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Recurrence coefficients of Toda-type orthogonal polynomials I. Asymptotic analysis.

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Recurrence coefficients of Toda-type orthogonal polynomials I. Asymptotic analysis.

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Abstract

We study the three-term recurrence coefficients β_n, γ_n, of polynomial sequences orthogonal with respect to a perturbed linear functional depending on a variable z. We obtain power series expansions in z and asymptotic expansions as $n \to \infty$.

We use our results to settle some conjectures proposed by Walter Van Assche and collaborators.

Keywords: Toda lattice, orthogonal polynomials, recurrence coefficients.
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1 Introduction

Let $\mathbb{N}_0$ denote the set

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = 0, 1, 2, \ldots,$$

and $\mathbb{F}$ the ring of formal power series on the variable z, $\mathbb{F} = \mathbb{C}[[z]]$. Suppose that $D_z : \mathbb{F} \to \mathbb{F}$ is a derivation (on the variable z), i.e., a linear operator satisfying the product rule

$$D_z(fg) = fD_z(g) + gD_z(f), \quad f, g \in \mathbb{F}.$$

If $\kappa(x, z)$ is an eigenfunction of D_z with eigenvalue x,

$$D_z \kappa(x, z) = x\kappa(x, z),$$

we define the vector space $\mathcal{S}_\kappa$ over the ring $\mathbb{F}$ generated by the monomial basis $\{x^n\}_{n \geq 0}$ and the function $\kappa(x, z)$,

$$\mathcal{S}_\kappa = \{p(x)\kappa(x, z) : p \in \mathbb{F}[x]\}.$$

Examples of D_z and $\kappa(x, z)$ include

$$D_t = \partial_t, \quad \kappa(x, t) = e^{xt},$$

where ∂_t denotes the derivative operator,

$$D_z = z\partial_z, \quad \kappa(x, z) = z^x,$$

and

$$D_w = w(1 - w)\partial_w, \quad \kappa(x, w) = \left(\frac{w}{w - 1}\right)^x$$

Note that we have

$$e^t = z \to \partial_t = z\partial_z,$$
$$\frac{w}{w - 1} = z \to w(1 - w)\partial_w = z\partial_z, \tag{1}$$

and in general

$$z = g(s) \to z\partial_z = \frac{g(s)}{g'(s)}\partial_s.$$

Thus, up to a suitable change of variables, we can take $D_z = z\partial_z$ and $\kappa(x, z) = z^x$.

If $\mathcal{L} \in \mathcal{S}_{\kappa}^{*}$ (the dual space) commutes with the derivation D_z, we define $L : \mathbb{F}[x] \rightarrow \mathbb{F}$ by

$$L[p] = \mathcal{L}[p(x)\,\kappa(x,z)], \quad p \in \mathbb{F}[x], \tag{2}$$

where we always think of L as acting on the variable x. A sequence $\{P_n\}_{n\geq 0}$, $\deg(P_n) = n$, is called an orthogonal polynomial sequence with respect to L if

$$L[P_k P_n] = h_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0, \quad h_n \neq 0, \tag{3}$$

where $\delta_{k,n}$ denotes the Kronecker delta. If $h_n = 1$, then $\{P_n\}_{n\geq 0}$ is called an orthonormal polynomial sequence.

Let's denote by $P_n(x; z)$, the sequence of monic polynomials orthogonal with respect to L. From (3) we see that

$$L[x P_k P_n] = 0, \quad k \neq n, n \pm 1,$$

and therefore the polynomials $P_n(x; z)$ satisfy the three term recurrence relation

$$x P_n(x;z) = P_{n+1}(x;z) + \beta_n(z) P_n(x;z) + \gamma_n(z) P_{n-1}(x;z) \tag{4}$$

with $P_{-1} = 0,\ P_0 = 1$. The coefficients β_n, γ_n are given by [12]

$$\beta_0 = \frac{\mu_1}{\mu_0}, \quad \gamma_0 = 0, \tag{5}$$

and

$$\beta_n = \frac{L[x P_n^2]}{h_n}, \quad \gamma_n = \frac{L[x P_n P_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}. \tag{6}$$

Note that using (3) we have

$$h_n = L[x^n P_n] = L[x P_n P_{n-1}] = \gamma_n h_{n-1},$$

and hence

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad n \in \mathbb{N}. \tag{7}$$

Orthogonal polynomials associated with deformed functionals have been studied by many authors, see [1], [3], [28], [29], [38]. Particular examples include continuous [11], [19], [23], discrete [5], [34], elliptic [36], [37], matrix

[2], [9], multiple [4], multivariate [6], q [10], and skew [24] orthogonal polynomials. Applications to combinatorics have also been considered in [7] and [25].

In this paper, we focus on linear functionals of the form

$$L[r] = \sum_{x=0}^{\infty} r(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad r \in \mathbb{F}[x], \tag{8}$$

where

$$\mathbf{a} = (a_1, \ldots, a_p), \quad \mathbf{b} = (b_1, \ldots, b_q), \quad p, q \in \mathbb{N}_0,$$

$$(\mathbf{a})_n = \prod_{i=1}^{p} (a_i)_n, \quad (\mathbf{b})_n = \prod_{i=1}^{q} (b_i)_n, \quad n \in \mathbb{N}_0,$$

and the Pochhammer polynomial $(x)_n$ is defined by $(x)_0 = 1$ and [27, 18:12]

$$(x)_n = \prod_{j=0}^{n-1} (x+j), \quad n \in \mathbb{N}. \tag{9}$$

These functionals are both particular cases of (2) with $\kappa(x,z) = z^x$, and also discrete semiclassical [18].

Our main objective is to analyze the asymptotic behavior of the three-term recurrence coefficients (6) as $n \to \infty$. We illustrate our results with several examples of discrete semiclassical polynomials of class $s \leq 2$. Among these we are able to prove some conjectures proposed by Walter Van Assche and collaborators in [20] and [32].

2 Orthogonal polynomials

Let $L : \mathbb{F}[x] \to \mathbb{F}$ be defined by (2). Then, we have

$$D_z L[p] = \mathcal{L}[D_z(p)\,\kappa(x,z)] + \mathcal{L}[x\kappa(x,z)\,p] = L[D_z(p)] + L[xp], \quad p \in \mathbb{F}[x].$$

If we denote the moments of L by

$$L[x^n] = \mu_n(z) \in \mathbb{F}, \tag{10}$$

we see that

$$D_z \mu_n = D_z L[x^n] = L[x x^n] = \mu_{n+1}, \quad n \in \mathbb{N}_0,$$

and conclude that

$$\mu_n = D_z^n \mu_0, \quad n \in \mathbb{N}_0. \tag{11}$$

If we write

$$P_n(x;z) = x^n - \sigma_n(z) x^{n-1} + q_n(x;z), \quad \deg(q_n) \leq n-2, \tag{12}$$

then we have $\sigma_0 = 0$, and using (4) we get

$$x^{n+1} - \sigma_n x^n + x q_n = x^{n+1} - \sigma_{n+1} x^n + q_{n+1} + \beta_n \left(x^n - \sigma_n x^{n-1} + q_n \right) + \gamma_n P_{n-1}.$$

Comparing coefficients of x^n, we obtain $-\sigma_n = -\sigma_{n+1} + \beta_n$, or

$$\beta_n = \sigma_{n+1} - \sigma_n. \tag{13}$$

The connection between σ_n, γ_n, h_n, and β_n is given in the next proposition.

Proposition 1 *Let h_n be defined by (3), β_n, γ_n be defined by (6), and σ_n be defined by (12). Then, we have*

$$D_z \sigma_n = \gamma_n \tag{14}$$

and

$$D_z \ln h_n = \beta_n. \tag{15}$$

Proof. From (12) we have

$$D_z P_n(x;z) = -D_z \sigma_n(z) x^{n-1} + D_z q_n(x;z),$$

and using (3) we get

$$L[P_{n-1} D_z P_n] = -D_z(\sigma_n) L\left[x^{n-1} P_{n-1}\right] = -D_z(\sigma_n) h_{n-1}. \tag{16}$$

On the other hand, since $L[P_n P_{n-1}] = 0$ and $\deg(D_z P_{n-1}) = n-2$,

$$\begin{aligned} 0 = D_z L[P_n P_{n-1}] &= L[P_{n-1} D_z P_n] + L[P_n D_z P_{n-1}] + L[x P_n P_{n-1}] \\ &= -D_z(\sigma_n) h_{n-1} + \gamma_n h_{n-1}, \end{aligned}$$

and we obtain (14). Since $\deg(D_z P_n) = n-1$ we have

$$D_z h_n = D_z L\left[P_n^2\right] = L[2 P_n D_z P_n] + L\left[x P_n^2\right] = L\left[x P_n^2\right] = \beta_n h_n,$$

and (15) follows. ■

As a direct consequence, we see that (β_n, γ_n) are solutions of the Toda lattice [33].

Corollary 2 *Toda equations. The coefficients of the 3-term recurrence relation (4) are solutions of the differential-difference equations*

$$D_z \beta_n = \Delta \gamma_n, \quad D_z \ln \gamma_n = \nabla \beta_n, \tag{17}$$

with initial conditions (5), where

$$\Delta f(n) = f(n+1) - f(n), \quad \nabla f(n) = f(n) - f(n-1). \tag{18}$$

Proof. From (13) and (14) we get

$$D_z \beta_n = D_z \sigma_{n+1} - D_z \sigma_n = \gamma_{n+1} - \gamma_n,$$

while (7) and (15) give

$$D_z \ln \gamma_n = D_z \ln h_n - D_z \ln h_{n-1} = \beta_n - \beta_{n-1}.$$

■

Let H_n be the $n \times n$ Hankel matrix defined by

$$(H_n)_{i,j} = \mu_{i+j}, \quad 0 \le i, j \le n-1,$$

and the Hankel determinants $\mathcal{H}_n$ be defined by $\mathcal{H}_0 = 1$ and $\mathcal{H}_n = \det(H_n)$. The following result gives a relation between σ_n and $\mathcal{H}_n$.

Proposition 3 *Let $\sigma_n(z)$ defined by (12) and*

$$\mathcal{H}_n(z) = \det_{0 \le i,j \le n-1} (\mu_{i+j}). \tag{19}$$

Then,

$$\mathcal{H}_n = \prod_{k=0}^{n-1} h_k, \tag{20}$$

and

$$\sigma_n = D_z \ln(\mathcal{H}_n). \tag{21}$$

Proof. Writing

$$P_n(x; z) = \sum_{k=0}^{n} \mathfrak{c}_{n,k}(z)\, x^k,$$

and using (3) and (10), we have

$$\sum_{k=0}^{n} \mu_{i+k} \mathfrak{c}_{n,k} = h_n \delta_{i,n}.$$

Using Cramer's rule, we get

$$\mathfrak{c}_{n,k} = (-1)^{k+n} h_n \frac{M_{k,n}}{\mathcal{H}_{n+1}},$$

where $M_{k,n}$ is the minor defined by

$$M_{k,l} = \det_{\substack{0 \leq i,j \leq n \\ i \neq k, j \neq l}} (\mu_{i+j}).$$

In particular, since $\mathfrak{c}_{n,n} = 1$, we have

$$h_n = \frac{\mathcal{H}_{n+1}}{M_{n,n}} = \frac{\mathcal{H}_{n+1}}{\mathcal{H}_n},$$

and (20) follows. From (13) and (15), we get

$$\sigma_{n+1} - \sigma_n = \beta_n = D_z \ln(h_n) = D_z \ln(\mathcal{H}_{n+1}) - D_z \ln(\mathcal{H}_n),$$

and since $\sigma_0 = 0 = D_z \ln(\mathcal{H}_0)$, we obtain (21). ■

The functions $\sigma_n, \beta_n, \gamma_n, h_n$, and $\mathcal{H}_n$ are known exactly in very few cases. Below we present two of the classical discrete orthogonal polynomials as examples.

Example 4 *The monic Charlier polynomials are defined by [27, 18.20.8]*

$$C_n(x;z) = (-z)^n \; {}_2F_0 \left[\begin{matrix} -n, -x \\ - \end{matrix} ; -\frac{1}{z} \right],$$

where ${}_pF_q$ denotes the generalized hypergeometric function defined by [27, 16.2]

$${}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(\mathbf{a})_k}{(\mathbf{b})_n} \frac{z^k}{k!}. \tag{22}$$

The first few $C_n(x;z)$ are given by $C_0(x;z) = 1$, $C_1(x;z) = x - z$,

$$C_2(x;z) = x^2 - (2z+1)x + z^2,$$
$$C_3(x;z) = x^3 - 3(z+1)x^2 + \left(3z^2 + 3z + 2\right)x - z^2,$$

and we can see that in this case the coefficients of $C_n(x;z)$ are polynomials in z.

The Charlier polynomials are orthogonal with respect to the linear functional

$$L[p] = \sum_{x=0}^{\infty} p(x) \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

from which we obtain the moments of L [17]

$$\mu_n(z) = e^z \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z^k, \quad n \in \mathbb{N}_0, \tag{23}$$

where $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind, defined by [27, 26.8.6]

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

The three-term recurrence coefficients are given by [27, 18.22(i)]

$$\beta_n(z) = n + z, \quad \gamma_n(z) = nz, \tag{24}$$

and using (7) and (13) we get

$$h_n(z) = n! z^n e^z, \quad \sigma_n(z) = \frac{n(n-1)}{2} + nz.$$

Finally, using (20) we obtain [14]

$$\mathcal{H}_n(z) = \left(\prod_{k=0}^{n-1} k! \right) z^{\frac{n(n-1)}{2}} e^{nz}.$$

Example 5 *The monic Meixner polynomials are defined by [27, 18.20.7]*

$$M_n(x;z) = (a)_n \left(1 - \frac{1}{z}\right)^{-n} {}_2F_1 \left[\begin{matrix} -n, -x \\ a \end{matrix} ; 1 - \frac{1}{z} \right],$$

where $a > 0$ and $z \in (0,1)$. The first few $M_n(x;z)$ are given by $M_0(x;z) = 1$,

$$M_1(x;z) = x + \frac{az}{z-1},$$
$$M_2(x;z) = x^2 + \frac{2az + z + 1}{z-1} x + a(a+1) \left(\frac{z}{z-1} \right)^2,$$

and we note that the coefficients of $M_n(x;z)$ *are rational functions of* z.

The Meixner polynomials are orthogonal with respect to the linear functional

$$L[p] = \sum_{x=0}^{\infty} p(x) \frac{(a)_x}{x!} z^x, \quad p \in \mathbb{F}[x],$$

and the moments of L *are given by [17]*

$$\mu_n(z) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (a)_k \ z^k (1-z)^{-(a+k)} \quad n \in \mathbb{N}_0. \tag{25}$$

The three-term recurrence coefficients are given by [27, 18.22(i)]

$$\beta_n(z) = \frac{n + (n+a) z}{1-z}, \quad \gamma_n(z) = \frac{(n+a-1) nz}{(1-z)^2}, \tag{26}$$

and using (7) and (13) we get

$$h_n(z) = \frac{(a)_n n! z^n}{(1-z)^{2n+a}}, \quad \sigma_n(z) = \frac{n(n-1)}{2} + \frac{(a+n-1) nz}{1-z}.$$

Finally, using (20) we obtain [14]

$$\mathcal{H}_n(z) = \left[\prod_{k=0}^{n-1} k! (a)_k \right] z^{\frac{n(n-1)}{2}} (1-z)^{-n(n+a-1)}.$$

Since the Charlier and Meixner polynomials have hypergeometric representations, and the function

$$\mu_0(z) = {}_pF_q \left(\begin{matrix} \mathbf{a} \\ \mathbf{b}+1 \end{matrix} ; z \right) \tag{27}$$

satisfies the differential equation [27, 16.8.3]

$$\left[D_z (D_z + b_1) \cdots (D_z + b_q) - z (D_z + a_1) \cdots (D_z + a_p) \right] \mu_0 = 0, \tag{28}$$

with $D_z = z\partial_z$, we see that the monic Charlier polynomials satisfy the ODE

$$\left[D_z^2 + (x - n - z) D_z + nz \right] C_n = 0, \tag{29}$$

and the monic Meixner polynomials satisfy the ODE

$$\left[D_z^2 + \left(x - \frac{n + az + nz}{1-z} \right) D_z + \frac{(n+a-1) nz}{(1-z)^2} \right] M_n = 0. \tag{30}$$

In the next result, we find an ODE for general polynomials.

Proposition 6 *The polynomials $P_n(x;z)$ defined by (3) satisfy the ODE*

$$D_z^2 P_n + (x - \beta_n) D_z P_n + \gamma_n P_n = 0. \tag{31}$$

Proof. If we write

$$D_z P_n = \sum_{k=1}^{n-1} v_k P_k,$$

then (16) and (14) give

$$v_{n-1} = \frac{1}{h_{n-1}} L[P_{n-1} D_z P_n] = -D_z \sigma_n = -\gamma_n.$$

Moreover, for all $k = 0, 1, \ldots, n-2$

$$0 = D_z L[P_n P_k] = L[P_k D_z P_n] + L[P_n D_z P_k] + L[x P_n P_k] = L[P_k D_z P_n] = h_k v_k,$$

and therefore we obtain

$$D_z P_n = -\gamma_n P_{n-1}. \tag{32}$$

From (4) and (32), we have

$$D_z P_n = -\gamma_n P_{n-1} = P_{n+1} + (\beta_n - x) P_n.$$

Using (13), we get

$$\begin{aligned} D_z^2 P_n &= D_z P_{n+1} + P_n D_z \beta_n + (\beta_n - x) D_z P_n \\ &= -\gamma_{n+1} P_n + (\gamma_{n+1} - \gamma_n) P_n + (\beta_n - x) D_z P_n \end{aligned}$$

and (31) follows. ∎

Using (24) in (31) we obtain (29), and using (26) in (31) we get (30).

Remark 7 *The convergence of the series (22) depends on the values of p and q. We have three different cases [27, 16.2]:*

1. *If $p < q + 1$, then ${}_pF_q$ is an entire function of z.*
2. *If $p = q+1$, then ${}_pF_q$ is analytic inside the unit circle, $|z| < 1$, and can be extended by analytic continuation to the cut plane $\mathbb{C} \setminus [1, \infty)$. Let*

$$\gamma = b_1 + \cdots + b_q - (a_1 + \cdots + a_{q+1}). \tag{33}$$

On the unit circle $|z|=1$, *the series (22) is*

(i) absolutely convergent if $\operatorname{Re}(\gamma)>0$,

(ii) convergent except at $z=1$ *if* $\operatorname{Re}(\gamma)\in(-1,0]$,

and

(iii) divergent if $\operatorname{Re}(\gamma)\leq -1$.

3. *If* $p>q+1$, *then* ${}_pF_q$ *diverges for all* $z\neq 0$, *up to* $a_i=-N$, *with* $N\in\mathbb{N}$ *for some* $1\leq i\leq p$. *In this case,* ${}_pF_q$ *becomes a polynomial of degree* N.

If the first moment of L is given by (27),we can use the formulas [27, 16.3.1]

$$\partial_z^n \ {}_pF_q\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array};z\right)=\frac{(\mathbf{a})_n}{(\mathbf{b})_n}\ {}_pF_q\left(\begin{array}{c}\mathbf{a}+n\\ \mathbf{b}+n\end{array};z\right),$$

and [30, 6.6]

$$(z\partial_z)^n=D_z^n=\sum_{k=0}^{n}\left\{\begin{matrix}n\\ k\end{matrix}\right\}z^k\partial_z^k, \tag{34}$$

in (11) and obtain [17]

$$\mu_n(z)=\sum_{k=0}^{n}\left\{\begin{matrix}n\\ k\end{matrix}\right\}z^k\ \frac{(\mathbf{a})_k}{(\mathbf{b})_k}\ {}_pF_q\left(\begin{array}{c}\mathbf{a}+k\\ \mathbf{b}+k\end{array};z\right).$$

Thus, the analytic properties (with respect to z) of all the functions $\mu_n,\sigma_n,\beta_n,\gamma_n,h_n$, and $\mathcal{H}_n$ depend on the analyticity of $\mu_0(z)$, and in view of Remark (7) this depends just on the parameters (p,q). In the next section, we find power series expansions for $\sigma_n(z)$.

3 Series expansion of $\sigma_n(z)$

In [15], we studied power series expansions of Hankel determinants of the form (19) with $\mu_n\in\mathbb{F}$. Below we state one of the main results we obtained.

Theorem 8 *Let the Hankel determinant* $\mathfrak{D}_n(z)$ *be defined by (19) and*

$$\mathfrak{D}_1(z)=\mu_0(z)=\sum_{k=0}^{\infty}c_kz^k\in\mathbb{F}. \tag{35}$$

Then, we have

$$\mathcal{H}_n(z) = z^{\binom{n}{2}} g_n(z) \prod_{j=0}^{n-1} \left[(j!)^2 c_j \right], \quad n \geq 2, \tag{36}$$

where

$$g_n(z) = 1 + \binom{n}{1}^2 \frac{c(n)}{c(n-1)} z + \left[\binom{n+1}{2}^2 \frac{c_{n+1}}{c(n-1)} + \binom{n}{2}^2 \frac{c(n)}{c_{n-2}} \right] z^2 + O\left(z^3\right), \quad z \to 0.$$

Using (21) we obtain the following result.

Corollary 9 *Let $\sigma_n(z)$ defined by (12). Then,*

$$\sigma_n(z) = \frac{n(n-1)}{2} + n^2 \frac{c(n)}{c(n-1)} z + O\left(z^2\right), \quad z \to 0. \tag{37}$$

We can now state one of our main results.

Theorem 10 *Let $\sigma_n(z)$ defined by (12). If we write*

$$\sigma_n(z) = \sum_{k=0}^{\infty} s_{n,k} z^k \in \mathbb{F}, \tag{38}$$

then we have

$$s_{n,0} = \frac{n(n-1)}{2}, \quad s_{n,1} = n^2 \frac{c(n)}{c(n-1)}, \tag{39}$$

and

$$s_{n,k} = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j)\, s_{n,k-j} \nabla\Delta s_{n,j}, \quad k \geq 2, \tag{40}$$

where the coefficients $c(n)$ were defined in (35).

Proof. The initial values (39), just follow from (37). From (13), (14), and (17) we get

$$D_z \ln(D_z \sigma_n) = D_z \ln(\gamma_n) = \beta_n - \beta_{n-1} = \sigma_{n+1} - 2\sigma_n + \sigma_{n-1}.$$

Using the difference operators (18), we can write

$$\sigma_{n+1} - 2\sigma_n + \sigma_{n-1} = \nabla\Delta\sigma_n,$$

and since we are using $D_z = z\partial_z$, we have

$$\sigma_n''(z) = \sigma_n'(z) \frac{\nabla\Delta\sigma_n(z) - 1}{z}. \tag{41}$$

Since

$$\nabla\Delta s_{n,0} = \nabla\Delta \frac{n(n-1)}{2} = 1,$$

we see that from (38) that

$$\frac{\nabla\Delta\sigma_n - 1}{z} = \sum_{k=1}^{\infty} \nabla\Delta s_{n,k} z^{k-1} = \sum_{k=0}^{\infty} \nabla\Delta s_{n,k+1} z^k.$$

Also,

$$\sigma_n'(z) = \sum_{k=1}^{\infty} k s_{n,k} z^{k-1} = \sum_{k=0}^{\infty} (k+1) s_{n,k+1} z^k,$$

and

$$\sigma_n''(z) = \sum_{k=2}^{\infty} k(k-1) s_{n,k} z^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) s_{n,k+2} z^k.$$

Comparing coefficients of z in (41) gives

$$(k+2)(k+1) s_{n,k+2} = \sum_{j=0}^{k} (k-j+1) s_{n,k-j+1} \nabla\Delta s_{n,j+1},$$

and (40) follows after shifting $k \to k-2$ and $j \to j-1$. ∎

Remark 11 *Note that we have*

$$s_{n,2} = \frac{s_{n,1}}{2} \nabla\Delta s_{n,1},$$

$$s_{n,3} = \frac{1}{3} s_{n,2} \nabla\Delta s_{n,1} + \frac{1}{6} s_{n,1} \nabla\Delta s_{n,2} = \frac{s_{n,1}}{6} \left[(\nabla\Delta s_{n,1})^2 + \nabla\Delta s_{n,2} \right],$$

and using induction we see that

$$s_{n,k} = \frac{s_{n,1}}{k!} \widetilde{s}_{n,k}.$$

From (13) and (14), we immediately obtain the following.

Corollary 12 *The coefficients of the 3-term recurrence relation (4) admit the power series*

$$\beta_n(z) = \sum_{k=0}^{\infty} \Delta s_{n,k} z^k, \quad \gamma_n(z) = \sum_{k=1}^{\infty} k s_{n,k} z^k, \tag{42}$$

where the coefficients $s_{n,k}$ *are defined in (38).*

In particular,

$$\beta_n(0) = n, \quad \gamma_n(0).$$

We now give some examples. We start with the discrete classical orthogonal polynomials [26].

Example 13 *Charlier polynomials. From (23) we have*

$$\mu_0(z) = e^z,$$

and using (35) we get

$$c(n) = \frac{1}{n!}.$$

Therefore, (39) gives

$$s_{n,1} = n,$$

and using (40) we conclude that

$$s_{n,k} = 0, \quad k \geq 2.$$

Hence, we obtain

$$\sigma_n(z) = \frac{n(n-1)}{2} + nz,$$

and

$$\beta_n = n + z, \quad \gamma_n = nz$$

in agreement with (24).

Example 14 *Meixner polynomials. From (23) we have*

$$\mu_0(z) = (1-z)^{-a},$$

and using (35) we get

$$c(n) = \frac{(a)_n}{n!}. \tag{43}$$

Therefore, (39) gives

$$s_{n,1} = n\left(n+a-1\right),$$

and using (40) we conclude that

$$s_{n,k} = s_{n,1}, \quad k \geq 2.$$

Hence, we obtain

$$\begin{aligned}\sigma_n\left(z\right) &= \frac{n\left(n-1\right)}{2} + n\left(n+a-1\right)\sum_{k=1}^{\infty} z^k \\ &= \frac{n\left(n-1\right)}{2} + n\left(n+a-1\right)\frac{z}{1-z},\end{aligned}$$

and

$$\beta_n = n + \left(2n+a\right)\frac{z}{1-z}, \quad \gamma_n = n\left(n+a-1\right)\frac{z}{\left(1-z\right)^2}.$$

in agreement with (26).

Next, we consider the discrete semiclassical orthogonal polynomials of class $s = 1$ [18].

Example 15 *Generalized Charlier polynomials. These polynomials were introduced in [22], and studied in [13], [18], [21], [32], [35]. They are orthogonal with respect to the linear functional*

$$L\left[p\right] = \sum_{x=0}^{\infty} p\left(x\right)\frac{1}{\left(b+1\right)_x}\frac{z^x}{x!}, \quad p \in \mathbb{F}\left[x\right],$$

and from (35) we get

$$c\left(n\right) = \frac{1}{\left(b+1\right)_n \; n!}.$$

Therefore, (39) gives

$$s_{n,1} = \frac{n}{n+b}$$

and using (40) we have

$$s_{n,2} = -\frac{b}{\left(n+b-1\right)_3} s_{n,1}, \quad s_{n,3} = -\frac{2b\left(n-b\right)}{(n+b)(n+b-2)_5} s_{n,1}.$$

From (42), we obtain

$$\beta_n(z) = n + \frac{bz}{(n+b)_1}\left[1 + \frac{3n^2 + (2b+3)n - b(b-1)}{(n+b-1)_4}z\right] + O\left(z^3\right),$$

and

$$\gamma_n(z) = s_{n,1}z\left[1 - \frac{2b}{(n+b-1)_3}z - \frac{6b(n-b)}{(n+b)(n+b-2)_5}z^2\right] + O\left(z^4\right),$$

as $z \to 0$.

Example 16 *Generalized Meixner polynomials. These polynomials were introduced in [31], and studied in [8], [13], [18], [32]. They are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x)\frac{(a)_x}{(b+1)_x}\frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

and from (35) we get

$$c(n) = \frac{(a)_n}{(b+1)_n\ n!}.$$

Therefore, (39) gives

$$s_{n,1} = \frac{n(n+a-1)}{n+b},$$

and using (40) we have

$$s_{n,2} = \frac{b(b+1-a)}{(n+b-1)_3}s_{n,1}, \quad s_{n,3} = \frac{b(b+1-a)(n-b)(n+2a-b-2)}{(n+b-2)_5(n+b)}s_{n,1}.$$

Note that if $a = b+1$, then $s_{n,k} = 0$, $k \geq 2$ since then we recover the Charlier polynomials.

From (42), we obtain

$$\beta_n(z) = n + \frac{n(n+2b+1)+ab}{n+b}z + O\left(z^2\right),$$

and

$$\gamma_n(z) = \frac{n(n+a-1)}{n+b}z + \frac{2b(b+1-a)n(n+a-1)}{(n+b)(n+b-1)_3}z^2 + O\left(z^3\right),$$

as $z \to 0$.

Example 17 *Generalized Krawtchouk polynomials. These polynomials were introduced in [18], and studied in [13]. They are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x)(a)_x (-N)_x \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

where $N \in \mathbb{N}$ and from (35) we get

$$c(n) = \frac{(a)_n (-N)_n}{n!}.$$

Therefore, (39) gives

$$s_{n,1} = n(n+a-1)(n-N-1),$$

and using (40) we have

$$\sigma_n(z) = \frac{n(n-1)}{2} + s_{n,1} z \left[1 + q_1(n) z + q_2(n) z^2\right] + O\left(z^4\right), \quad z \to 0,$$

where

$$\begin{aligned} q_1(n) &= 3n - N - 2 + a, \\ q_2(n) &= 12n^2 - 8(N-a+2)n + 5N - 5a - 3Na + N^2 + a^2 + 6. \end{aligned}$$

From (42), we obtain

$$\beta_n(z) = n + \left[3n^2 + n(-1+2a-2N) - aN\right] z + O\left(z^2\right),$$

and

$$\gamma_n(z) = s_{n,1} z \left[1 + 2q_1(n) z + 3q_2(n) z^2\right] + O\left(z^4\right),$$

as $z \to 0$.

Note that $\gamma_{N+1}(z) = 0$, and from (7) it follows that $h_{N+1}(z) = 0$. Therefore, in this case the polynomials $P_n(x;z)$ are a finite family for $0 \leq n \leq N$.

Example 18 *Generalized Hahn polynomials of type I. These polynomials were introduced in [18], and studied in [13], [16], [20]. They are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x) \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

with $a_1, a_2 \neq b+1$, *and from (35) we get*

$$c(n) = \frac{(a_1)_n (a_2)_n}{(b+1)_n \; n!}. \tag{44}$$

Therefore, (39) gives

$$s_{n,1} = \frac{n(n+a_1-1)(n+a_2-1)}{n+b}$$

and using (40) we have

$$\sigma_n(z) = \frac{n(n-1)}{2} + s_{n,1} z \left[1 + \frac{q_1(n)}{(n+b-1)_3} z \right] + O\left(z^3\right), \quad z \to 0,$$

where

$$q_1(n) = n^3 + 3bn^2 + \left(3b^2 - 1\right) n + \left[(a_1 + a_2 - 2) b + a_1 + a_2 - a_1 a_2 - 2\right] b.$$

From (42), we obtain

$$\beta_n(z) = n + \frac{q_2(n)}{n+b} z + O\left(z^2\right),$$

and

$$\gamma_n(z) = s_{n,1} z \left[1 + \frac{2q_1(n)}{(n+b-1)_3} z \right] + O\left(z^3\right),$$

as $z \to 0$, *where*

$$q_2(n) = 2n^3 + (3b + a_1 + a_2 + 1) n^2 + (a_1 - b + a_2 + 2ba_1 + 2ba_2 - 1) n + ba_1 a_2.$$

Remark 19 *It is clear from (40) that if* $s_{n,1} \in \mathbb{C}(n)$ *(i.e., is a rational function of* n*), then* $s_{n,k} \in \mathbb{C}(n)$ *for all* $k \geq 1$. *This will be the case for all families of orthogonal polynomials for which* $\mu_0(z)$ *is a hypergeometric function.*

Note that from the previous examples we have, as $n \to \infty$

$$\begin{aligned}
\text{Meixner:} &\quad s_{n,1} \sim n^2, \quad s_{n,k} \sim n^2, \quad k \geq 2, \\
\text{Generalized Charlier:} &\quad s_{n,1} \sim 1, \quad s_{n,k} \sim n^{-2k+1}, \quad k \geq 2, \\
\text{Generalized Meixner:} &\quad s_{n,1} \sim n, \quad s_{n,k} \sim n^{-k}, \quad k \geq 2, \\
\text{Generalized Krawtchouk:} &\quad s_{n,1} \sim n^3, \quad s_{n,k} \sim n^{k+2}, \quad k \geq 2, \\
\text{Generalized Hahn:} &\quad s_{n,1} \sim n^2, \quad s_{n,k} \sim n^2, \quad k \geq 2.
\end{aligned}$$

Therefore, it seems that in some cases the coefficients $s_{n,k}$ form an asymptotic sequence as $n \to \infty$. We show that this is the case in the next section.

4 Asymptotic analysis

We begin with a simple lemma.

Lemma 20 *Suppose that for* $j \geq 1$

$$s_{n,j} \sim n^{\theta_j} \sum_{l \geq 0} A_{j,l} n^{-l}, \quad n \to \infty, \tag{45}$$

with $A_{j,0} \neq 0$. *Then, for all* $1 \leq j \leq k-1$,

$$s_{n,k-j} \nabla \Delta s_{n,j} \sim 2n^{\theta_{k-j}+\theta_j} \left[\sum_{m \geq 2} n^{-m} \sum_{l=2}^{m} \sum_{i=1}^{\frac{l}{2}} A_{k-j,m-l} A_{j,l-2i} \binom{\theta_j + 2i - l}{2i} \right], \quad n \to \infty. \tag{46}$$

Proof. First, we observe that

$$\nabla \Delta n^{\theta} = 2 \sum_{i \geq 1} \binom{\theta}{2i} n^{\theta - 2i}, \tag{47}$$

since

$$\nabla \Delta n^{\theta} = (n+1)^{\theta} + (n-1)^{\theta} - 2n^{\theta} = -2n^{\theta} + \sum_{i \geq 0} \binom{\theta}{i} \left[1 + (-1)^i \right] n^{\theta - i}.$$

Using (47) in (45), we get

$$\nabla \Delta s_{n,j} \sim 2 \sum_{l \geq 0} A_{j,l} \sum_{i \geq 1} \binom{\theta_j - l}{2i} n^{\theta_j - l - 2i}, \quad n \to \infty,$$

and changing the index of summation to $m = l + 2i$, we have

$$\nabla \Delta s_{n,j} \sim 2n^{\theta_j} \sum_{m \geq 2} n^{-m} \sum_{i=1}^{\frac{m}{2}} A_{j,m-2i} \binom{\theta_j + 2i - m}{2i}, \quad n \to \infty. \tag{48}$$

Computing the Cauchy product of $s_{n,k-j}$ and $\nabla \Delta s_{n,j}$, we obtain (46). ■

We now have all the elements to prove our main result.

Theorem 21 *Let* $c(n)$ *be defined by (35) and the coefficients* $s_{n,k}$ *be defined by (40). If*

$$n^2 \frac{c(n)}{c(n-1)} \sim n^{\theta_1} \sum_{l \geq 0} A_{1,l} n^{-l}, \quad n \to \infty, \tag{49}$$

then:

(i) *If* $\theta_1 \neq 0, 1$, *then for all* $k \geq 2$

$$s_{n,k} \sim A_{k,0} n^{(\theta_1 - 2)k + 2}, \quad n \to \infty, \tag{50}$$

where

$$A_{k,0} = \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) A_{k-j,0} A_{j,0} \binom{\theta_j}{2}. \tag{51}$$

(ii) *If* $\theta_1 = 0$, *then for all* $k \geq 2$

$$s_{n,k} \sim -\frac{A_{1,1}}{2} \frac{(A_{1,0})^{k-1} 4^k}{k!} \left(-\frac{1}{2}\right)_k n^{-2k+1}, \quad n \to \infty.$$

(iii) *If* $\theta_1 = 1$, *then for all* $k \geq 2$

$$s_{n,k} \sim A_{1,2} (A_{1,0})^{k-1} \; n^{-k}, \quad n \to \infty.$$

Proof. Using (46) in (40), we have

$$n^{\theta_k} \sum_{l \geq 0} A_{k,l} n^{-l} = \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) n^{\theta_{k-j} + \theta_j} \sum_{m \geq 2} n^{-m} \sum_{l=2}^{m} \sum_{i=1}^{\frac{l}{2}} A_{k-j,m-l} A_{j,l-2i} \binom{\theta_j + 2i - l}{2i}. \tag{52}$$

For $k = 2$, we get

$$\begin{aligned} n^{\theta_2} \sum_{l \geq 0} A_{2,l} n^{-l} &= n^{2\theta_1} \sum_{m \geq 2} n^{-m} \sum_{l=2}^{m} \sum_{i=1}^{\frac{l}{2}} A_{1,m-l} A_{1,l-2i} \binom{\theta_1 + 2i - l}{2i} \\ &= n^{2\theta_1 - 2} \left\{ (A_{1,0})^2 \binom{\theta_1}{2} + A_{1,0} A_{1,1} (\theta_1 - 1)^2 n^{-1} + \left[(A_{1,0})^2 \binom{\theta_1}{4} \right. \right. \\ &\quad \left. \left. + (A_{1,1})^2 \binom{\theta_1 - 1}{2} + A_{1,2} A_{1,0} \left(\theta_1^2 - 3\theta_1 + 3\right) \right] n^{-2} + O\left(n^{-3}\right) \right\}, \end{aligned}$$

and therefore we need to consider three cases.

(i) $\theta_1 \neq 0, 1$. In this case,

$$\theta_2 = 2\theta_1 - 2, \quad A_{2,0} = (A_{1,0})^2 \binom{\theta_1}{2},$$

and a simple induction argument shows that

$$\theta_k = k\theta_1 - 2(k-1), \quad k \geq 1.$$

We conclude from (52) that the leading coefficient satisfies

$$A_{k,0} = \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) A_{k-j,0} A_{j,0} \binom{\theta_j}{2}, \quad k \geq 2,$$

since

$$\theta_{k-j} + \theta_j - \theta_k = 2, \quad 1 \leq j \leq k-1, \quad k \geq 2.$$

(ii) $\theta_1 = 0$. In this case,

$$\theta_2 = -3, \quad A_{2,0} = A_{1,0} A_{1,1},$$

and we can show by induction that

$$\theta_k = -2k + 1, \quad k \geq 2.$$

Hence,

$$\theta_k = -2k + 1 + \delta_{k,1}, \quad k \geq 1,$$

and if $k \geq 2$ we get

$$\theta_{k-j} + \theta_j - \theta_k = 1 + \delta_{k-j,1} + \delta_{j,1} = \begin{cases} 1, & 2 \leq j \leq k-2 \\ 2, & j = 1, k-1 \end{cases}. \tag{53}$$

From (52) we see that for $k \geq 3$

$$n^{\theta_k} \sum_{l=0}^{1} A_{k,l} n^{-l} = \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) n^{\theta_{k-j} + \theta_j - 2} A_{k-j,0} A_{j,0} \binom{\theta_j}{2}, \tag{54}$$

and we conclude from (53) that

$$A_{k,0} = \frac{2}{k(k-1)} A_{1,0} A_{k-1,0} \binom{\theta_{k-1}}{2} = 2 \frac{2k-3}{k} A_{1,0} A_{k-1,0}, \quad k \geq 3.$$

Thus,

$$A_{k,0} = -\frac{A_{1,1}}{2} \frac{(A_{1,0})^{k-1} 4^k}{k!} \left(-\frac{1}{2}\right)_k, \quad k \geq 2.$$

(iii) $\theta_1 = 1$. In this case we have

$$\theta_2 = -2, \quad A_{2,0} = A_{1,0}A_{1,2},$$

and using induction we get

$$\theta_k = -k, \quad k \geq 2.$$

Therefore,

$$\theta_k = -k + 2\delta_{k,1}, \quad k \geq 1,$$

and if $k \geq 2$ we have

$$\theta_{k-j} + \theta_j - \theta_k = 2\left(\delta_{k-j,1} + \delta_{j,1}\right) = \begin{cases} 0, & 2 \leq j \leq k-2 \\ 2, & j = 1, k-1 \end{cases} . \tag{55}$$

Using (54), we obtain

$$A_{k,0} = \frac{2}{k\left(k-1\right)} A_{1,0}A_{k-1,0} \binom{\theta_{k-1}}{2} = A_{1,0}A_{k-1,0}, \quad k \geq 3,$$

and hence

$$A_{k,0} = A_{1,2}\left(A_{1,0}\right)^{k-1}, \quad k \geq 2.$$

■

We now specialize our main result to the case when $\mu_0(z)$ is a hypergeometric function.

Corollary 22 *Suppose that the first moment $\mu_0(z)$ is given by (27). Then:*

(i) *If $p = q - 1$ and $m \geq 1$,*

$$\sigma_n(z) = \sum_{k=0}^{m} s_{n,k} z^k + O\left(n^{-2m-1}\right), \quad n \to \infty. \tag{56}$$

(ii) *If $p = q$ and $m \geq 1$,*

$$\sigma_n(z) = \sum_{k=0}^{m} s_{n,k} z^k + O\left(n^{-m-1}\right), \quad n \to \infty. \tag{57}$$

(iii) *If* $p < q - 1$ *and* $m \geq 1$,

$$\sigma_n(z) = \sum_{k=0}^{m} s_{n,k} z^k + O\left(n^{-(q-p+1)m-(q-p-1)}\right), \quad n \to \infty. \tag{58}$$

Proof. All we need to observe is that if $\mu_0(z)$ is given by (27), then

$$c(n) = \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n \; n!},$$

and therefore

$$s_{n,1} = n^2 \frac{c(n)}{c(n-1)} \sim n^{p-q+1}, \quad n \to \infty.$$

■

Remark 23 *If* $p > q + 1$, *then* $\theta_1 > 2$ *and therefore*

$$\theta_{k+1} - \theta_k = \theta_1 - 2 > 0, \quad k \geq 1.$$

Thus, in this case $\{s_{n,k}\}_{k \geq 1}$ *is not an asymptotic sequence as* $n \to \infty$. *This agrees with Remark 7, since* ${}_pF_q$ *is divergent for* $p > q+1$. *For these orthogonal polynomials we need to take* $a_1 = -N$, *for some* $N \in \mathbb{N}$, *and they will be finite families, with* $0 \leq n \leq N$.

Example 24 *Generalized Charlier. In this case,* $(p, q) = (0, 1)$ *and from (56) we get*

$$\sigma_n = \frac{n(n-1)}{2} + z - \frac{bz}{n} + \frac{b^2 z}{n^2} - \frac{b(b^2+z)z}{n^3} + O\left(n^{-4}\right), \quad n \to \infty.$$

From (42), we obtain

$$\beta_n = n + \frac{bz}{n^2} - \frac{b(2b+1)z}{n^3} + O\left(n^{-4}\right), \quad n \to \infty,$$

and

$$\gamma_n = z - \frac{bz}{n} + \frac{b^2 z}{n^2} - \frac{bz(b^2+2z)}{n^3} + O\left(n^{-4}\right), \quad n \to \infty.$$

These results agree with the limiting values conjectured from numerical experiments by Smet and Van Assche in [32].

Example 25 *Generalized Meixner. In this case,* $(p,q) = (1,1)$ *and from (57) we get as* $n \to \infty$

$$\sigma_n = \frac{n^2}{2} + \left(z - \frac{1}{2}\right) n + (a-b-1)\, z - \frac{(a-b-1)\, bz}{n} - \frac{(a-b-1)\, bz\,(z-b)}{n^2} + O\left(n^{-3}\right).$$

From (42), we obtain

$$\beta_n = n + z + \frac{(a-b-1)\, bz}{n^2} - \frac{(a-b-1)\, bz\,(2b+1-2z)}{n^3} + O\left(n^{-4}\right),$$

and

$$\gamma_n = zn + (a-b-1)\, z - \frac{(a-b-1)\, bz}{n} + \frac{(a-b-1)\, bz\,(b-2z)}{n^2} + O\left(n^{-3}\right),$$

as $n \to \infty$. *These results agree with the limiting values conjectured from numerical experiments by Smet and Van Assche in [32].*

In ([13]), we studied the discrete semiclassical orthogonal polynomials of class $s = 2$. We named the families based on the (p,q) parameters for the hypergeometric representation of the first moment $\mu_0(z)$.

Example 26 *Polynomials of type* $(0,2)$. *These polynomials are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x)\, \frac{1}{(b_1+1)_x\ (b_2+1)_x}\, \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

and therefore

$$s_{n,1} = \frac{n}{(n+b_1)(n+b_2)}.$$

Using (58), we have as $n \to \infty$

$$\sigma_n(z) = \frac{n^2}{2} - \frac{n}{2} + \frac{z}{n} - \frac{(b_1+b_2)\, z}{n^2} + \frac{(b_1^2+b_2^2+b_1 b_2)\, z}{n^3} + O\left(n^{-4}\right),$$

and from (42), we obtain

$$\beta_n = n - \frac{z}{n^2} + \frac{(2b_1+2b_2+1)\, z}{n^3} + O\left(n^{-4}\right),$$

and

$$\gamma_n = \frac{z}{n} - \frac{(b_1+b_2)\, z}{n^2} + \frac{(b_1^2+b_2^2+b_1 b_2)\, z}{n^3} + O\left(n^{-4}\right),$$

as $n \to \infty$.

Example 27 *Polynomials of type* $(1,2)$*. These polynomials are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x) \frac{(a)_x}{(b_1+1)_x \ (b_2+1)_x} \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

and therefore

$$s_{n,1} = \frac{n(n+a-1)}{(n+b_1)(n+b_2)}.$$

Using (56), we have as $n \to \infty$

$$\sigma_n = \frac{n^2}{2} - \frac{n}{2} + z + \frac{(a-1-b_1-b_2)z}{n} + O\left(n^{-2}\right)$$

and from (42), we obtain

$$\beta_n = n - \frac{(a-1-b_1-b_2)z}{n^2} + O\left(n^{-3}\right),$$

and

$$\gamma_n = z + \frac{(a-1-b_1-b_2)z}{n} + O\left(n^{-2}\right)$$

as $n \to \infty$.

Example 28 *Polynomials of type* $(2,2)$*. These polynomials are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x) \frac{(a_1)_x (a_2)_x}{(b_1+1)_x \ (b_2+1)_x} \frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

and therefore

$$s_{n,1} = \frac{n(n+a_1-1)(n+a_2-1)}{(n+b_1)(n+b_2)}.$$

Using (57), we have as $n \to \infty$

$$\sigma_n(z) = \frac{n^2}{2} + \left(z - \frac{1}{2}\right) n + (a_1+a_2-2-b_1-b_2) z + O\left(n^{-1}\right),$$

and from (42), we obtain

$$\beta_n = n + z + O\left(n^{-2}\right),$$

and

$$\gamma_n = zn + (a_1+a_2-2-b_1-b_2) z + O\left(n^{-1}\right),$$

as $n \to \infty$.

Remark 29 *If $\theta_1 = 2$, then (39) and (50) give*

$$s_{n,k} \sim A_{k,0} n^2, \quad n \to \infty,$$

for all $k \geq 0$. From (51) we get

$$A_{k,0} = \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) A_{k-j,0} A_{j,0},$$

and therefore $A_{k,0} = (A_{1,0})^k$ for all $k \geq 1$. If $A_{1,0} = 1$, it follows that

$$\sigma_n(z) = \left(\frac{1}{2} + \sum_{k=1}^{\infty} A_{1,0}^k z^k \right) n^2 + O(n) = \left(\frac{1}{2} - \frac{z}{z-1} \right) n^2 + O(n), \quad n \to \infty.$$

We conclude that in this case, the natural variable to use is $w = \frac{z}{z-1}$.

4.1 The variable w

In this section, we "translate" our previous results to the variable $w = \frac{z}{z-1}$.

Theorem 30 *Let $\sigma_n(z)$ defined by (12). If we write*

$$\sigma_n(w) = \sum_{k=0}^{\infty} \xi_{n,k} w^k,$$

with $w = \frac{z}{z-1}$, then we have

$$\xi_{n,0} = \frac{n(n-1)}{2}, \quad \xi_{n,1} = -n^2 \frac{c_n}{c(n-1)}, \tag{59}$$

and

$$\xi_{n,k} = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \xi_{n,k-j} (\nabla \Delta \xi_{n,j} + k\delta_{1,j}), \quad k \geq 2, \tag{60}$$

where the coefficients $c(n)$ were defined in (35). In particular,

$$\xi_{n,2} = \xi_{n,1} \left(1 + \frac{1}{2} \nabla \Delta \xi_{n,1} \right). \tag{61}$$

Proof. If we use the identity [27, 26.3.4]

$$\sum_{k=0}^{\infty} \binom{n+k}{k} w^k = (1-w)^{-n-1},$$

we have

$$\left(\frac{w}{w-1}\right)^n = (-w)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} w^k. \tag{62}$$

Using (62) in (37), we get

$$\sigma_n(w) = \frac{n(n-1)}{2} - n^2 \frac{c(n)}{c(n-1)} w + O\left(w^2\right), \quad w \to 0,$$

and (59) follows. Using (1) in (41), we obtain

$$(1-w)\,\sigma_n''(w) - 2\sigma_n'(w) = \sigma_n'(w)\,\frac{\nabla\Delta\sigma_n(w) - 1}{w},$$

and since

$$\frac{\nabla\Delta\sigma_n(w) - 1}{w} = \sum_{k=1}^{\infty} \nabla\Delta\xi_{n,k} w^{k-1} = \sum_{k=0}^{\infty} \nabla\Delta\xi_{n,k+1} w^k,$$

we see that

$$(1-w)\,\sigma_n''(w) = \sigma_n'(w) \sum_{k=0}^{\infty} \left(2\delta_{k,0} + \nabla\Delta\xi_{n,k+1}\right) w^k.$$

Comparing coefficients of w, we conclude that

$$\xi_{n,k} = \xi_{n,k-1} + \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j)\,\xi_{n,k-j} \nabla\Delta\xi_{n,j}.$$

■

Remark 31 *If we use (1) in (14), we see that*

$$\gamma_n = z\sigma_n'(z) = w(1-w)\,\sigma_n'(w),$$

and therefore

$$\gamma_n = (1-w) \sum_{k=1}^{\infty} k\xi_{n,k} w^k.$$

Theorem 32 *Let $c(n)$ be defined by (35) and the coefficients $\xi_{n,k}$ be defined by (40). If*

$$-\frac{c(n)}{c(n-1)} \sim \sum_{l\geq 0} B_{1,l} n^{-l}, \quad n\to\infty, \tag{63}$$

we have:

(i) *If $B_{1,0}\neq -1$, then for all $k\geq 1$*

$$\xi_{n,k} = B_{1,0}\left(1+B_{1,0}\right)^{k-1} n^2 + O(n), \quad n\to\infty.$$

(ii) *If $B_{1,0}=-1$ and $B_{1,3}\neq 0$, then for all $k\geq 2$,*

$$\xi_{n,k} = O\left(n^{-k+1}\right), \quad n\to\infty. \tag{64}$$

Proof. Suppose that for $j\geq 1$

$$\xi_{n,j} \sim n^{\tau_j} \sum_{l\geq 0} B_{j,l} n^{-l}, \quad n\to\infty,$$

with $B_{j,0}\neq 0$. Using (46) in (60), we obtain

$$n^{\tau_k}\sum_{l\geq 0} B_{k,l} n^{-l} \sim n^{\tau_{k-1}} \sum_{l\geq 0} B_{k-1,l} n^{-l} \tag{65}$$

$$+\frac{2}{k(k-1)}\sum_{j=1}^{k-1}(k-j)\, n^{\tau_{k-j}+\tau_j}\sum_{m\geq 2} n^{-m}\sum_{l=2}^{m}\sum_{i=1}^{\frac{l}{2}} B_{k-j,m-l} B_{j,l-2i}\binom{\tau_j+2i-l}{2i}.$$

From (59) and (63) we see that

$$\xi_{n,1} \sim n^2 \sum_{l\geq 0} B_{1,l} n^{-l}, \quad n\to\infty,$$

and using (61) we get

$$\xi_{n,2} = \left(1+B_{1,0}\right)\left(B_{1,0}n^2 + B_{1,1}n + B_{1,2}\right) + \left(1+2B_{1,0}\right) B_{1,3} n^{-1} + O\left(n^{-2}\right), \quad n\to\infty.$$

Thus, we need to consider two cases.

(i) $B_{1,0} \neq -1$. In this case, $\tau_2 = 2$, and it follows from (65) that $\tau_k = \tau_{k-1} = 2, \quad k \geq 2$. To leading order, (65) gives

$$B_{k,0} = B_{k-1,0} + \frac{2}{k(k-1)} \sum_{j=1}^{k-1} (k-j) B_{k-j,0} B_{j,0},$$

and therefore

$$B_{k,0} = B_{1,0} (1 + B_{1,0})^{k-1}.$$

(ii) $B_{1,0} = -1$, $B_{1,3} \neq 0$. We now have $\tau_2 = -1$, and we can show by induction that $\tau_k = -k+1$, $k \geq 2$. Hence, $\tau_k = -k+1+2\delta_{1,k}$, $k \geq 1$ and we observe that

$$\tau_{k-j} + \tau_j - \tau_k = 1 + 2(\delta_{k-j,1} + \delta_{j,1}) = \begin{cases} 1, & 2 \leq j \leq k-2 \\ 3, & j = 1, k-1 \end{cases}. \tag{66}$$

Using (66) in (65), we have for $k \geq 3$

$$B_{k,0} = B_{k-1,0} n + B_{k-1,1} + B_{k-1,0} B_{1,0} n + \frac{k+2}{k} B_{1,0} B_{k-1,1} + \frac{k-2}{k} B_{k-1,0} B_{1,1},$$

and since $B_{1,0} = -1$,

$$B_{k,0} = -\frac{2}{k} B_{k-1,1} + \frac{k-2}{k} B_{k-1,0} B_{1,1}, \quad k \geq 3. \tag{67}$$

■

Example 33 *Meixner polynomials. From (43), we see that*

$$\frac{c(n)}{c(n-1)} = -\frac{n+a-1}{n} = -1 + \frac{1-a}{n}. \tag{68}$$

Therefore, we are in case (ii) of Theorem 32. Using (68) in (59) we get

$$\xi_{n,1} = -n(n+a-1),$$

and (60) gives

$$\xi_{n,k} = 0, \quad k \geq 2.$$

Hence, we conclude that

$$\sigma_n(w) = \frac{1}{2} n(n-1) - n(n+a-1) w = \left(\frac{1}{2} - w\right) n^2 + \left(w - aw - \frac{1}{2}\right) n,$$

and

$$\beta_n(w) = n - (2n+a)w, \quad \gamma_n(w) = n(n+a-1)w(w-1),$$

in agreement with (26).

Example 34 *Generalized Hahn polynomials of type I. From (44), we see that*

$$-\frac{c(n)}{c(n-1)} = -\frac{(n+a_1-1)(n+a_2-1)}{n(n+b)} - 1 + \frac{b+2-a_1-a_2}{n} \qquad (69)$$
$$-(b+1-a_1)(b+1-a_2)\sum_{k=0}^{\infty}\frac{(-b)^k}{n^{k+2}}.$$

Since $B_{1,0} = -1$ and $B_{1,3} \neq 0$, we are in case (ii) of Theorem 32. Using (69) in (59), we get $B_{1,1} = b+2-a_1-a_2$ and

$$B_{1,k} = -(b+1-a_1)(b+1-a_2)(-b)^k, \quad k \geq 2.$$

Thus, (60) gives

$$\xi_{n,2} = -b(b+1-a_1)(b+1-a_2)\left[n^{-1} - (4b+2-a_1-a_2)n^{-2}\right] + O\left(n^{-3}\right),$$

and

$$\xi_{n,3} = -b(b+1-a_1)(b+1-a_2)(3b+2-a_1-a_2)n^{-2} + O\left(n^{-3}\right),$$

as $n \to \infty$, in agreement with (64). Note that

$$B_{3,0} = \left[-\frac{2}{3}(4b+2-a_1-a_2) - \frac{1}{3}(b+2-a_1-a_2)\right]b(b+1-a_1)(b+1-a_2),$$

in agreement with (67).

We conclude that

$$\sigma_n(w) = \left(\frac{1}{2}-w\right)n^2 + \left[(b+2-a_1-a_2)w - \frac{1}{2}\right]n - (b+1-a_1)(b+1-a_2)w$$
$$-b(b+1-a_1)(b+1-a_2)w(w-1)n^{-1} + O\left(n^{-2}\right),$$

$$\beta_n = (1-2w)n + (b+1-a_1-a_2)w + b(b+1-a_1)(b+1-a_2)w(w-1)n^{-2} + O\left(n^{-3}\right),$$

and

$$\frac{\gamma_n}{w(w-1)} = n^2 - (b+2-a_1-a_2)\,n + (b+1-a_1)(b+1-a_2)$$
$$+\, b(b+1-a_1)(b+1-a_2)(2w-1)\,n^{-1} + O\left(n^{-2}\right),$$

as $n \to \infty$.

Remark 35 *In ([20]), the authors considered the sequences* (x_n, y_n)*, defined by*

$$x_n = \beta_n - \frac{n + (n+a_1+a_2)\,z - (b+1)}{1-z} \tag{70}$$
$$= \beta_n + (2w-1)\,n + b + 1 + (a_1+a_2-b-1)\,w,$$

and

$$y_n = \frac{1-z}{z}\gamma_n - \sigma_n - (a_1+a_2)\,n - \frac{1}{2}n(n-1) \tag{71}$$
$$= -\frac{\gamma_n}{w} - \sigma_n - (a_1+a_2)\,n - \frac{1}{2}n(n-1).$$

Using the results from our previous example, we see that

$$x_n = b + 1 + b(b+1-a_1)(b+1-a_2)\,w(w-1)\,n^{-2} + O\left(n^{-3}\right),$$

and

$$y_n = -(b+1)\,n + (b+1-a_1)(b+1-a_2) - b(b+1-a_1)(b+1-a_2)(w-1)^2\,n^{-1} + O\left(n^{-2}\right).$$

These expansions agree with the limiting values conjectured from numerical experiments by Filipuk and Van Assche.

Example 36 *Polynomials of type* $(3,2)$*. These polynomials were introduced in [13], and are orthogonal with respect to the linear functional*

$$L[p] = \sum_{x=0}^{\infty} p(x)\,\frac{(a_1)_x\,(a_2)_x\,(a_3)_x}{(b_1+1)_x\ (b_2+1)_x}\,\frac{z^x}{x!}, \quad p \in \mathbb{F}[x],$$

with $b_1 \neq b_2$ *and* $a_i \neq b_j + 1$, $1 \leq i \leq 3$, $1 \leq j \leq 2$*. We have*

$$-\frac{c(n)}{c(n-1)} = -\frac{(n+a_1-1)(n+a_2-1)(n+a_3-1)}{n(n+b_1)(n+b_2)} \tag{72}$$
$$-1 + \frac{b_1+b_2+3-a_1-a_2-a_3}{n} - \sum_{k=0}^{\infty} \frac{\eta_k(b_2) - \eta_k(b_1)}{b_2 - b_1}\,n^{-k-2},$$

where

$$\eta_k(x) = (-x)^k \prod_{j=1}^{3} (x + 1 - a_j).$$

Since $B_{1,0} = -1$ and $B_{1,3} \neq 0$, we are in case (ii) of Theorem 32 and therefore $\xi_{n,2} = O(n^{-1})$, $n \to \infty$. We conclude that

$$\sigma_n(w) = \left(\frac{1}{2} - w\right) n^2 + \left[(b_1 + b_2 + 3 - a_1 - a_2 - a_3)w - \frac{1}{2}\right] n - \frac{\eta_0(b_2) - \eta_0(b_1)}{b_2 - b_1} w + O\left(n^{-1}\right),$$

$$\beta_n = (1 - 2w)\, n - (a_1 + a_2 + a_3 - b_1 - b_2 - 2)w + O\left(n^{-1}\right),$$

and

$$\frac{\gamma_n}{w(w-1)} = n^2 - (b_1 + b_2 + 3 - a_1 - a_2 - a_3)n + \frac{\eta_0(b_2) - \eta_0(b_1)}{b_2 - b_1} + O\left(n^{-1}\right)$$

as $n \to \infty$.

5 The shifted lattice

In [32] and [20], the authors consider shifted linear functionals of the form

$$\widetilde{L}[r] = \sum_{x=0}^{\infty} r(x) \frac{(\mathbf{a})_{x-b}}{(b+1)_{x-b}} \frac{z^{x-b}}{(x-b)!}, \quad r \in \mathbb{F}[x]. \tag{73}$$

The moments of L and $\widetilde{L}$ are related by

$$\mu_n(z) = \sum_{k \geq 0} k^n c(k-b)\ z^{k-b} = z^{-b} \widetilde{\mu}_n(z), \quad n \in \mathbb{N}_0,$$

and the Hankel determinants by

$$z^{nb}\mathcal{H}_n = z^{nb} \det_{0 \leq i,j \leq n-1} (\mu_{i+j}) = \det_{0 \leq i,j \leq n-1} \left(z^b \mu_{i+j}\right) = \det_{0 \leq i,j \leq n-1} (\widetilde{\mu}_{i+j}) = \widetilde{\mathcal{H}}_n.$$

Therefore, we see from (36) that

$$\mathcal{H}_n(z) = z^{\binom{n}{2} - nb}\, \widetilde{g}_n(z) \prod_{j=0}^{n-1} \left[(j!)^2 c(j-b)\right], \quad n \geq 2,$$

where

$$\widetilde{g}_n(z) = 1 + \binom{n}{1}^2 \frac{c(n-b)}{c(n-b-1)} z + O\left(z^2\right).$$

We conclude that

$$\widetilde{\sigma}_n(z) = \frac{n(n-1)}{2} - nb + n^2 \frac{c(n-b)}{c(n-b-1)} z + O\left(z^2\right), \quad z \to 0, \tag{74}$$

and we can apply the results of Theorem 21 if we replace $n^2 \frac{c(n)}{c(n-1)}$ by

$$n^2 \frac{c(n-b)}{c(n-b-1)}.$$

Let's look at some examples.

Example 37 *Generalized Charlier polynomials on the shifted lattice. We have*

$$n^2 \frac{c(n-b)}{c(n-b-1)} = \frac{n}{n-b} = 1 + \frac{b}{n} + \frac{b^2}{n^2} + O\left(n^3\right), \quad n \to \infty,$$

and

$$\widetilde{\sigma}_n(z) = \frac{n^2}{2} - \left(b + \frac{1}{2}\right) n + z + \frac{bz}{n} + \frac{b^2 z}{n^2} + \frac{bz\left(b^2 + z\right)}{n^3} + O\left(n^{-4}\right), \quad n \to \infty.$$

From (42), we obtain

$$\widetilde{\beta}_n = n - b - \frac{bz}{n^2} + \frac{(1-2b)\, bz}{n^3} + O\left(n^{-4}\right), \quad n \to \infty$$

and

$$\widetilde{\gamma}_n = z + \frac{bz}{n} + \frac{b^2 z}{n^2} + \frac{bz\left(b^2 + 2z\right)}{n^3} + O\left(n^{-4}\right), \quad n \to \infty.$$

These results agree with the limiting values conjectured from numerical experiments by Smet and Van Assche in [32].

Example 38 *Generalized Meixner polynomials on the shifted lattice. We have*

$$n^2 \frac{c(n-b)}{c(n-b-1)} = \frac{n(n+a-b-1)}{n-b} = n + a - 1 + \frac{b(a-1)}{n} + \frac{b^2(a-1)}{n^2} + O\left(n^3\right),$$

and

$$\widetilde{\sigma}_n(z) = \frac{n^2}{2} + \left(z - b - \frac{1}{2}\right) n + (a-1)\, z + \frac{(a-1)\, bz}{n} + \frac{(a-1)\, bz\,(z+b)}{n^2} + O\left(n^3\right),$$

as $n \to \infty$. *From (42), we get*

$$\widetilde{\beta}_n = n + z - b - \frac{(a-1)\, bz}{n^2} - \frac{(a-1)bz(2b+2z-1)}{n^3} + O\left(n^{-4}\right),$$

and

$$\widetilde{\gamma}_n = zn + (a-1)\, z + \frac{(a-1)\, bz}{n} + \frac{(a-1)\, bz\,(2z+b)}{n^2} + O\left(n^3\right),$$

as $n \to \infty$. *These results agree with the limiting values conjectured from numerical experiments by Smet and Van Assche in [32].*

Example 39 *Generalized Hahn polynomials of type I on the shifted lattice. We have*

$$-n^2 \frac{c\,(n-b)}{c\,(n-b-1)} = -\frac{n\,(n+a_1-b-1)\,(n+a_2-b-1)}{n-b} = -n^2$$
$$-(a_1+a_2-b-2)\, n - (a_1-1)\,(a_2-1) - \frac{(a_1a_2-a_1-a_2+1)\, b}{n} + O\left(n^{-2}\right),$$

and

$$\widetilde{\sigma}_n(w) = \left(\frac{1}{2} - w\right) n^2 + \left(2w - wa_1 - wa_2 + bw - b - \frac{1}{2}\right) n$$
$$+ (a_1 + a_2 - a_1a_2 - 1)w + \frac{(a_1-1)(a_2-1)b(w-1)w}{n} + O\left(n^{-2}\right),$$

as $n \to \infty$. *We conclude that*

$$\beta_n = (1-2w)\, n + w - wa_1 - wa_2 + bw - b - \frac{(a_1-1)\,(a_2-1)\, bw\,(w-1)}{n^2} + O\left(n^{-3}\right),$$

and

$$\frac{\gamma_n}{w\,(w-1)} = n^2 + (a_1+a_2-b-2)n + (a_1-1)\,(a_2-1) + \frac{(a_1-1)\,(a_2-1)\, b(2w-1)}{n} + O\left(n^{-2}\right),$$

as $n \to \infty$.

In terms of the sequences (x_n, y_n) *defined by (70) and (71, we get*

$$x_n = 1 - \frac{(a_1 - 1)(a_2 - 1)\, bw\, (w-1)}{n^2} + O\left(n^{-3}\right), \quad n \to \infty,$$

and

$$y_n = -n + (a_1 - 1)(a_2 - 1) + \frac{(a_1 - 1)(a_2 - 1)\, b\, (w-1)^2}{n} + O\left(n^{-2}\right), \quad n \to \infty.$$

These results agree with the limiting values conjectured from numerical experiments by Filipuk and Van Assche in [20].

6 Conclusions

We have analyzed the three-term recurrence coefficients (β_n, γ_n) of orthogonal polynomials associated to a perturbed linear functional depending on a variable z. The functions $\beta_n(z), \gamma_n(z)$ satisfy the Toda system

$$D_z \beta_n = \Delta \gamma_n, \quad D_z \ln \gamma_n = \nabla \beta_n,$$

and we have obtained asymptotic expansions of $\beta_n(z), \gamma_n(z)$ as $n \to \infty$.

We have shown that our methods can be used to prove some conjectures stated by Walter Van Assche and collaborators.

In follow-up papers, we will use our results to obtain nonlinear ODEs for the functions $\beta_n(z), \gamma_n(z)$, and we will analyze the polynomials $P_n(x; z)$ asymptotically as $n \to \infty$.

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