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DK-Report No. 2019-04

04 2019

A-4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

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# Discrete semiclassical orthogonal polynomials of class 2

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May 1, 2019

## Abstract

In this contribution, discrete semiclassical orthogonal polynomials of class  $s \leq 2$  are studied. By considering all possible solutions of the Pearson equation, we obtain the canonical families in each class. We also consider limit relations between these and other families of orthogonal polynomials.

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# 1 Introduction

Semiclassical linear functionals with respect to the derivative operator have received increasing attention in the literature of orthogonal polynomials taking into account their connections with many interesting problems in mathematical physics and numerical quadratures. From the so called Pearson equation associated with those linear functionals, ladder operators for the corresponding sequences of orthogonal polynomials can be deduced from the structure relations they satisfy (see [36],[37], [39]). As a direct consequence, a simple computation allows to deduce the corresponding holonomic equations, i.e. second order linear differential equations with polynomial coefficients of bounded degrees (depending on the degree of the polynomial) associated with the class of the linear functional (see [30], [39]).

The case of semiclassical linear functionals with respect to the difference operator in a uniform lattice is related to discretizations of holonomic equations, a fact pointed out in the classical case by Nikiforov, Uvarov and Suslov in a stimulating monograph on classical discrete orthogonal polynomials [42].

In both situations, a key problem is the classification of semiclassical linear functionals by using a hierarchy according to their class. This represents an alternative method to the Askey tableau, where classical orthogonal polynomials appear as a hierarchy of hypergeometric polynomials (see [31]).

Examples of semiclassical linear functionals with respect to the derivative operator when the class is either  $s = 1$  (see [9]) or  $s = 2$  (see [35]) have been studied in the literature. Nevertheless, some of them are related to perturbations of classical linear functionals by the addition of Dirac functionals, or their derivatives, supported on convenient points ([2], [33]). The important fact is that these orthogonal polynomials, the so called Krall-type orthogonal polynomials, are eigenfunctions of higher order differential operators. They are related to bispectral problems and have been intensively used in the generation of exceptional polynomials, an useful tool in the study of integrable systems.

In the discrete case, there is a limited development in these topics (see [34]). The case  $s = 0$  has been deeply studied by many authors (see [1], [24]) and the approach to the case  $s = 1$  has been done by Dominici and Marcellán in a recent paper (see [19]). Some examples of semiclassical discrete polynomials for  $s = 1$  appeared in [47].

The aim of this contribution is twofold. First, we deal with three different perturbations of discrete linear functionals based on the so called linear spectral transformations (Uvarov, Christoffel and Geronimus) [50], truncations and symmetrization processes, respectively. Second, from the above transformations we are able to generate semiclassical linear functionals of class  $s = 1$  and  $s = 2$ . Some of them appear in a natural way in the literature but others are new. We put attention only in the representation of the corresponding Stieltjes functions and, as a consequence, we deduce the corresponding class. Notice that an interesting and open problem is to analyze the coefficients of the three term recurrence relation they satisfy, in particular the so called Laguerre-Freud equations. For generalized Charlier polynomials such equations appeared in the monograph by W. Van Assche [49] as well as in [29]. They lead to limiting cases of the discrete Painlevé equation corresponding to  $D_4^c$  in the Sakai's classification. A generalization of the Krawtchouk polynomials and the fifth Painlevé equation are studied in [10]. Some interesting examples of semiclassical extensions of Meixner polynomials appeared in [11] and [22]. They yield a limit case of asymmetric discrete Painlevé IV equations. Discrete orthogonal polynomials with respect to the hypergeometric weight have been analyzed in [21] and [17].

The structure of the paper is the following. In Section 2, a basic background about discrete semiclassical linear functionals is given. In particular, the class is defined from the degrees of polynomials appearing in the discrete Pearson equation they satisfy. We emphasize the role of the Stieltjes function associated with every linear functional. In the semiclassical case, the corresponding Stieltjes function satisfies a first order linear difference equation with polynomial coefficients. They provide the key information about the class of the linear functional. Section 3 deals with the three canonical perturbations of linear functionals we will deal in the sequel. The corresponding Stieltjes functions are deduced. In Section 4, classical discrete linear functionals are revisited. In Section 5 we analyze semiclassical linear functionals of

class  $s = 1$  in such a way that the results given in [19] are completed. Finally, in Section 6, we study examples of discrete semiclassical linear functionals of class  $s = 2$  generated by the three canonical perturbations introduced in Section 3.

## 2 Basic background on discrete semiclassical linear functionals

Let  $\mathbb{N}_0$  denote the set  $\mathbb{N} \cup \{0\} = 0, 1, 2, \dots$ . Throughout the paper we will use the notation

$$\mathbf{a} = (a_1, \dots, a_p), \quad \mathbf{b} = (b_1, \dots, b_q), \quad p, q \in \mathbb{N}_0,$$

$$x + \mathbf{a} = \prod_{i=1}^p (x + a_i), \quad x + \mathbf{b} = \prod_{i=1}^q (x + b_i), \quad (1)$$

and

$$(\mathbf{a})_n = \prod_{i=1}^p (a_i)_n, \quad (\mathbf{b})_n = \prod_{i=1}^q (b_i)_n, \quad n \in \mathbb{N}_0,$$

where the *Pochhammer polynomial*  $(x)_n$  is defined by  $(x)_0 = 1$  and [43, 18:12]

$$(x)_n = \prod_{j=0}^{n-1} (x + j), \quad n \in \mathbb{N}. \quad (2)$$

From (2), we immediately obtain the recurrence

$$(x)_{n+1} = (x + n)(x)_n, \quad n \in \mathbb{N}_0. \quad (3)$$

On the other hand, if we consider the analytic continuation with  $\alpha \in \mathbb{C}$ , then

$$(x)_\alpha = \frac{\Gamma(x + \alpha)}{\Gamma(x)}, \quad -(x + \alpha) \notin \mathbb{N}_0, \quad (4)$$

where  $\Gamma(x)$  denotes the Gamma function [44, Chapter 5].

In this work, we deal with linear functionals defined in  $\mathbb{C}[x]$ , the linear space of polynomials with complex coefficients. Indeed,

$$L[r] = \sum_{x=0}^{\infty} r(x) \varrho(x), \quad r \in \mathbb{C}[x], \quad (5)$$

where the weight function  $\varrho : \mathbb{N}_0 \rightarrow \mathbb{C}$  is

$$\varrho(x) = \frac{(\mathbf{a})_x}{(\mathbf{b} + 1)_x} \frac{z^x}{x!}. \quad (6)$$

Using (3), we see that the weight function  $\varrho(x)$  satisfies the *Pearson equation*

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{\eta(x)}{\sigma(x+1)}, \quad (7)$$

where the polynomials  $\eta(x), \sigma(x)$  are

$$\eta(x) = z(x + \mathbf{a}), \quad \sigma(x) = x(x + \mathbf{b}). \quad (8)$$

Note that from (1) we have  $\deg(\eta) = p$  and  $\deg(\sigma) = q + 1$  and we assume they are coprime.

The linear functional  $L$  (and also  $\varrho$ ) is said to be *semiclassical*. The class of  $L$  (and also  $\varrho$ ) is the non-negative integer number

$$s = \max \{ \deg(\sigma) - 2, \deg(\sigma - \eta) - 1 \}.$$

The linear functionals of class  $s = 0$  are called *discrete classical [24]* linear functionals.

The number  $s$  depends on  $p, q$ , and  $z$ . It follows from (8) that we have four cases to consider:

1.  $p > q + 1$ . In this case,  $\deg(\sigma - \eta) - 1 = p - 1 > q$  and  $s = \max \{q - 1, p - 1\} = p - 1$ .
2.  $p < q + 1$ . In this case,  $\deg(\sigma - \eta) - 1 = q$  and  $s = \max \{q - 1, q\} = q$ .
3.  $q + 1 = p, z \neq 1$ . In this case,  $\deg(\sigma - \eta) - 1 = q$  and  $s = \max \{q - 1, q\} = q$ .
4.  $q + 1 = p, z = 1$ . In this case,  $\deg(\sigma - \eta) - 1 = q - 1$  and  $s = \max \{q - 1, q - 1\} = q - 1$ .

Thus, we conclude that

$$s = \begin{cases} p - 1, & p > q + 1, \\ q, & p < q + 1, \\ q, & p = q + 1, \quad z \neq 1, \\ q - 1, & p = q + 1, \quad z = 1. \end{cases} \quad (9)$$

Note that we can also write  $s$  as

$$s = \begin{cases} \max \{ \deg(\sigma) - 1, \deg(\sigma - \eta) - 1 \}, & \deg(\sigma - \eta) \neq \deg(\sigma) - 1, \\ \deg(\sigma) - 2, & \deg(\sigma - \eta) = \deg(\sigma) - 1. \end{cases} \quad (10)$$

The *Stieltjes transform* of the linear functional  $L$  is defined by

$$S(t) = L \left[ \frac{1}{t-x} \right]. \quad (11)$$

Here the linear functional is acting on  $x$ . The connection between  $\eta(x), \sigma(x)$  and  $S(t)$  is given by the following result.

**Lemma 1** *The Stieltjes transform  $S(t)$  of the linear functional  $L$  satisfies the difference equation*

$$\sigma(t+1)S(t+1) - \eta(t)S(t) = \xi(t), \quad (12)$$

where  $\xi(t)$  is a polynomial of degree  $s$  that is the class of the linear functional.

**Proof.** From (11), we have

$$\begin{aligned} \sigma(t+1)S(t+1) - \eta(t)S(t) &= \sigma(t+1) \sum_{x=0}^{\infty} \frac{\varrho(x)}{t+1-x} - \eta(t) \sum_{x=0}^{\infty} \frac{\varrho(x)}{t-x} \\ &= \frac{\sigma(t+1)}{t+1} + \sigma(t+1) \sum_{x=1}^{\infty} \frac{\varrho(x)}{t+1-x} - \eta(t) \sum_{x=0}^{\infty} \frac{\varrho(x)}{t-x} \\ &= \frac{\sigma(t+1)}{t+1} + \sigma(t+1) \sum_{x=0}^{\infty} \frac{\varrho(x+1)}{t-x} - \eta(t) \sum_{x=0}^{\infty} \frac{\varrho(x)}{t-x}. \end{aligned}$$

Using the Pearson equation (7), we get

$$\begin{aligned} \sigma(t+1)S(t+1) - \eta(t)S(t) &= \frac{\sigma(t+1)}{t+1} + \sigma(t+1) \sum_{x=0}^{\infty} \frac{\eta(x)}{\sigma(x+1)} \frac{\varrho(x)}{t-x} - \eta(t) \sum_{x=0}^{\infty} \frac{\varrho(x)}{t-x} \\ &= \frac{\sigma(t+1)}{t+1} + \sum_{x=0}^{\infty} \frac{\sigma(t+1)\eta(x) - \eta(t)\sigma(x+1)}{\sigma(x+1)} \frac{\varrho(x)}{t-x}. \end{aligned}$$

But

$$\lambda_x(t) = \frac{\sigma(t+1)\eta(x) - \eta(t)\sigma(x+1)}{t-x}$$



is a polynomial in  $t$  (with coefficients depending on  $x$ ) of degree  $\deg(\sigma - \eta) - 1$ . Since  $\eta(x) \neq \sigma(x + 1)$  (otherwise  $\varrho$  is a constant function), we have  $\deg(\sigma - \eta) \neq \deg(\sigma) - 1$ .

Therefore,

$$\xi(t) = \frac{\sigma(t+1)}{t+1} + \sum_{x=0}^{\infty} \lambda_x(t) \frac{\varrho(x)}{\sigma(x+1)} \quad (13)$$

is a polynomial in  $t$  of degree  $\max\{\deg(\sigma) - 1, \deg(\sigma - \eta) - 1\}$ . From (10) we conclude that  $\deg(\xi) = s$ . ■

Let the *falling factorial polynomials* be defined by  $\phi_0(x) = 1$  and

$$\phi_n(x) = \prod_{j=0}^{n-1} (x - j), \quad n \in \mathbb{N}. \quad (14)$$

Comparing (14) with (2), we see that

$$\phi_n(x) = (-1)^n (-x)_n = (x - n + 1)_n, \quad n \in \mathbb{N}_0, \quad (15)$$

or, using (4),

$$\phi_n(x) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = \frac{x!}{(x-n)!} = n! \binom{x}{n}, \quad n \in \mathbb{N}_0. \quad (16)$$

We define the *moments* of  $L$  with respect to the basis  $\{\phi_n(x)\}_{n \geq 0}$  by

$$\nu_n = L[\phi_n], \quad n \in \mathbb{N}_0.$$

From (6) and (16), we have

$$\nu_n = \sum_{x=n}^{\infty} \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{(x-n)!} = \sum_{x=0}^{\infty} \frac{(\mathbf{a})_{x+n}}{(\mathbf{b}+1)_{x+n}} \frac{z^{x+n}}{x!}.$$

If we use the recurrence [43, 18:5:12]

$$(x)_{n+m} = (x)_n (x+n)_m, \quad n, m \in \mathbb{N}_0, \quad (17)$$

we get

$$\nu_n = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \sum_{x=0}^{\infty} \frac{(\mathbf{a}+n)_x}{(\mathbf{b}+1+n)_x} \frac{z^x}{x!} = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} {}_pF_q \left( \begin{matrix} \mathbf{a}+n \\ \mathbf{b}+1+n \end{matrix}; z \right), \quad (18)$$

where  ${}_pF_q$  denotes the *generalized hypergeometric function* defined by [44, 16.2]

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \quad (19)$$

The moments of discrete semiclassical orthogonal polynomials of class  $s \leq 1$  were studied in [18].

**Remark 2** *The convergence of the series (19) depends on the values of  $p$  and  $q$ . Three different situations appear [44, 16.2]:*

1. *If  $p < q + 1$ , then  ${}_pF_q$  is an entire function of  $z$ .*
2. *If  $p = q + 1$ , then  ${}_pF_q$  is analytic inside the unit circle,  $|z| < 1$ , and can be extended by analytic continuation to the cut plane  $\mathbb{C} \setminus [1, \infty)$ . Let*

$$\gamma = b_1 + \cdots + b_q - (a_1 + \cdots + a_{q+1}). \quad (20)$$

*On the unit circle  $|z| = 1$ , the series (19) is*

- (i) absolutely convergent if  $\operatorname{Re}(\gamma) > 0$ ,*
  - (ii) convergent except at  $z = 1$  if  $\operatorname{Re}(\gamma) \in (-1, 0]$ ,*
- and*
- (iii) divergent if  $\operatorname{Re}(\gamma) \leq -1$ .*

3. *If  $p > q + 1$ , then  ${}_pF_q$  diverges for all  $z \neq 0$ , up to  $a_i = -N$ , with  $N \in \mathbb{N}$  for some  $1 \leq i \leq p$ . In this case,  ${}_pF_q$  becomes a polynomial of degree  $N$ .*

### 3 Modification of functionals

In this section, we describe three ways in order to change the class of a linear functional.

### 3.1 Rational spectral transformations

Let  $L$  be a linear functional and  $S(t)$  denote its Stieltjes transform introduced in (11). A *rational spectral transformation* of  $S(t)$  is defined by [50]

$$\tilde{L} \left[ \frac{1}{t-x} \right] = \tilde{S}(t) = \frac{A(t)S(t) + B(t)}{C(t)S(t) + D(t)},$$

where  $A(t), B(t), C(t), D(t)$  are polynomials such that  $A(z)D(z) - B(z)C(z) \neq 0$ . See also [13], [23], [32], [45], [48], and [50], among others.

It was shown in [50] what are the families of semiclassical orthogonal polynomials related by *spectral linear transformations* (with  $C = 0$ ). They can be written as a composition of two basic transformations:

1. The *Christoffel transformation*, (see [12],[13], [14], [46]) is defined by

$$\frac{L_C[r]}{\lambda_C} = L[(x-\omega)r(x)], \quad r \in \mathbb{C}[x], \quad (21)$$

or, equivalently, by

$$\frac{S_C(z)}{\lambda_C} = (z-\omega)S(z) - L[1], \quad (22)$$

where  $L(x-\omega) \neq 0, n \geq 0$ , and

$$\lambda_C = \frac{L_C[1]}{L[x-\omega]}.$$

2. The *Geronimus transformation* (see [15], [16], [26], [27], [32], [38]) is defined by

$$\frac{L_G[r]}{\lambda_G} = L \left[ \frac{r(x)}{x-\omega} \right] + Mr(\omega), \quad r \in \mathbb{C}[x], \quad (23)$$

or, equivalently, by

$$(z-\omega) \frac{S_G(z)}{\lambda_G} = S(z) - S(\omega) + M, \quad (24)$$

with

$$\lambda_G = \frac{L_G[1]}{M - S(\omega)},$$

where we assume that  $S(z)$  is analytic at  $z = \omega$  and  $M - S(\omega) \neq 0$ .

On the other hand, the *Uvarov transformation* (see [6], [33], [48]) is defined by

$$\frac{L_U[r]}{\lambda_U} = L[r] + Mr(\omega), \quad r \in \mathbb{C}[x], \quad (25)$$

or by

$$\frac{S_U(z)}{\lambda_U} = S(z) + \frac{M}{z - \omega}, \quad (26)$$

where  $M + L[1] \neq 0$  and

$$\lambda_U = \frac{L_U[1]}{L[1] + M}.$$

Note that since

$$r(x) = \sum_{k=0}^{\deg(r)} \frac{r^{(k)}(\omega)}{k!} (x - \omega)^k, \quad r \in \mathbb{C}[x],$$

we can rewrite (23) as

$$\frac{L_G[r]}{\lambda_G} = [M - S(\omega)]r(\omega) + \sum_{k=1}^{\deg(r)} \frac{r^{(k)}(\omega)}{k!} L[(x - \omega)^k],$$

which is well defined for all  $r \in \mathbb{C}[x]$ .

We denote by  $L_{T_2T_1}$ ,  $S_{T_2T_1}$ ,  $\varrho_{T_2T_1}$ , etc., the object obtained by applying the transformation  $T_1$  followed by the transformation  $T_2$  (in other words, by the composition  $T_2 \circ T_1$ ).

Let us assume that  $\lambda_U = \lambda_C = \lambda_G = 1$ . If we apply a Geronimus transformation followed by a Christoffel transformation, we get

$$L_{CG}[r] = L_G[(x - \omega)r(x)] = L\left[\frac{(x - \omega)r(x)}{x - \omega}\right] + M[(x - \omega)r(x)]_{x=\omega} = L[r]. \quad (27)$$

On the other hand, changing the order in the composition, we obtain

$$L_{GC}[r] = L_C\left[\frac{r(x)}{x - \omega}\right] + Mr(\omega) = L\left[\frac{(x - \omega)r(x)}{x - \omega}\right] + Mr(\omega) = L[r] + Mr(\omega) = L_U[r]. \quad (28)$$

Thus, the Christoffel and Geronimus transformations are almost inverses of each other in the sense that  $L_{CG}$  is the identity and  $L_{GC}$  is an Uvarov transformation.

### 3.1.1 Uvarov transformations

Let  $L$  be given by (5) and consider the linear functional  $L_U$  defined by (25) with

$$L_U [1] = L [1] + M \neq 0.$$

It follows that the moments of  $L_U$  with respect to the basis  $\{\phi_n(x)\}_{n \geq 0}$  are

$$\nu_n^U = \nu_n + M\phi_n(\omega), \quad n \in \mathbb{N}_0. \quad (29)$$

Using (26) in (12), we obtain

$$\sigma(t+1) \left[ S_U(t+1) - \frac{M}{t+1-\omega} \right] - \eta(t) \left[ S_U(t) - \frac{M}{t-\omega} \right] = \xi(t). \quad (30)$$

Therefore, the Stieltjes transform of  $L_U$  satisfies the difference equation

$$(t+1-\omega)(t-\omega)\sigma(t+1)S_U(t+1) - (t+1-\omega)(t-\omega)\eta(t)S_U(t) = \xi_U(t),$$

where

$$\xi_U(t) = (t-\omega)(t-\omega+1)\xi(t) + M[(t-\omega)\sigma(t+1) - (t-\omega+1)\eta(t)],$$

and the modified weight function  $\varrho_U(x)$  satisfies the Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{(x-\omega)(x+1-\omega)\eta(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)}.$$

We conclude that if  $L$  was of class  $s$ , then  $L_U$  will be of class at most  $s+2$ .

Notice that we have two special cases to consider:

1. If  $\sigma(\omega) = 0$  and  $\eta(\omega) \neq 0$ , with  $\sigma(x) = (x-\omega)\sigma_1(x)$ , then

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{(x-\omega)(x+1-\omega)\eta(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)} = \frac{(x-\omega)\eta(x)}{(x-\omega)\sigma(x+1)},$$

$$\sigma(t+1)S_U(t+1) - \eta(t)S_U(t) = \xi(t) + M \left[ \sigma_1(t+1) - \frac{\eta(t)}{t-\omega} \right], \quad (31)$$

and  $L_U$  will be of class  $s+1$ .

2. If  $\sigma(\omega) \neq 0$  and  $\eta(\omega) = 0$ , with  $\eta(x) = (x - \omega)\eta_1(x)$ , then

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{(x-\omega)(x+1-\omega)\eta(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)} = \frac{(x+1-\omega)\eta(x)}{(x+1-\omega)\sigma(x+1)},$$

$$\sigma(t+1)S_U(t+1) - \eta(t)S_U(t) = \xi(t) + M \left[ \frac{\sigma(t+1)}{t+1-\omega} - \eta_1(t) \right],$$

and  $L_U$  will be of class  $s+1$ .

Perturbations of discrete semiclassical functionals by Dirac masses were studied in [2], [6], [24], [28], as well as in [3], [4], and [5].

### 3.1.2 Christoffel transformations

Let  $L$  be given by (5), and consider the functional  $L_C$  defined by (21), with

$$L_C[1] = L[x - \omega] \neq 0.$$

It follows that the moments of  $L_C$  with respect to the basis  $\{\phi_n(x)\}_{n \geq 0}$  are

$$\nu_n^C = \nu_{n+1} + (n - \omega)\nu_n, \quad n \in \mathbb{N}_0 \quad (32)$$

since we see from (14) that

$$\phi_{n+1}(x) = (x - n)\phi_n(x) = (x - \omega)\phi_n(x) + (\omega - n)\phi_n(x). \quad (33)$$

Using (22) in (12), we obtain

$$\sigma(t+1) \frac{S_C(t+1) + \nu_0}{t+1-\omega} - \eta(t) \frac{S_C(t) + \nu_0}{t-\omega} = \xi(t).$$

Therefore, the Stieltjes transform of  $L_C$  satisfies the difference equation

$$(t - \omega)\sigma(t+1)S_C(t+1) - (t+1-\omega)\eta(t)S_C(t) = \xi_C(t),$$

where

$$\xi_C(t) = (t - \omega)(t - \omega + 1)\xi(t) - [(t - \omega)\sigma(t+1) - (t - \omega + 1)\eta(t)]\nu_0,$$

and the modified weight function  $\varrho_C(x)$  satisfies the Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{(x+1-\omega)\eta(x)}{(x-\omega)\sigma(x+1)}.$$

We conclude that if  $L$  was of class  $s$ , then  $L_C$  will be of class at most  $s+1$ .

Note that we have two special cases to consider:

1. If  $\sigma(\omega) = 0$  and  $\eta(\omega) \neq 0$ , with  $\sigma(x) = (x - \omega)\sigma_1(x)$ , then

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{\eta(x)}{(x - \omega)\sigma_1(x+1)},$$

$(t - \omega)\sigma_1(t+1)S_C(t+1) - \eta(t)S_C(t) = (t - \omega)\xi(t) - [(t - \omega)\sigma_1(t+1) - \eta(t)]\nu_0$ ,  
and  $L_C$  could be of class  $s$ .

2. If  $\sigma(\omega) \neq 0$  and  $\eta(\omega) = 0$ , with  $\eta(x) = (x - \omega)\eta_1(x)$ , then

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{(x+1 - \omega)\eta_1(x)}{\sigma(x+1)},$$

$\sigma(t+1)S_C(t+1) - (t+1 - \omega)\eta_1(t)S_C(t) = (t - \omega + 1)\xi(t) - [\sigma(t+1) - (t - \omega + 1)\eta_1(t)]\nu_0$ ,  
and  $L_C$  could be of class  $s$ .

Christoffel transformations in the discrete case have been studied in [46].

### 3.1.3 Geronimus transformations

Let  $L$  be given by (5), and consider the functional  $L_G$  defined by (23), with

$$L_G[1] = M - S(\omega) \neq 0.$$

Since

$$S(\omega) = \sum_{x=0}^{\infty} \frac{1}{\omega - x} \frac{(\mathbf{a})_x}{(\mathbf{b} + 1)_x} \frac{z^x}{x!}$$

is a meromorphic function, we need  $\omega \notin \mathbb{N}_0$  for  $L_G[1]$  to be well defined.

It follows from (23) that the moments of  $L_G$  with respect to the basis  $\{\phi_n(x)\}_{n \geq 0}$  satisfy

$$\nu_{n+1}^G + (n - \omega)\nu_n^G = \nu_n, \quad n \in \mathbb{N}_0, \quad (34)$$

since from (33) we see that

$$L_G[\phi_{n+1} - (\omega - n)\phi_n] = L_G[(x - \omega)\phi_n(x)] = L[\phi_n].$$

As it is well known, the general solution of the initial value problem

$$y_{n+1} = c_n y_n + g_n, \quad y_{n_0} = y_0,$$

is [20, 1.2.4]

$$y_n = y_0 \prod_{j=n_0}^{n-1} c_j + \sum_{k=n_0}^{n-1} \left( g_k \prod_{j=k+1}^{n-1} c_j \right).$$

Solving (34) with  $\nu_0^G = M - S(\omega)$ , we get

$$\nu_n^G = \phi_n(\omega) \left[ \nu_0^G + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right],$$

since from (14) we have

$$\prod_{j=0}^{n-1} (\omega - j) = \phi_n(\omega), \quad \prod_{j=k+1}^{n-1} (\omega - j) = \frac{\phi_n(\omega)}{\phi_{k+1}(\omega)}.$$

Replacing (24) in (12), we obtain

$$\sigma(t+1) [(t+1-\omega) S_G(t+1) - \nu_0^G] - \eta(t) [(t-\omega) S_G(t) - \nu_0^G] = \xi(t).$$

Therefore, the Stieltjes transform of  $L_G$  satisfies the difference equation

$$(t+1-\omega) \sigma(t+1) S_G(t+1) - (t-\omega) \eta(t) S_G(t) = \xi_G(t),$$

where

$$\xi_G(t) = \xi(t) + [\sigma(t+1) - \eta(t)] \nu_0^G,$$

and the modified weight function  $\varrho_G(x)$  satisfies the Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{(x-\omega) \eta(x)}{(x+1-\omega) \sigma(x+1)}.$$

We conclude that if  $L$  was of class  $s$ , then  $L_G$  will be of class at most  $s+1$ .

**Remark 3** For all these transformations, it is understood that

$$\begin{aligned} \eta(\omega) = 0 &\Rightarrow \frac{\eta(x)}{x-\omega} \times (x-\omega) = \eta(x), \\ \sigma(\omega) = 0 &\Rightarrow \frac{x+1-\omega}{\sigma(x+1)} \times \frac{1}{x+1-\omega} = \frac{1}{\sigma(x+1)}. \end{aligned}$$



Thus, if we write

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{(x-\omega)\eta(x)}{(x+1-\omega)\sigma(x+1)} = \frac{\eta_G(x)}{\sigma_G(x+1)},$$

then clearly  $\eta_G(\omega) = 0 = \sigma_G(\omega)$ . Hence,

$$\frac{\varrho_{CG}(x+1)}{\varrho_{CG}(x)} = \frac{\eta_G(x)}{\sigma_G(x+1)} \times \frac{x-\omega}{x+1-\omega} = \frac{\eta(x)}{\sigma(x+1)} = \frac{\varrho(x+1)}{\varrho(x)},$$

in agreement with (27).

On the other hand, if we write

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{(x+1-\omega)\eta(x)}{(x-\omega)\sigma(x+1)} = \frac{\eta_C(x)}{\sigma_C(x+1)},$$

then  $\eta_G(\omega)\sigma_G(\omega) \neq 0$ , and therefore

$$\begin{aligned} \frac{\varrho_{GC}(x+1)}{\varrho_{GC}(x)} &= \frac{\eta_C(x)}{\sigma_C(x+1)} \times \frac{x+1-\omega}{x-\omega} \\ &= \frac{(x-\omega)(x+1-\omega)\eta(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)} = \frac{\varrho_U(x+1)}{\varrho_U(x)}, \end{aligned}$$

in agreement with (28).

## 3.2 Truncated linear functionals

Let  $N \in \mathbb{N}_0$  and define the *truncated functional*  $L_T$  of the linear functional  $L$  by

$$L_T[r] = \sum_{x=0}^N r(x) \varrho(x), \quad r \in \mathbb{C}[x], \quad (35)$$

where  $\varrho(x)$  satisfies (7) with  $\eta(N) \neq 0$ . We define the truncated weight function  $\varrho_T(x)$  by

$$\varrho_T(x) = \varrho(x) [1 - H(x - N)], \quad (36)$$

where  $H(x)$  is the Heaviside function [44, 1.16(iv)] defined by

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Multiplying (36) by  $\frac{1}{(N-x)!}$ , we obtain

$$\frac{1}{(N-x)!} \varrho_T(x) = \frac{1}{(N-x)!} \varrho(x), \quad x \in \mathbb{N}_0. \quad (37)$$

Using (37) in (7), we get

$$(N-x) \frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{\frac{1}{(N-x-1)!} \varrho_T(x+1)}{\frac{1}{(N-x)!} \varrho_T(x)} = \frac{\frac{1}{(N-x-1)!} \varrho(x+1)}{\frac{1}{(N-x)!} \varrho(x)} = (N-x) \frac{\eta(x)}{\sigma(x+1)}.$$

We conclude that the modified weight function  $\varrho_T(x)$  satisfies the Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{\eta(x)(x-N)}{\sigma(x+1)(x-N)}, \quad (38)$$

and  $L_T$  will be of class  $s+1$ .

The moments of  $L_T$  with respect to the basis  $\{\phi_n(x)\}_{n \geq 0}$  are

$$\nu_n^T(z) = \sum_{x=0}^N \phi_n(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad n \in \mathbb{N}_0.$$

A similar calculation leading to (18), gives

$$\nu_n^T(z) = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \sum_{x=0}^{N-n} \frac{(\mathbf{a}+n)_x}{(\mathbf{b}+1+n)_x} \frac{z^x}{x!}, \quad n \in \mathbb{N}_0. \quad (39)$$

Using the identity [44, 16.2.4]

$$\sum_{x=0}^K \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!} = \frac{(\mathbf{a})_K}{(\mathbf{b}+1)_K} \frac{z^K}{K!} {}_{q+2}F_p \left[ \begin{matrix} -K, 1, -K - \mathbf{b} \\ 1 - K - \mathbf{a} \end{matrix} ; \frac{(-1)^{p+q+1}}{z} \right], \quad K \in \mathbb{N},$$

we can rewrite (39) as

$$\nu_n^T(z) = \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} \frac{z^N}{(N-n)!} {}_{q+2}F_p \left[ \begin{matrix} n - N, 1, -N - \mathbf{b} \\ 1 - N - \mathbf{a} \end{matrix} ; \frac{(-1)^{p+q+1}}{z} \right]. \quad (40)$$

### 3.3 Symmetrized functionals

Let  $\varrho(x)$  be given by (6) and  $m \in \mathbb{N}$ . We consider the symmetric weight function  $\varrho_\zeta(x)$  defined by

$$\varrho_\zeta(x) = C\varrho(x+m)\varrho(-x+m), \quad (41)$$

where  $C$  is a constant factor to be determined. Replacing (7) in (41), we get

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{\varrho(x+1+m)\varrho(-x-1+m)}{\varrho(x+m)\varrho(-x+m)} = \frac{\eta(x+m)\sigma(-x+m)}{\sigma(x+m+1)\eta(-x-1+m)}.$$

Therefore,

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{x+m+\mathbf{a}}{(x+m+\mathbf{b}+1)(x+m+1)} \frac{(m+\mathbf{b}-x)(m-x)}{m+\mathbf{a}-x-1},$$

or

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(-1)^{p+q+1}(x+m+\mathbf{a})(x-m-\mathbf{b})(x-m)}{(x+1-m-\mathbf{a})(x+1+m+\mathbf{b})(x+1+m)}. \quad (42)$$

Note that, from (42), we have  $\varrho_\zeta(x) = \varrho_1(x+m)$ , with

$$\frac{\varrho_1(x+1)}{\varrho_1(x)} = \frac{(-1)^{p+q+1}(x+\mathbf{a})(x-2m-\mathbf{b})(x-2m)}{(x+1-2m-\mathbf{a})(x+1+\mathbf{b})(x+1)}.$$

If we write  $N = 2m$ , then

$$\frac{\varrho_1(x+1)}{\varrho_1(x)} = \frac{(-1)^{p+q+1}(x+\mathbf{a})(x-N-\mathbf{b})(x-N)}{(x+1-N-\mathbf{a})(x+1+\mathbf{b})(x+1)},$$

and it follows that  $\varrho_1(x)$  (and  $\varrho_\zeta$ ) is of class  $s_1$  with

$$s_1 = \begin{cases} p+q, & \text{if } q+p \text{ is even} \\ p+q-1, & \text{if } q+p \text{ is odd} \end{cases}.$$

If  $\mathbf{a} = (\mathbf{a}_1, -N)$ , then

$$\frac{\varrho_1(x+1)}{\varrho_1(x)} = \frac{(-1)^{q+1+p}(x+\mathbf{a}_1)x-N}{(x+1+\mathbf{b})(x+1)} \frac{x-N}{x+1} \frac{(x-N-\mathbf{b})(x-N)}{x+1-N-\mathbf{a}_1},$$

and since the term  $\frac{x-N}{x+1}$  is repeated, we consider the reduced Pearson equation

$$\begin{aligned}\frac{\varrho_2(x+1)}{\varrho_2(x)} &= \frac{(-1)^{q+1+p} (x+\mathbf{a}_1) (x-N-\mathbf{b}) (x-N)}{(x+1+\mathbf{b}) (x+1) (x+1-N-\mathbf{a}_1)} \\ &= \frac{(-1)^{q+1+p} (x+\mathbf{a}) (x-N-\mathbf{b})}{(x+1+\mathbf{b}) (x+1-N-\mathbf{a})},\end{aligned}$$

and  $\varrho_\zeta(x) = \varrho_2(x+m)$ . In this case  $\varrho_2(x)$  (and  $\varrho_\zeta$ ) is of class  $s_2$  with

$$s_2 = \begin{cases} p+q-1, & \text{if } q+p \text{ is even} \\ p+q-2, & \text{if } q+p \text{ is odd} \end{cases}.$$

As a conclusion, if  $\varrho(x)$  satisfies (7), then we have  $\varrho_\zeta(x) = \rho(x+m)$  where

$$\rho(x) = \begin{cases} \frac{(\mathbf{a})_x (z_0)^x (-N)_x (-N-\mathbf{b})_x}{(\mathbf{b}+1)_x x! (1-N-\mathbf{a})_x}, & \eta(N) \neq 0, \\ \frac{(\mathbf{a})_x (z_0)^x (-N-\mathbf{b})_x}{(\mathbf{b}+1)_x (1-N-\mathbf{a})_x}, & \eta(N) = 0, \end{cases}$$

and

$$z_0 = (-1)^{p+q+1}, \quad N = 2m.$$

We define the *symmetrized functional*  $L_\zeta$  of  $L_\rho$  by

$$L_\zeta[r] = \sum_{x=-m}^m r(x) \varrho_\zeta(x) = \sum_{x=0}^N r(x-m) \rho(x) = L_\rho[r(x-m)].$$

Hence, the moments of  $L_\zeta$  on the basis

$$\bar{\phi}_n(x) = \phi_n(x+m), \quad n \in \mathbb{N}_0,$$

are

$$\nu_n^\zeta = \sum_{x=-m}^m \bar{\phi}_n(x) \varrho_\zeta(x) = \sum_{x=0}^N \phi_n(x) \rho(x) = L_\rho[\phi_n], \quad n \in \mathbb{N}_0, \quad (43)$$

and the Stieltjes transform of  $L_\zeta$  is

$$S_\zeta(t) = \sum_{x=-m}^m \frac{\varrho_\zeta(x)}{t-x} = \sum_{x=0}^N \frac{\rho(x)}{t+m-x} = S_\rho(t+m).$$

It follows that if the Stieltjes transform of  $L_\rho$  satisfies the difference equation

$$\sigma_\rho(t+1)S_\rho(t+1) - \eta_\rho(t)S_\rho(t) = \xi_\rho(t),$$

then

$$\sigma_\rho(t+m+1)S_\rho(t+m+1) - \eta_\rho(t+m)S_\rho(t+m) = \xi_\rho(t+m).$$

Therefore

$$\sigma_\varsigma(t+1)S_\varsigma(t+1) - \eta_\varsigma(t)S_\varsigma(t) = \xi_\rho(t+m), \quad (44)$$

since

$$\frac{\eta_\varsigma(x)}{\sigma_\varsigma(x+1)} = \frac{\varrho_\varsigma(x+1)}{\varrho_\varsigma(x)} = \frac{\rho(x+m+1)}{\rho(x+m)} = \frac{\eta_\rho(x+m)}{\sigma_\rho(x+m+1)}.$$

Symmetric orthogonal polynomials of a discrete variable were studied in [7], [8], [40], [41].

### 3.4 Summary

Let's list all the transformations that we will use in the sequel. For simplicity, we will not consider compositions of them.

1. Uvarov transformation:  $s \rightarrow s+2$

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{(x-\omega)(x+1-\omega)}{(x-\omega)(x+1-\omega)}, \quad \eta(\omega)\sigma(\omega) \neq 0.$$

2. Reduced Uvarov transformation:  $s \rightarrow s+1$

(a)

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{x+1-\omega}{x+1-\omega}, \quad \eta(\omega) = 0.$$

(b)

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{x-\omega}{x-\omega}, \quad \sigma(\omega) = 0.$$

3. Christoffel transformation:  $s \rightarrow s+1$

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{x+1-\omega}{x-\omega}, \quad \eta(\omega)\sigma(\omega) \neq 0.$$

4. Geronimus transformation:  $s \rightarrow s + 1$

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{x-\omega}{x+1-\omega}, \quad \eta(\omega-1)\sigma(\omega+1) \neq 0.$$

5. Truncation at  $x = N : s \rightarrow s + 1$

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{\eta(x)}{\sigma(x+1)} \times \frac{x-N}{x-N}, \quad \eta(N) \neq 0.$$

6. Symmetrization on the interval  $[-m, m]$ :

(a)

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{\eta(m+x)}{\sigma(m+x+1)} \times \frac{\sigma(m-x)}{\eta(m-x-1)}, \quad \eta(2m) \neq 0.$$

(b)

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{\eta(m+x)}{\sigma(m+x+1)} \times \frac{\sigma_1(m-x)}{\eta_1(m-x-1)}, \quad \eta(2m) = 0, \quad (45)$$

where

$$\eta(x) = (x-2m)\eta_1(x), \quad \sigma(x) = x\sigma_1(x).$$

## 4 Semiclassical polynomials of class 0 (classical polynomials)

In this section, we consider the families of discrete classical polynomials. We have 3 main cases, corresponding to  $(p, q) = (0, 0), (1, 0), (2, 1)$ . There are also 3 symmetrized subcases.

We use the notation  $(p, q; N)$  to denote the family such that one of the parameters in the numerator is a non-negative integer, and  $(p, q; N, 1)$  if in addition the value of  $z$  is equal to 1.

For each polynomial, we list the linear functional, the Pearson equation satisfied by the weight function, the moments computed from (18), and the difference equation satisfied by the Stieltjes transform, using (13) and (44).

## 4.1 (0, 0) : Charlier polynomials

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z}{x+1}.$$

Moments

$$\nu_n(z) = z^n {}_0F_0 \left[ \begin{matrix} - \\ - \end{matrix} ; z \right] = z^n e^z, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1)S(t+1) - zS(t) = \nu_0.$$

## 4.2 (1, 0) : Meixner polynomials

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) (a)_x \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a)}{x+1}.$$

Moments

$$\nu_n(z) = z^n (a)_n {}_1F_0 \left[ \begin{matrix} a+n \\ - \end{matrix} ; z \right] = z^n (a)_n (1-z)^{-a-n}, \quad n \in \mathbb{N}_0,$$

where we choose the principal branch  $z \in \mathbb{C} \setminus [1, \infty)$ .

Stieltjes transform difference equation

$$(t+1)S(t+1) - z(t+a)S(t) = (1-z)\nu_0.$$

### 4.3 $(1, 0; N)$ : Krawtchouk polynomials

These polynomials are a particular case of the Meixner polynomials, with  $-a = N \in \mathbb{N}$ .

Linear functional

$$L[r] = \sum_{x=0}^N r(x) (-N)_x \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x-N)}{x+1}.$$

Moments

$$\nu_n(z) = z^n (-N)_n (1-z)^{N-n}, \quad z \neq 1, \quad n \in \mathbb{N}_0. \quad (46)$$

Stieltjes transform difference equation

$$(t+1)S(t+1) - z(t-N)S(t) = (1-z)\nu_0.$$

**Remark 4** *Let's consider the symmetrized Krawtchouk polynomials. Since  $\eta(N) = 0$ , we use (45) and obtain*

$$\frac{\varrho_\varsigma(x+1)}{\varrho_\varsigma(x)} = \frac{z(x-m)}{x+m+1} \frac{(x+m+1)(-x+m)}{(x-m)z(-x-1-m)} = \frac{x-m}{x+m+1}.$$

*Hence, the symmetrized Krawtchouk polynomials are shifted Krawtchouk polynomials with  $z = 1$ . But from (43) and (46) we see that*

$$\nu_n^\varsigma = \nu_n(1) = 0, \quad n \in \mathbb{N}_0,$$

*and, therefore, we need to discard this example.*

#### 4.3.1 Symmetrized Charlier polynomials

Special values

$$N \rightarrow 2m, \quad z \rightarrow -1, \quad x \rightarrow x+m.$$

Weight function

$$\begin{aligned} \varrho_\varsigma(x) &= \varrho(x+m) = (-2m)_{x+m} \frac{(-1)^{x+m}}{(x+m)!} \\ &= (-2m)_m \frac{(-1)^m}{m!} \frac{(-m)_x}{(m+1)_x} (-1)^x = \binom{2m}{m} \frac{(-m)_x}{(m+1)_x} (-1)^x. \end{aligned}$$



Linear functional

$$L_\varsigma[r] = \binom{2m}{m} \sum_{x=-m}^m r(x) \frac{(-m)_x}{(m+1)_x} (-1)^x.$$

Pearson equation

$$\frac{\varrho_\varsigma(x+1)}{\varrho_\varsigma(x)} = \frac{-(x-m)}{x+m+1}. \quad (47)$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\nu_n^\varsigma = L_\varsigma[\phi(x+m)] = (-1)^n (-2m)_n 2^{2m-n} = \nu_n(-1), \quad n \in \mathbb{N}_0, \quad (48)$$

where  $\nu_n(z)$  are the moments of the Krawtchouk polynomials defined in (46).

Stieltjes transform difference equation

$$(t+m+1)S_\varsigma(t+1) + (t-m)S_\varsigma(t) = 2\nu_0^\varsigma.$$

**Remark 5** *These polynomials were studied in [8]. The Pearson equation (47) is the same as equation (11) in that paper, with  $m = c$ . The authors used the weight function*

$$\rho(x) = \frac{1}{\Gamma(x+m+1)\Gamma(-x+m+1)} = \frac{1}{(m!)^2} \frac{(-m)_x}{(m+1)_x} (-1)^x,$$

and, therefore,

$$\varrho_\varsigma(x) = (2m)!\rho(x). \quad (49)$$

The moments (48) are the same as those appearing in equation (15), if we use (15) and write

$$\nu_n^\varsigma = (-1)^n (-2m)_n 2^{2m-n} = 2^{2m-n} \phi_n(2m),$$

after taking the scaling (49) into account.

#### 4.4 $(2, 1; N, 1)$ : Hahn polynomials

Linear functional

$$L[r] = \sum_{x=0}^N r(x) \frac{(a)_x (-N)_x}{(b+1)_x} \frac{1}{x!}.$$

Pearson equation

$$\frac{\eta(x)}{\sigma(x+1)} = \frac{\varrho(x+1)}{\varrho(x)} = \frac{(x+a)(x-N)}{(x+b+1)(x+1)}.$$

Moments

$$\nu_n = \frac{(-N)_n (a)_n}{(b+1)_n} {}_2F_1 \left( \begin{matrix} -N+n, a+n \\ b+1+n \end{matrix}; 1 \right), n \in \mathbb{N}_0.$$

Using the Chu–Vandermonde Identity [44, 15.4.24],

$${}_2F_1 \left( \begin{matrix} -n, b \\ c \end{matrix}; 1 \right) = \frac{(c-b)_n}{(c)_n},$$

we get

$$\nu_n = \frac{(-N)_n (a)_n (b+1-a)_{N-n}}{(b+1)_n (b+1+n)_{N-n}}, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1)(t+b+1)S(t+1) - (t-N)(t+a)S(t) = (b+1-a+N)\nu_0.$$

#### 4.4.1 Symmetrized Meixner polynomials

Special values

$$b \rightarrow -N-a, \quad N \rightarrow 2m, \quad x \rightarrow x+m.$$

Linear functional

$$L_\zeta[r] = \binom{2m}{m} \frac{(a)_m}{(a+m)_m} \sum_{x=-m}^m r(x) \frac{(a+m)_x (-m)_x}{(-m-a+1)_x (m+1)_x}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(x+a+m)(x-m)}{(x+1-m-a)(x+1+m)}. \quad (50)$$

Moments

$$\nu_n^\zeta = \frac{(-N)_n (a)_n (1-N-2a)_{N-n}}{(1-N-a)_n (n+1-N-a)_{N-n}}, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+m+1)(t-m-a+1)S_\zeta(t+1) - (t-m)(t+m+a)S_\zeta(t) = (1-2a)\nu_0^\zeta.$$

**Remark 6** If we use (4) and (41) to write the weight function  $\varrho_\zeta(x)$  in terms of Gamma functions, we have

$$\varrho_\zeta(x) = \varrho_\zeta(0) \left[ \frac{m!}{(a)_m} \right]^2 \frac{(a)_{x+m}}{(x+m)!} \frac{(a)_{-x+m}}{(-x+m)!} = \frac{(2m)! \Gamma(x+m+a) \Gamma(-x+m+a)}{(a)_{2m} \Gamma(x+m+1) \Gamma(-x+m+1)}.$$

This agrees (up to a normalization factor) with the weight function considered by the authors in [8] (equation 27), if we set  $\delta_1 = -m$  and  $\delta_2 = m+a$ . Note that the condition  $\delta_2 > -\delta_1$  is satisfied if  $a > 0$  (the positive-definite case).

#### 4.4.2 Symmetrized generalized Charlier polynomials

See section 5.1 for a definition of the Generalized Charlier polynomials.

Special values

$$a \rightarrow -N - b, \quad N \rightarrow 2m, \quad x \rightarrow x + m.$$

Linear functional

$$L_\zeta[r] = \binom{2m}{m} \frac{(b+1+m)_m}{(b+1)_m} \sum_{x=-m}^m r(x) \frac{(-m-b)_x (-m)_x}{(b+1+m)_x (m+1)_x}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(x-m-b)(x-m)}{(x+m+b+1)(x+m+1)}.$$

Moments

$$\nu_n^\zeta = \frac{(-2m)_n (-2m-b)_n (2m+2b+1)_{2m-n}}{(b+1)_n (b+1+n)_{2m-n}}, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+m+1)(t+m+b+1)S_\zeta(t+1) - (t-m)(t-m-b)S_\zeta(t) = (2b+1+4m)\nu_0^\zeta.$$

**Remark 7** If we use (4) and (41) to write the weight function  $\varrho_\zeta(x)$  in terms of Gamma functions, we have

$$\begin{aligned} \varrho_\zeta(x) &= \varrho_\zeta(0) [(b+1)_m m!]^2 \frac{1}{(b+1)_{x+m} (x+m)!} \frac{1}{(b+1)_{-x+m} (-x+m)!} \\ &= \frac{(b+1)_{2m} (2m)!}{\Gamma(x+m+b+1) \Gamma(x+m+1) \Gamma(-x+m+b+1) \Gamma(-x+m+1)}. \end{aligned}$$

This agrees (up to a normalization factor) with the weight function considered by the authors in [8] (equation 26).

## 5 Semiclassical polynomials of class 1

In this section, we consider all families of polynomials of class 1. We have 5 main cases, corresponding to

$$(p, q) = (0, 1), (1, 1), (2, 0), (2, 1), (3, 2).$$

There are also 14 subcases.

### 5.1 $(0, 1)$ : Generalized Charlier polynomials

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{1}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z}{(x+b+1)(x+1)}.$$

Moments

$$\nu_n(z) = \frac{z^n}{(b+1)_n} {}_0F_1 \left[ \begin{matrix} - \\ b+1+n \end{matrix} ; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1)(t+b+1)S(t+1) - zS(t) = (t+b+1)\nu_0 + \nu_1.$$

### 5.2 $(1, 1)$ : Generalized Meixner polynomials

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a)}{(x+1)(x+b+1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(a)_n}{(b+1)_n} {}_1F_1 \left[ \begin{matrix} a+n \\ b+1+n \end{matrix}; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1)(t+b+1)S(t+1) - z(t+a)S(t) = (t+b+1-z)\nu_0 + \nu_1. \quad (51)$$

### 5.2.1 Reduced-Uvarov Charlier polynomials

Let  $\omega = 0$ .

Special values

$$a \rightarrow 0, \quad b \rightarrow -1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) \frac{z^x}{x!} + Mr(0).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{zx}{(x+1)x}.$$

Moments

$$\nu_n^U(z) = z^n e^z + M\phi_n(0), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$t[(t+1)S_U(t+1) - zS_U(t)] = (t-z)\nu_0^U + \nu_1^U. \quad (52)$$

Since

$$(t-z)\nu_0^U + \nu_1^U = (t-z)(e^z + M) + ze^z = t\nu_0 + M(t-z),$$

where  $\nu_0 = e^z$  is the first moment of the Charlier polynomials, we can rewrite (52) in the  $M$ -dependent form

$$(t+1)S_U(t+1) - zS_U(t) = \nu_0 + M\left(1 - \frac{z}{t}\right),$$

which agrees with (31).

### 5.2.2 Christoffel Charlier polynomials

Let  $\omega \notin \{0, z\}$ .

Special values

$$a \rightarrow -\omega + 1, \quad b \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x - \omega) \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{z(x+1-\omega)}{(x+1)(x-\omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega) \nu_n = (n + z - \omega) z^n e^z, \quad n \in \mathbb{N}_0. \quad (53)$$

Stieltjes transform difference equation

$$(t - \omega)(t + 1) S_C(t + 1) - (t + 1 - \omega) z S_C(t) = \nu_0 [(z - \omega)t + (\omega^2 - z\omega + z)].$$

Note that if we use (53), we have

$$\nu_0^C = (z - \omega) \nu_0, \quad \nu_1^C = (1 + z - \omega) z \nu_0,$$

and therefore

$$(t - \omega)(t + 1) S_C(t + 1) - (t + 1 - \omega) z S_C(t) = (t - \omega - z) \nu_0^C + \nu_1^C,$$

which agrees with (51) as  $a \rightarrow 1 - \omega$ ,  $b \rightarrow -\omega - 1$ .

### 5.2.3 Geronimus Charlier polynomials

Let  $\omega \notin \{-1\} \cup \mathbb{N}_0$ .

Special values

$$a \rightarrow -\omega, \quad b \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{z(x-\omega)}{(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + e^z \sum_{k=0}^{n-1} \frac{z^k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1-\omega)\sigma(t+1)S_G(t+1) - (t-\omega)\eta(t)S_G(t) = \nu_0 + \nu_0^G(t+1-z).$$

Note that if we use (34), we have

$$\nu_1^G - \omega\nu_0^G = \nu_0,$$

and therefore we can write

$$(t+1)(t+1-\omega)S_G(t+1) - z(t-\omega)S_G(t) = (t+1-\omega-z)\nu_0^G + \nu_1^G,$$

which agrees with (51) as  $a \rightarrow -\omega$ ,  $b \rightarrow -\omega$ .

## 5.2.4 Truncated Charlier polynomials

Special values

$$a \rightarrow -N, \quad b \rightarrow -N-1.$$

Linear functional

$$L_T[r] = \sum_{x=0}^N r(x) \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{z(x-N)}{(x+1)(x-N)}.$$

Moments

$$\nu_n^T(z) = \frac{z^N}{(N-n)!} {}_2F_0 \left[ \begin{matrix} n-N, 1 \\ - \\ -\frac{1}{z} \end{matrix} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t-N)[(t+1)S_T(t+1) - zS_T(t)] = (t-N-z)\nu_0^T + \nu_1^T.$$

**Remark 8** *The performance of the modified Chebyshev algorithm used to compute the moments of these polynomials was studied in [25] (example 4.3).*

### 5.3 $(2, 0; N)$ : Generalized Krawtchouk polynomials

Linear functional

$$L[r] = \sum_{x=0}^N r(x) (a)_x (-N)_x \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a)(x-N)}{x+1}.$$

Moments

$$\nu_n(z) = z^n (-N)_n (a)_n {}_2F_0 \left[ \begin{matrix} -N+n, a+n \\ - \end{matrix} ; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+1)S(t+1) - z(t+a)(t-N)S(t) = [1 - (t+a-N)z]\nu_0 - z\nu_1.$$

### 5.4 $(2, 1)$ : Generalized Hahn polynomials of type I

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(a_1)_x (a_2)_x z^x}{(b+1)_x x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a_1)(x+a_2)}{(x+b+1)(x+1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(a_1)_n (a_2)_n}{(b+1)_n} {}_2F_1 \left[ \begin{matrix} a_1+n, a_2+n \\ b+1+n \end{matrix} ; z \right], \quad n \in \mathbb{N}_0,$$

where we choose the principal branch  $z \in \mathbb{C} \setminus [1, \infty)$ . Note that since

$$b+1+n - (a_1+n+a_2+n) \leq 0, \quad n \geq b - a_1 - a_2 + 1,$$

$\nu_n(1)$  is undefined for  $n \geq b - a_1 - a_2 + 1$ .

Stieltjes transform difference equation

$$\begin{aligned} & (t+1)(t+b+1)S(t+1) - z(t+a_1)(t+a_2)S(t) \\ & = [t+b+1 - z(t+a_1+a_2)]\nu_0 + (1-z)\nu_1. \end{aligned}$$



### 5.4.1 Reduced-Uvarov Meixner polynomials

Let  $(\omega, \Omega) \in \{(0, 0), (-a, a + 1)\}$ .

Special values

$$a_1 \rightarrow \Omega, \quad a_2 \rightarrow a, \quad b \rightarrow \Omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) (a)_x \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x+\Omega)(x+a)}{(x+\Omega)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n (a)_n (1-z)^{-a-n} + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+\Omega)[(t+1)S_U(t+1) - z(t+a)S_U(t)] = [(1-z)(t+\Omega) - az]\nu_0^U + (1-z)\nu_1^U.$$

Stieltjes transform  $M$ -dependent difference equation

$$(t+1)S_U(t+1) - z(t+a)S_U(t) = (1-z)\nu_0 + M \left[ \frac{t+1}{t-\omega+1} - \frac{z(t+a)}{t-\omega} \right].$$

### 5.4.2 Christoffel Meixner polynomials

Let  $\omega \notin \{-a, 0, az(1-z)^{-1}\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega + 1, \quad b \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x-\omega) (a)_x \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{z(x+a)(x+1-\omega)}{(x+1)(x-\omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n-\omega)\nu_n = z^n (a)_n (1-z)^{-a-n-1} (az + \omega z + n - \omega), \quad n \in \mathbb{N}_0. \quad (54)$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-\omega)(t+1)S_C(t+1) - (t+1-\omega)z(t+a)S_C(t) \\ &= \nu_0 [(az - \omega + \omega z)t + (\omega^2 + az + \omega z - az\omega - z\omega^2)]. \end{aligned}$$

Note that if we use (54), we have

$$\nu_0^C = \frac{az + \omega z - \omega}{1-z}\nu_0, \quad \nu_1^C = \frac{za(az + \omega z + 1 - \omega)}{(1-z)^2}\nu_0,$$

and therefore we can also write

$$\begin{aligned} & (t-\omega)(t+1)S_C(t+1) - (t+1-\omega)z(t+a)S_C(t) \\ &= [t-\omega - z(t+a-\omega+1)]\nu_0^C + (1-z)\nu_1^C. \end{aligned}$$

### 5.4.3 Geronimus Meixner polynomials

Let  $\omega \notin \{-1, 1-a\} \cup \mathbb{N}_0$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega, \quad b \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x-\omega} (a)_x \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{z(x+a)(x-\omega)}{(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + (1-z)^{-a} \sum_{k=0}^{n-1} \frac{z^k (a)_k (1-z)^{-k}}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+1)(t-\omega+1)S_G(t+1) - z(t+a)(t-\omega)S_G(t) \\ &= (1-z)\nu_0 + \nu_0^G [(1-z)t + (1-az)]. \end{aligned}$$

Note that if we use (34), we have

$$\nu_1^G - \omega\nu_0^G = \nu_0,$$

and therefore we can also write

$$\begin{aligned} & (t+1)(t-\omega+1)S_G(t+1) - z(t+a)(t-\omega)S_G(t) \\ &= [(1-z)t + z\omega - \omega - az + 1]\nu_0^G + (1-z)\nu_1^G. \end{aligned}$$

#### 5.4.4 Truncated Meixner polynomials

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -N, \quad b \rightarrow -N - 1.$$

Linear functional

$$L_T[r] = \sum_{x=0}^N r(x) (a)_x \frac{z^x}{x!}.$$

Moments

$$\nu_n^T = (a)_N \frac{z^N}{(N-n)!} {}_2F_1 \left[ \begin{matrix} n-N, 1 \\ 1-N-a \end{matrix}; \frac{1}{z} \right], \quad n \in \mathbb{N}_0.$$

Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{z(x+a)(x-N)}{(x-N)(x+1)}.$$

Stieltjes transform difference equation

$$(t-N)[(t+1)S_T(t+1) - z(t+a)S_T(t)] = [(t-N)(1-z) - az]\nu_0^T + (1-z)\nu_1^T.$$

### 5.4.5 Symmetrized Generalized Krawtchouk polynomials

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -2m, \quad b \rightarrow -2m - a, \quad z \rightarrow -1, \quad x \rightarrow x + m.$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(a+m)_x (-m)_x}{(-m-a+1)_x (m+1)_x} (-1)^x,$$

where

$$\varrho_\zeta(0) = (-1)^m \binom{2m}{m} \frac{(a)_m}{(a+m)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{-(x+a+m)(x-m)}{(x+1-a-m)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\nu_n^\zeta = (-1)^n \frac{(a)_n (-2m)_n}{(-2m-a)_n} {}_2F_1 \left[ \begin{matrix} -2m+n, a+n \\ -2m-a+n \end{matrix}; -1 \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+m+1)(t-m-a+1)S_\zeta(t+1) + (t-m)(t+m+a)S_\zeta(t) = (2t-2m+1)\nu_0^\zeta + 2\nu_1^\zeta.$$

## 5.5 $(3, 2; N, 1)$ : Generalized Hahn polynomials of type II

Linear functional

$$L[r] = \sum_{x=0}^N r(x) \frac{(a_1)_x (a_2)_x (-N)_x}{(b_1+1)_x (b_2+1)_x x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{(x+a_1)(x+a_2)(x-N)}{(x+b_1+1)(x+b_2+1)(x+1)}.$$

Moments

$$\nu_n = \frac{(-N)_n (a_1)_n (a_2)_n}{(b_1 + 1)_n (b_2 + 1)_n} {}_3F_2 \left[ \begin{matrix} -N + n, a_1 + n, a_2 + n \\ b_1 + 1 + n, b_2 + 1 + n \end{matrix} ; 1 \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t + b_1 + 1)(t + b_2 + 1)(t + 1)S(t + 1) - (t + a_1)(t + a_2)(t - N)S(t) \\ &= [(N + 2 + b_1 + b_2 - a_1 - a_2)t + b_1 + b_2 + b_1 b_2 - a_1 - a_2 - a_1 a_2 + Na_1 + Na_2 + 1] \nu_0 \\ &+ (N + b_1 + b_2 - a_1 - a_2 + 1) \nu_1. \end{aligned}$$

### 5.5.1 Reduced-Uvarov Hahn polynomials

Let  $(\omega, \Omega) \in \{(0, 0), (-a, a + 1), (-b, b), (N, -N + 1)\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow \Omega, \quad b_1 \rightarrow \Omega - 1, \quad b_2 \rightarrow a.$$

Linear functional

$$L_U[r] = \sum_{x=0}^N r(x) \frac{(a)_x (-N)_x}{(b + 1)_x} \frac{1}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x + 1)}{\varrho_U(x)} = \frac{(x + a)(x + \Omega)(x - N)}{(x + b + 1)(x + \Omega)(x + 1)}.$$

Moments

$$\nu_n^U = \frac{(-N)_n (a)_n (b + 1 - a)_{N-n}}{(b + 1)_n (b + 1 + n)_{N-n}} + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t + \Omega)[(t + b + 1)(t + 1)S_U(t + 1) - (t + a)(t - N)]S_U(t) \\ &= [(N - a + b + 1)t + (N - a + b + 1)\Omega + aN] \nu_0^U + (N + b - a) \nu_1^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned} & (t + b + 1)(t + 1)S_U(t + 1) - (t - N)(t + a)S_U(t) \\ &= (N + b - a + 1) \nu_0 + M \left[ \frac{(t + b + 1)(t + 1)}{t - \omega + 1} - \frac{(t - N)(t + a)}{t - \omega} \right]. \end{aligned}$$

### 5.5.2 Christoffel Hahn polynomials

Let  $\omega \notin \left\{ -a, -b, 0, N, -\frac{aN}{b-a+N} \right\}$ .  
Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega + 1, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x - \omega) \frac{(a)_x (-N)_x}{(b+1)_x} \frac{1}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{(x+a)(x-N)(x+1-\omega)}{(x+b+1)(x+1)(x-\omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega) \nu_n = \left[ n - \omega - \frac{(a+n)(N-n)}{b-a+N-n} \right] \nu_n, \quad n \in \mathbb{N}_0. \quad (55)$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega)(t + b + 1)(t + 1)S_C(t + 1) - (t - \omega + 1)(t - N)(t + a)S_C(t) \\ &= [(\omega a - N\omega - b\omega - Na)t + \omega a - N\omega + \omega^2 + N\omega^2 - a\omega^2 + b\omega^2 - Na + Na\omega] \nu_0. \end{aligned}$$

Note that if we use (55), we have

$$\nu_0^C = -\frac{(a + \omega)N + (b - a)\omega}{b - a + N} \nu_0, \quad \nu_1^C = \frac{(Na + N\omega - a\omega + b\omega - b - \omega)Na}{(b - a + N)(b - a + N - 1)} \nu_0,$$

and therefore we can also write

$$\begin{aligned} & (t - \omega)(t + b + 1)(t + 1)S_C(t + 1) - (t - \omega + 1)(t - N)(t + a)S_C(t) \\ &= [(N - a + b)t + Na - N\omega + a\omega - b\omega + N - a - \omega] \nu_0^C + (N - a + b - 1) \nu_1^C. \end{aligned}$$

### 5.5.3 Geronimus Hahn polynomials

Let  $\omega \notin \{-b-1, -1, 1-a, N+1\} \cup \mathbb{N}_0$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega, \quad b \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x-\omega} \frac{(a)_x (-N)_x}{(b+1)_x} \frac{1}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{(x+a)(x-N)(x-\omega)}{(x+b+1)(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\xi_G(t) = \xi(t) + [\sigma(t+1) - \eta(t)] \nu_0^G,$$

$$\begin{aligned} & (t-\omega+1)(t+b+1)(t+1)S_G(t+1) - (t-\omega)(t-N)(t+a)S_G(t) \\ &= (b+1-a+N)\nu_0 + [(N-a+b+2)t+b+Na+1]\nu_0^G. \end{aligned}$$

Note that if we use (34), we have

$$\nu_1^G - \omega\nu_0^G = \nu_0,$$

and therefore we can also write

$$\begin{aligned} & (t-\omega+1)(t+b+1)(t+1)S_G(t+1) - (t-\omega)(t-N)(t+a)S_G(t) \\ &= [(N-a+b+2)t+Na-N\omega+a\omega-b\omega+b-\omega+1]\nu_0^G + (b-1-a+N)\nu_1^G. \end{aligned}$$

### 5.5.4 Symmetrized Hahn polynomials

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -N - b, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -N - a, \quad N \rightarrow 2m, \quad x \rightarrow x + m.$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(m+a)_x (-m-b)_x (-m)_x}{(1-m-a)_x (m+b+1)_x (m+1)_x},$$

where

$$\varrho_\zeta(0) = (-1)^m \binom{2m}{m} \frac{(a)_m}{(a+m)_m} \frac{(b+1+m)_m}{(b+1)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(x+m+a)(x-m-b)(x-m)}{(x+1-m-a)(x+1+m+b)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\nu_n^\zeta = \frac{(-2m)_n (a)_n (-2m-b)_n}{(b+1)_n (-2m-a+1)_n} {}_3F_2 \left[ \begin{matrix} -2m+n, a+n, -2m-b+n \\ b+1+n, -2m-a+1+n \end{matrix}; 1 \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+1-m-a)(t+1+m+b)(t+1+m) S_\zeta(t+1) - (t+m+a)(t-m-b)(t-m) S_\zeta(t) \\ &= [2(1+b-a+m)t + 2am - 2bm - 2m^2 - a + b + 1] \nu_0^\zeta + (1+2m-2a+2b) \nu_1^\zeta. \end{aligned}$$

### 5.5.5 Symmetrized polynomials of type $(3, 0; N)$

For a definition of the polynomials of type  $(3, 0; N)$ , see Section 6.4.

Special values

$$b_1 \rightarrow -N - a_1, \quad b_2 \rightarrow -N - a_2, \quad N \rightarrow 2m, \quad x \rightarrow x + m.$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(a_1+m)_x (a_2+m)_x (-m)_x}{(1-a_1-m)_x (1-a_2-m)_x (m+1)_x},$$



where

$$\varrho_\zeta(0) = (-1)^m \binom{2m}{m} \frac{(a_1)_m}{(a_1+m)_m} \frac{(a_2)_m}{(a_2+m)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(x+a_1+m)(x+a_2+m)(x-m)}{(x+1-a_1-m)(x+1-a_2-m)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\nu_n^\zeta = \frac{(-2m)_n (a_1)_n (a_2)_n}{(1-a_1-2m)_n (1-a_2-2m)_n} {}_3F_2 \left[ \begin{matrix} -2m+n, a_1+n, a_2+n \\ 1-a_1-2m+n, 1-a_2-2m+n \end{matrix} ; 1 \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+1-a_1-m)(t+1-a_2-m)(t+1+m) S_\zeta(t+1) \\ & - (t+a_1+m)(t+a_2+m)(t-m) S_\zeta(t) \\ & = [-2(m+a_1+a_2-1)t + 2m^2 - 2m + 1 + (2m-1)(a_1+a_2)] \nu_0^\zeta \\ & + (-2m+1-2a_1-2a_2) \nu_1^\zeta. \end{aligned}$$

## 6 Semiclassical polynomials of class 2

In this section, we consider all families of polynomials of class 2. We have 7 main cases, corresponding to

$$(p, q) = (0, 2), (1, 2), (2, 2), (3, 0), (3, 1), (3, 2), (4, 3).$$

There are also 25 subcases.

### 6.1 Polynomials of type (0,2)

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{1}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z}{(x+b_1+1)(x+b_2+1)(x+1)}.$$

Moments

$$\nu_n(z) = \frac{z^n}{(b_1 + 1)_n (b_2 + 1)_n} {}_0F_2 \left[ \begin{matrix} - \\ b_1 + 1 + n, b_2 + 1 + n \end{matrix}; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t + b_1 + 1)(t + b_2 + 1)(t + 1)S(t + 1) - zS(t) \\ &= (t + b_1 + 1)(t + b_2 + 1)\nu_0 + (t + b_1 + b_2 + 2)\nu_1 + \nu_2. \end{aligned}$$

## 6.2 Polynomials of type (1,2)

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(a)_x}{(b_1 + 1)_x (b_2 + 1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x + 1)}{\varrho(x)} = \frac{z(x + a)}{(x + b_1 + 1)(x + b_2 + 1)(x + 1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(a)_n}{(b_1 + 1)_n (b_2 + 1)_n} {}_1F_2 \left[ \begin{matrix} a + n \\ b_1 + 1 + n, b_2 + 1 + n \end{matrix}; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t + b_1 + 1)(t + b_2 + 1)(t + 1)S(t + 1) - z(t + a)S(t) \\ &= [(t + b_1 + 1)(t + b_2 + 1) - z]\nu_0 + (t + b_1 + b_2 + 2)\nu_1 + \nu_2. \end{aligned}$$

### 6.2.1 Reduced-Uvarov Generalized Charlier polynomials

Let  $\omega \in \{0, -b\}$ .

Special values

$$a \rightarrow -\omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) \frac{1}{(b+1)_x} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x-\omega)}{(x+b+1)(x-\omega)(x+1)}.$$

Moments

$$\nu_n^U(z) = \frac{z^n}{(b+1)_n} {}_0F_1 \left[ \begin{matrix} - \\ b+1+n \end{matrix} ; z \right] + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} (t-\omega)[(t+b+1)(t+1)S_U(t+1) - zS_U(t)] \\ = [(t+b+1)(t+b) - z]\nu_0^U + (t+2b+1)\nu_1^U + \nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$(t+b+1)(t+1)S_U(t+1) - zS_U(t) = (t+b+1)\nu_0 + \nu_1 + M \left[ \frac{(t+b+1)(t+1)}{t-\omega+1} - \frac{z}{t-\omega} \right].$$

### 6.2.2 Christoffel generalized Charlier polynomials

Let  $\omega \notin \left\{ -b, 0, \frac{\nu_1}{\nu_0} \right\}$ .

Special values

$$a \rightarrow -\omega + 1, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x-\omega) \frac{1}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{z(x+1-\omega)}{(x+b+1)(x+1)(x-\omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega) \nu_n, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega)(t + b + 1)(t + 1) S_C(t + 1) - z(t - \omega + 1) S_C(t) \\ &= [-\omega t^2 + (z - \omega - b\omega + \omega^2)t + (z - z\omega + \omega^2 + b\omega^2)] \nu_0 + (t - \omega)(t - \omega + 1) \nu_1. \end{aligned}$$

### 6.2.3 Geronimus generalized Charlier polynomials

Let  $\omega \notin \{-b - 1, -1\} \cup \mathbb{N}_0$ .

Special values

$$a \rightarrow -\omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} \frac{1}{(b + 1)_x} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x + 1)}{\varrho_G(x)} = \frac{z(x - \omega)}{(x + b + 1)(x + 1)(x + 1 - \omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega + 1)(t + b + 1)(t + 1) S_G(t + 1) - (t - \omega) z S_G(t) \\ &= (t + b + 1) \nu_0 + \nu_1 + [(t + b + 1)(t + 1) - z] \nu_0^G. \end{aligned}$$

### 6.2.4 Truncated generalized Charlier polynomials

Special values

$$a \rightarrow -N, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -N - 1.$$

Linear functional

$$L_T[r] = \sum_{x=0}^N r(x) \frac{1}{(b + 1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{z(x-N)}{(x+b+1)(x-N)(x+1)}.$$

Moments

$$\nu_n^T(z) = \frac{1}{(b+1)_N} \frac{z^N}{(N-n)!} {}_3F_0 \left[ \begin{matrix} n-N, 1, -N-b \\ - \end{matrix} ; \frac{1}{z} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-N)[(t+b+1)(t+1)S_T(t+1) - zS_T(t)] \\ &= [(t+b+1)(t-N) - z]\nu_0^T + (t+b+1-N)\nu_1^T + \nu_2^T. \end{aligned}$$

### 6.3 Polynomials of type (2,2)

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(a_1)_x (a_2)_x}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a_1)(x+a_2)}{(x+b_1+1)(x+b_2+1)(x+1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(a_1)_n (a_2)_n}{(b_1+1)_n (b_2+1)_n} {}_2F_2 \left[ \begin{matrix} a_1+n, a_2+n \\ b_1+1+n, b_2+1+n \end{matrix} ; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+b_1+1)(t+b_2+1)(t+1)S(t+1) - z(t+a_1)(t+a_2)S(t) \\ &= [(t+b_1+1)(t+b_2+1) - (t+a_1+a_2)z]\nu_0 + (t+b_1+b_2+2-z)\nu_1 + \nu_2. \end{aligned}$$

### 6.3.1 Uvarov Charlier polynomials

Let  $\omega \neq 0$ .

Special values

$$a_1 \rightarrow -\omega, \quad a_2 \rightarrow -\omega + 1, \quad b_1 \rightarrow -\omega - 1, \quad b_2 \rightarrow -\omega.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x-\omega)(x-\omega+1)}{(x-\omega)(x-\omega+1)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n e^z + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-\omega)(t-\omega+1)[(t+1)S_U(t+1) - zS_U(t)] \\ &= [(t-\omega)(t-\omega+1) - (t-2\omega+1)z]\nu_0^U + (t-z-2\omega+1)\nu_1^U + \nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$(t+1)S_U(t+1) - zS_U(t) = \nu_0 + M \left( \frac{t+1}{t+1-\omega} - \frac{z}{t-\omega} \right).$$

### 6.3.2 Reduced-Uvarov generalized Meixner polynomials

Let  $(\omega, \Omega) \in \{(-a, a+1), (-b, b), (0, 0)\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow \Omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow \Omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x+a)(x+\Omega)}{(x+b+1)(x+\Omega)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n \frac{(a)_n}{(b+1)_n} {}_1F_1 \left[ \begin{matrix} a+n \\ b+1+n \end{matrix}; z \right] + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+\Omega)[(t+b+1)(t+1)S_U(t+1) - z(t+a)S_U(t)] \\ &= [(t+b+1)(t+\Omega) - (t+a+\Omega)z]\nu_0^U + (t+b+1+\Omega-z)\nu_1^U + \nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned} & (t+b+1)(t+1)S_U(t+1) - z(t+a)S_U(t) \\ &= (t+b+1-z)\nu_0 + \nu_1 + M \left[ \frac{(t+b+1)(t+1)}{t+1-\omega} - \frac{z(t+a)}{t-\omega} \right]. \end{aligned}$$

### 6.3.3 Christoffel generalized Meixner polynomials

Let  $\omega \notin \left\{ -a, -b, 0, \frac{\nu_1}{\nu_0} \right\}$ .  
Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega + 1, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x)(x-\omega) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{z(x+a)(x+1-\omega)}{(x+1)(x+b+1)(x-\omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n-\omega)\nu_n, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega)(t + b + 1)(t + 1)S_C(t + 1) - z(t - \omega + 1)(t + a)S_C(t) \\ &= [-\omega t^2 + (z\omega - b\omega - \omega + \omega^2 + az)t + z\omega + \omega^2 + b\omega^2 - z\omega^2 + az - az\omega] \nu_0 \\ &+ (t - \omega)(t - \omega + 1)\nu_1. \end{aligned}$$

### 6.3.4 Geronimus generalized Meixner polynomials

Let  $\omega \notin \{-b - 1, -1, 1 - a\} \cup \mathbb{N}_0$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} \frac{(a)_x}{(b + 1)_x} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x + 1)}{\varrho_G(x)} = \frac{z(x + a)(x - \omega)}{(x + 1)(x + b + 1)(x + 1 - \omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega + 1)(t + b + 1)(t + 1)S_G(t + 1) - z(t - \omega)(t + a)S_G(t) \\ &= (t + b + 1 - z)\nu_0 + \nu_1 + [(t + b + 1)(t + 1) - z(t + a)]\nu_0^G. \end{aligned}$$

### 6.3.5 Truncated generalized Meixner polynomials

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -N, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -N - 1.$$



Linear functional

$$L_T[r] = \sum_{x=0}^N r(x) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{z(x+a)(x-N)}{(x+b+1)(x-N)(x+1)}.$$

Moments

$$\nu_n^T(z) = \frac{(a)_N}{(b+1)_N} \frac{z^N}{(N-n)!} {}_3F_1 \left[ \begin{matrix} n-N, 1, -N-b \\ 1-N-a \end{matrix} ; \frac{1}{z} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-N)[(t+b+1)(t+1)S_T(t+1) - z(t+a)S_T(t)] \\ &= [(t+b+1)(t-N) - (t+a-N)z]\nu_0^T + (t+b-N+1-z)\nu_1^T + \nu_2^T. \end{aligned}$$

## 6.4 Polynomials of type (3,0;N)

Linear functional

$$L[r] = \sum_{x=0}^N r(x) \frac{(a_1)_x (a_2)_x (-N)_x}{x!} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a_1)(x+a_2)(x-N)}{x+1}.$$

Moments

$$\nu_n(z) = z^n (-N)_n (a_1)_n (a_2)_n {}_3F_0 \left[ \begin{matrix} -N+n, a_1+n, a_2+n \\ - \end{matrix} ; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+1)S(t+1) - z(t+a_1)(t+a_2)(t-N)S(t) \\ &= [-zt^2 - (a_1+a_2-N)zt + z(Na_1+Na_2-a_1a_2) + 1]\nu_0 \\ & - (t+a_1+a_2+1-N)z\nu_1 - z\nu_2. \end{aligned}$$

## 6.5 Polynomials of type (3,1;N)

Linear functional

$$L[r] = \sum_{x=0}^N r(x) \frac{(a_1)_x (a_2)_x (-N)_x z^x}{(b+1)_x x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a_1)(x+a_2)(x-N)}{(x+b+1)(x+1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(-N)_n (a_1)_n (a_2)_n}{(b+1)_n} {}_3F_1 \left[ \begin{matrix} -N+n, a_1+n, a_2+n \\ b+1+n \end{matrix}; z \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (x+b+1)(t+1)S(t+1) - z(t+a_1)(t+a_2)(t-N)S(t) \\ &= [-zt^2 + t - (a_1+a_2-N)zt + (Na_1+Na_2-a_1a_2)z + b+1] \nu_0 \\ &+ [1 - (t+a_1+a_2+1-N)z] \nu_1 - z\nu_2. \end{aligned}$$

### 6.5.1 Reduced-Uvarov generalized Krawtchouk polynomials

Let  $(\omega, \Omega) \in \{(-a, a+1), (0, 0), (N, -N+1)\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow \Omega, \quad b \rightarrow \Omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^N r(x) (a)_x (-N)_x \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x+a)(x+\Omega)(x-N)}{(x+\Omega)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n (-N)_n (a)_n {}_2F_0 \left[ \begin{matrix} -N + n, a + n \\ - \end{matrix} ; z \right] + M \phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t + \Omega) [(t + 1) S_U(t + 1) - z(t + a)(t - N) S_U(t)] \\ &= [-zt^2 + t - (a + \Omega - N)zt + (Na + N\Omega - a\Omega)z + \Omega] \nu_0^U \\ &+ [1 - (t + a + \Omega + 1 - N)z] \nu_1^U - z\nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned} & (t + 1) S_U(t + 1) - z(t + a)(t - N) S_U(t) \\ &= [1 - (t + a - N)z] \nu_0 - z\nu_1 + M \left[ \frac{t + 1}{t - \omega + 1} - \frac{z(t + a)(t - N)}{t - \omega} \right]. \end{aligned}$$

### 6.5.2 Christoffel generalized Krawtchouk polynomials

Let  $\omega \notin \left\{ -a, 0, N, \frac{\nu_1}{\nu_0} \right\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega + 1, \quad b \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x - \omega) (a)_x (-N)_x \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x + 1)}{\varrho_C(x)} = \frac{z(x + a)(x - N)(x + 1 - \omega)}{(x + 1)(x - \omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega) \nu_n, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega)(t + 1) S_C(t + 1) - z(t - \omega + 1)(t + a)(t - N) S_C(t) \\ &= [z\omega t^2 - (\omega - z\omega + z\omega^2 + Nz\omega - az\omega + Naz)t] \nu_0 \\ &+ (\omega^2 - Nz\omega + az\omega + Nz\omega^2 - az\omega^2 - Naz + Naz\omega) \nu_0 - z(t - \omega)(t - \omega + 1) \nu_1. \end{aligned}$$

### 6.5.3 Geronimus generalized Krawtchouk polynomials

Let  $\omega \notin \{-1, 1 - a, 1 + N\} \cup \mathbb{N}_0$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega, \quad b \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} (a)_x (-N)_x \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{z(x+a)(x-N)(x-\omega)}{(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega + 1)(t + 1)S_G(t + 1) - z(t - \omega)(t + a)(t - N)S_G(t) \\ & = (-zt + Nz - az + 1)\nu_0 - z\nu_1 + [t + 1 - z(t + a)(t - N)]\nu_0^G. \end{aligned}$$

## 6.6 Polynomials of type (3,2)

Linear functional

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(a_1)_x (a_2)_x (a_3)_x}{(b_1 + 1)_x (b_2 + 1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{z(x+a_1)(x+a_2)(x+a_3)}{(x+b_1+1)(x+b_2+1)(x+1)}.$$

Moments

$$\nu_n(z) = z^n \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1+1)_n (b_2+1)_n} {}_3F_2 \left[ \begin{matrix} a_1+n, a_2+n, a_3+n \\ b_1+1+n, b_2+1+n \end{matrix}; z \right], \quad n \in \mathbb{N}_0,$$

where we choose the principal branch  $z \in \mathbb{C} \setminus [1, \infty)$ . Note that since

$$b_1+1+n+b_2+1+n-(a_1+n+a_2+n+a_3+n) \leq 0, \quad n \geq b_1+b_2-a_2-a_3-a_1+2,$$

$\nu_n(1)$  is undefined for  $n \geq b_1+b_2-a_2-a_3-a_1+2$ .

Stieltjes transform difference equation

$$(t+b_1+1)(t+b_2+1)(t+1)S(t+1)-z(t+a_1)(t+a_2)(t+a_3)S(t) = \sum_{k=0}^2 \xi_k t^k,$$

where

$$\begin{aligned} \xi_2 &= (1-z)\nu_0, & \xi_1 &= [e_1(\mathbf{b}+1) - e_1(\mathbf{a})z]\nu_0 + (1-z)\nu_1, \\ \xi_0 &= [e_2(\mathbf{b}+1) - e_2(\mathbf{a})z]\nu_0 + [e_1(\mathbf{b}+1) - e_1(\mathbf{a})z - z]\nu_1 + (1-z)\nu_2, \end{aligned}$$

and  $e_1, e_2, e_3$  denote the *elementary symmetric polynomials* defined by

$$e_1(\mathbf{x}) = \sum_i x_i, \quad e_2(\mathbf{x}) = \sum_{i<j} x_i x_j, \quad e_3(\mathbf{x}) = \sum_{i<j<k} x_i x_j x_k.$$

### 6.6.1 Uvarov Meixner polynomials

Let  $\omega \notin \{-a, 0\}$ .

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -\omega, \quad a_3 \rightarrow -\omega + 1, \quad b_1 \rightarrow -\omega - 1, \quad b_2 \rightarrow -\omega.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) (a)_x \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x+a)(x-\omega)(x-\omega+1)}{(x-\omega)(x-\omega+1)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n (a)_n (1-z)^{-a-n} + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-\omega)(t-\omega+1)[(t+1)S_U(t+1) - z(t+a)S_U(t)] \\ &= [(1-z)t^2 + (-az + 2\omega z - 2\omega - z + 1)t + \omega(z + \omega - z\omega + 2az - 1) - az] \nu_0^{U\zeta} \\ &+ [(1-z)t - 2z + 2z\omega - 2\omega - az + 1] \nu_1^U + (1-z) \nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$(t+1)S_U(t+1) - z(t+a)S_U(t) = (1-z)\nu_0 + M \left[ \frac{t+1}{t+1-\omega} - \frac{z(t+a)}{t-\omega} \right].$$

### 6.6.2 Symmetrized generalized Meixner polynomials

Special values

$$\begin{aligned} a_1 &\rightarrow a, & a_2 &\rightarrow -b - 2m, & a_3 &\rightarrow -2m, \\ b_1 &\rightarrow -a - 2m, & b_2 &\rightarrow b - 2m, & z &\rightarrow -1, & x &\rightarrow x + m. \end{aligned}$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(a+m)_x (-b-m)_x (-m)_x}{(-a-m+1)_x (b+1-m)_x (m+1)_x} \frac{(-1)^x}{(m+1)_x},$$

where

$$\varrho_\zeta(0) = \binom{2m}{m} \frac{(a)_m (b+1+m)_m}{(a+m)_m (b+1-2m)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{-(x+a+m)(x-b-m)(x-m)}{(x-a-m+1)(x+b+1+m)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\nu_n^\zeta = \frac{(-1)^n (a)_n (-b-2m)_n (-2m)_n}{(-a-2m+1)_n (b-2m+1)_n} {}_3F_2 \left[ \begin{matrix} a+n, -b-2m+n, -2m+n \\ -a-2m+1+n, b-2m+1+n \end{matrix}; -1 \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-a-m+1)(t+b+1+m)(t+1+m)S_\zeta(t+1) + (t+a+m)(t-b-m)(t-m)S_\zeta(t) \\ &= [2t^2 + 2(1-m)t + b-a-2ab-4am+1] \nu_0^\zeta + (2t-4m+3) \nu_1^\zeta + 2\nu_2^\zeta. \end{aligned}$$

### 6.6.3 Reduced-Uvarov generalized Hahn polynomials of type I

Let  $\{\omega, \Omega\} \in \{(-a_1, a_1 + 1), (-a_2, a_2 + 1), (-b, b), (0, 0)\}$ .

Special values

$$a_3 \rightarrow \Omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow \Omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^{\infty} r(x) \frac{(a_1)_x (a_2)_x z^x}{(b+1)_x x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{z(x+a_1)(x+a_2)(x+\Omega)}{(x+b+1)(x+\Omega)(x+1)}.$$

Moments

$$\nu_n^U(z) = z^n \frac{(a_1)_n (a_2)_n}{(b+1)_n} {}_2F_1 \left[ \begin{matrix} a_1+n, a_2+n \\ b+1+n \end{matrix}; z \right] + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+\Omega)[(t+b+1)(t+1)S_U(t+1) - z(t+a_1)(t+a_2)S_U(t)] \\ &= [(1-z)t^2 + (b+\Omega - z\Omega - za_1 - za_2 + 1)t - (\Omega a_1 + \Omega a_2 + a_1 a_2)z + (b+1)\Omega] \nu_0^U \\ &+ [(1-z)t + 1 - z(\Omega + a_1 + a_2 + 1) + \Omega + b] \nu_1^U + (1-z) \nu_2^U. \end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned} & (t+b+1)(t+1)S_U(t+1) - z(t+a_1)(t+a_2)S_U(t) = [t+b+1 - z(t+a_1+a_2)] \nu_0 \\ &+ (1-z) \nu_1 + M \left[ \frac{(t+b+1)(t+1)}{t-\omega+1} - \frac{z(t+a_1)(t+a_2)}{t-\omega} \right]. \end{aligned}$$

### 6.6.4 Christoffel generalized Hahn polynomials of type I

Let  $\omega \notin \left\{ -a_1, -a_2, -b, 0, \frac{\nu_1}{\nu_0} \right\}$ .

Special values

$$a_3 \rightarrow -\omega + 1, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x) (x - \omega) \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x+1)}{\varrho_C(x)} = \frac{z(x+a_1)(x+a_2)(x+1-\omega)}{(x+b+1)(x+1)(x-\omega)}, \quad n \in \mathbb{N}_0.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega) \nu_n.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega)(t + b + 1)(t + 1) S_C(t + 1) - z(t - \omega + 1)(t + a_1)(t + a_2) S_C(t) \\ &= [(z - 1)\omega t^2 - (\omega + b\omega - z\omega - \omega^2 + z\omega^2 - z\omega a_1 - z\omega a_2 - za_1 a_2) t] \nu_0 \\ &+ [\omega^2(b + 1) - z(\omega - 1)(\omega a_1 + \omega a_2 + a_1 a_2)] \nu_0 - (z - 1)(t - \omega)(t - \omega + 1) \nu_1. \end{aligned}$$

### 6.6.5 Geronimus generalized Hahn polynomials of type I

Let  $\omega \notin \{-b - 1, -1, 1 - a_1, 1 - a_2\} \cup \mathbb{N}_0$ .

Special values

$$a_3 \rightarrow -\omega, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{z(x+a_1)(x+a_2)(x-\omega)}{(x+b+1)(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega + 1)(t + b + 1)(t + 1) S_G(t + 1) - z(t - \omega)(t + a_1)(t + a_2) S_G(t) \\ &= [(1 - z)t + (b - za_1 - za_2 + 1)] \nu_0 + (1 - z) \nu_1 \\ &+ [(t + b + 1)(t + 1) - z(t + a_1)(t + a_2)] \nu_0^G. \end{aligned}$$



### 6.6.6 Truncated generalized Hahn polynomials of type I

Special values

$$a_3 \rightarrow -N, \quad b_1 \rightarrow b, \quad b_2 \rightarrow -N - 1.$$

Linear functional

$$L_T[r] = \sum_{x=0}^N r(x) \frac{(a_1)_x (a_2)_x z^x}{(b+1)_x x!}.$$

Pearson equation

$$\frac{\varrho_T(x+1)}{\varrho_T(x)} = \frac{z(x+a_1)(x+a_2)(x-N)}{(x+b+1)(x-N)(x+1)}.$$

Moments

$$\nu_n^T(z) = \frac{(a_1)_N (a_2)_N}{(b+1)_N} \frac{z^N}{(N-n)!} {}_3F_2 \left[ \begin{matrix} n-N, 1, -N-b \\ 1-N-a_1, 1-N-a_2 \end{matrix}; \frac{1}{z} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t-N)[(t+b+1)(t+1)S_U(t+1) - z(t+a_1)(t+a_2)S_U(t)] \\ &= [(1-z)t^2 + (b-N+zM - za_1 - za_2 + 1)t - (-Na_1 - Na_2 + a_1a_2)z - (b+1)N] \nu_0^U \\ &+ [(1-z)t + 1 - z(-N + a_1 + a_2 + 1) - N + b] \nu_1^U + (1-z) \nu_2^U. \end{aligned}$$

### 6.6.7 Symmetrized polynomials of type (0, 2)

Special values

$$a_1 \rightarrow -b_1 - 2m, \quad a_2 \rightarrow -b_2 - 2m, \quad a_3 \rightarrow -2m, \quad z \rightarrow -1, \quad x \rightarrow x + m.$$

Linear functional

$$L_\varsigma[r] = \varrho_\varsigma(0) \sum_{x=-m}^m r(x) \frac{(-b_1 - m)_x (-b_2 - m)_x (-m)_x (-1)^x}{(b_1 + 1 + m)_x (b_2 + 1 + m)_x (m + 1)_x},$$

where

$$\varrho_\varsigma(0) = \binom{2m}{m} \frac{(b_1 + 1 + m)_m (b_2 + 1 + m)_m}{(b_1 + 1)_m (b_2 + 1)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{-(x-b_1-m)(x-b_2-m)(x-m)}{(x+b_1+1+m)(x+b_2+1+m)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$ .

For  $n \in \mathbb{N}_0$

$$\nu_n^\zeta = \frac{(-1)^n (-2m)_n (-b_1-2m)_n (-b_2-2m)_n}{(b_1+1)_n (b_2+1)_n} {}_3F_2 \left[ \begin{matrix} -2m+n, -b_1-2m, -b_2-2m+n \\ b_1+1+n, b_2+1+n \end{matrix} ; -1 \right].$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+b_1+1+m)(t+b_2+1+m)(t+1+m) S_\zeta(t+1) \\ & + (t-b_1-m)(t-b_2-m)(t-m) S_\zeta(t) \\ & = [2t^2 + 2(1-m)t + 8m^2 + 2m(2b_1+2b_2+1) + b_1+b_2+2b_1b_2+1] \nu_0^\zeta \\ & + (2t-4m+3) \nu_1^\zeta + 2\nu_2^\zeta. \end{aligned}$$

## 6.7 Polynomials of type (4,3;N,1)

Linear functional

$$L[r] = \sum_{x=0}^N r(x) \frac{(a_1)_x (a_2)_x (a_3)_x (-N)_x}{(b_1+1)_x (b_2+1)_x (b_3+1)_x x!}.$$

Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{(x+a_1)(x+a_2)(x+a_3)(x-N)}{(x+b_1+1)(x+b_2+1)(x+b_3+1)(x+1)}.$$

Moments

For  $n \in \mathbb{N}_0$

$$\nu_n = \frac{(-N)_n (a_1)_n (a_2)_n (a_3)_n}{(b_1+1)_n (b_2+1)_n (b_3+1)_n} {}_4F_3 \left[ \begin{matrix} -N+n, a_1+n, a_2+n, a_3+n \\ b_1+1+n, b_2+1+n, b_3+1+n \end{matrix} ; 1 \right].$$

Stieltjes transform difference equation

$$\begin{aligned} & (t+b_1+1)(t+b_2+1)(t+b_3+1)(t+1) S(t+1) \\ & - (t+a_1)(t+a_2)(t+a_3)(t-N) S(t) = \sum_{k=0}^2 \xi_k t^k, \end{aligned}$$

where

$$\begin{aligned}\xi_2 &= [e_1(\mathbf{b} + 1) - e_1(\mathbf{a})] \nu_0, & \xi_1 &= [e_2(\mathbf{b} + 1) - e_2(\mathbf{a})] \nu_0 + [e_1(\mathbf{b} + 1) - e_1(\mathbf{a}) - 1] \nu_1, \\ \xi_0 &= [e_3(\mathbf{b} + 1) - e_3(\mathbf{a})] \nu_0 + [e_2(\mathbf{b} + 1) - e_2(\mathbf{a}) - e_1(\mathbf{a}) - 1] \nu_1 + [e_1(\mathbf{b} + 1) - e_1(\mathbf{a}) - 2] \nu_2,\end{aligned}$$

with  $a_4 = -N$ .

If we use the relations

$$e_1(\mathbf{a}, -N) = e_1(\mathbf{a}) - N, \quad e_2(\mathbf{a}, -N) = e_2(\mathbf{a}) - Ne_1(\mathbf{a}), \quad e_3(\mathbf{a}, -N) = e_3(\mathbf{a}) - Ne_2(\mathbf{a}),$$

we obtain expressions for the coefficients  $\xi_k$  explicitly depending on  $N$

$$\begin{aligned}\xi_2 &= [e_1(\mathbf{b} + 1) - e_1(\mathbf{a}) + N] \nu_0, \\ \xi_1 &= [e_2(\mathbf{b} + 1) - e_2(\mathbf{a}) + Ne_1(\mathbf{a})] \nu_0 + [e_1(\mathbf{b} + 1) - e_1(\mathbf{a}) + N - 1] \nu_1, \\ \xi_0 &= [e_3(\mathbf{b} + 1) - e_3(\mathbf{a}) + Ne_2(\mathbf{a})] \nu_0 + [e_2(\mathbf{b} + 1) - e_2(\mathbf{a}) + (N - 1)(e_1(\mathbf{a}) + 1)] \nu_1 \\ &\quad + [e_1(\mathbf{b} + 1) - e_1(\mathbf{a}) + N - 2] \nu_2.\end{aligned}$$

### 6.7.1 Uvarov Hahn polynomials

Let  $\omega \notin \{-a, -b, 0, N\}$ .

Special values

$$\begin{aligned}a_1 &\rightarrow a, & a_2 &\rightarrow -\omega, & a_3 &\rightarrow -\omega + 1, \\ b_1 &\rightarrow b, & b_2 &\rightarrow -\omega - 1, & b_3 &\rightarrow -\omega.\end{aligned}$$

Linear functional

$$L_U[r] = \sum_{x=0}^N r(x) \frac{(a)_x (-N)_x}{(b+1)_x} \frac{1}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{(x+a)(x-\omega)(x-\omega+1)(x-N)}{(x+b+1)(x-\omega)(x-\omega+1)(x+1)}.$$

Moments

$$\nu_n^U = \frac{(-N)_n (a)_n (b+1-a)_{N-n}}{(b+1)_n (b+1+n)_{N-n}} + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned}
& (t - \omega)(t - \omega + 1) [(t + b + 1)(t + 1) S_U(t + 1) - (t + a)(t - N) S_U(t)] \\
&= [(N - a + b + 1)t^2 + (N - a + b - 2\omega - 2N\omega + 2a\omega - 2b\omega + Na + 1)t] \nu_0^{U\zeta} \\
&+ (a\omega - N\omega - \omega - b\omega + \omega^2 + N\omega^2 - a\omega^2 + b\omega^2 + Na - 2Na\omega) \nu_0^{U\zeta} \\
&+ [(N - a + b)t + 2N - 2a + b - 2N\omega + 2a\omega - 2b\omega + Na - 1] \nu_1^U + (-a - 1 + b + N) \nu_2^U.
\end{aligned}$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned}
& (t + b + 1)(t + 1) S_U(t + 1) - (t + a)(t - N) S_U(t) \\
&= (b + 1 - a + N) \nu_0 + M \left[ \frac{(t + b + 1)(t + 1)}{t + 1 - \omega} - \frac{(t + a)(t - N)}{t - \omega} \right].
\end{aligned}$$

### 6.7.2 Symmetrized generalized Hahn polynomials of type I

Special values

$$\begin{aligned}
a_3 &\rightarrow -b - N, & b_1 &\rightarrow -a_1 - N, & b_2 &\rightarrow -a_2 - N, \\
b_3 &\rightarrow b, & N &\rightarrow 2m, & x &\rightarrow x + m.
\end{aligned}$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(a_1 + m)_x (a_2 + m)_x (-b - m)_x (-m)_x}{(-a_1 - m + 1)_x (-a_2 - m + 1)_x (b + 1 + m)_x (m + 1)_x} \frac{1}{(m + 1)_x},$$

where

$$\varrho_\zeta(0) = \binom{2m}{m} \frac{(a_1)_m (a_2)_m (b + 1 + m)_m}{(a_1 + m)_m (a_2 + m)_m (b + 1)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x + 1)}{\varrho_\zeta(x)} = \frac{(x + a_1 + m)(x + a_2 + m)(x - b - m)(x - m)}{(x + 1 - a_1 - m)(x + 1 - a_2 - m)(x + 1 + b + m)(x + 1 + m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$ .

For  $n \in \mathbb{N}_0$

$$\nu_n^\zeta = \frac{(a_1)_n (a_2)_n (-b-2m)_n (-2m)_n}{(-a_1-2m+1)_n (-a_2-2m+1)_n (b+1)_n} \\ \times {}_4F_3 \left[ \begin{matrix} -2m+n, a_1+n, a_2+n, -b-2m+n \\ -a_1-2m+1+n, -a_2-2m+1+n, b+1+n \end{matrix} ; 1 \right].$$

Stieltjes transform difference equation

$$(t+1-a_1-m)(t+1-a_2-m)(t+1+b+m)(t+1+m)S_\zeta(t+1) \\ - (t+a_1+m)(t+a_2+m)(t-b-m)(t-m)S_\zeta(t) \\ = [2(b-a_1-a_2+1)(t+1-m)t+t(t+1)]\nu_0^\zeta \\ + [b-m-2bm-m^2+1-(b+1)(a_1+a_2)+(2b+4m+1)a_1a_2]\nu_0^\zeta \\ + [(2t+3-4m)(b-a_1-a_2)+2(t+1-m)]\nu_1^\zeta + (2b+1-2a_1-2a_2)\nu_2^\zeta.$$

### 6.7.3 Reduced-Uvarov generalized Hahn polynomials of type II

Let  $\{\omega, \Omega\} \in \{(-a_1, a_1+1), (-a_2, a_2+1), (-b_1, b_1), (-b_2, b_2), (0, 0), (N, -N+1)\}$ .

Special values

$$a_3 \rightarrow \Omega, \quad b_3 \rightarrow \Omega - 1.$$

Linear functional

$$L_U[r] = \sum_{x=0}^N r(x) \frac{(-N)_x (a_1)_x (a_2)_x}{(b_1+1)_x (b_2+1)_x x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_U(x+1)}{\varrho_U(x)} = \frac{(x+a_1)(x+a_2)(x+\Omega)(x-N)}{(x+b_1+1)(x+b_2+1)(x+\Omega)(x+1)}.$$

Moments

$$\nu_n^U = \frac{(-N)_n (a_1)_n (a_2)_n}{(b_1+1)_n (b_2+1)_n} {}_3F_2 \left[ \begin{matrix} -N+n, a_1+n, a_2+n \\ b_1+1+n, b_2+1+n \end{matrix} ; 1 \right] + M\phi_n(\omega), \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$(t+\Omega)[(t+b_1+1)(t+b_2+1)(t+1)S_U(t+1) - (t+a_1)(t+a_2)(t-N)S_U(t)] \\ = [(N-a_1-a_2+b_1+b_2+2)(t+\Omega)t + (b_1+1)(b_2+1)t + N(a_1+a_2)t - a_1a_2t]\nu_0^U \\ + [(b_1+1)(b_2+1)\Omega + a_1a_2(N-\Omega) + (a_1+a_2)\Omega N]\nu_0^U \\ + [(N-a_1-a_2+b_1+b_2+1)(t+\Omega) + N+b_1+b_2+b_1b_2 + (N-1)(a_1+a_2) - a_1a_2]\nu_1^U \\ + (N+b_1+b_2-a_1-a_2)\nu_2^U.$$

Stieltjes transform  $M$ -dependent difference equation

$$\begin{aligned}
& (t + b_1 + 1)(t + b_2 + 1)(t + 1)S_U(t + 1) - (t + a_1)(t + a_2)(t - N)S_U(t) \\
& = [(N + 2 + b_1 + b_2 - a_1 - a_2)t + b_1 + b_2 + b_1b_2 + (N - 1)(a_1 + a_2) - a_1a_2 + 1]\nu_0 \\
& + (N + b_1 + b_2 - a_1 - a_2 + 1)\nu_1 \\
& + M \left[ \frac{(t + b_1 + 1)(t + b_2 + 1)(t + 1)}{t - \omega + 1} - \frac{(t + a_1)(t + a_2)(t - N)}{t - \omega} \right].
\end{aligned}$$

#### 6.7.4 Christoffel generalized Hahn polynomials of type II

Let  $\omega \notin \left\{ -a_1, -a_2, -b_1, -b_2, 0, N, \frac{\nu_1}{\nu_0} \right\}$ .

Special values

$$a_3 \rightarrow -\omega + 1, \quad b_3 \rightarrow -\omega - 1.$$

Linear functional

$$L_C[r] = \sum_{x=0}^{\infty} r(x)(x - \omega) \frac{(a_1)_x (a_2)_x (-N)_x}{(b_1 + 1)_x (b_2 + 1)_x} \frac{1}{x!}.$$

Pearson equation

$$\frac{\varrho_C(x + 1)}{\varrho_C(x)} = \frac{(x + a_1)(x + a_2)(x - N)(x + 1 - \omega)}{(x + b_1 + 1)(x + b_2 + 1)(x + 1)(x - \omega)}.$$

Moments

$$\nu_n^C(z) = \nu_{n+1} + (n - \omega)\nu_n, \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned}
& (t - \omega)(t + b_1 + 1)(t + b_2 + 1)(t + 1)S_C(t + 1) - (t - \omega + 1)(t + a_1)(t + a_2)(t - N)S_C(t) \\
& = -[\omega(N + b_1 + b_2 + 1) - (\omega - 1)(a_1 + a_2)]t^2\nu_0 + \omega(\omega - 1)(N + b_1 + b_2 + 1)t\nu_0 \\
& + [(-N + \omega)a_1a_2 + (3\omega - N\omega - \omega^2 - 1)(a_1 + a_2) + \omega(\omega - b_1b_2)]t\nu_0 \\
& + [\omega(\omega - 1)(N - 1)(a_1 + a_2) + (\omega - 1)(N - \omega)a_1a_2 + \omega^2(b_2 + 1)(b_1 + 1)]\nu_0 \\
& + (N - a_1 - a_2 + b_1 + b_2 + 1)(t - \omega)(t - \omega + 1)\nu_1.
\end{aligned}$$

### 6.7.5 Geronimus generalized Hahn polynomials of type II

Let  $\omega \notin \{-b_1 - 1, -b_2 - 1, -1, 1 - a_1, 1 - a_2, 1 + N\} \cup \mathbb{N}_0$ .

Special values

$$a_3 \rightarrow -\omega, \quad b_3 \rightarrow -\omega.$$

Linear functional

$$L_G[r] = \sum_{x=0}^{\infty} \frac{r(x)}{x - \omega} \frac{(a_1)_x (a_2)_x (-N)_x}{(b_1 + 1)_x (b_2 + 1)_x} \frac{1}{x!} + Mr(\omega).$$

Pearson equation

$$\frac{\varrho_G(x+1)}{\varrho_G(x)} = \frac{(x+a_1)(x+a_2)(x-N)(x-\omega)}{(x+b_1+1)(x+b_2+1)(x+1)(x+1-\omega)}.$$

Moments

$$\nu_n^G(z) = \phi_n(\omega) \left[ M - S(\omega) + \sum_{k=0}^{n-1} \frac{\nu_k}{\phi_{k+1}(\omega)} \right], \quad n \in \mathbb{N}_0.$$

Stieltjes transform difference equation

$$\begin{aligned} & (t - \omega + 1)(t + b_1 + 1)(t + b_2 + 1)(t + 1)S_G(t + 1) - (t - \omega)(t + a_1)(t + a_2)(t - N)S_G(t) \\ &= [(N + 2 + b_1 + b_2 - a_1 - a_2)t + (b_1 + 1)(b_2 + 1) + (N - 1)(a_1 + a_2) - a_1a_2]\nu_0 \\ &+ (N + b_1 + b_2 - a_1 - a_2 + 1)\nu_1 \\ &+ [(N - a_1 - a_2 + b_1 + b_2 + 3)t^2 + (2b_1 + 2b_2 + Na_1 + Na_2 - a_1a_2 + b_1b_2 + 3)t]\nu_0^G \\ &+ (b_1 + b_2 + b_1b_2 + Na_1a_2 + 1)\nu_0^G. \end{aligned}$$

### 6.7.6 Symmetrized polynomials of type (1, 2)

Special values

$$a_1 \rightarrow a, \quad a_2 \rightarrow -b_1 - N, \quad a_3 \rightarrow -b_2 - N, \quad b_3 \rightarrow -a - N, \quad N \rightarrow 2m, \quad x \rightarrow x + m.$$

Linear functional

$$L_\zeta[r] = \varrho_\zeta(0) \sum_{x=-m}^m r(x) \frac{(a+m)_x (-b_1-m)_x (-b_2-m)_x (-m)_x}{(b_1+1+m)_x (b_2+1+m)_x (-a-m+1)_x (m+1)_x} \frac{1}{x!},$$

where

$$\varrho_\zeta(0) = \binom{2m}{m} \frac{(a)_m (b_1 + 1 + m)_m (b_1 + 1 + m)_m}{(a + m)_m (b_1 + 1)_m (b_2 + 1)_m}.$$

Pearson equation

$$\frac{\varrho_\zeta(x+1)}{\varrho_\zeta(x)} = \frac{(x+a+m)(x-b_1-m)(x-b_2-m)(x-m)}{(x+b_1+1+m)(x+b_2+1+m)(x+1-a-m)(x+1+m)}.$$

Moments on the basis  $\{\bar{\phi}_n(x)\}_{n \geq 0}$

$$\begin{aligned} \nu_n^\zeta &= \frac{(-2m)_n (a)_n (-b_1 - 2m)_n (-b_2 - 2m)_n}{(1-a-2m)_n (b_1+1)_n (b_2+1)_n} \\ &\times {}_4F_3 \left[ \begin{matrix} -2m+n, a+n, -b_1-2m, -b_2-2m+n \\ 1-a-2m+n, b_1+1+n, b_2+1+n \end{matrix} ; 1 \right], \quad n \in \mathbb{N}_0. \end{aligned}$$

Stieltjes transform difference equation

$$\begin{aligned} &(t+1-a-m)(t+b_1+1+m)(t+b_2+1+m)(t+1+m)S_\zeta(t+1) \\ &\quad - (t+a+m)(t-b_1-m)(t-b_2-m)(t-m)S_\zeta(t) \\ &= [(4m+3+2b_1+2b_2-2a)t^2 + 2(m-1)(a-b_1-b_2)t] \nu_0^\zeta \\ &+ [(-4m^2+2m+3)t - (18a+4ab_1+4ab_2+1)m - (b_1+b_2+1)(a-1) - b_1b_2(2a-1)] \nu_1^\zeta \\ &\quad + [(2t-4m+3)(b_1+b_2-a) + 2(t+2mt+1) + 4m(1-2m)] \nu_1^\zeta \\ &\quad + (4m+1+2b_1+2b_2-2a) \nu_2^\zeta. \end{aligned}$$

## 7 Conclusions

We have considered all solutions of the Pearson equation

$$\frac{\varrho(x+1)}{\varrho(x)} = \frac{\eta(x)}{\sigma(x+1)},$$

with  $\deg(\sigma) \leq 4$  and  $\deg(\sigma - \eta) \leq 3$ . We deduce 15 canonical cases, from which 42 subcases can be obtained and we relate those to rational spectral transformations (Uvarov, Christoffel, and Geronimus), truncations, and symmetrizations.



For each family, we have listed a representation of the moments and the first order linear difference equations satisfied by their Stieltjes transform.

Extensions to the class  $s = 3$  are possible, but the complexity of the formulas will demand new forms of notation in order to be able to type the results in a comprehensive way.

## Acknowledgements

This work of the first author was done while visiting the Johannes Kepler Universität Linz and supported by the strategic program "Innovatives OÖ– 2010 plus" from the Upper Austrian Government. The second author acknowledges financial support of Dirección general de Investigación, Ministerio de Economía, Industria y Competitividad of Spain, grant MTM2015-65888. C4-2-P.

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