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Mehler-Heine type formulas for the Krawtchouk polynomials

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Abstract

We derive Mehler–Heine type asymptotic expansions for the Krawtchouk polynomials. These formulas provide good approximations for the polynomials in the neighborhood of $x = 0$, and determine the asymptotic limit of their zeros as the degree n goes to infinity.

Keywords: Mehler-Heine formulas, discrete orthogonal polynomials.
MSC-class: 41A30 (Primary), 33A65, 33A15, 44A15 (Secondary)

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1 Introduction

Let $\mathbb{N}_0$ denote the set

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = 0, 1, 2, \ldots.$$

The monic Krawtchouk polynomials are defined by [54, 18.20.6]

$$K_n(x; p, N) = p^n (-N)_n \; {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} ; \frac{1}{p} \right), \tag{1}$$

where $p \in [0, 1], \quad n, N \in \mathbb{N}_0, \quad n \leq N, \; {}_rF_s$ denotes the generalized hypergeometric function [54, Chapter 16]

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$

and $(x)_k$ the Pochhammer symbol (or rising factorial) [54, 5.2.4],

$$(x)_z = \frac{\Gamma(x+z)}{\Gamma(x)}, \quad x, z \in \mathbb{C}, \tag{2}$$

where in general we need $-(x+z) \notin \mathbb{N}_0$. Note that when $x = 0$, we have

$$K_n(0; p, N) = p^n (-N)_n. \tag{3}$$

If we set $p = 1$ in (1) and use the Chu–Vandermonde identity [54, 15.4.24],

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; 1 \right) = \frac{(c-b)_n}{(c)_n}, \tag{4}$$

we get

$$K_n(x; 1, N) = (x - N)_n. \tag{5}$$

Using the identity [54, 15.8.7]

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z \right) = z^n \frac{(c-b)_n}{(c)_n} \; {}_2F_1\left(\begin{matrix} -n, 1-c-n \\ 1+b-c-n \end{matrix} ; 1 - z^{-1} \right)$$

in (1), we obtain

$$K_n(x; p, N) = (x - N)_n \; {}_2F_1\left(\begin{matrix} -n, 1+N-n \\ 1+N-n-x \end{matrix} ; 1 - p \right). \tag{6}$$

Setting $p = 0$ in (6) and using (4), we get

$$K_n(x; 0, N) = (x - N)_n \frac{(-x)_n}{(1 + N - n - x)_n},$$

and using the identity [53, 18:5:1]

$$(-x)_n = (-1)^n (x + 1 - n)_n, \tag{7}$$

we conclude that

$$K_n(x; 0, N) = (-1)^n (-x)_n = (x - n + 1)_n. \tag{8}$$

The Krawtchouk polynomials are one of the families of discrete classical orthogonal polynomials [52]. They satisfy the orthogonality relation

$$\sum_{x=0}^{N} K_n(x; p, N) K_m(x; p, N) \binom{N}{x} p^x (1 - p)^{N-x} = (n!)^2 \binom{N}{n} [p(1 - p)]^n \delta_{n,m},$$

the three-term recurrence relation

$$xK_n = K_{n+1} + (Np + n - 2np) K_n + np(1 - p)(N - n + 1) K_{n-1},$$

and have the generating function

$$\sum_{n=0}^{\infty} K_n(x; p, N) \frac{t^n}{n!} = [1 + (1 - p)t]^x (1 - pt)^{N-x},$$

from which we obtain the symmetry relation

$$K_n(x; p, N) = (-1)^n K_n(N - x; 1 - p, N).$$

The Krawtchouk polynomials are important in the study of the Hamming scheme of classical coding theory [35], [39], [46], [55], [60], [62]. Lloyd's theorem [42] states that if a perfect code exists in the Hamming metric, then the Krawtchouk polynomial must have integral zeros [5], [10], [38]. Not surprisingly, the zeros of $K_n(x; p, N)$ have been the subject of extensive research [3], [8], [15], [26], [27], [29], [32], [36], [66], [72].

The Krawtchouk polynomials also have applications in probability theory [19], queueing models [14], stochastic processes [56], quantum mechanics [4], [43], [71], face recognition systems [1], combinatorics [18], and biology [31].

Multivariate [17], [20], [22], [23], [50], [58], [59], [68], and q extensions [6], [7], [21], [24], [25], [34], [37], [61], [65], [63], [64], have also been considered.

The asymptotic behavior of the Krawtchouk polynomials have been studied by many authors, including [9], [13], [30], [40], [49], [57], [67].

In this article, we focus on a very special type of asymptotic analysis in a region around the smallest zero of the Krawtchouk polynomials. These so-called Mehler–Heine type formulas were introduced by Heinrich Heine in 1861 [28] and Gustav Mehler [48] in 1868 to analyze the asymptotic behavior of Legendre polynomials. See Watson's book [69, 5.71] for some historical remarks. Mehler–Heine type formulas are very important in the field of Sobolev orthogonal polynomials, see [2], [41], [44], [45], [51], [47].

In [12], we studied Mehler–Heine type formulas for the Charlier and Meixner polynomials, and extended our results to full asymptotic expansions in [11]. Although it seems that one could apply these results to the Krawtchouk polynomials using the relation

$$K_n(x;p,N) = M_n\left(x;\frac{p}{p-1},-N\right),$$

where $M_n(x;z,a)$ denotes the monic Meixner polynomials defined by [54, 18.20.7]

$$M_n(x;z,a) = (a)_n \left(\frac{z}{z-1}\right)^n {}_2F_1\left(\begin{matrix} -n,-x \\ a \end{matrix};1-\frac{1}{z}\right),$$

we see that this presents many problems because for the Meixner polynomials (i) $z \in (0,1)$, (ii) $a > 0$, and (iii) $a = O(1)$, while for the Krawtchouk polynomials we need (i') $z \in (-\infty, 0)$, (ii') $a < 0$, and (iii') $|a| > n$.

Thus, we should use a different approach based on the asymptotic analysis of the differential equation satisfied by $K_n(x;p,N)$ as a function of p. A similar idea was followed by Dunster in [16] to study the asymptotic behavior of the Charlier polynomials.

2 Asymptotic approximation

Using the identity [54, 15.8.1]

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = (1-z)^{c-a-b}\ {}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix};z\right)$$

in (6), we obtain

$$\frac{K_n(x;p,N)}{(x-N)_n \; p^{n-x}} = {}_2F_1\left(\begin{array}{c} N-x+1,-x \\ N-n-x+1 \end{array};1-p\right).$$

If we set $n = rN, r = O(1)$ and let $N \to \infty$, we obtain [33, 1.4.5]

$$\begin{aligned} \lim_{N\to\infty} \frac{K_n(x;p,N)}{(x-N)_n \; p^{n-x}} &= \lim_{N\to\infty} {}_2F_1\left(\begin{array}{c} N-x+1,-x \\ N-rN-x+1 \end{array};1-p\right) \\ &= {}_1F_0\left(\begin{array}{c} -x \\ - \end{array};\frac{1-p}{1-r}\right) = \left(1-\frac{1-p}{1-r}\right)^x. \end{aligned}$$

Thus, if $0 \leq r < p \leq 1$,

$$\frac{K_n(x;p,N)}{(x-N)_n \; p^{n-x}} \sim \left(\frac{p-r}{1-r}\right)^x, \quad N \to \infty. \tag{9}$$

From (1) we see that $K_n(x;p,N)$ satisfies the ODE

$$p(1-p)\frac{d^2 f}{dp^2} + [x-n+1-(N-2n+2)p]\frac{df}{dp} + n(N-n+1)f = 0, \tag{10}$$

while (5) and (8) give the boundary conditions

$$f(0) = (x-n+1)_n, \quad f(1) = (x-N)_n.$$

Setting

$$n = rN, \quad r = O(1), \quad 0 < r < 1, \quad N = \varepsilon^{-1}, \quad \varepsilon > 0,$$

in (10), we have

$$\varepsilon^2 p(1-p)\frac{d^2 f}{dp^2} + \varepsilon[(x+1-2p)\varepsilon + 2pr - r - p]\frac{df}{dp} + r(\varepsilon+1-r)f = 0. \tag{11}$$

Replacing

$$f(p) = \exp\left[\varepsilon^{-1}\psi(p)\right]\xi(p;x) \tag{12}$$

in (11) we obtain, to leading order,

$$[(p-1)\psi'(p) + 1 - r][p\psi'(p) - r] = 0,$$

with solutions

$$\psi_1(p) = r\ln(p) + C_1, \quad p > 0, \tag{13}$$

and

$$\psi_0(p) = (r-1)\ln(1-p) + C_0, \quad p < 1. \tag{14}$$

Using (12) and (13) in (11), we get

$$\varepsilon p^2 (1-p)\xi_1'' + [\varepsilon(x+1-2p) + r - p]\, p\xi_1' + rx\xi_1 = 0, \tag{15}$$

and using (12) and (14) in (11), we obtain

$$\begin{gathered}\varepsilon p(1-p)^2 \xi_0'' + [\varepsilon(x+1-2p) + p - r](1-p)\xi_0' \\ + (x+1-rx-p)\xi_0 = 0.\end{gathered} \tag{16}$$

It is clear from the analysis so far that the solution

$$f_1(p) = (x-N)_n \exp\left[\varepsilon^{-1}\psi_1(p)\right]\xi_1(p;x), \tag{17}$$

should satisfy the boundary condition at $p = 1$, and

$$f_0(p) = (x-n+1)_n \exp\left[\varepsilon^{-1}\psi_0(p)\right]\xi_0(p;x) \tag{18}$$

the boundary condition at $p = 0$. Hence, as $N \to \infty$

$$\begin{gathered}K_n(x;p,N) \sim k_0(x)(x-n+1)_n (1-p)^{n-N}\xi_0(p;x) \\ +k_1(x)(x-N)_n \; p^n \xi_1(p;x), \quad 0 \le p < r,\end{gathered} \tag{19}$$

and

$$K_n(x;p,N) \sim (x-N)_n \; p^n \xi_1(p;x), \quad r < p \le 1, \tag{20}$$

where

$$\xi_0(0;x) = 1, \quad \xi_1(1;x) = 1, \tag{21}$$

and $k_0(x), k_1(x)$ are unknown functions. In order to satisfy the condition (3), we need

$$k_1(0) = \xi_1(p;0) = 1. \tag{22}$$

2.1 The function ξ_1

In this section, we assume that $r < p \leq 1$. If we let $\varepsilon \to 0$ in (15), we get

$$(r-p)\, py' + r\,(p-r+x)\, y = 0,$$

with solution

$$y\,(p) = C\left(1-\frac{r}{p}\right)^{x}.$$

Using (21) and (22), we see that

$$\xi_1\,(p;x) \sim (1-r)^{-x}\left(1-\frac{r}{p}\right)^{x}, \quad \varepsilon \to 0,$$

in agreement with (9).

To find higher order terms, we set

$$\xi_1\,(p;x) = (1-r)^{-x}\left(1-\frac{r}{p}\right)^{x} g\,(p)\,, \quad g\,(1) = 1,$$

in (15) and obtain the ODE

$$\begin{aligned}\varepsilon p\,(1-p)\,(p-r)^2 g'' + \varepsilon\,(p-r)\left(p-r+2pr+px+rx-2p^2-2prx\right) g' \\ -\,(p-r)^3\, g' + \varepsilon r\,(1-r)\, x\,(x-1)\, g = 0.\end{aligned}$$

If we fix some $m \in \mathbb{N}$ and define

$$g\,(p) = \sum_{k=0}^{m} g_k\,(p)\, \varepsilon^k, \quad g_0\,(p) = 1, \tag{23}$$

then

$$(p-r)^3 \frac{dg_k}{dp} = U_p g_{k-1}, \quad g_k\,(1) = 0, \quad k \in \mathbb{N}, \tag{24}$$

where U_p denotes the differential operator

$$\begin{aligned}U_p &= p\,(1-p)\,(p-r)^2 \frac{d^2}{dp^2} \\ &\quad + (p-r)\left(p-r+2pr+px+rx-2p^2-2prx\right)\frac{d}{dp} \\ &\quad + r\,(1-r)\, x\,(x-1)\,.\end{aligned}$$

Solving (24) with initial condition $g_k(1) = 0$, we get

$$g_k(p) = \int_1^p \frac{U_s g_{k-1}(s)}{(s-r)^3} ds, \quad k \in \mathbb{N}. \tag{25}$$

For $k = 1$, we have

$$U_p g_0 = r(1-r)x(x-1),$$

and therefore

$$g_1(p) = \int_1^p \frac{r(1-r)x(x-1)}{(s-r)^3} ds = \frac{rx(x-1)(p-1)(p-2r+1)}{2(1-r)(p-r)^2}. \tag{26}$$

The next functions $g_k(p)$, $k \geq 2$ can be easily computed using (25), but the expressions become increasingly cumbersome, so we shall not list them here.

Since from (7) we have

$$(x-N)_n = (-1)^n \frac{\Gamma(N-x+1)}{\Gamma(N-x-n+1)}, \tag{27}$$

we can use Stirling's formula [54, 5.11.1]

$$\ln\Gamma(z) \sim z[\ln(z) - 1] - \frac{1}{2}\ln(z) + \frac{1}{2}\ln(2\pi) + \frac{1}{12z}, \quad z \to \infty, \tag{28}$$

and obtain

$$(-1)^n (x-N)_n \sim (1-r)^{x-\frac{1}{2}} \exp\left[\frac{(r-1)\ln(1-r) - (1+\ln\varepsilon)r}{\varepsilon}\right].$$

Using this approximation in (9) gives

$$(-1)^n K_n(x;p,N) \sim \frac{1}{\sqrt{1-r}} \exp\left[\frac{\tau_1(p)}{\varepsilon}\right] \left(1 - \frac{r}{p}\right)^x, \quad \varepsilon \to 0, \tag{29}$$

where

$$\tau_1(p) = (r-1)\ln(1-r) + [\ln(p) - \ln(\varepsilon) - 1] r. \tag{30}$$

2.2 The function ξ_0

In this section, we assume that $0 \leq p < r$. If we let $\varepsilon \to 0$ in (16), we get

$$(p-r)(1-p)\,y' + [(1-r)\,x + 1 - p]\,y = 0,$$

with solution

$$y(p) = C(1-p)^{x}(r-p)^{-x-1}.$$

Using (21), we see that

$$\xi_0(p;x) \sim (1-p)^{x}\left(1-\frac{p}{r}\right)^{-x-1}, \quad \varepsilon \to 0.$$

To find higher order terms, we set

$$\xi_0(p;x) = (x-n+1)_n\ (1-p)^{x}\left(1-\frac{p}{r}\right)^{-x-1} h(p), \quad h(0) = 1,$$

in (16) and obtain the ODE

$$\varepsilon p(1-p)(p-r)^2 h'' + \varepsilon(p-r)(x+1)(2pr-p-r)\,h' + (p-r)^3 h' \\ + \varepsilon(x+1)\left(rx+p+r-r^2x-2pr\right)h = 0.$$

If we fix $m \in \mathbb{N}$, and define

$$h(p) = \sum_{k=0}^{m} h_k(p)\,\varepsilon^k, \quad h_0(p) = 1, \tag{31}$$

then

$$(r-p)^3 \frac{dh_k}{dp} = V_p h_{k-1}, \quad h_k(0) = 0, \quad k \in \mathbb{N}, \tag{32}$$

where V_p denotes the differential operator

$$V_p = p(1-p)(p-r)^2 \frac{d^2}{dp^2} + (p-r)(x+1)(2pr-p-r)\frac{d}{dp} \\ + (x+1)\left(p-2pr+r+rx-r^2x\right).$$

Solving (32) with initial condition $h_k(0) = 0$, we get

$$h_k(p) = \int_0^p \frac{V_s h_{k-1}(s)}{(r-s)^3}\,ds, \quad k \in \mathbb{N}. \tag{33}$$

For $k=1$, we have

$$V_p h_0 = (x+1)\left(p - 2pr + r + rx - r^2 x\right),$$

and therefore

$$h_1(p) = (x+1)\int_0^p \frac{s - 2sr + r + rx - r^2 x}{(r-s)^3} ds$$
$$= \frac{p(x+1)}{(p-r)^2}\left(px\frac{r-1}{2r} - p - rx + x + 1\right).$$

Similarly, the functions $h_k(p)$, $k \geq 2$ can be easily computed using (33).

Since from (7) we have

$$(x-n+1)_n = (-1)^n \frac{\Gamma(n-x)}{\Gamma(-x)},$$

we can use Stirling's formula (28), and obtain

$$(-1)^n (x-n+1)_n \sim \frac{\sqrt{2\pi} r^{-x-\frac{1}{2}}}{\Gamma(-x)} \exp\left[\frac{(\ln r - \ln\varepsilon - 1) r}{\varepsilon} + \left(x + \frac{1}{2}\right)\ln(\varepsilon)\right].$$

Using this approximation in (19) gives

$$(-1)^n K_n(x;p,N) \sim \frac{k_1(x)}{\sqrt{1-r}} \exp\left[\frac{\tau_1(p)}{\varepsilon}\right]\left(1 - \frac{r}{p}\right)^x \tag{34}$$
$$+ k_0(x)\sqrt{2\pi\varepsilon r}\frac{[\varepsilon(1-p)]^x}{\Gamma(-x)} \exp\left[\frac{\tau_0(p)}{\varepsilon}\right](r-p)^{-x-1},$$

where

$$\tau_0(p) = (r-1)\ln(1-p) + [\ln(r) - \ln(\varepsilon) - 1] r. \tag{35}$$

The formula (34) is undefined at $p = r$ for all $x > -1$. Therefore, in the next section we shall find a new asymptotic approximation in the neighborhood of $p = r$.

2.3 The turning point at $p = r$

If we set

$$p = r + u\sqrt{a\varepsilon}, \quad a > 0, \quad u = O(1), \tag{36}$$

in (11), we have

$$\varepsilon\left[\varepsilon a u^2+\sqrt{a\varepsilon}\,(2r-1)\,u+r\,(r-1)\right]\frac{d^2 f}{du^2}$$
$$+\sqrt{\varepsilon}\left[2\varepsilon^{\frac{3}{2}}au+\varepsilon\sqrt{a}\,(2r-x-1)+\sqrt{\varepsilon}au\,(1-2r)+2\sqrt{a}r\,(1-r)\right]\frac{df}{du} \quad (37)$$
$$-\left(\varepsilon+1-r\right)^2 arf=0.$$

Replacing

$$f\,(u)=\exp\left[\frac{\phi\,(u)}{\sqrt{\varepsilon}}\right]\eta\,(u) \quad (38)$$

in (37) and letting $\varepsilon\to 0$, we get

$$\left(\phi'-\sqrt{a}\right)^2=0,$$

with solution

$$\phi\,(u)=u\sqrt{a}+C. \quad (39)$$

Using (38) and (39) in (37) and letting $\varepsilon\to 0$, we obtain

$$r\,(1-r)\,\eta''-a\left[u\,(2r-1)\,\eta'+\left(au^2+r-x-1\right)\eta\right]=0. \quad (40)$$

Setting $a=r\,(1-r)$ and

$$\eta\,(u)=\exp\left(\frac{2r-1}{4}u^2\right)\lambda\,(u)\,,$$

in (40), we see that $\lambda\,(u)$ satisfies

$$\lambda''=\left(\frac{u^2}{4}-x-\frac{1}{2}\right)\lambda,$$

whose linearly independent solutions are the parabolic cylinder functions $D_x\,(u)\,,D_x\,(-u)$ [70, 16.5].

Hence, we conclude that

$$\eta\,(u)=\exp\left(\frac{2r-1}{4}u^2\right)\left[C_p\,(x)\,D_x\,(u)+C_m\,(x)\,D_x\,(-u)\right], \quad (41)$$

where $C_p\,(x)$ and $C_m\,(x)$ are functions to be determined. Combining (38), (39), and (41), we get

$$f\,(u)=\left[C_p\,(x)\,D_x\,(u)+C_m\,(x)\,D_x\,(-u)\right]\exp\left[\tau_2\,(u)+\frac{u^2}{4}\right], \quad (42)$$

where

$$\tau_2(u) = u\sqrt{\frac{r(1-r)}{\varepsilon}} + \frac{1}{2}u^2(r-1).$$

Using (36) in (30), we obtain

$$\frac{\tau_1(p)}{\varepsilon} \sim \frac{\tau_1(r)}{\varepsilon} + \tau_2(u), \quad p \to r^+,$$

and since

$$1 - \frac{r}{p} = \frac{p-r}{p} \sim \frac{u}{r}\sqrt{r(1-r)\varepsilon}, \quad p \to r^+,$$

we see from (29) that

$$(-1)^n K_n(x;p,N) \sim \frac{1}{\sqrt{1-r}}\left(u\sqrt{\frac{1-r}{r}\varepsilon}\right)^x \times \exp\left[\frac{\tau_1(r)}{\varepsilon} + \tau_2(u)\right], \quad p \to r^+. \tag{43}$$

Using the asymptotic approximation [70, 16.5]

$$D_x(u) \sim \exp\left(-\frac{u^2}{4}\right)u^x, \quad u \to \infty,$$

in (42) and comparing with (43), we conclude that $C_m(x) = 0$ and

$$C_p(x) = \frac{1}{\sqrt{1-r}}\left(\frac{1-r}{r}\varepsilon\right)^{\frac{x}{2}} \exp\left[\frac{\tau_1(r)}{\varepsilon}\right].$$

Therefore,

$$(-1)^n K_n(x;p,N) \sim \frac{1}{\sqrt{1-r}}\left(\frac{1-r}{r}\varepsilon\right)^{\frac{x}{2}} \times \exp\left[\frac{\tau_1(r)}{\varepsilon} + \tau_2(u) + \frac{u^2}{4}\right] D_x(u), \quad \varepsilon \to 0. \tag{44}$$

Using (36) in (35), we obtain

$$\frac{\tau_0(p)}{\varepsilon} \sim \frac{\tau_0(r)}{\varepsilon} + \tau_2(u) + \frac{u^2}{2}, \quad p \to r^-,$$

and since

$$(1-p)^x (r-p)^{-x-1} \sim (1-r)^x \left(-u\sqrt{r(1-r)\varepsilon}\right)^{-x-1}, \quad p \to r^-,$$

we see from (34) that as $p \to r^-$

$$(-1)^n K_n(x;p,N) \sim \frac{k_1(x)}{\sqrt{1-r}} \left(\frac{1-r}{r}\varepsilon\right)^{\frac{x}{2}} \exp\left[\frac{\tau_1(r)}{\varepsilon} + \tau_2(u)\right] u^x \tag{45}$$
$$+\frac{\sqrt{2\pi}}{\Gamma(-x)} \frac{k_0(x)}{\sqrt{1-r}} \left(\frac{1-r}{r}\varepsilon\right)^{\frac{x}{2}} \exp\left[\frac{\tau_0(r)}{\varepsilon} + \tau_2(u) + \frac{u^2}{2}\right] (-u)^{-x-1}.$$

Using the asymptotic approximation [70, 16.52]

$$D_x(u) \sim \exp\left(-\frac{u^2}{4}\right)(-u)^x + \frac{\sqrt{2\pi}}{\Gamma(-x)} \exp\left(\frac{u^2}{4}\right)(-u)^{-x-1}, \quad u \to -\infty$$

in (44) and comparing with (45), we conclude that

$$k_1(x) = (-1)^x, \quad k_0(x) = 1, \tag{46}$$

since $\tau_1(r) = \tau_0(r)$. Using (46) in (19) gives

$$K_n(x;p,N) \sim (x-n+1)_n (1-p)^{n-N+x} \left(1-\frac{p}{r}\right)^{-x-1} h(p)$$
$$+(x-N)_n \, p^{n-x} \left(\frac{r-p}{1-r}\right)^x g(p), \quad 0 \leq p < r,$$

for $N \to \infty$, with $g(p)$, $h(p)$ defined by (23) and (31) respectively.

3 Summary of results

Theorem 1 *Let $K_n(x;p,N)$ be defined by (1), with $p \in [0,1]$, $x = O(1)$, $n = rN$, $r \in (0,1)$, and $\varepsilon = N^{-1}$. Then, as $\varepsilon \to 0$ we have the following asymptotic approximations:*

(i) *For $0 < r < p \leq 1$,*

$$(-1)^n K_n(x;p,N) \sim \frac{1}{\sqrt{1-r}} \exp\left[\frac{\tau_1(p)}{\varepsilon}\right] \left(1-\frac{r}{p}\right)^x \sum_{k=0}^{m} g_k(p)\varepsilon^k, \tag{47}$$

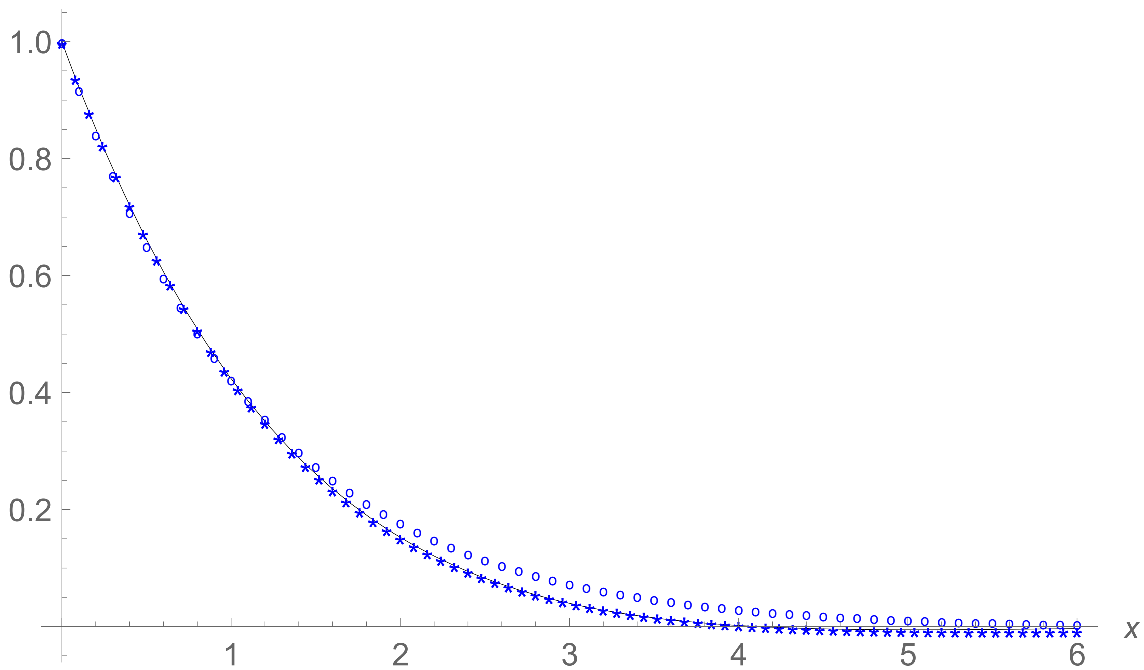

Figure 1: A plot of the scaled polynomial $K_{10}(x; 0.347, 50)$ (solid line), the one term approximation (ooo), and the two term approximation (***).

where

$$\tau_1(p) = (r-1)\ln(1-r) + [\ln(p) - \ln(\varepsilon) - 1] r,$$

$$g_0(p) = 1, \quad g_k(p) = \int_1^p \frac{U_s g_{k-1}(s)}{(s-r)^3} ds, \quad k \in \mathbb{N},$$

and the differential operator U_p *is defined by*

$$U_p = p(1-p)(p-r)^2 \frac{d^2}{dp^2} + (p-r)\left(p - r + 2pr + px + rx - 2p^2 - 2prx\right)\frac{d}{dp} + r(1-r)x(x-1).$$

In Figure 1, we plot $K_{10}(x; 0.347, 50)$ *and the approximation (47) for* $m = 0, 1$. *Both functions are divided by* $p^n(-N)_n \simeq 9.43 \times 10^{11}$. *Note that in this case* $r = 0.2 < p$ *and* $\varepsilon = 0.02$.

(ii) *For* $0 \le p < r < 1$,

$$(-1)^n K_n(x;p,N) \sim \frac{\sqrt{2\pi\varepsilon r}}{r-p} \frac{\varepsilon^x}{\Gamma(-x)} \exp\left[\frac{\tau_0(p)}{\varepsilon}\right] \times \left(\frac{1-p}{r-p}\right)^x \sum_{k=0}^{m} h_k(p)\,\varepsilon^k + \frac{1}{\sqrt{1-r}} \exp\left[\frac{\tau_1(p)}{\varepsilon}\right] \left(\frac{r}{p}-1\right)^x \sum_{k=0}^{m} g_k(p)\,\varepsilon^k, \tag{48}$$

where

$$\tau_0(p) = (r-1)\ln(1-p) + [\ln(r) - \ln(\varepsilon) - 1]\,r,$$

$$h_0(p) = 1, \quad h_k(p) = \int_0^p \frac{V_s h_{k-1}(s)}{(r-s)^3} ds, \quad k \in \mathbb{N},$$

and the differential operator V_p *is defined by*

$$V_p = p(1-p)(p-r)^2 \frac{d^2}{dp^2} + (p-r)(x+1)(2pr-p-r)\frac{d}{dp} + (x+1)\left(p - 2pr + r + rx - r^2 x\right).$$

In Figure 2, we plot $K_{40}(x;0.347,50)$ *and the approximation (47) for* $m = 0,1,2$. *All functions are divided by the scaling factor*

$$(-1)^n \Gamma(n-x)(1-p)^{x+n-N} \left(1-\frac{p}{r}\right)^{-x-1} \frac{\Gamma(x+1)}{\pi}.$$

Note that in this case $r = 0.8 > p$ *and* $\varepsilon = 0.02$.

(iii) *For* $p = r + O(\sqrt{\varepsilon})$,

$$(-1)^n K_n(x;p,N) \sim \frac{1}{\sqrt{1-r}} \left(\frac{1-r}{r}\varepsilon\right)^{\frac{x}{2}} \exp\left[\frac{\tau_1(r)}{\varepsilon} + \tau_2(u) + \frac{u^2}{4}\right] D_x(u), \tag{49}$$

where $D_x(u)$ *denotes the Parabolic Cylinder function,*

$$p = r + u\sqrt{r(1-r)\varepsilon}, \quad u = O(1),$$

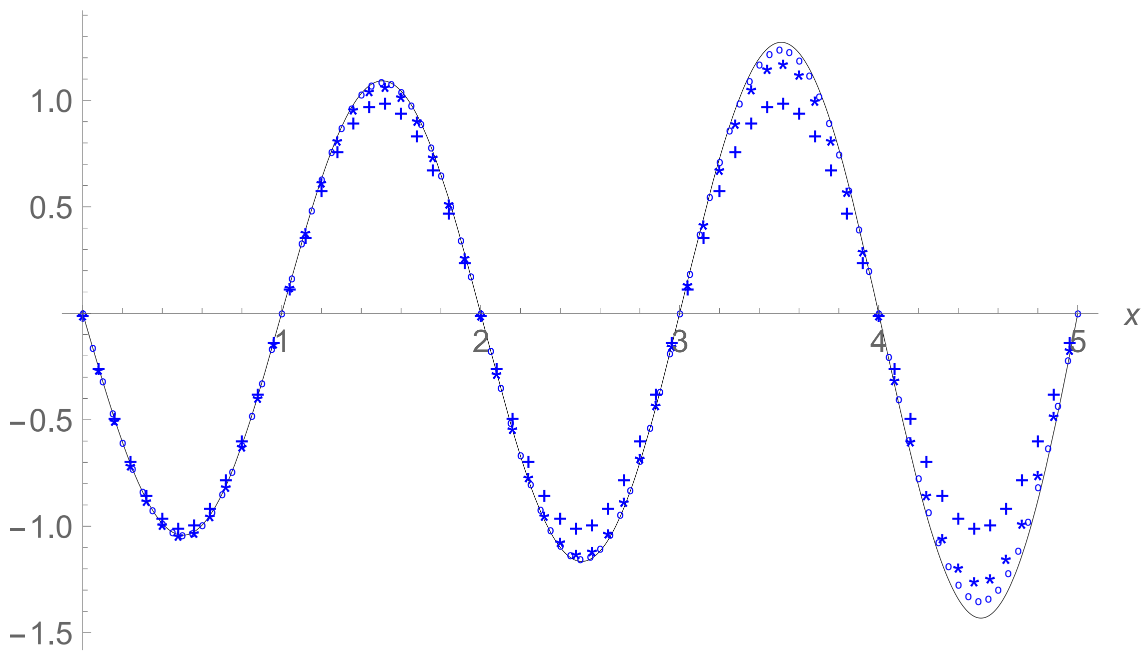

Figure 2: A plot of the scaled polynomial $K_{40}\,(x; 0.347, 50)$ (solid line), the one term approximation (+++), the two term approximation (***), and the three term approximation (ooo).

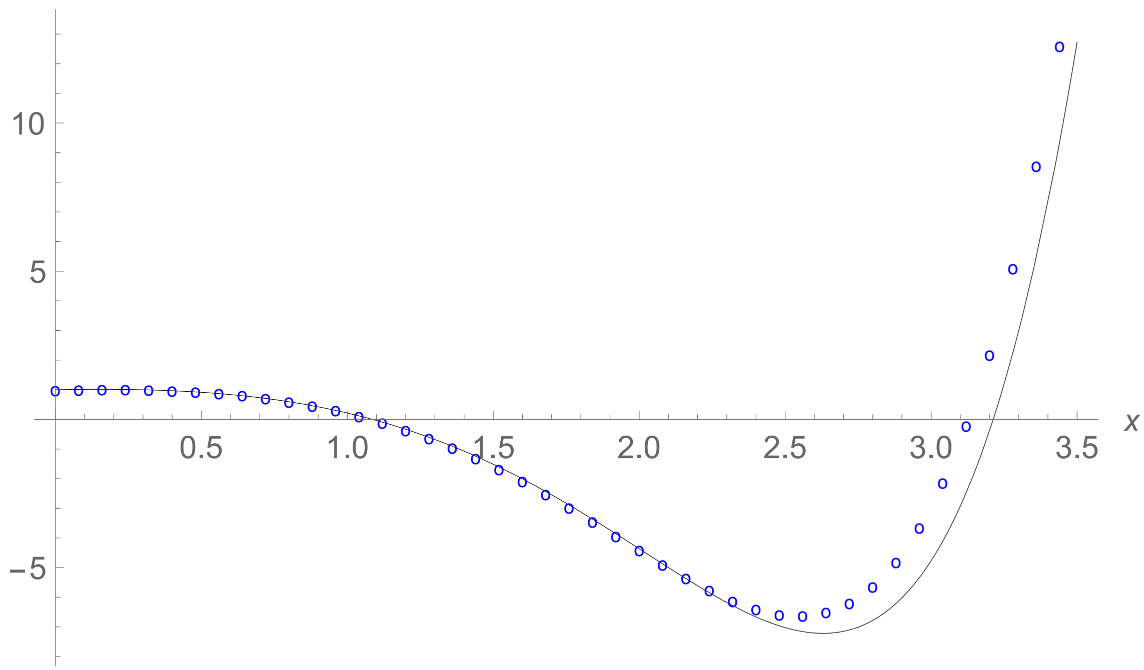

Figure 3: A plot of the scaled polynomial $K_{17}(x; 0.347, 50)$ (solid line) and its approximation (ooo).

and

$$\tau_2(u) = u\sqrt{\frac{r(1-r)}{\varepsilon}} + \frac{1}{2}u^2(r-1).$$

In Figure 3, we plot $K_{17}(x; 0.347, 50)$ *and the approximation (49). Both functions are divided by the scaling factor* $N^{\frac{x}{2}} p^n (x-N)_n$*. Note that in this case* $r = 0.34 < p$*,* $u = 0.1044$ *and* $\varepsilon = 0.02$*.*

In Figure 4, we plot $K_{18}(x; 0.347, 50)$ *and the approximation (49). Both functions are divided by the scaling factor* $N^{\frac{x}{2}} p^n (x-N)_n$*. Note that in this case* $r = 0.36 > p$*,* $u = -0.1915$ *and* $\varepsilon = 0.02$*.*

Remark 2 *Note that from (30) and (35) we see that*

$$\tau_1(p) - \tau_0(p) = (r-1)\ln\left(\frac{1-p}{1-r}\right) + r\ln\left(\frac{r}{p}\right) \leq 0, \quad r \in [0,1],$$

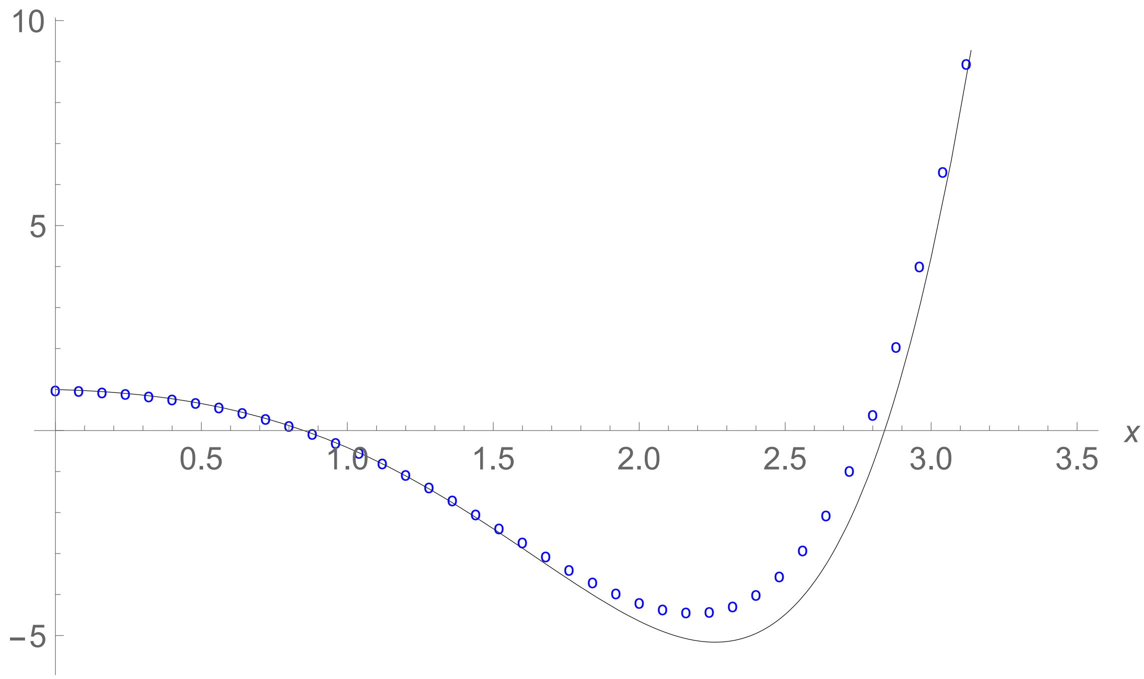

Figure 4: A plot of the scaled polynomial $K_{18}(x; 0.347, 50)$ (solid line) and its approximation (ooo).

with equality when $p = r$. *Thus, from (34) we obtain*

$$\exp\left[-\frac{\tau_0(p)}{\varepsilon}\right](-1)^n K_n(x;p,N) \sim \frac{\sqrt{2\pi\varepsilon r}}{r-p}\frac{\varepsilon^x}{\Gamma(-x)}\left(\frac{1-p}{r-p}\right)^x + \frac{1}{\sqrt{1-r}}\exp\left[\frac{\tau_1(p)-\tau_0(p)}{\varepsilon}\right]\left(\frac{r}{p}-1\right)^x, \quad \varepsilon \to 0, \tag{50}$$

and the second term is exponentially small, except when $x \to 0$.

In Figure 5, we plot $K_{40}(x;0.347,50)$, *the one term approximation*

$$(x-n+1)_n (1-p)^{n-N+x}\left(1-\frac{p}{r}\right)^{-x-1},$$

and the composite approximation

$$(x-n+1)_n (1-p)^{n-N+x}\left(1-\frac{p}{r}\right)^{-x-1} + (x-N)_n\ p^{n-x}\left(\frac{r-p}{1-r}\right)^x,$$

in the small interval $[0, 10^{-8}]$, *to show the need for the second term when* $x \to 0$. *All functions are divided by* $p^n(-N)_n \simeq 3.43 \times 10^{39}$.

We now have all the elements needed to state the Mehler–Heine type formulas for the Krawtchouk polynomials.

Corollary 3 *With the same definitions as in Theorem 1, we have:*

(i) *For* $0 < r < p \leq 1$,

$$\lim_{N\to\infty}\frac{K_n(x;p,N)}{(x-N)_n\ p^n} = \left[\frac{p-r}{(1-r)p}\right]^x.$$

(ii) *For* $0 \leq p < r < 1$,

$$\lim_{N\to\infty}\frac{(-1)^n K_n(x;p,N)}{\Gamma(n-x)\ (1-p)^{n-N}} = \frac{(1-p)^x}{\Gamma(-x)}\left(1-\frac{p}{r}\right)^{-x-1}.$$

(iii) *For* $p = r + O(\sqrt{\varepsilon})$,

$$\lim_{N\to\infty}\frac{N^{\frac{x}{2}}K_n(x;p,N)}{(x-N)_n\ p^n} = [r(1-r)]^{-\frac{x}{2}}\exp\left(\frac{u^2}{4}\right)D_x(u).$$

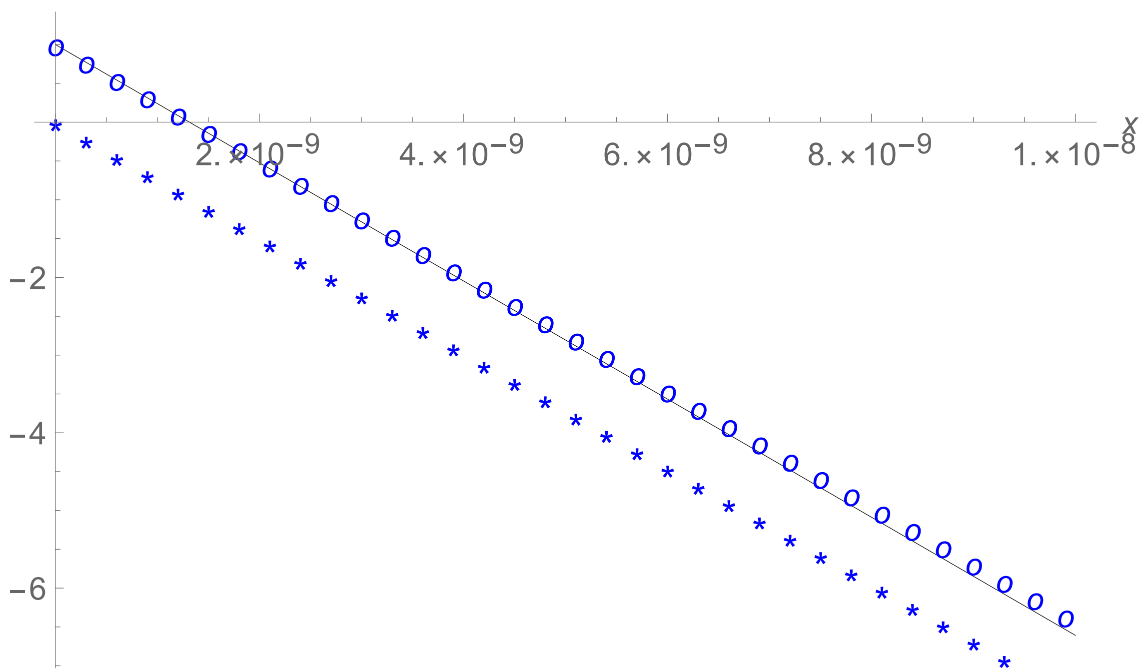

Figure 5: A plot of the scaled polynomial $K_{40}\left(x;0.347,50\right)$ (solid line), the one term approximation (***), and the composite approximation (ooo).

4 Zeros

If we set $x = 1$ in (1), we have

$$K_n(1; p, N) = p^n (-N)_n \left(1 - \frac{n}{Np}\right),$$

and hence

$$K_n(1; p, N) = 0, \quad n = Np. \tag{51}$$

Note that this agrees with (44), since [54, 12.7.2]

$$\exp\left(\frac{u^2}{4}\right) D_1(u) = u,$$

and therefore if $u = 0$ (i.e., $n = Np$), we get (51).

For the special case $p = \frac{1}{2}$ (the so called binary Krawtchouk polynomials), we can use the identity [54, 15.4.30]

$${}_2F_1\left(\begin{matrix} a, 1-a \\ b \end{matrix}; \frac{1}{2}\right) = \frac{2^{1-b}\sqrt{\pi}\,\Gamma(b)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}$$

in (6), and obtain

$$K_n\left(x; \frac{1}{2}, 2n\right) = (x - 2n)_n \; \frac{2^{x-n}\sqrt{\pi}\,\Gamma(n + 1 - x)}{\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(n + 1 - \frac{1}{2}x\right)} = (-2)^n \left(\frac{1-x}{2}\right)_n .$$

Hence, $K_n\left(x; \frac{1}{2}, 2n\right)$ has zeros at the odd integers $x = 2k - 1$, $k = 1, \ldots, n$.

If we use the two-term expansion

$$\frac{K_n(x; p, N)}{(x - N)_n \; p^n} \sim \left(\frac{p - r}{1 - r}\frac{1}{p}\right)^x \left[1 + g_1(p) N^{-1}\right], \quad N \to \infty,$$

where $g_1(p)$ was defined in (26), we can solve for x and obtain an approximation for the smallest zero of $K_n(x; p, N)$ in the region $r < p \leq 1$,

$$x_1 \sim (p - r)\sqrt{\frac{2(1 - r)N}{(1 - p)(p - 2r + 1)r}} + \frac{1}{2}, \quad N \to \infty.$$

For example, when $n = 10$, $N = 50$, and $p = 0.347$, we get $x_1 \simeq 4.23$, while the exact value is $x_1 = 4.11$.

If we expand (50) as $x \to 0$, we have

$$\exp\left[-\frac{\tau_0(p)}{\varepsilon}\right](-1)^n K_n(x;p,N) \sim \frac{\sqrt{2\pi\varepsilon r}}{r-p}(-x)+\frac{1}{\sqrt{1-r}}\exp\left[\frac{\tau_1(p)-\tau_0(p)}{\varepsilon}\right],$$

and therefore the smallest zero of $K_n(x;p,N)$ in the region $0 \leq p < r < 1$ is asymptotically given by

$$x_1 \sim \frac{r-p}{\sqrt{2\pi\varepsilon r(1-r)}}\exp\left[\frac{\tau_1(p)-\tau_0(p)}{\varepsilon}\right], \quad \varepsilon \to 0. \tag{52}$$

For example, when $\varepsilon = \frac{1}{50}$, $p = 0.347$, and $r = \frac{40}{50}$, the approximation (52) gives $x_1 \simeq 1.35 \times 10^{-9}$ while the exact solution is $x_1 = 1.31 \times 10^{-9}$.

5 Conclusions

We have found asymptotic expansions for the monic Krawtchouk polynomials $K_n(x;p,N)$, valid for $n = O(N)$, $x = O(1)$ and all values of $p \in [0,1]$. We have also obtained asymptotic approximations for the smallest zero. In a sequel, we plan to use similar methods to study the Hahn polynomials.

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