follows from (52) that the Meixner polynomials are classical.

Setting $k = 1, 0, -1$ in (72), we obtain

$$\alpha_0(n+1) - \alpha_0(n) = (1 + \beta_{n+1} - \beta_n)\,\alpha_1 + \gamma_{n+2}\alpha_2(n) - \gamma_n\alpha_2(n-1)\,,$$

$$\alpha_{-1}(n+1) - \alpha_{-1}(n) = \alpha_0(n) + \gamma_{n+1}\alpha_1(n) - \gamma_n\alpha_1(n-1)\,,$$

$$\alpha_{-2}(n+1) - \alpha_{-2}(n) = (1 + \beta_{n-1} - \beta_n)\,\alpha_{-1}(n) + \gamma_n\alpha_0(n) - \gamma_n\alpha_0(n-1)\,,$$

and from (73) we see that

$$\alpha_2 = 0, \quad \alpha_{-2} = 0, \quad \alpha_1 = z, \quad \alpha_{-1} = \frac{h_n}{h_{n-1}} = \gamma_n,$$

where we have used (33).

Thus, we have

$$\alpha_0(n+1) - \alpha_0(n) = z\,(1 + \beta_{n+1} - \beta_n)\,,$$

$$\gamma_{n+1} - \gamma_n = \alpha_0(n) + z\,(\gamma_{n+1} - \gamma_n)\,,$$

$$0 = (1 + \beta_{n-1} - \beta_n)\,\gamma_n + \gamma_n\left[\alpha_0(n) - \alpha_0(n-1)\right],$$

and we conclude that

$$\alpha_0(n) = \alpha_0(0) + z\,(n + \beta_n - \beta_0)\,,$$

$$\alpha_0(n) = (1 - z)\,(\gamma_{n+1} - \gamma_n)\,,$$

$$\alpha_0(n) = \alpha_0(0) - n - \beta_0 + \beta_n.$$

Using (74), we get

$$\alpha_k(0) = z\frac{h_0}{h_k}\,(J + aI)_{k,0}\,, \quad \alpha_{-k}(k) = J_{k,0},$$

and therefore

$$z\,(\beta_0 + a) = \alpha_0(0) = \beta_0.$$

Hence,

$$\beta_0 = a\frac{z}{1 - z},$$

and

$$\alpha_0(n) = z\,(n + a + \beta_n) = (1 - z)\,(\gamma_{n+1} - \gamma_n) = -n + \beta_n,$$

from which it follows that

$$\beta_n = \frac{n + z\,(a+n)}{1-z}, \quad \gamma_n = \gamma_0 + \frac{n\,(n+a-1)\,z}{(1-z)^2}.$$

Since $\gamma_0 = 0$, *we recover (49) and (50).*

Moreover, from (71) we have

$$z\,(x+a)\,p_n\,(x+1) = \gamma_n p_{n-1}\,(x) + (\beta_n - n)\,p_n\,(x) + z p_{n+1}\,(x)\,, \tag{78}$$

and from (76)

$$x p_n\,(x-1) = z\gamma_n p_{n-1}\,(x) + (\beta_n - n)\,p_n\,(x) + \frac{h_n}{h_{n+1}}\gamma_{n+1} p_{n+1}\,(x)\,.$$

But

$$\frac{h_{n+1}}{h_n} = \gamma_{n+1},$$

and therefore

$$x p_n\,(x-1) = z\gamma_n p_{n-1}\,(x) + (\beta_n - n)\,p_n\,(x) + p_{n+1}\,(x)\,. \tag{79}$$

$$z\,(x+a)\,M_n\,(x+1) + [n - x - z\,(x+a+n)]\,M_n\,(x) + x M_n\,(x-1) = 0,$$

Adding (78) and (79), we obtain

$$z\,(x+a)\,p_n\,(x+1) + x p_n\,(x-1) = (1+z)\,\gamma_n p_{n-1}\,(x) + 2\,(\beta_n - n)\,p_n\,(x) + (1+z)\,p_{n+1}\,(x)\,,$$

and from (39),

$$(1+z)\,\gamma_n p_{n-1}\,(x) + (1+z)\,p_{n+1}\,(x) = (1+z)\,(x - \beta_n)\,p_n\,(x)\,.$$

Thus, we obtain the difference equation [61, 18.22.12]

$$\begin{aligned} z\,(x+a)\,p_n\,(x+1) + x p_n\,(x-1) &= [(1+z)\,(x-\beta_n) + 2\,(\beta_n - n)]\,p_n\,(x) \\ &= [z\,(x+a+n) + x - n]\,p_n\,(x)\,. \end{aligned}$$

Remark 43 *Difference equations for discrete orthogonal polynomials using a matrix approach were derived in [71].*

Remark 44 *If we use (63), (64), and (77) in (66), we obtain*

$$\frac{A_{n,k}}{z} = [S(J+aI)]_{n,k} = \sum_j \left(\frac{z}{z-1}\right)^{n-j-1+\delta_{n,j}} \frac{n!}{j!}\chi(n \geq j)\left[\delta_{j+1,k} + (\beta_j + a)\,\delta_{j,k} + \gamma_j \delta_{j-1,k}\right]$$

$$= \left(\frac{z}{z-1}\right)^{n-k+\delta_{n,k-1}} \frac{n!}{(k-1)!}\chi(n \geq k-1) + (\beta_k + a)\left(\frac{z}{z-1}\right)^{n-k-1+\delta_{n,k}} \frac{n!}{k!}\chi(n \geq k)$$

$$+ \gamma_{k+1}\left(\frac{z}{z-1}\right)^{n-k-2+\delta_{n,k+1}} \frac{n!}{(k+1)!}\chi(n \geq k+1),$$

and

$$B_{n,k} = \left(S^{-1}J\right)_{n,k} = \sum_j \frac{(-1)^{1+\delta_{n,j}}}{(z-1)^{n-j-1+\delta_{n,j}}} \frac{n!}{j!}\chi(n \geq j)\left[\delta_{j+1,k} + \beta_j \delta_{j,k} + \gamma_j \delta_{j-1,k}\right]$$

$$= \frac{(-1)^{1+\delta_{n,k-1}}}{(z-1)^{n-k+\delta_{n,k-1}}} \frac{n!}{(k-1)!}\chi(n \geq k-1) + \beta_k \frac{(-1)^{1+\delta_{n,k}}}{(z-1)^{n-k-1+\delta_{n,k}}} \frac{n!}{k!}\chi(n \geq k)$$

$$+ \gamma_{k+1} \frac{(-1)^{1+\delta_{n,k+1}}}{(z-1)^{n-k-2+\delta_{n,k+1}}} \frac{n!}{(k+1)!}\chi(n \geq k+1).$$

The only nonzero terms are

$$\frac{A_{n,n+1}}{z} = 1, \quad \frac{A_{n,n}}{z} = n + \beta_n + a, \quad \frac{A_{n,n-1}}{z} = \frac{z}{z-1} n(n-1) + (\beta_{n-1} + a)\,n + \gamma_n,$$

and

$$B_{n,n+1} = 1, \quad B_{n,n} = -n + \beta_n, \quad B_{n,n-1} = \frac{-1}{z-1} n(n-1) - n\beta_{n-1} + \gamma_n.$$

But from (68), we have

$$1 = B_{n,n+1} = \frac{h_n}{h_{n+1}} A_{n+1,n} = \frac{z}{\gamma_{n+1}}\left[\frac{z}{z-1} n(n+1) + (\beta_n + a)(n+1) + \gamma_{n+1}\right],$$

$$-n + \beta_n = B_{n,n} = A_{n,n} = z(n + \beta_n + a),$$

and

$$\frac{n(n-1)}{1-z} - n\beta_{n-1} + \gamma_n = B_{n,n-1} = \frac{h_n}{h_{n-1}} A_{n-1,n} = z\gamma_n.$$

Solving for β_n and γ_n, we obtain once again (49) and (50).

4 Conclusions

We have presented an introduction to a theory of polynomials $\overrightarrow{p} \in \mathbb{C}[x]^{\infty \times 1}$, orthogonal with respect to a linear functional $\mathfrak{L} : \mathbb{C}[x] \rightarrow \mathbb{C}$, based on the assumption that the Gram matrix

$$G = \mathfrak{L}\left[\overrightarrow{q}\ \overrightarrow{q}^T\right] \in \mathbb{C}^{\infty \times \infty}$$

admits the LDL decomposition $G = CHC^T$ for some monic basis $\overrightarrow{q} \in \mathbb{C}[x]^{\infty \times 1}$, where $C \in \mathbb{C}^{\infty \times \infty}$ is a unit upper triangular matrix and $H \in \mathbb{C}^{\infty \times \infty}$ is a nonsingular diagonal matrix. The polynomials $\overrightarrow{p}$ are defined by $\overrightarrow{p} = C^{-1}\overrightarrow{q}$, and satisfy the orthogonality condition

$$\mathfrak{L}\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = H.$$

The advantages of this approach are manifold, including the simplification of many proofs, and the shining of new light on many formulas that are standard in the theory of orthogonal polynomials.

Many other papers have explored the same topic, especially in the fields of mathematical physics, random matrices, and integrable systems. In most cases, the authors have used the monomial basis$(\overrightarrow{q})_i = x^i$ and studied orthogonal polynomials that are eigenfunctions of differential operators.

In this work, we have used the basis of falling factorials, and consider orthogonal polynomials with respect to a functional that satisfies a Person equation for the shift operator (called discrete semiclassical polynomials). We have illustrated our methodology using the family of Meixner polynomials, because this is a case were the formulas can be evaluated explicitly, and some of them can be compared with classical results.

Much is left to be done, and we plan to expand the theory in further articles. Directions to be considered include Toda systems, discrete and continuous Painlevé equations for the 3-term recurrence coefficients, higher order difference equations, and linear functionals with added point masses.

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