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Linearizing Differential Equations Riccati Solutions as D^n-Finite Functions

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Linearizing Differential Equations
Riccati Solutions as D^n-Finite Functions*

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Abstract

D-finite (or holonomic) functions are a class of formal power series that satisfy linear differential equations with polynomial coefficients. The finite representation of these functions (using a differential equation and some initial conditions) boosted the development of algorithms working symbolically over them. This has been recently extended to the DD-finite class (functions satisfying linear differential equations with D-finite coefficients) and implemented some closure properties. It was also proved that DD-finite functions (and also their generalization to the D^n-finite functions) are differentially algebraic. In this document we show how solutions to some non-linear differential equations (starting with the Riccati differential equation) are always D^n-finite functions for some n and proposed some ideas to set the difference between D^n-finite functions and differentially algebraic functions.

1 Introduction

A formal power series $f(x) = \sum_{k\geq 0} a_k x^k$ is called D-finite, if it satisfies a linear differential equation with polynomial coefficients [11, 18, 19]. The most commonly used special functions [1, 5, 16] are of this type as well as many generating functions of combinatorial sequences. D-finite functions are not only closed under certain operations, but these closure properties can be executed algorithmically. A key is the finite description of D-finite functions in terms of the polynomial coefficients and sufficiently many initial values. Given D-finite representations, the defining differential equation for the antiderivative, the derivative, addition, multiplication, algebraic substitution, etc. can be computed algorithmically. This has been implemented in several computer algebra systems [17, 13, 3, 12, 10]. These implementations can be used to automatically prove and derive results on holonomic functions [9].

Following the notation in [7], given a computable differential ring R, we can define a *differentially definable function over* R to be a formal power series that satisfies a linear differential equation with coefficients in R. Let $D(R)$ denote these functions. We have then that D-finite functions are $D(K[x])$ and if we set R to be the set of D-finite functions we arrive to the DD-finite functions. In general, as it was shown in [7], the same closure properties hold.

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Definition 1. *Let R be a non-trivial differential subring of $K[[x]]$ and $R[\partial]$ the ring of linear differential operators over R. We call $f \in K[[x]]$* differentially definable over *R if there is a non-zero operator $\mathcal{A} \in R[\partial]$ that annihilates f, i.e., $\mathcal{A} \cdot f = 0$. By $\mathrm{D}(R)$ we denote the set of all $f \in K[[x]]$ that are differentially definable over R. We define the* order of f w.r.t. R *as the minimal order of the operators that annihilate f (i.e., the minimal ∂-degree of $\mathcal{A} \in R[\partial]$ such that $\mathcal{A} \cdot f = 0$).*

Theorem 2. *Let R be a non-trivial differential subring of $K[[x]]$, F its field of fractions and $f, g \in \mathrm{D}(R)$ with orders d_1 and d_2, respectively. Then:*

1. *$f' \in \mathrm{D}(R)$ with order at most d_1.*
2. *Any antiderivative of f is in $\mathrm{D}(R)$ with order at most $d_1 + 1$.*
3. *$f + g \in \mathrm{D}(R)$ with order at most $d_1 + d_2$.*
4. *$fg \in \mathrm{D}(R)$ with order at most $d_1 d_2$.*
5. *If $r \in R$ and $r(0) \neq 0$, then its multiplicative inverse $1/r$ in $K[[x]]$ is in $\mathrm{D}(R)$ with order 1.*
6. *If a is algebraic over F with degree p, then $a \in \mathrm{D}(R)$ with order at most p.*

Proof. For 1-5, see [7]. For 6 see [8]. □

In fact, as addition, multiplication and derivation are closed on the differentially definable functions over any ring, we can iterate this process obtaining what we call the chain of D^n-finite functions.

$$K[x] \subset \mathrm{D}(K[x]) \subset \mathrm{D}^2(K[x]) \subset \ldots \subset \mathrm{D}^n(K[x]) \subset \ldots$$

It has been recently proven that this chain is proper ($\mathrm{D}^n(K[x]) \subsetneq \mathrm{D}^{n+1}(K[x])$) and close under composition of power series. It was also shown in the same paper that all D^n-finite functions are differentially algebraic over $K[x]$, i.e., they satisfy a non-linear differential equation with polynomial coefficients.

Theorem 3. *Let $f \in \mathrm{D}^n(K[x])$ and $g \in \mathrm{D}^m(K[x])$ with $g(0) = 0$. Then $(f \circ g) \in \mathrm{D}^{n+m}(K[x])$.*

Theorem 4. *Let $f \in K[[x]]$ be differentially algebraic over $\mathrm{D}^n(K[x])$. Then f is differentially algebraic over $\mathrm{D}^{n-1}(K[x])$.*

Constructive proofs of these two theorems were given in [8].

The inverse problem, i.e., knowing which differentially algebraic functions are D^n-finite for some natural n, is still open. The first natural question is: *is there any differentially algebraic function that is not D^n-finite for any n?*. This question was recently answered [15], showing that solutions to the equation $y' = y^3 - y^2$ can not be D^n-finite for any n. This proof, based in Differential Galois theory [20, 4], uses the fact that any finite set of solutions to that equation is algebraically independent over $K(x)$.

The following question arises then naturally: *is there any necessary or sufficient condition for a differentially algebraic function to be D^n-finite for some n?* In this document we study some non-trivial example of differentially algebraic functions whose solutions are D^n-finite.

In particular, we will show that there is no bound neither for the order or the degree of a non-linear equation from which we can assure their solutions are not D^n-finite.

In section 2 we study the classical Riccati differential equation. Then in section 3 we consider its higher order generalization. Finally in section 4 we study the separable first order differential equations.

2 The Riccati differential equation

The Riccati differential equation [5] is a non-linear differential equation of order 1 and degree 2 that can be generically written as

$$y' = cy^2 + by + a, \tag{1}$$

some some arbitrary functions a, b and c.

It is known that the change of variables $y = \frac{-v'}{cv}$ linearize the differential equation, obtaining that y satisfies (1) if and only if v satisfies the second order linear differential equation

$$v'' - \left(b - \frac{c'}{c}\right) v' + (ac)v = 0. \tag{2}$$

This leads naturally to the following result:

Proposition 5. *Let a, b be D^n-finite functions and c be D^{n-1}-finite with $c(0) \neq 0$. Then any power series solution of (1) is D^{n+2}-finite. More precisely, it is a quotient of D^{n+1}-finite functions.*

Proof. Let y be a power series solution of (1). Then y is uniquely determined by the value $y(0)$. From here, we can deduce that the associated function v solution of (2) must have

$$v(0) \neq 0, \qquad v'(0) = -y(0)v(0)c(0).$$

Starting from $c \in \mathrm{D}^{n-1}(K[x])$, and $c(0) \neq 0$, we have that $c'/c \in \mathrm{D}^n(K[x])$ using Theorem 2(5,1,4). This shows that all coefficients in equation (2) are D^n-finite. Hence $v \in \mathrm{D}^{n+1}(K[x])$ and we have $y = \frac{-v'}{cv} \in \mathrm{D}^{n+2}(K[x])$. □

This result is currently implemented in the Sage package *dd_functions* [6], in the function *RiccatiD*. This method receives as input the functions a, b, c and the initial condition $y(0)$ and computes the appropriate representation computing the corresponding v such that $v(0) = 1$.

3 Higher order Riccati equations

A generalization to the Riccati equation can be achieved by increasing the order of the differential equation but keeping the property that an appropriate substitution of the type $y = v'/cv$ linearize the equation. These kind of equations (also called Riccati differential equations) are defined with the following recursive definition [2].

For a given y we define the differential operator $L_y = \partial + cy$. We say that y is a nth order Riccati function if it satisfies

$$(L_y^n \cdot y) + \sum_{i=1}^{n-1} \alpha_i (L_y^i \cdot y) + \alpha_0 = 0. \tag{3}$$

In the particular case of $n = 1$ (i.e., first order Riccati equation) we recover the original Riccati differential equation (1).

In this higher order case, if c is a constant and we make the change of variables $y = v'/cv$ we obtain that y is solution to (3) if and only if v satisfies the linear differential equation

$$v^{(n)} + \alpha_{n-1}v^{(n-1)} + \dots \alpha_0 v = 0.$$

Corollary 6. *Let α_i be D^n-finite functions for $i = 0, ..., n$. Then any solution to the nth order Riccati differential equation for any constant c is D^{n+2}-finite.*

We can remove the condition over c and allow a non-constant coefficient in L_y. We can perform the same change of variables and still control the coefficients of the final differential equation.

Proposition 7. *Let $L_y = \partial + cy$ be a differential operator and R a differential ring such that $c^{(k)} \in R$ for all k and $1/c \in R$. Let v be defined by the functional relation $y = \frac{v'}{cv}$. Then for all $n \in \mathbb{N}$ there is an operator $M_n \in R[\partial]$ such that*

$$L_y^n \cdot y = \frac{M_n \cdot v}{cv}.$$

Proof. For $n = 0$ we have clearly the result, with $M_0 = \partial$.

Assume the result true for n. Then we have that:

$$(L_y^{n+1} \cdot y) = L_y \cdot \left(L_y^n \cdot y\right) = L_y \cdot \left(\frac{M_n \cdot v}{cv}\right).$$

On the other hand, we already have that $L_y = \partial + cy = \partial + v'/v$, so we can easily compute $L_y^{n+1} \cdot y$:

$$\begin{aligned}
L_y^{n+1} \cdot y &= \partial \cdot \left(\frac{M_n \cdot v}{cv}\right) + \frac{v'}{v}\frac{M_n \cdot v}{cv} \\
&= \frac{cv((\partial M_n) \cdot v) - (c'v + cv')(M_n \cdot v)}{c^2v^2} + \frac{v'}{v}\frac{M_n \cdot v}{cv} \\
&= \frac{(\partial M_n) \cdot v}{cv} - \frac{(c'/c)M_n \cdot v}{cv} \\
&= \frac{1}{cv}\left(\partial M_n - (c'/c)M_n\right) \cdot v.
\end{aligned}$$

Hence we have $M_{n+1} = \partial M_n - (c'/c)M_n \in R[\partial]$. □

Now we can show a complete result for the higher order Riccati differential equation:

Corollary 8. *Let $c(x)$ be a D^{n-1}-finite function with $c(0) \neq 0$ and $\alpha_0, ..., \alpha_n$ be D^n-finite functions. Then the solution to the higher order Riccati differential equation*

$$\sum_{i=1}^{n} \alpha_i(L_y^i \cdot y) + \alpha_0 = 0,$$

is D^{n+2}-finite. In fact, it is a quotient between two D^{n+1}-finite functions.

Proof. Applying Proposition 7, we have that for $v(x)$ defined with $y(x) = v'(x)/c(x)v(x)$, $y(x)$ is solution to the differential equation if and only if $v(x)$ satisfies the differential equation

$$\sum_{i=1}^{n} \alpha_i \frac{M_i \cdot v}{vc} + \alpha_0 = 0,$$

but, multiplying everything by $c(x)v(x)$, this is equivalent to

$$\left(\sum_{i=1}^{n} \alpha_i M_i + c\alpha_0\right) \cdot v = 0. \tag{4}$$

Since $c(x)$ is D^{n-1}-finite, the coefficients of all M_i are D^n-finite, and since all the α_i are also D^n-finite, then we obtain that all the coefficients of the differential operator in equation 4 are D^n-finite, making $v(x)$ a D^{n+1}-finite function and, finally, $y(x)$ a D^{n+2}-finite function. □

4 Separable first order equations

In this section we consider a first order *separable* differential equation. Those equations can be generically written in the form:

$$y' = g(x)f(y), \tag{5}$$

This kind of equations have been widely studied [14] and it is known that solutions of equation (5) satisfies the functional relation

$$G(x) = \int \frac{dy}{f(y)}, \tag{6}$$

for some $G(x)$ satisfying $G'(x) = g(x)$. This leads naturally to the following result:

Proposition 9. *Let $f(x) \in K[[x]]$ such that for $F(x) = \int dx/f(x)$, it satisfies $F^{-1}(x) \in \mathrm{D}^n(K[x])$, and $g(x) \in \mathrm{D}^m(K[x])$. Then any solution of the separable differential equation* (5) *is D^{n+m}-finite.*

Proof. By hypothesis, we have that $y(x)$ satisfies the functional relation $G(x) = F(y)$. If we compose this equation with $F^{-1}(x)$ we obtain:

$$y(x) = F^{-1}(G(x)),$$

and, by Theorem 3, the composition of a D^n-finite function ($F^{-1}(x)$) and a D^m-finite function ($G(x)$) is D^{n+m}-finite. □

Example 10. *Let $p(x) = \prod_{i=0}^{n}(x - \alpha_i)$, with $\alpha_i \neq \alpha_j$ for all $i \neq j$ and $\alpha_i \in \mathbb{Q}$ for all i. Then all the solutions of the non-linear differential equation $y'(x) = p(y)$ are DD-finite.*

Proof. Using the notation of Proposition 9, we have for this example that, for some rational numbers a_i:

$$F(x) = \log\left(\prod_{i=1}^{n}(x - \alpha_i)^{a_i}\right).$$

Let $B(x)$ be the argument of the logarithm. It is clear that $B(x)$ is an algebraic function. Hence we have that:

$$F^{-1}(x) = B^{-1}(\exp(x)).$$

$B^{-1}(x)$ is D-finite since it is algebraic, and $\exp(x)$ is clearly D-finite too. Thus we conclude $F^{-1}(x) \in \mathrm{D}^2(K[x])$ since it is the composition of two D-finite functions.

Using now Proposition 9, we have that any solution to $y' = p(y)$ is DD-finite. □

Example 11. *Let $p(x) = (x-\alpha)^n$ for some α algebraic over $\mathbb{Q}$ and $n \in \mathbb{Q} \setminus \{1\}$. Then any solution to $y' = p(y)$ is D-finite. In fact, y is algebraic over $\mathbb{Q}[x]$.*

Proof. Using again the notation from Proposition 9, in this example we have

$$F(x) = \frac{1}{(-n+1)(x-\alpha)^{n-1}},$$

which is clearly an algebraic function. Then $F^{-1}(x)$ is also algebraic. Hence any solution y to $y' = p(y)$ is of the form $y(x) = F^{-1}(x+C)$, and it is algebraic. So in particular, y is D-finite. □

One would expect that intermediate cases (with some common roots) will be somewhere in between the cases explained in Examples 10 and 11. If we go for a degree 3 polynomial with two different roots (α_1 and α_2), classical books [14] present that solutions satisfy the relation:

$$\ln\left(\frac{(y-\alpha_2)^{\alpha_2}}{(y-\alpha_1)^{\alpha_1}}\right) = (\alpha_2 - \alpha_1)(C + x),$$

for some constant C. This made us think that solutions will be again DD-finite. However, a recent result shows that the solutions of the differential equation $y' = y^3 - y^2$ are not D^n-finite for any n (see Section 7 in [15]).

5 Conclusions

In this document we have studied some relations between the set of differentially algebraic functions and the chain of D^n-finite functions. More precisely, we have checked that for some particular types of non-linear differential equations it is possible to linearize them and obtain a D^n-finite differential equation.

In the Riccati differential equation, we use an appropriate change of variables that make the non-linear equation linear. This leads to some open questions that we will consider in the future:

1. Which change of variables are admissible for building D^n-finite functions?
2. For which differentially algebraic functions is there a chain of admissible change of variables?
3. When can a D^n-finite function be decompose as a composition of two simpler functions?

On the separable case, we solved the non-linear equation and then analyze the objects involved in that expression. For this solutions it is really common to work with functional inverses. We now know that it may be that, for a D-finite functions, its functional inverse is not D^n-finite (see the differential equation $y' = y^3 - y^2$). But we also know some other examples where both a function and its inverse are in the chain (see the tangent function). These remarks leads naturally to the following open questions

1. When does a differentially algebraic function has a D^n-finite inverse?
2. When a D^n-finite function has a D^m-finite inverse?
3. Is there any computable criteria to assure $f(x)$ and $f^{-1}(x)$ are in the same D^n-finite ring?

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