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An Implementation of Radu's Ramanujan-Kolberg Algorithm

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Abstract

In 2015 Cristian-Silviu Radu designed an algorithm to detect identities of a class studied by Ramanujan and Kolberg. This class includes the famous identities by Ramanujan which provide a witness to the divisibility properties of $p(5n+4)$, $p(7n+5)$. We give an implementation of this algorithm using Mathematica. The basic theory is first described, and an outline of the algorithm is briefly given, in order to describe the functionality and utility of our package. We thereafter give multiple examples of applications to recent work in partition theory. In many cases we have used our package to derive alternate proofs of various identities or congruences; in other cases we have improved previously established identities, and in at least one case we have confirmed a standing conjecture.

1 Introduction

Given some $n \in \mathbb{Z}_{\geq 0}$, we define a partition of n as a weakly decreasing sequence of positive integers which sum to n. Thus, the number 4 has 5 different partitions: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$. We define $p(n)$ as the number of partitions of n. Thus, $p(4) = 5$ (we define $p(0) = 1$).

The function $p(n)$ has been seriously studied since 1748 [13], when Euler identified the generating function for $p(n)$ (with q a formal indeterminate):

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}. \tag{1}$$

However, almost nothing was known of the arithmetic properties of $p(n)$ before the twentieth century. One of the first major breakthroughs in this area came from Ramanujan [31]:

Theorem 1.

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \cdot \prod_{m=1}^{\infty} \frac{(1-q^{5m})^5}{(1-q^m)^6}, \tag{2}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 49q \cdot \prod_{m=1}^{\infty} \frac{(1-q^{7m})^7}{(1-q^m)^8} + 7 \cdot \prod_{m=1}^{\infty} \frac{(1-q^{7m})^3}{(1-q^m)^4}. \tag{3}$$

These are among the most iconic results in partition theory. They are particularly remarkable in that they reveal arithmetic information about $p(n)$:

Corollary. *For all* $n \in \mathbb{Z}_{\geq 0}$,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7.$$

Moreover, the overall form of the identities conveys a deep relationship in the underlying theory of modular functions.

Nearly 40 years later, Kolberg realized [18] that these identities of Ramanujan could, with a very slight generalization, be extended to include a much larger variety of similar identities for $p(5n+j)$, $p(7n+j)$, and $p(3n+j)$. For instance,

$$\left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right) \tag{4}$$
$$=25q\prod_{m=1}^{\infty}\frac{(1-q^{5m})^{10}}{(1-q^m)^{12}} + 2\prod_{m=1}^{\infty}\frac{(1-q^{5m})^{4}}{(1-q^m)^{6}},$$

$$\left(\sum_{n=0}^{\infty} p(7n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+3)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+4)q^n\right) \tag{5}$$
$$=117649q^4\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{21}}{(1-q^m)^{24}} + 50421q^3\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{17}}{(1-q^m)^{20}}$$
$$+ 8232q^2\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{13}}{(1-q^m)^{16}} + 588q\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{9}}{(1-q^m)^{12}}$$
$$+ 15\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{5}}{(1-q^m)^{8}},$$

$$\left(\sum_{n=0}^{\infty} p(3n)q^n\right)\left(\sum_{n=0}^{\infty} p(3n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(3n+2)q^n\right) \tag{6}$$
$$=9q\prod_{m=1}^{\infty}\frac{(1-q^{3m})(1-q^{9m})^{6}}{(1-q^m)^{10}} + 2\prod_{m=1}^{\infty}\frac{(1-q^{3m})(1-q^{9m})^{3}}{(1-q^m)^{7}}.$$

There are many different approaches by which these sorts of identities may be derived. Kolberg, for example, proved each of the examples above (including Ramanujan's results) by manipulation of certain formal power series. We will study them using the theory of modular functions.

The principle behind these identities is that by changing variables to $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, the generating function for $p(n)$ is (very nearly) the multiplicative inverse of the Dedekind η function. This allows us to isolate and express the series

$$\sum_{n=0}^{\infty} p(mn+j)q^n,\ j,m \in \mathbb{Z}_{\geq 0},\ 0 \leq j < m,\ 1 \leq m.$$

in terms of linear combinations of η (with a fractional input). We can then take advantage of the symmetric properties of η to construct a modular function using $\sum_{n\geq 0} p(mn+j)q^n$, over an appropriately chosen congruence subgroup $\Gamma_0(N)$. Finally, we can ask whether this modular function is a linear combination of suitably defined eta quotients, by manipulating and studying its behavior at the cusps of $\Gamma_0(N)$.

What makes this a particularly powerful approach from a computational standpoint is that certain theorems from complex analysis impose a finiteness condition on the behavior of any modular function near a cusp of its corresponding subgroup. This allows us to check the equality of two given modular functions by checking the equality of only a finite number of coefficients.

Cristian-Silviu Radu recognized [29] that this approach could be used to construct an algorithm to compute identities in the form of those discovered by Ramanujan and Kolberg above. Indeed, he designed an algorithm which takes any arithmetic function $a(n)$ with generating function

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M}\prod_{m=1}^{\infty}(1-q^{\delta m})^{r_\delta},$$

with $r_\delta \in \mathbb{Z}$ for all $\delta|M$, and a generating function for $a(mn+j)$, with $0 \leq j \leq m-1$. From here, and an appropriately chosen $N \in \mathbb{Z}_{\geq 2}$, the algorithm attempts to produce a set $P_{m,r}(j) \subseteq \{0, 1, 2, ..., m-1\}$ with member j; an integer-valued vector $s = (s_\delta)_{\delta|N}$; and some $\alpha \in \mathbb{Z}$ such that

$$\begin{aligned} f_{LHS} :=& f_{LHS}(s, N, M, r, m, j)(\tau) \\ =& q^{\alpha}\prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta} \cdot \prod_{j'\in P_{m,r}(j)}\sum_{n=0}^{\infty} p(mn+j')q^n \end{aligned} \tag{7}$$

is a modular function with a single pole at ∞ over the subgroup $\Gamma_0(N)$.

From here, a basis for the $\mathbb{Q}$-algebra generated by all eta quotient modular functions with a single pole at ∞ over $\Gamma_0(N)$ can be constructed. We can check membership of f_{LHS} in this algebra by examining only its principal part over q (including its constant term).

This paper summarizes our successful implementation of Radu's algorithm. Section 2.1 will provide a very brief review the basic theory, and in Sections 2.2 to 2.3 an outline of our software package's structure will be given, following the design of Radu's algorithm. Due to matters of space, we cannot provide more than a short description of the algorithm, or the underlying theory. We highly recommend that this paper be read as a companion to [29] and [28]. We have changed the notation of these papers: notably, we denote an arithmetic progression with the letters m, j, rather than m, t to avoid confusion with the use of t as a modular function. We have also denoted by $h_{m,j}$ what would be referred to in [29] and [28] as $g_{m,j}$, and have generally reserved the letter g to denote an arbitrary eta quotient.

In addition to some small notational changes from Radu's original work, we have also designed separate procedures, which account for various theoretical or computational difficulties. We discuss these separate procedures in Section 2.4.

We refer the reader to [11], [20], and especially [17] for a more comprehensive treatment of the theory of modular forms.

Apart from a description of the basic features of our package, the bulk of our paper will be examples computed by our software package. We cover the classic cases of Ramanujan, Kolberg, and Zuckerman in Sections 3.1, 3.2. In Section 3.3-3.4 we show examples which Radu has previously computed, and which we have given slight improvements to. In Sections 3.5-3.9 we give applications of our package to recently discovered identities and congruences. In many cases we are able to improve previous results. In one case (Section 3.5.2) we prove a conjecture by Xia. Section 4 explains the availability of the package, as well as its installation.

2 Background

2.1 Basic Theory

Let $N \in \mathbb{Z}_{>0}$. We will denote $\mathbb{H}$ as the upper half complex plane, and we will let $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$ (except in Section 3.10, wherein we will use $z \in \mathbb{H}$) . Hereafter, we will use the notation

$$(q^a; q^b)_\infty := \prod_{m=0}^{\infty} (1 - q^{bm+a}).$$

In particular,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty}. \tag{8}$$

We now give only the necessary preliminaries for an understanding of the RK algorithm and its underlying theory.

Denote $\mathrm{SL}(2, \mathbb{Z})$ to be the set of all 2×2 integer matrices with determinant 1.

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}.$$

Furthermore, we let $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : N|c \right\},$$

Definition 1. Let $a/c \in \mathbb{Q} \cup \{\infty\}$. The cusp over $\Gamma_0(N)$ represented a/c is the coset

$$\Gamma_0(N) \cdot \frac{a}{c}.$$

If $a_1/c_1 \in \Gamma_0(N) \cdot \frac{a}{c}$, then a_1/c_1 represents the same cusp as a/c.

Notice that because $\Gamma_0(N)$ is a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ [11, Section 1.2], any congruence subgroup admits only a finite number of distinct cusps.

Definition 2. Let $q = e^{2\pi i \tau}$, with $\tau \in \mathbb{H}$, and suppose that $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function for all $\tau \in \mathbb{H}$. In this case, f is a modular function over $\Gamma_0(N)$ if the following conditions apply:

1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have
$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau);$$

2. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we have
$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n=n_\gamma(f)}^{\infty} \alpha_\gamma(n) q^{\gcd(c^2,N)n/N},$$
with $n_\gamma(f) \in \mathbb{Z}$, and $\alpha_\gamma(n) \in \mathbb{C}$ for all $n \geq n_\gamma(f)$.

Here, we refer to $n_\gamma(f) = n_{a/c}(f)$ as the order of f at the cusp represented by by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, respectively by a/c, over $\Gamma_0(N)$. If $n_{a/c}(f) < 0$, then f is said to have a pole at a/c, with principal part

$$\sum_{n=n_{a/c}(f)}^{-1} \alpha_\gamma(n) q^{\gcd(c^2,N)n/N}.$$

If $n_{a/c}(f) > 0$, then f is said to have a zero at a/c.

We now give an extremely important theorem in the subject of modular forms, upon which the entirety of our paper relies:

Theorem 2. *Let $N \in \mathbb{Z}_{\geq 1}$. If f is a modular function with nonnegative order at every cusp of $\Gamma_0(N)$, then f must be a constant.*

A proof can be found in [17, Chapter 2, Theorem 7]. Its usefulness becomes clear upon comparing any two modular functions. If f, g are both modular functions over $\Gamma_0(N)$, and their principal parts at each of their poles match, then $f - g$ must be a modular function with no poles at any cusp. This forces $f - g$ to be a constant. If their constants also match, then f and g must be equal, since $f - g = 0$.

The question of equality between modular functions can therefore be reduced to the question of comparing their finite principal parts and constants—which can of course be quickly reduced to the question of comparing polynomials.

Hereafter, we will denote $\mathcal{M}(N)$ as the set of all modular functions over $\Gamma_0(N)$. We also define $\mathcal{M}^\infty(N)$ as the set of all modular functions f over $\Gamma_0(N)$ in which $n_\gamma(f) \geq 0$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})\backslash\Gamma_0(N)$. Finally, for any set $\mathcal{S} \subseteq \mathcal{M}(N)$, and any field $\mathbb{K} \subseteq \mathbb{C}$, define $\mathcal{S}_\mathbb{K}$ as the set of functions in $f \in \mathcal{S}$ whose coefficients at infinity are all members of $\mathbb{K}$, i.e., $\alpha_\mathrm{I}(n) \in \mathbb{K}$ for all $n \geq n_\mathrm{I}(f)$.

Finally, we will define the eta function and its extraordinary properties:

Definition 3. For $\tau \in \mathbb{H}$, let

$$\eta(\tau) = q^{1/24}(q;q) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Theorem 3. *For any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, *we have*

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a,b,c,d)\left(-i(c\tau + d)\right)^{1/2} \eta(\tau),$$

with $z^{1/2}$ *defined in terms of its principal branch, and* $\epsilon(a,b,c,d)$ *a certain 24th root of unity.*

The near-modular symmetry of η enables us to construct a very large number of modular functions over $\Gamma_0(N)$. For example, it can be shown that

$$\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6 \in \mathcal{M}(5).$$

Definition 4. An eta quotient over $\Gamma_0(N)$ is an object of the form

$$\prod_{\lambda | N} \eta(\lambda\tau)^{s_\lambda} \in \mathcal{M}(N).$$

Denote $\mathcal{E}(N)$ as the set of all eta quotients over $\Gamma_0(N)$. We denote $\mathcal{E}^{\infty}(N) = \mathcal{E}(N) \cap \mathcal{M}^{\infty}(N)$. Finally, for any set $\mathcal{S}$ of functions over $\mathbb{C}$, denote

$$\langle \mathcal{S} \rangle_{\mathbb{K}} := \left\{ \sum_{u=1}^{v} r_u \cdot g_u : g_u \in \mathcal{S}, r_u \in \mathbb{K} \right\}.$$

It is easy to see that $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{K}}$ fulfills the conditions of a $\mathbb{K}$-algebra.

We will want to determine whether a given $f \in \mathcal{M}(N)$ can be expressed as a linear combination of eta quotients, i.e., whether $f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$. To do this directly, we would be forced to have a complete set of generators for $\langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$, and to study the behavior of f at each cusp of $\Gamma_0(N)$.

To simplify the problem, we introduce the following theorem:

Theorem 4. *For every* $N \in \mathbb{Z}_{\geq 2}$, *there exists a function* $\mu \in \mathcal{E}^{\infty}(N)$ *which has positive order at every cusp of* $\Gamma_0(N)$ *except* ∞.

A proof can be found in [29, Lemma 20].

This theorem is useful in that, for any $f \in \mathcal{M}(N)$, there exists a $\mu \in \mathcal{E}^{\infty}(N)$ and a sufficiently large $k_1 \in \mathbb{Z}_{\geq 0}$, such that $\mu^{k_1} \cdot f \in \mathcal{M}^{\infty}(N)$. Then we need only examine the single principal part of $\mu^{k_1} \cdot f$.

On the other hand, the elements of $\mathcal{E}^{\infty}(N)$ also have only a single principal part to examine; moreover, as we shall see, $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$ contains a very precise algebra structure which can be adapted to check membership for any given element of $\mathcal{M}^{\infty}(N)$. If $\mu^{k_1} \cdot f \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$ for a specific k_1, then we must have

$\mu^{k_1} \cdot f \in \mathcal{M}^\infty(N)_\mathbb{Q} \cap \langle \mathcal{E}(N) \rangle_\mathbb{Q}$. However, the converse is not necessarily true. In other words, we know that

$$\mathcal{M}^\infty(N)_\mathbb{Q} \cap \langle \mathcal{E}(N) \rangle_\mathbb{Q} \supseteq \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q},$$

but we have not yet established that

$$\mathcal{M}^\infty(N)_\mathbb{Q} \cap \langle \mathcal{E}(N) \rangle_\mathbb{Q} = \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}.$$

Current evidence suggests that this equality holds, and we strongly suspect that it is true. Unfortunately, we are as of yet unable to prove it. However, Radu was able [29, Lemma 28] to establish a weaker theorem:

Theorem 5. *Given some $N \in \mathbb{Z}_{\geq 2}$ and a $\mu \in \mathcal{E}^\infty(N)$ with positive order at every cusp except ∞, there exists a $k_0 \in \mathbb{Z}_{\geq 0}$ such that*

$$\mu^{k_0} \cdot \left(\mathcal{M}^\infty(N)_\mathbb{Q} \cap \langle \mathcal{E}(N) \rangle_\mathbb{Q} \right) \subseteq \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}.$$

The ambiguity of whether $k_0 = 0$ will become important later. But what is important for the time being is that an upper bound for k_0 is at least computable [29, Proof of Lemma 28]. With the previous two theorems, in order to check whether $f \in \langle \mathcal{E}(N) \rangle_\mathbb{Q}$, we need only check the equivalent statement that

$$\mu^{k_0+k_1} \cdot f \in \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}.$$

2.2 Membership Algorithm

Suppose that for some $N \in \mathbb{Z}_{\geq 2}$ we have a function $f \in \mathcal{M}^\infty(N)_\mathbb{Q}$. We know that we can expand f as the following:

$$f = \frac{c(-m_1)}{q^{m_1}} + \frac{c(-m_1+1)}{q^{m_1-1}} + ... + \frac{c(-1)}{q} + c(0) + \sum_{n=1}^{\infty} c(n) q^n. \tag{9}$$

Here we will refer to $\mathrm{pord}(f) := m_1$ as the minimal exponent of f.

Moreover, we can identify $c(-m_1)$, the leading coefficient of f, with the notation $\mathrm{LC}(f) := c(-m_1)$.

We now need to define an algorithm to check the potential membership of a given f in $\langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}$. We can take advantage of the very precise algebra basis which $\langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}$ admits.

Theorem 6. *For any $N \in \mathbb{Z}_{\geq 2}$, $\mathcal{E}^\infty(N)$ is a finitely generated monoid. Moreover, there exist functions $t, g_1, g_2, ..., g_v \in \mathcal{M}^\infty(N)$ such that*

$$\mathrm{pord}(t) = v, \tag{10}$$
$$\mathrm{pord}(g_i) < \mathrm{pord}(g_j), \ \textit{for } 1 \leq i < j \leq v \tag{11}$$
$$\mathrm{pord}(g_i) \not\equiv \mathrm{pord}(g_j) \pmod{\mathrm{pord}(t)}, \ \textit{for } 1 \leq i < j \leq v \tag{12}$$
$$\mathrm{pord}(g_i) \not\equiv 0 \pmod{\mathrm{pord}(t)}, \ \textit{for } 1 \leq i \leq v \tag{13}$$
$$\langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q} = \langle 1, g_1, ..., g_v \rangle_{\mathbb{Q}[t]}, \tag{14}$$

The proof can be found in [29, Sections 2.1, 2.2]. Given any $N \in \mathbb{Z}_{\geq 2}$, the corresponding monoid generators of $\mathcal{E}^{\infty}(N)$ can be computed through a terminating algorithm [29, Lemma 25].

PROCEDURE: `etaGenerators` (Eta Monoid Generators)

INPUT:
$N \in \mathbb{Z}_{\geq 2}$

OUTPUT:
$\{\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_r\}$ such that $\left\{\mathcal{E}_1^{k_1} \cdot \mathcal{E}_2^{k_2} \cdot ... \cdot \mathcal{E}_r^{k_r} : k_1, k_2, ..., k_r \in \mathbb{Z}_{\geq 0}\right\} = \mathcal{E}^{\infty}(N)$.

Similarly, the corresponding basis elements of $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$ can be computed through a terminating algorithm [29, Theorem 16].

PROCEDURE: `AB` (Eta Algebra Basis)

INPUT:
$N \in \mathbb{Z}_{\geq 2}$

OUTPUT:
$t, g_1, g_2, ..., g_v \in \mathcal{M}^{\infty}(N)$ such that conditions (10)-(14) are satisfied.

The algebra basis algorithm is the most immediately important for the outline of Radu's algorithm. However, we mention the monoid algorithm because it will prove useful in later examples.

We now suppose that $f \in \mu^{k_0} \cdot \mathcal{M}^{\infty}(N)_{\mathbb{Q}}$. To determine whether $f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$, we need only determine whether $f \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$. The previous theorem reduces this to the problem of checking whether

$$f \in \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]} \,. \tag{15}$$

By Theorem 2.1, we need only examine the principal parts and constants of f to determine whether (15) is correct.

Because the orders of the functions g_j give a complete set of representatives of the residue classes modulo v, we know that $m_1 \equiv \mathrm{pord}(g_{j_1}) \pmod{v}$, for some j_1 with $1 \leq j_1 \leq v$.

Suppose first that $m_1 \geq \mathrm{pord}(g_{j_1})$. Let g_{j_1} have the expansion

$$g_{j_1} = \frac{b_1(-n_1)}{q^{n_1}} + \frac{b_1(-n_1+1)}{q^{n_1-1}} + ... + \frac{b_1(-1)}{q} + b_1(0) + \sum_{n=1}^{\infty} b_1(n) q^n. \tag{16}$$

Then clearly, we can write

$$f_1 = f - \frac{c(-m_1)}{\mathrm{LC}\left(g_{j_1} \cdot t^{\frac{m_1-n_1}{v}}\right)} \cdot g_{j_1} \cdot t^{\frac{m_1-n_1}{v}}, \tag{17}$$

$$\mathrm{pord}\,(f_1) = m_2 < m_1. \tag{18}$$

We can identify m_2 as the residue of the order of another g_{j_2} modulo v. If again we have $m_2 \geq \text{pord}(g_{j_2})$, we may similarly construct a function f_2 subtract a product of g_{j_2} with a suitable power of t to reduce the resulting order still further.

In this way, we may construct a sequence of functions $(f_l)_{l\geq 1}$, with decreasing absolute order at infinity. One possible result of this process is that we find some $k \in \mathbb{Z}_{>1}$ such that $f_{k-1} \in \mathbb{Q}[[q]]$, having no negative powers of q—that is,

$$f_{k-1} = c_{k-1}(0) + \sum_{n=1}^{\infty} c_{k-1}(n)q^n. \tag{19}$$

Of course, $c_{k-1}(0) \in \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$, so that $f_k = f_{k-1} - c_{k-1}(0)$ has no principal part and no constant. In this case, we have shown that the principal part and constant of f can be constructed through combinations of the principal parts and constants of 1, $g_1, g_2, ..., g_v, t$. Since we only need to match the principal parts and constants, we can conclude that $f \in \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$.

On the other hand, let us suppose that before the principal part is completely reduced, we produce a function f_l such that $\text{pord}(f_l) = m_l < \text{pord}(g_{j_l})$. In this case, no power of g_{j_l} can reduce the order of f_l, and no other element in our basis can have a matching order modulo v. We must immediately conclude that the principal part of f cannot be reduced in terms of the principal parts of $\langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$. Of course, this implies that $f \notin \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$.

As we reduce the principal part of f, we can collect the terms

$$\frac{c(-m_l)}{\text{LC}\left(g_{j_l} \cdot t^{\frac{m_l - n_l}{v}}\right)} \cdot g_{j_l} \cdot t^{\frac{m_l - n_l}{v}}$$

into a set $\mathcal{V}$ of v polynomials, each a sum of all the terms which use the same element g_{j_l}. In the event that we can completely reduce the principal part of f, $\mathcal{V}$ represents the basis decomposition of f over $\langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$. Below, let $\text{Princ}(f)$ be the principal part of f (including its constant):

<u>PROCEDURE</u>: `MW` (Membership Witness)

<u>INPUT</u>:

- $N \in \mathbb{Z}_{\geq 2}$,
- $t, g_1, g_2, ..., g_v \in \mathcal{M}^{\infty}(N)$ satisfying (10)-(14),
- $\text{Princ}(f)$, for some $f \in \mathcal{M}^{\infty}(N)$.

<u>OUTPUT</u>:

IF $f \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$, RETURN $\{p_0, p_1, ..., p_k\} \subseteq \mathbb{Q}[x]$ such that

$f = \displaystyle\sum_{k=0}^{v} g_k \cdot p_k(t)$ with $g_0 = 1$;

ELSE, PRINT "NO MEMBERSHIP".

2.3 Main Procedure

The previous two sections discussed how to determine whether $f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$, for some modular function f. We now need to construct the modular function f_{LHS} discussed in Section 1. Let us take an arithmetic function $a(n)$ with the generating function

$$F_r(\tau) = \sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta}, \tag{20}$$

with $r = (r_\delta)_{\delta|M}$ an integer-valued vector. Suppose we are interested in a possible RK identity for $a(mn + j)$, with $0 \leq j < m$.

In [28, Section 2], Radu demonstrates that

$$\begin{aligned} &q^{\frac{24j+\sum_{\delta|M}\delta\cdot r_\delta}{24m}} \sum_{n=0}^{\infty} a(mn+j)q^n \\ =&\frac{1}{m}\sum_{\lambda=0}^{m-1} e^{-\frac{2\pi i\kappa\lambda}{24m}\left(24j+\sum_{\delta|M}\delta\cdot r_\delta\right)} \prod_{\delta|M}\eta\left(\delta\cdot\frac{\tau+\kappa\lambda}{m}\right)^{r_\delta}, \end{aligned}$$

with $\kappa = \gcd(m^2 - 1, 24)$. Therefore, if we define

$$h_{m,j}(\tau) = q^{\frac{24j+\sum_{\delta|M}\delta\cdot r_\delta}{24m}} \sum_{n=0}^{\infty} a(mn+j)q^n, \tag{21}$$

then the functional equation on η gives $h_{m,j}(\tau)$ a rough modular symmetry with respect to $\Gamma_0(N)$, for a suitably chosen $N \in \mathbb{Z}_{\geq 2}$. However, due to the imperfect symmetry of the modularity of η, it is extremely unlikely that $h_{m,j}(\tau)$ will have a perfect modular symmetry. Indeed, [28, Theorem 2.14], for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ with } a > 0, c > 0, \text{ and } \gcd(a, 6) = 1,$$

$$h_{m,j}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\cdot(-i\,(c\tau+d))^{\sum_{\delta|M} r_\delta/2}\, h_{m,j'}(\tau),$$

with $\rho := \rho(a, b, c, d, M, r, m, j)$ a certain root of unity, and j' an integer, which can be computed precisely, with $0 \leq j' < m$.

Because m serves as an upper bound for all possible j', we can take a product over all possible $h_{m,j'}$ that can be derived from $h_{m,j}$ as a result of a transformation over $\Gamma_0(N)$. Denote the set of all possible j' produced in this manner as $P_{m,r}(j)$. Then a transformation over $\Gamma_0(N)$ will send $\prod_{j'\in P_{m,r}(j)} h_{m,j'}(\tau)$ to itself, multiplied by

$$\prod_{j'\in P_{m,r}(j)} \rho\cdot(-i\,(c\tau+d))^{\sum_{\delta|M} r_\delta/2}. \tag{22}$$

To cancel the unwanted factors (22), we can construct a specific

$$\prod_{\delta|N} \eta(\delta\tau)^{s_\delta},$$

with an integer-valued vector $s = (s_\delta)_{\delta|N}$. This product of eta factors will produce the multiplicative inverse of the factors we wish to cancel. The vector s is chosen so that a transformation over $\Gamma_0(N)$ will produce the multiplicative inverse of the factors (22).

Moreover, we also adjust s so as to push the order of $\prod_{j'} h_{m,j'}$ at every cusp of $\Gamma_0(N)$ to the nonnegative integers. That is, we incorporate the function μ^{k_1} into our system s. The reasoning behind this will become clear shortly.

We can obtain s as the solution to a system of equations and inequalities found in [29, Theorems 45, 47]. Such a vector is guaranteed to exist for an appropriately chosen $N \in \mathbb{Z}_{\geq 2}$ [29, Lemma 48].

Multiplying this eta quotient by our product of $h_{m,j'}$ factors, we obtain

$$\begin{aligned} f_{LHS} :=& f_{LHS}(s, N, M, r, m, j)(\tau) & (23)\\ =& \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \cdot \prod_{j' \in P_{m,r}(j)} h_{m,j'}(\tau) \in \mathcal{M}^\infty(N)_\mathbb{Q}. & (24)\end{aligned}$$

We compute the set of possible solutions, and then select the optimal vector such that f_{LHS} will have minimal order at ∞. This is why we incorporate μ^{k_1} into our s vector: doing so will greatly simplify our later calculations, since a smaller total order on the left hand side of our prospective identity ensures that fewer computation time will be needed to determine membership of f_{LHS} (we completely ignore μ^{k_0} for the time being; see Section 2.4.3).

We now define f_1 as our prefactor, together with the fractional powers of q taken in each $h_{m,j'}$. This gives us another way to write f_{LHS}:

$$f_1(s, N, M, r, m, j) = \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \cdot q^{\sum_{j' \in P_{m,r}(j)} \frac{24j' + \sum_{\delta|M} \delta \cdot r_\delta}{24m}}, \tag{25}$$

$$\begin{aligned} f_{LHS}(s, N, M, r, m, j) &= \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \cdot \prod_{j' \in P_{m,r}(j)} h_{m,j'}(\tau) & (26)\\ &= f_1(s, N, M, r, m, j) \cdot \prod_{j' \in P_{m,r}(j)} \left(\sum_{n=0}^{\infty} a(mn + j') q^n \right). & (27)\end{aligned}$$

At last, we come to the question of how to program f_{LHS} into a computer. Because we have previously established that f_{LHS} has only one pole over $\Gamma_0(N)$, we only need to examine its principal part and constant.

Notice that f_1 has a principal part in q, and $\prod_{j' \in P_{m,r}(j)} \left(\sum_{n=0}^{\infty} a(mn + j')q^n\right)$ has no principal part in q. To take the full principal part and constant of f_{LHS}, we need only take the principal part of f_1, and every term of the form $a(mn + j')q^n$, with $n \leq \text{pord}(f_1)$.

Let us take $\text{pord}(f_1) = n_1$, and write

$$\begin{aligned} f_1 &= \sum_{n=-n_1}^{\infty} c(n)q^n \\ &= \frac{c(-n_1)}{q^{n_1}} + \frac{c(-n_1)+1}{q^{n_1-1}} + ... + \frac{c(1)}{q} + c(0) + \sum_{n=1}^{\infty} c(n)q^n, \\ f_1^{(-)} &:= \frac{c(-n_1)}{q^{n_1}} + \frac{c(-n_1)+1}{q^{n_1-1}} + ... + \frac{c(1)}{q} + c(0). \end{aligned}$$

Because $0 \leq j' \leq m-1$, we define

$$L := \sum_{n=0}^{m\cdot(\text{pord}(f_1)+1)} a(n)q^n,$$

with an extra multiple of m defined so that we have $a(m \cdot \text{pord}(f_1) + j')$ defined for all necessary j'. We need not consider any larger values of $a(n)$.

Now define

$$f_{LHS}^{(-)} := \text{Princ}\left(f_1^{(-)} \cdot \prod_{j' \in P_{m,r}(j)} \left(\sum_{n=0}^{m} [mn+j']L \cdot q^n \right) \right),$$

where by $[k]f(q)$ we mean the coefficient of q^k in the expansion of $f(q)$ about $q=0$, and by $\text{Princ}(f)$ we mean the principal part of f (including its constant term). We see that $f_{LHS}^{(-)}$ is a polynomial in q^{-1}. In particular, $f_{LHS}^{(-)}$ is finite, and can therefore be examined by a computer.

At last we define our main procedure. We want to determine whether our constructed $f_{LHS} \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$. We construct [29, Section 2.1] the functions $t, g_1, g_2, ..., g_v \in \mathcal{M}^{\infty}(N)$, satisfying conditions (10)-(14):

$$\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} = \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}.$$

We may now use our `MW` procedure to check whether $f_{LHS} \in \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]}$ by examining $f_{LHS}^{(-)}$.

Notice that we cannot merely construct the principal parts of the functions t, g_l, and disregard the rest of each function. We reduce $f_{LHS}^{(-)}$ by subtracting monomials of the form $g_l \cdot t^n$; terms other than the principal parts of t, g_l will influence the overall principal part of the product. We must therefore be careful to construct the complete principal part of each $g_l \cdot t^n$.

If `MW` returns "NO MEMBERSHIP", then the suspected identity does not exist—at least over $\Gamma_0(N)$. One may attempt a different N to find an identity. Otherwise, `MW` will return

$$\{p_0, p_1, ..., p_v\} \subseteq \mathbb{Q}[x], \tag{28}$$

and we have the complete identity

$$f_1(s,N,M,r,m,j)\cdot\prod_{j'\in P_{m,r}(j)}\left(\sum_{n=0}^{\infty}a(mn+j')q^n\right)=\sum_{k=0}^{v}g_k\cdot p_k(t). \tag{29}$$

Finally, we make note of an application so ubiquitous that we include it in our main procedure. We will attempt to extract the GCD of all of the coefficients of the p_k. Mathematica has a GCD procedure. If all of the coefficients of the p_k are integers, the procedure returns the GCD, which we will denote here as $\mathcal{D}$. On the other hand, if there exists some $K\in\mathbb{Z}_{\geq 2}$ such that the coefficients are elements in $\frac{1}{K}\mathbb{Z}$, then the GCD procedure will return $\frac{1}{K}\mathcal{D}$, with $\mathcal{D}$ defined as the GCD of the coefficients with the factor $1/K$ removed.

Our procedure, `RKDelta[`N,M,r,m,j`]`, takes as input an $N\in\mathbb{Z}_{\geq 2}$ which defines the congruence subgroup $\Gamma_0(N)$ to work over; a generating function (defined by M and r), an arithmetic progression $mn+j$, with $0\leq j\leq m$.

PROCEDURE: `RKDelta` (Ramanujan–Kolberg Implementation, Case Delta)

INPUT:

$$M\in\mathbb{Z}_{\geq 1}, \tag{30}$$
$$r=(r_\delta)_{\delta|M}, r_\delta\in\mathbb{Z} \tag{31}$$
$$m,j\in\mathbb{Z}\text{ such that }0\leq j<m. \tag{32}$$
$$N\in\mathbb{Z}_{\geq 2},\ \text{satisfying }\Delta^*\ \text{[29, Definition 35]} \tag{33}$$

OUTPUT:

$$\texttt{\{N,M,r,m,j\}} = \{N,M,r,m,j\} \tag{34}$$
$$\prod_{\delta|\mathtt{M}}(\mathtt{q}^\delta;\mathtt{q}^\delta)_\infty^{\mathtt{r}_\delta}=\sum_{\mathtt{n}=0}^{\infty}\mathtt{a(n)q^n} \tag{35}$$
$$\boxed{\mathtt{f_1(q)}\cdot\prod_{\mathtt{j}\in\mathtt{P_{m,r}(j)}}\sum_{\mathtt{n}=0}^{\infty}\mathtt{a(mn+j')q^n}=\sum_{\mathtt{g}\in\mathrm{AB}}\mathtt{g}\cdot\mathtt{p_g(t)}} \tag{36}$$
$$\mathtt{P_{m,r}(j)} = P_{m,r}(j) \tag{37}$$
$$\mathtt{f_1(q)} = f_1(q) \tag{38}$$
$$\mathtt{t} = t \tag{39}$$
$$\mathtt{AB} = \{1,g_1,g_2,...,g_v\} \tag{40}$$
$$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\} = \{p_1,p_{g_1},...,p_{g_v}\} \tag{41}$$
$$\texttt{Common Factor} = \mathcal{D} \tag{42}$$

Line (34) returns the input (30) to (33). Lines (35), (36) are unsubstituted expressions, indicating the form of a potential RK identity; they are meant to serve as a guide for the remainder of the output.

The following lines give appropriate substitutions found by the algorithm. If a vector s cannot be found, then line (38) will return

$$\texttt{f}_1\texttt{(q) = Select Another N}$$

indicating that we are unable to construct the necessary modular function over the given $\Gamma_0(N)$. Similarly, if $f_{LHS} \notin \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}$, then line (41) will return

$$\{\texttt{p}_\texttt{g}\texttt{(t):g} \in \texttt{AB}\} \texttt{ = No Membership}$$

Otherwise, the corresponding membership witness is returned.

Finally, if a greatest common factor exists and is greater than one, then $\mathcal{D}$ is returned in line (42); otherwise, the line will return

```
Common Factor = None
```

2.4 Some Remarks

2.4.1 Delta

To make use of the techniques discussed so far, we must find an appropriate $N \in \mathbb{Z}_{\geq 2}$ such that a generating function defined by $M, r = (r_\delta)_{\delta|M}$, and an arithmetic progression $mn+j$, $0 \leq j < m$, can be effectively studied over $\Gamma_0(N)$. The key criterion, called the Δ^* criterion by Radu [29, Definitions 34, 35], is checked with the procedure `Delta[N, M, r, m, j]`.

PROCEDURE: `Delta`

INPUT:

$$N \in \mathbb{Z}_{\geq 2} \tag{43}$$
$$M \in \mathbb{Z}_{\geq 1} \tag{44}$$
$$r = (r_\delta)_{\delta|M}, r_\delta \in \mathbb{Z} \tag{45}$$
$$m, j \in \mathbb{Z} \text{ such that } 0 \leq j < m. \tag{46}$$

OUTPUT:

IF Δ^* IS SATISFIED, RETURN TRUE,
ELSE, RETURN FALSE

Radu also includes a case in his algorithm in which the Δ^* criterion may be disregarded [29, Section 3.1, Second Case]. We have prepared a prototype of this second condition; however, we are as yet unable to find any identities from this condition. We include here only the case in which the Δ^* criterion is necessary.

At any rate, for any given $M, r = (r_\delta)_{\delta|M}, m, j$ with $0 \leq j < m$, there must exist an $N \in \mathbb{Z}_{\geq 2}$ such that the Δ^* criterion is satisfied [29, Section 3.1]. It is generally convenient to work with the smallest possible N that satisfies the criterion. However, we will see in subsequent examples that the smallest possible case is not always the most useful.

We will therefore leave the criterion for establishing N as separate from the main algorithm, and define N as part of the input.

2.4.2 `RKDeltaMan`

We also include a slightly modified implementation that we refer to as `RKDeltaMan`. This procedure is nearly identical to that used for Radu's algorithm, except that the algebra basis is included in the input. This is often helpful because, as we will see in some examples, construction of the algebra basis for $\langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}$ is often inefficient. If we already have a suitable algebra basis calculated (perhaps from a database, or a general study of eta quotient spaces), we may easily shorten the computation time.

2.4.3 `RKDeltaE`

Regarding the value of k_0 in Theorem 5, we very strongly suspect that k_0 may always be set to 0, and that therefore

$$\mathcal{M}^\infty(N)_\mathbb{Q} \cap \langle \mathcal{E}(N) \rangle_\mathbb{Q} = \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}.$$

for all $N \in \mathbb{Z}_{\geq 2}$. This is important, because the computation of a bound for k_0 is costly, and increases the runtime of our package. We therefore include the procedure `RKDeltaE` in addition to `RKDelta` command. The two commands are nearly identical, except that `RKDeltaE` includes the power μ^{k_0} in our prefactor.

We also include the procedure `RKDeltaEMan`, which is identical to `RKDeltaMan`, except that it includes μ^{k_0}.

In the examples below, we use the procedures `RKDelta`, `RKDeltaMan`.

3 Examples

We now give an overview of applications of our package. Except for Sections 3.1-3.2, which cover the classic cases, each of our examples is chosen from contemporary work done in partition theory over the last ten years—in most cases, within the last five years. Our proofs are of course based on the computational theory of modular functions. In many cases, these results may be proved with more elementary methods, and we happily invite the interested reader to attempt them.

3.1 Ramanujan's Classics

The most obvious examples to check are the classic identities of Ramanujan and Kolberg for $p(5n + 4)$ and $p(7n + 5)$.

The generating function for $p(n)$ is of course $1/(q;q)_\infty$, which can be described by setting $M = 1$, $r = (-1)$. If we now take $m = 5$, guess $N = 5$, and take $j = 4$, then we have

```
 In[1] = RKDelta[5, 1, {-1}, 5, 4]
Out[1] =
```

$\{N, M, r, m, j\} = \{5, 1, \{-1\}, 5, 4\}$

$$\prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n$$

$$\boxed{f_1(q) \cdot \prod_{j \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)}$$

$P_{m,r}(j) = \{4\}$

$f_1(q) = \dfrac{((q;q)_\infty)^6}{((q^5;q^5)_\infty)^5}$

$t = \dfrac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$

AB =

$\{1\}$

$\{p_g(t) : g \in AB\}$ =

$\{5\}$

Common Factor = 5

We see that $P_{m,r}(j) = \{4\}$, indicating that our left hand side will only contain the series $\sum_{n \geq 0} p(5n+4)q^n$. With f_1, we have the left hand side of any possible identity as

$$f_{LHS} = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} p(5n+4) q^n \in \mathcal{M}^\infty(5).$$

In this case our algebra basis is extremely simple:

$$\langle \mathcal{E}^\infty(5) \rangle_{\mathbb{Q}} = \langle 1 \rangle_{\mathbb{Q}[t]} = \mathbb{Q}[t],$$

with

$$t = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}.$$

Because the basis contains only the identity, we only need a single polynomial in t. In this case, the polynomial is 5.

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} p(5n+4) q^n = 5.$$

A quick rearrangement gives us (2)

Similarly, taking $m = 7$, $j = 5$, and guessing $N = 7$, we have

In[2] = RKDelta$[7, 1, \{-1\}, 7, 5]$

Out[2] =

$\{\mathtt{N,M,r,m,j}\}$ = $\{7, 1, \{-1\}, 7, 5\}$

$$\prod_{\delta|\mathtt{M}} (\mathtt{q}^{\delta};\mathtt{q}^{\delta})_{\infty}^{\mathtt{r}_{\delta}} = \sum_{\mathtt{n}=0}^{\infty} \mathtt{a(n)q^n}$$

$$\boxed{\mathtt{f_1(q)} \cdot \prod_{\mathtt{j}\in \mathtt{P_{m,r}(j)}} \sum_{\mathtt{n}=0}^{\infty} \mathtt{a(mn + j')q^n} = \sum_{\mathtt{g}\in \mathrm{AB}} \mathtt{g} \cdot \mathtt{p_g(t)}}$$

$\mathtt{P_{m,r}(j)}$ = $\{5\}$

$\mathtt{f_1(q)}$ = $\dfrac{((q;q)_{\infty})^8}{q((q^7;q^7)_{\infty})^7}$

t = $\dfrac{((q;q)_{\infty})^4}{q((q^7;q^7)_{\infty})^4}$

AB =

$\{1\}$

$\{\mathtt{p_g(t)} : \mathtt{g} \in \mathtt{AB}\}$ =

$\{49 + 7t\}$

Common Factor = 7

This gives us

$$\frac{(q;q)_{\infty}^8}{q(q^7;q^7)_{\infty}^7} \sum_{n=0}^{\infty} p(7n+5)q^n = 49 + 7\frac{(q;q)_{\infty}^4}{q(q^7;q^7)_{\infty}^4},$$

which yields (3) on rearrangement.

In the following examples, we will omit the first three lines of output from each example for the sake of brevity.

3.2 Classic Identities by Kolberg and Zuckerman

A large number of classic analogues to Ramanujan's results have been found. We start with an identity discovered by Zuckerman [40] for $p(13n + 6)$.

Theorem 7.

$$\begin{aligned}\sum_{n=0}^{\infty} p(13n+6)q^n =& 11\frac{(q^{13};q^{13})_{\infty}}{(q;q)_{\infty}^2} + 468q\frac{(q^{13};q^{13})_{\infty}^3}{(q;q)_{\infty}^4} + 6422q^2\frac{(q^{13};q^{13})_{\infty}^5}{(q;q)_{\infty}^6}\\ &+ 43940q^3\frac{(q^{13};q^{13})_{\infty}^7}{(q;q)_{\infty}^8} + 171366q^4\frac{(q^{13};q^{13})_{\infty}^9}{(q;q)_{\infty}^{10}}\\ &+ 371293q^5\frac{(q^{13};q^{13})_{\infty}^{11}}{(q;q)_{\infty}^{12}} + 371293q^6\frac{(q^{13};q^{13})_{\infty}^{13}}{(q;q)_{\infty}^{14}}.\end{aligned}$$

```
In[3] = RKDelta[13, 1, {−1}, 13, 6]
Out[3] =
```

$\mathtt{P_{m,r}(j)} = \{6\}$

$\mathtt{f_1(q)} = \dfrac{((q;q)_\infty)^{14}}{q^6((q^{13};q^{13})_\infty)^{13}}$

$\mathtt{t} = \dfrac{((q;q)_\infty)^{2}}{q((q^{13};q^{13})_\infty)^{2}}$

AB =

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$ =

$\{371293 + 371293t + 171366t^2 + 43940t^3 + 6422t^4 + 468t^5 + 11t^6\}$

Common Factor = None

We will now use our algorithm to derive the identities which Kolberg found [18] for $p(5n+j)$, $p(7n+j)$, and $p(3n+j)$.

Starting with $p(5n+j)$ for $0 \le j \le 4$, if we take $N = 5$ once more, and set $j = 1$, [18, (4.2)] we have

```
In[4] = RKDelta[5, 1, {−1}, 5, 1]
Out[4] =
```

$\mathtt{P_{m,r}(j)} = \{1, 2\}$

$\mathtt{f_1(q)} = \dfrac{((q;q)_\infty)^{12}}{((q^{5};q^{5})_\infty)^{10}}$

$\mathtt{t} = \dfrac{((q;q)_\infty)^{6}}{q((q^{5};q^{5})_\infty)^{6}}$

AB =

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$ =

$\{25 + 2t\}$

Common Factor = None

Working over the same congruence subgroup $\Gamma_0(5)$, we keep the same algebra basis and t. The most notable difference is that we have the product

$$\left(\sum_{n\geq 0} p(5n+1)q^n\right)\left(\sum_{n\geq 0} p(5n+2)q^n\right)$$

on the left hand side. Our right-hand side is given as a more complicated $25+2t$, and we have

$$\frac{(q;q)_\infty^{12}}{(q^5;q^5)_\infty^{10}}\left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right) = 25 + 2\frac{(q;q)_\infty^{6}}{q(q^5;q^5)_\infty^{6}}.$$

We can similarly examine $j = 3$ [18, (4.3)] and derive the identity

$$\frac{(q;q)_\infty^{12}}{(q^5;q^5)_\infty^{10}}\left(\sum_{n=0}^{\infty}p(5n+3)q^n\right)\left(\sum_{n=0}^{\infty}p(5n)q^n\right)=25+3\frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}.$$

On the other hand, we can set $m = 7, j = 1, N = 7$, [18, (5.2)] and we will derive

```
 In[5] = RKDelta[7, 1, {−1}, 7, 1]
Out[5] =
```

$\mathtt{P_{m,r}(j)} = \{1, 3, 4\}$

$\mathtt{f_1(q)} = \dfrac{((q;q)_\infty)^{24}}{q((q^7;q^7)_\infty)^{21}}$

$\mathtt{t} = \dfrac{((q;q)_\infty)^4}{q((q^7;q^7)_\infty)^4}$

`AB =`

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g}\in \mathtt{AB}\} =$

$\{117649 + 50421t + 8232t^2 + 588t^3 + 15t^4\}$

`Common Factor = None`

and the identity

$$\frac{(q;q)_\infty^{24}}{q^4(q^7;q^7)_\infty^{21}}\left(\sum_{n=0}^{\infty}p(7n+1)q^n\right)\left(\sum_{n=0}^{\infty}p(7n+3)q^n\right)\left(\sum_{n=0}^{\infty}p(7n+4)q^n\right)$$
$$=117649+50421\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}+8232\frac{(q;q)_\infty^8}{q^2(q^7;q^7)_\infty^8}+588\frac{(q;q)_\infty^{12}}{q^3(q^7;q^7)_\infty^{12}}+15\frac{(q;q)_\infty^{16}}{q^4(q^7;q^7)_\infty^{16}}.$$

The corresponding identity for $p(7n + 2)$ [18, (5.3)] can be easily found.

Finally, we set $m = 3$, $j = 1$, $N = 9$, [18, (3.4)] and derive

```
 In[6] = RKDelta[9, 1, {−1}, 3, 1]
Out[6] =
```

$\mathtt{P_{m,r}(j)} = \{0, 1, 2\}$

$\mathtt{f_1(q)} = \dfrac{((q;q)_\infty)^{10}}{q(q^3;q^3)_\infty((q^9;q^9)_\infty)^6}$

$\mathtt{t} = \dfrac{((q;q)_\infty)^3}{q((q^9;q^9)_\infty)^3}$

`AB =`

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g}\in \mathtt{AB}\} =$

$\{9 + 2t\}$

`Common Factor = None`

And we have

$$\frac{(q;q)^{10}}{q(q^3;q^3)(q^9;q^9)^6}\left(\sum_{n=0}^{\infty}p(3n)q^n\right)\left(\sum_{n=0}^{\infty}p(3n+1)q^n\right)\left(\sum_{n=0}^{\infty}p(3n+2)q^n\right)$$
$$=9+2\frac{(q;q)^3}{q(q^9;q^9)^3}.$$

3.3 Radu's Identity for 11

A substantial amount of work has been done attempting a witness identity for $p(11n+6) \equiv 0 \pmod{11}$. We will show one interesting attempt by Radu, though we hasten to add that a great deal of work has been done by others on the problem (for an interesting approach, see [14]). If we were to attempt to find such an identity for $M = 1$, $r = (-1)$, $m = 11$, $N = 11$, $j = 6$, then our algorithm returns

In[7] = RKDelta$[11, 1, \{-1\}, 11, 6]$

Out[7] =

$\mathtt{P}_{\mathtt{m,r}}\mathtt{(j)} = \{6\}$

$\mathtt{f_1(q)} = \frac{(q;q)_\infty^{12}}{q^4(q^{11};q^{11})_\infty^{11}}$

$\mathtt{t} = \frac{(q;q)_\infty^{12}}{q^5(q^{11};q^{11})_\infty^{12}}$

AB =

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$ =

No Membership

Common Factor = None

Our membership witness returns a null result, indicating that our constructed modular function does not lie within $\langle \mathcal{E}^\infty(11)\rangle_{\mathbb{Q}}$.

If we were to take $N = 22$, however, we get

In[8] = RKDelta$[22, 1, \{-1\}, 11, 6]$

Out[8] =

$\mathtt{P}_{\mathtt{m,r}}\mathtt{(j)} = \{6\}$

$\mathtt{f_1(q)} = \frac{(q;q)_\infty^{12}(q^2;q^2)_\infty^2(q^{11};q^{11})_\infty^{11}}{q^{14}(q^{22};q^{22})_\infty^{22}}$

$$\mathtt{t} = -\frac{1}{8}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}}+\frac{1}{11}\frac{(q^2;q^2)_\infty^8(q^{11};q^{11})_\infty^4}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^8}$$
$$+\frac{3}{88}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7}$$

```
AB =
```

$$\{1, -\frac{1}{8}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}} + \frac{2}{11}\frac{(q^2;q^2)_\infty^{8}(q^{11};q^{11})_\infty^{4}}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^{8}} + \frac{5}{88}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7},$$

$$\frac{5}{4}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}} - \frac{3}{11}\frac{(q^2;q^2)_\infty^{8}(q^{11};q^{11})_\infty^{4}}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^{8}} + \frac{1}{44}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7}\}$$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$ =

$$\{6776 + 9427t + 15477t^2 + 13332t^3 + 1078t^4, -9581 + 594t + 5390t^2 + 187t^3, -6754 + 5368t + 2761t^2 + 11t^3\}$$

```
Common Factor = 11
```

Our procedure returns a variation on a result that Radu already computed [29]. The result is as tantalizing as it is annoying. It has a form resembling the classic witness identities which Ramanujan discovered for his congruences of $p(5n+4), p(7n+5)$ by $5, 7$, respectively. In particular, the coefficients of t in the membership witness are all divisible by 11, indicating a potential witness identity.

However, it is not obvious that the functions in our algebra basis have integer coefficients in their expansions around $q = 0$. In particular, the prevalence of 11 throughout the denominators of each function makes the overall congruence of $p(11n+6)$ modulo 11 far from obvious. Peter Paule was the first to realize this [25, Discussion, pp. 541-542], and successfully demonstrated that the functions g_l in the algebra basis do in fact have integer coefficients.

3.4 An Identity for Broken 2-Diamond Partitions

Broken k-diamond partitions, denoted by $\Delta_k(n)$, were defined by Andrews and Paule in 2007 [4]. They conjectured that

Theorem 8. *For all* $n \in \mathbb{Z}_{\geq 0}$,

$$\Delta_2(25n+14) \equiv \Delta_2(25n+24) \equiv 0 \pmod 5.$$

This was subsequently proved in 2008 by Chan [7]. In 2015 Radu was able [29] to give a proof by studying another arithmetic function with a simpler generating function. Our complete implementation allows us to verify these congruences by directly examining the generating function for $\Delta_2(n)$.

We take $N = 10, M = 10, r = (-3, 1, 1, -1), m = 25, j = 14$. Our package returns

```
 In[9] = RKDelta[10, 10, {-3, 1, 1, -1}, 25, 14]
Out[9] =
```

$\mathtt{P_{m,r}(j)}$ = $\{14, 24\}$

$$\mathtt{f_1(q)} = \frac{(q;q)_\infty^{126}(q^5;q^5)_\infty^{70}}{q^{58}(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^{190}}$$

$$\texttt{t = } \frac{(q^2;q^2)_\infty (q^5;q^5)^5_\infty}{q(q;q)_\infty (q^{10};q^{10})^5_\infty}$$

```
AB =
{1}
{p_g(t):g ∈ AB} =
{...}
Common Factor = 25
```

The membership witness returns a lengthy result, with terms of the order of 10^{76}. However, the computation time is short—less than 40 seconds with a 2.6 GHz Intel Processor on a modest laptop. The complete witness is available, and easily computed, at [35, RaduRKexamples.nb].

Each term in the membership witness is divisible by 25. By expanding the generating function for $\Delta_2(n)$, one determines that $\Delta_2(14) = 10445$, and that $\Delta_2(49) = 1022063815$.

Because each of these numbers is divisible by 5 but not by 25, therefore $\sum_{n\geq 0} \Delta_2(25n + 14)$, $\sum_{n\geq 0} \Delta_2(25n + 24)$ must each be divisible by exactly one power of 5. This completes the proof.

3.5 Congruences with Overpartitions

An enormous amount of work has been published in recent years on the congruence properties of overpartition functions, and our package has a great deal of utility in this subject. We will examine three distinct problems here: two will involve the standard overpartition function $\bar{p}(n)$, and one will involve an overpartition function with additional restrictions $A_m(n)$. In each case, we are able to make substantial improvements to previously established results.

As a preliminary, an overpartition of n is a partition of n in which the first occurrence of a part may or may not be "marked." Generally, this "mark" is denoted with an overline (hence the term "overpartition"). For example, the number 3 has 8 overpartitions:

$$\begin{aligned}
&3,\\
&\bar{3},\\
&2+1,\\
&\bar{2}+1,\\
&2+\bar{1},\\
&\bar{2}+\bar{1},\\
&1+1+1,\\
&\bar{1}+1+1.
\end{aligned}$$

We denote the number of overpartitions of n by $\bar{p}(n)$. The generating function for $\bar{p}(n)$ has the form

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)^2_\infty}$$

Part of the appeal of $\bar{p}(n)$ is the simplicity of the combinatoric interpretation, given the relative complexity of its generating function [10].

3.5.1 Congruences Over $\bar{p}(n)$

We will begin by giving some remarkable improvements to previously established congruences over $\bar{p}(n)$. Moreover, we have the opportunity to apply our "manual" procedure, and use the connection of modular functions with the topology of associated Riemann surfaces in order to construct a suitable algebra basis.

In 2016 Dou and Lin showed [12] that

$$\bar{p}(80n+8) \equiv \bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv \bar{p}(80n+72) \equiv 0 \pmod{5}. \quad (47)$$

Hirschhorn in 2016 [16], and Chern and Dastidar in 2018 [8] have studied these congruences as well, with the latter improving these congruences:

$$\bar{p}(80n+8) \equiv \bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv \bar{p}(80n+72) \equiv 0 \pmod{25}.$$

Chern and Dastidar go on to point out that

$$\bar{p}(135n+63) \equiv \bar{p}(135n+117) \equiv 0 \pmod{5}.$$

However, a quick computation of each of these sequences of overpartition numbers reveals much more. For instance,

n	$\bar{p}(80n+8)$
0	100
1	8638130600
2	350865646632400
3	1512900775311002400
4	1919738036947929590800
5	1092453314947897908542800
6	348534368588210202093102600
7	71377855377904690816918291600
8	10261762697785410674339371853700

A very much stronger congruence clearly suggests itself. We are able to make the following substantial improvements in each case:

Theorem 9.

$$\bar{p}(80n+8) \equiv \bar{p}(80n+72) \equiv 0 \pmod{100},$$
$$\bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv 0 \pmod{200}.$$

Theorem 10.

$$\bar{p}(135n+63) \equiv \bar{p}(135n+117) \equiv 0 \pmod{40}.$$

Our package can be used to demonstrate each of these, though with some adjustments. In the case of $\bar{p}(80n+j)$, we are forced to work over the congruence subgroup $\Gamma_0(40)$. The generating set $\mathcal{G}_0(40)$ of the corresponding monoid $\mathcal{E}^\infty(40)$ of monopolar eta quotients can be computed with relative ease using `etaGenerators`; however, the set is nevertheless extremely large, and our procedure to compute the algebra basis using `AB` would be extremely inefficient.

We can remedy the problem by taking advantage of the Weierstrass gap theorem, (see [38, Part 2, Section 17] for a classical introduction to the subject; see [26] for a more modern treatment of the theorem). We use [11, Theorem 3.1.1] to compute the genus of the corresponding modular curve $X_0(40)$ as 3, which implies that all monopolar modular functions with a single pole at ∞ over $\Gamma_0(40)$ must have order 4 or greater. Radu's refinement of Newmann's conjecture [27, Conjecture 9.4] suggests that a suitable combination of eta quotients will yield functions in $\langle\mathcal{E}^\infty(40)\rangle_{\mathbb{Q}}$ with orders 4, 5, 6, 7. Such a set of functions would be a sufficient algebra basis for $\langle\mathcal{E}^\infty(40)\rangle_{\mathbb{Q}}$.

In this case, we are lucky, because a simple ordering of $\mathcal{G}_0(40)$ by the order of the elements at ∞ reveals that

$$\begin{aligned}
\mathcal{G}_0(40)[1] =& \frac{(q^4;q^4)^3_\infty (q^{20};q^{20})_\infty}{q^4(q^8;q^8)_\infty(q^{40};q^{40})^3_\infty},\\
\mathcal{G}_0(40)[4] =& \frac{(q^2;q^2)^3_\infty (q^5;q^5)_\infty (q^{20};q^{20})^2_\infty}{q^5(q;q)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})^4_\infty},\\
\mathcal{G}_0(40)[7] =& \frac{(q^2;q^2)^6_\infty (q^5;q^5)^2_\infty (q^8;q^8)_\infty (q^{20};q^{20})^3_\infty}{q^6(q;q)^2_\infty(q^4;q^4)^3_\infty(q^{10};q^{10})^2_\infty(q^{40};q^{40})^5_\infty},\\
\mathcal{G}_0(40)[17] =& \frac{(q;q)^2_\infty (q^5;q^5)^2_\infty (q^8;q^8)^2_\infty (q^{20};q^{20})^3_\infty}{q^7(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})^6_\infty}.
\end{aligned}$$

Here for any ordered, enumerable set $\mathcal{S}$, we define the term $\mathcal{S}[j]$ as the jth term in the ordering of $\mathcal{S}$.

We can then define our algebra basis as

$$\begin{aligned}
T &= \mathcal{G}_0(40)[1],\\
\mathrm{Ab40} &= \{T, \{1, \mathcal{G}_0(40)[4], \mathcal{G}_0(40)[7], \mathcal{G}_0(40)[17]\}\}.
\end{aligned}$$

Since we computed our algebra basis separately, we may now employ the manual case of our package, `RKDeltaMan` (See Section 2.4.2):

`In[10] =` $\mathtt{RKDeltaMan}[40, 2, \{-2, 1\}, 80, 8, \mathtt{Ab40}]$

`Out[10] =`

$\mathtt{P_{m,r}(j)} = \{8, 72\}$

$$\mathtt{f_1(q)} = \frac{(q;q)^{333}_\infty (q^8;q^8)^{66}_\infty (q^{10};q^{10})^{36}_\infty (q^{20};q^{20})^{165}_\infty}{q^{400}(q^2;q^2)^{168}_\infty (q^4;q^4)^{31}_\infty (q^5;q^5)^{65}_\infty (q^{40};q^{40})^{334}_\infty}$$

$$\mathtt{t} = \frac{(q^4;q^4)^3_\infty (q^{20};q^{20})_\infty}{q^4(q^8;q^8)_\infty(q^{40};q^{40})^3_\infty}$$

```
AB =
```

$$\{1, \frac{(q^2;q^2)_\infty^3(q^5;q^5)_\infty(q^{20};q^{20})_\infty^2}{q^5(q;q)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^4},$$

$$\frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty^2(q^8;q^8)_\infty(q^{20};q^{20})_\infty^3}{q^6(q;q)_\infty^2(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty^2(q^{40};q^{40})_\infty^5},$$

$$\frac{(q;q)_\infty^2(q^5;q^5)_\infty^2(q^8;q^8)_\infty^2(q^{20};q^{20})_\infty^3}{q^7(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^6}\}$$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$ =

{...}

```
Common Factor = 10000
```

The membership witness is too lengthy to present in this article. The complete output of the algorithm can be found in [35, OverpartitionExamples.nb]. It is trivial to compute $\bar{p}(80n+8)$, $\bar{p}(80n+72)$ for a handful of small n in order to demonstrate that neither is divisible by 2^3 or 5^3. Since the left hand side consists of a prefactor (with initial coefficient 1) and a product of the form

$$\left(\sum_{n\geq 0}\bar{p}(80n+8)q^n\right)\left(\sum_{n\geq 0}\bar{p}(80n+72)q^n\right),$$

with neither factor divisible by 2^3 or 5^3, the only remaining possibility is that each factor is divisible by $2^2 \cdot 5^2 = 100$.

An almost identical output is produced for

```
In[11] = RKDeltaMan[40, 2, {−2, 1}, 80, 52, Ab40]
```

but with an output of 40000 for congruences. This is also available at [35, OverpartitionExamples.nb]. We may show that $\bar{p}(80n+52)$, $\bar{p}(80n+68)$ are each divisible by 200, in a similar manner to the case of $\bar{p}(80n+8)$, $\bar{p}(80n+72)$.

Finally, we consider the case of $\bar{p}(135n+63), \bar{p}(135n+117)$. We may similarly construct an algebra basis manually. In this case, the most convenient congruence subgroup to work over is $\Gamma_0(30)$ ($N = 30$). The genus of $X_0(30)$ is 3, but we are at a slight disadvantage: there are eta quotients in $\mathcal{E}^\infty(30)$ with orders 4, 6, and 7, but none with order 5. But we can construct a difference of eta quotients, each with order 6, to produce a function of order 5. If we order the generators of $\mathcal{E}^\infty(30)$ by order at ∞, then

$$\mathcal{G}_0(30)[1] = \frac{(q;q)_\infty(q^6;q^6)_\infty^6(q^{10};q^{10})_\infty^2(q^{15};q^{15})_\infty^3}{q^4(q^2;q^2)_\infty^2(q^3;q^3)_\infty^3(q^5;q^5)_\infty(q^{30};q^{30})_\infty^6},$$

$$\mathcal{G}_0(30)[4] - \mathcal{G}_0(30)[3] = \frac{(q^2;q^2)_\infty^4(q^{10};q^{10})_\infty^4(q^{15};q^{15})_\infty^4}{q^6(q;q)_\infty^2(q^5;q^5)_\infty^2(q^{30};q^{30})_\infty^8} - \frac{(q;q)_\infty(q^6;q^6)_\infty^2(q^{10};q^{10})_\infty^{10}(q^{15};q^{15})_\infty^5}{q^6(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^5;q^5)_\infty^5(q^{30};q^{30})_\infty^{10}},$$

$$\mathcal{G}_0(30)[2] = \frac{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^5)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})^3_\infty}{q^6(q^3;q^3)_\infty(q^{30};q^{30})^7_\infty},$$

$$\mathcal{G}_0(30)[6] = \frac{(q;q)_\infty(q^5;q^5)^2_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})^3_\infty}{q^7(q^{30};q^{30})^8_\infty}.$$

The orders here are (respectively) $4, 5, 6, 7$, again sufficient for an algebra basis:

$$\begin{aligned} T &= \mathcal{G}_0(30)[1], \\ G_1 &= \mathcal{G}_0(30)[4] - \mathcal{G}_0(30)[3] \\ G_2 &= \mathcal{G}_0(30)[2] \\ G_3 &= \mathcal{G}_0(30)[6] \\ \mathrm{Ab30} &= \{T, \{1, G_1, G_2, G_3\}\}. \end{aligned}$$

Employing `RKDeltaMan` once again, we get

`In[11] =` RKDeltaMan$[30, 2, \{-2, 1\}, 135, 63,$ Ab30$]$

`Out[11] =`

$\mathtt{P_{m,r}(j)} = \{63, 117\}$

$$\mathtt{f_1(q)} = \frac{(q;q)^{653}_\infty(q^6;q^6)^{235}_\infty(q^{10};q^{10})^{272}_\infty(q^{15};q^{15})^{358}_\infty}{q^{507}(q^2;q^2)^{359}_\infty(q^3;q^3)^{275}_\infty(q^5;q^5)^{226}_\infty(q^{30};q^{30})^{656}_\infty}$$

$$\mathtt{t} = \frac{(q;q)_\infty(q^6;q^6)^6_\infty(q^{10};q^{10})^2_\infty(q^{15};q^{15})^3_\infty}{q^4(q^2;q^2)^2_\infty(q^3;q^3)^3_\infty(q^5;q^5)_\infty(q^{30};q^{30})^6_\infty}$$

`AB =`

$$\Big\{1, \frac{(q^2;q^2)^4_\infty(q^{10};q^{10})^4_\infty(q^{15};q^{15})^4_\infty}{q^6(q;q)^2_\infty(q^5;q^5)^2_\infty(q^{30};q^{30})^8_\infty} - \frac{(q;q)_\infty(q^6;q^6)^2_\infty(q^{10};q^{10})^{10}_\infty(q^{15};q^{15})^5_\infty}{q^6(q^2;q^2)^2_\infty(q^3;q^3)_\infty(q^5;q^5)^5_\infty(q^{30};q^{30})^{10}_\infty},$$
$$\frac{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^5)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})^3_\infty}{q^6(q^3;q^3)_\infty(q^{30};q^{30})^7_\infty},$$
$$\frac{(q;q)_\infty(q^5;q^5)^2_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})^3_\infty}{q^7(q^{30};q^{30})^8_\infty}\Big\}$$

$\{\mathtt{p_g(t)} : \mathtt{g} \in \mathtt{AB}\} =$

$\{\ldots\}$

$\mathtt{Common\ Factor} = \frac{1600}{3}$

Once again, the membership witness is too large to present here. It can be found in its entirety at [35, OverpartitionExamples.nb]. However, the fractional common factor emerges because each polynomial p_g in the witness has integer coefficients, except for p_{G_1}, which is a polynomial over $\frac{1}{3}\mathbb{Z}$. Because the remaining polynomials have integer coefficients (and all of the eta quotients involved have integer-coefficient expansions), we can conclude that G_1 has coefficients divisible by 3. At any rate, this makes no difference for congruences with respect to powers of 2 or 5.

We may again quickly demonstrate that $\bar{p}(135n+63), \bar{p}(135n+117)$ are not divisible by 2^4 or 5^2, indicating that they must each be divisible by $2^3 \cdot 5 = 40$.

As we have previously mentioned, there are almost certainly simpler proofs of these congruences. In any case, it is striking that these stronger congruences were not at least conjectured, given how many people studied the sequences in (47), and how clearly these congruences are revealed when even a handful cases are actually computed.

3.5.2 A Conjecture For $\bar{p}(n)$

In 2015 Xia conjectured [39] that

$$\bar{p}(96n+76) \equiv 0 \pmod{3^5}$$

for all $n \in \mathbb{Z}_{\geq 0}$. We have not only confirmed this conjecture, but extended it:

Theorem 11.

$$\bar{p}(96n+76) \equiv 0 \pmod{2^3 3^5}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

In[12] = RKDelta$[24, 2, \{-2, 1\}, 96, 76]$

Out[12] =

$P_{m,r}(j) = \{76\}$

$$f_1(q) = \frac{(q;q)_\infty^{213}(q^6;q^6)_\infty^{33}(q^8;q^8)_\infty^{77}(q^{12};q^{12})_\infty^{113}}{q^{150}(q^2;q^2)_\infty^{107}(q^3;q^3)_\infty^{64}(q^4;q^4)_\infty^{37}(q^{24};q^{24})_\infty^{227}}$$

$$t = \frac{(q^6;q^6)_\infty^3(q^8;q^8)_\infty}{q^2(q^2;q^2)_\infty(q^{24};q^{24})_\infty^3}$$

AB =

$$\{1, \frac{(q^6;q^6)_\infty^3(q^8;q^8)_\infty}{q^2(q^2;q^2)_\infty(q^{24};q^{24})_\infty^3} + \frac{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty(q^4;q^4)_\infty^3}{q^3(q^2;q^2)_\infty^2(q^{24};q^{24})_\infty^4}\}$$

$\{p_g(t) : g \in AB\}$ =

$\{...\}$

Common Factor = 1944

The theorem is then established, since $1944 = 2^3 \cdot 3^5$. The full identity can be found at [35, OverpartitionExamples.nb]

3.5.3 A Restricted Overpartition Function

Let $A_m(n)$ be the number of overpartitions of n in which only the parts not divisible by m may be overlined. Then it can be showed that [21]

$$\sum_{n=0}^{\infty} A_m(n)q^n = \frac{(q^2;q^2)_\infty(q^m;q^m)_\infty}{(q;q)_\infty^2(q^{2m};q^{2m})_\infty}.$$

In 2016, Munagi and Sellers give a variety of interesting congruences for $A_m(n)$.

For instance, [21, Corollary 4.4, Theorem 4.5]:

Theorem 12.

$$A_3(3n+1) \equiv 0 \pmod{2},$$
$$A_3(3n+2) \equiv 0 \pmod{4}.$$

Both of these can be proved quickly with our package. For example, to prove $A_3(3n+1) \equiv 0 \pmod{2}$:

In[13] = RKDelta$[6, 6, \{-2, 1, 1, -1\}, 3, 1]$

Out[13] =

$\mathtt{P_{m,r}(j)} = \{1\}$

$\mathtt{f_1(q)} = \dfrac{(q;q)_\infty^3 (q^2;q^2)_\infty (q^3;q^3)_\infty^6}{q(q^6;q^6)_\infty^9}$

$\mathtt{t} = \dfrac{(q;q)_\infty^5 (q^3;q^3)_\infty}{q(q^2;q^2)_\infty (q^6;q^6)_\infty^5}$

AB =

$\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$ =

$\{16 + 2t\}$

Common Factor = 2

On the other hand, [21, Theorem 4.7, Theorem 4.9] $A_3(27n+26) \equiv 0 \pmod{3}$, and $A_9(27n+24) \equiv 0 \pmod{3}$. Using our package, we can prove more:

Theorem 13.

$$A_3(27n+26) \equiv 0 \pmod{12},$$
$$A_9(27n+24) \equiv 0 \pmod{24}.$$

For example, to show that $A_9(27n+24) \equiv 0 \pmod{24}$:

In[14] = RKDelta$[6, 18, \{-2, 1, 0, 0, 1, -1\}, 27, 24]$

Out[14] =

$\mathtt{P_{m,r}(j)} = \{24\}$

$\mathtt{f_1(q)} = \dfrac{(q;q)_\infty^{47} (q^3;q^3)_\infty^{12}}{q^9 (q^2;q^2)_\infty^7 (q^6;q^6)_\infty^{51}}$

$\mathtt{t} = \dfrac{(q;q)_\infty^5 (q^3;q^3)_\infty}{q(q^2;q^2)_\infty (q^6;q^6)_\infty^5}$

AB =

$\{1\}$

$\{p_g(t) : g \in AB\} =$
$\{7703510787293184 + 5456653474332672t$
$+ 1649478582927360t^2 + 276646783352832t^3$
$+ 27989228519424t^4 + 1735943602176t^5$
$+ 63885293568t^6 + 1269340416t^7 + 10941888t^8 + 22056t^9\}$
Common Factor = 24

We expect that a very large variety of other congruences and associated results for overpartition functions still await discovery. Those researchers who study partitions outside of the theory of modular forms (e.g., from the perspective of q-series or combinatorial approaches) may find our package extremely useful. As in the case with our first example of $\bar{p}(n)$, our implementation can be used to give optimal congruences (that might otherwise be missed), from which more elementary proofs may be attempted.

3.6 Some Identities by Baruah and Sarmah

For $r \in \mathbb{Z}$, define

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r.$$

In 2013 Baruah and Sarmah [6] gave a large variety of results for $p_r(n)$, all of which are accessible through our package. One especially interesting example, [6, Theorem 2.1, (2.10)] is not a congruence, but rather a simple identity:

Theorem 14.

$$p_8(3n+1) = 0.$$

We can verify this by taking $M = 1, r = (8), m = 4, j = 3, N = 4$:

In[15] = RKDelta$[4, 1, \{8\}, 4, 3]$
Out[15] =
$P_{m,r}(j) = \{3\}$
$f_1(q) = \dfrac{(q^2;q^2)_\infty^{12}}{q(q;q)_\infty^4(q^4;q^4)_\infty^{16}}$
$t = \dfrac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$
AB =
$\{1\}$
$\{p_g(t) : g \in AB\} =$
$\{0\}$
Common Factor = 0

Baruah and Sarmah list several congruences [6, Theorem 5.1] which may easily be proved. For example:

Theorem 15.

$$\begin{aligned} p_{-4}(4n+3) &\equiv 0 \pmod{8}, \\ p_{-8}(4n+3) &\equiv 0 \pmod{64}, \\ p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) &\equiv 0 \pmod{5}, \\ p_{-4}(5n+3) \equiv p_{-4}(5n+4) &\equiv 0 \pmod{5}. \end{aligned}$$

We prove the first case by setting $M = 1, r = (-4), m = 4, j = 3, N = 8$.

In[16] = RKDelta$[8, 1, \{-4\}, 4, 3]$
Out[16] =

P$_{m,r}$(j) = $\{3\}$

f$_1$(q) = $\dfrac{(q;q)_\infty^{19}(q^4;q^4)_\infty^{15}}{q^4(q^2;q^2)_\infty^8(q^8;q^8)_\infty^{22}}$

t = $\dfrac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}$

AB =

$\{1\}$

$\{$p$_g$(t):g $\in$ AB$\}$ =

$\{512t + 1408t^2 + 480t^3 + 40t^4\}$

Common Factor = 8

The other cases of this theorem can be proved similarly.

In another example, they prove [6, Theorem 5.1, (5.3)] that $p_{-8}(8n+7) \equiv 0 \pmod{2^9}$, but we prove even more:

Theorem 16.

$$p_{-8}(8n+7) \equiv 0 \pmod{2^{11}}.$$

We set $N = 4$:

In[17] = RKDelta$[4, 1, \{-8\}, 8, 7]$
Out[17] =

P$_{m,r}$(j) = $\{7\}$

f$_1$(q) = $\dfrac{(q;q)_\infty^{84}}{q^8(q^2;q^2)_\infty^4(q^4;q^4)_\infty^{72}}$

t = $\dfrac{(q;q)_\infty^{8}}{q(q^4;q^4)_\infty^8}$

AB =

$\{1\}$

$\{\texttt{p}_\texttt{g}\texttt{(t):g} \in \texttt{AB}\}$ =

$\{576460752303423488 + 162129586585337856t$
$+ 18718085951258624t^2 + 1139094046375936t^3$
$+ 38970385760256t^4 + 737593524224t^5 + 7041187840t^6$
$+ 27033600t^7 + 22528t^8\}$

Common Factor = 2048

3.7 5-Regular Bipartitions

In 2016 Liuquan Wang developed [36] a large class of interesting congruences for the 5-regular bipartition function $B_5(n)$, with the generator

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty^2}.$$

Among many results were the following:

$$B_5(4n+3) \equiv 0 \pmod 5,$$
$$B_5(5n+2) \equiv B_5(5n+3) \equiv B_5(5n+4) \equiv 0 \pmod 5,$$
$$B_5(20n+7) \equiv B_5(20n+19) \equiv 0 \pmod{25}.$$

We are able to make the following improvements:

Theorem 17.

$$B_5(4n+3) \equiv 0 \pmod{10},$$
$$B_5(5n+2) \equiv B_5(5n+3) \equiv B_5(5n+4) \equiv 0 \pmod 5,$$
$$B_5(20n+7) \equiv B_5(20n+19) \equiv 0 \pmod{100}.$$

In[18] = RKDelta$[20, 5, \{-2, 2\}, 4, 3]$

Out[18] =

{N,M,r,m,j} = $\{20, 5, \{-2, 2\}, 4, 3\}$

$\texttt{P}_{\texttt{m,r}}\texttt{(j)}$ = $\{3\}$

$$\texttt{f}_1\texttt{(q)} = \frac{(q;q)_\infty^6 (q^2;q^2)_\infty (q^4;q^4)_\infty (q^{10};q^{10})_\infty^7}{q^7 (q^5;q^5)_\infty^2 (q^{20};q^{20})_\infty^{13}}$$

$$\texttt{t} = \frac{(q^4;q^4)_\infty^4 (q^{10};q^{10})_\infty^2}{q^2 (q^2;q^2)_\infty^2 (q^{20};q^{20})_\infty^4}$$

AB =

$$\{1, \frac{(q^4;q^4)_\infty (q^5;q^5)_\infty^5}{q^3 (q;q)_\infty (q^{20};q^{20})_\infty^5} - \frac{(q^4;q^4)_\infty^4 (q^{10};q^{10})_\infty^2}{q^2 (q^2;q^2)_\infty^2 (q^{20};q^{20})_\infty^4}\}$$

$\{\texttt{p}_\texttt{g}\texttt{(t):g} \in \texttt{AB}\}$ =

$\{50 - 40t - 50t^2 + 40t^3, -50 + 40t + 10t^2\}$

Common Factor = 10

In[19] = RKDelta$[5, 5, \{-2, 2\}, 5, 2]$

Out[19] =

{N,M,r,m,j} = $\{5, 5, \{-2, 2\}, 5, 2\}$

$P_{m,r}(j)$ = $\{2, 4\}$

$f_1(q)$ = $\frac{(q;q)_\infty^{20}}{q^2(q^5;q^5)_\infty^{20}}$

t = $\frac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$

AB =

$\{1\}$

$\{p_g(t) : g \in AB\}$ =

$\{15625 + 2500t + 100t^2\}$

Common Factor = 25

In[20] = RKDelta$[5, 5, \{-2, 2\}, 5, 3]$

Out[20] =

{N,M,r,m,j} = $\{5, 5, \{-2, 2\}, 5, 3\}$

$P_{m,r}(j)$ = $\{2, 4\}$

$f_1(q)$ = $\frac{(q;q)_\infty^{20}}{q^2(q^5;q^5)_\infty^{20}}$

t = $\frac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$

AB =

$\{1\}$

$\{p_g(t) : g \in AB\}$ =

$\{125 + 10t\}$

Common Factor = 25

In[21] = RKDelta$[10, 5, \{-2, 2\}, 20, 7]$

Out[21] =

{N,M,r,m,j} = $\{10, 5, \{-2, 2\}, 20, 7\}$

$P_{m,r}(j)$ = $\{7, 19\}$

$f_1(q)$ = $\frac{(q;q)_\infty^{77}(q^5;q^5)_\infty^{31}}{q^{27}(q^2;q^2)_\infty^{21}(q^{10};q^{10})_\infty^{87}}$

t = $\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$

`AB =`

$\{1\}$

$\{\texttt{p}_\texttt{g}\texttt{(t):g} \in \texttt{AB}\}$ `=`

$\{7388718138654720000t^2 + 15300803812130816 0000t^3$
$+ 125773135101296640000 0t^4 + 567549966474543104000 0t^5$
$+ 1650785764142743552000 0t^6 + 3408076787261898752000 0t^7$
$+ 5326685609492742144000 0t^8 + 6593718894911815680000 0t^9$
$+ 6670059753802039296000 0t^{10} + 5631416251164131328000 0t^{11}$
$+ 4023422763472519168000 0t^{12} + 2452781616685121536000 0t^{13}$
$+ 1280206744138547200000 0t^{14} + 571466042076299264000 0t^{15}$
$+ 216909878598172672000 0t^{16} + 69183948012019712000 0t^{17}$
$+ 18185075641339904000 0t^{18} + 3817570020433920000 0t^{19}$
$+ 607589073453056000 0t^{20} + 68009246675568000 0t^{21}$
$+ 4908094274568000 0t^{22} + 208348592196000 0t^{23} + 4690827635000 0t^{24}$
$+ 48340609000 0t^{25} + 181297000 0t^{26} + 119000 0t^{27}\}$

`Common Factor =` 10000

3.8 Some Congruences Related to the Tau Function

Please note that in this section we will assume $q = e^{2\pi i z}$ with $z \in \mathbb{H}$, to avoid confusion with τ, which will be used to identify a certain arithmetic function.

Ramanujan's tau function is defined as the coefficient $\tau(n)$ of the discriminant modular form:

$$\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)^{24} = \eta(z)^{24}.$$

This function is defined by taking the 24th power of Dedekind's eta function, and is among the most studied objects in the theory of modular forms. In particular, numerous interesting congruences have been found. Many classic examples include the following, discovered by Ramanujan [32]:

Theorem 18.

$$\tau(7n+m) \equiv 0 \pmod{7}$$

for $m \in \{0,3,5,6\}$.

Our algorithm can easily handle each of these cases. For example, we take the case of $\tau(7n)$ (notice that we study $(q;q)_\infty^{24}$, rather than with the proper generator for $\tau(n)$; because of this, we need to examine the progression $7n+6$):

```
In[22] = RKDelta[7, 1, {24}, 7, 6]
Out[22] =
```

{N,M,r,m,j} = $\{7, 1, \{24\}, 7, 6\}$

$P_{m,r}(j) = \{6\}$

$f_1(q) = \dfrac{1}{q^6(q^7;q^7)_\infty^{24}}$

$t = \dfrac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}$

AB =

$\{1\}$

$\{p_g(t) : g \in AB\}$ =

$\{-1977326743 - 16744t^6\}$

Common Factor = 7

We will give a more recent example discovered by Koustav Banerjee [5]:

Theorem 19.

$$\tau(8(14n+k)) \equiv 0 \pmod{2^3 \cdot 3 \cdot 5 \cdot 11},$$

for all $n \in \mathbb{Z}_{\geq 0}$ and k an odd integer mod 14.

This may be broken up into three distinct RK identities. We give the case of $112n+56$ (here shifted to $112n+55$)

```
In[23] = RKDelta[14, 1, {24}, 112, 55]
Out[23] =
```

{N,M,r,m,j} = $\{14, 1, \{24\}, 112, 55\}$

$P_{m,r}(j) = \{55\}$

$f_1(q) = \dfrac{(q^2;q^2)_\infty^{12}(q^7;q^7)_\infty^{30}}{q^{25}(q;q)_\infty^6(q^{14};q^{14})_\infty^{60}}$

$t = \dfrac{(q^2;q^2)_\infty(q^7;q^7)_\infty^7}{q^2(q;q)_\infty(q^{14};q^{14})_\infty^7}$

AB =

$\{1, \dfrac{(q^2;q^2)_\infty^8(q^7;q^7)_\infty^4}{q^3(q;q)_\infty^4(q^{14};q^{14})_\infty^8} - 4\dfrac{(q^2;q^2)_\infty(q^7;q^7)_\infty^7}{q^2(q;q)_\infty(q^{14};q^{14})_\infty^7}\}$

$\{p_g(t) : g \in AB\}$ =

$\{1483245480837120 + 22804899267870720t$

$- 281353127146291200t^2 + 4813307313059266560t^3$

$- 21171154911363072000t^4 - 33478635786731520000t^5$

$$
\begin{aligned}
&+ 845098635118510080t^6 + 77358598094131200t^7 \\
&- 25371836549283840t^8 - 1132615297820160t^9 \\
&- 512964938787840t^{10} - 114993988032000t^{11} - 349389680640t^{12}, \\
&- 1483245480837120 - 6489198978662400t + 990900684041748480t^2 \\
&- 151791226737131520t^3 - 1234180893392240640t^4 \\
&+ 461934380423577600t^5 - 65498418207129600t^6 \\
&+ 2233732210913280t^7 + 170807954042880t^8 + 855016378191360t^9 \\
&- 4703322624000t^{10} - 1414533120t^{11}\}
\end{aligned}
$$

`Common Factor =` 591360

The congruence here is even stronger than in the more general case, since $591360 = 2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

3.9 An Identity Related to Rogers–Ramanujan Subpartitions

We finish with an application of our package to studying infinite families of congruences. In 2017 Choi, Kim, and Lovejoy discovered a congruence [9, Proposition 6.4], based on a subpartition function studied by Kolitsch [19].

For any partition λ, define the corresponding Rogers–Ramanujan subpartition of λ as the unique subpartition of λ with a maximal number of parts, in which the parts are nonrepeating, nonconsecutive, and larger than the remaining parts of λ. For example, the partition $8+5+3+2+2+1+1+1$ contains the Rogers–Ramanujan subpartition $8+5+3$, whereas the partition $8+8+2+2+1+1+1$ contains the empty Rogers–Ramanujan subpartition.

Let us define $R_l(n)$ as the number of partitions of n which contain a Rogers–Ramanujan subpartition of length l, and then

$$A(n) = \sum_{l\geq 0} l \cdot R_l(n).$$

In [9] the following was demonstrated:

Theorem 20. *For $n \in \mathbb{Z}_{\geq 0}$,*

$$A(25n+9) \equiv A(25n+14) \equiv A(25n+24) \equiv 0 \pmod 5.$$

This was proved by connecting $A(n)$ with the coefficient $a(n)$ of

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^3 (q^4;q^4)_\infty^2}.$$

The authors of [9] pointed to other suspected congruences and compared the generating function for $a(n)$ with that of $c\phi_2(n)$. From this, they conjectured the existence of an infinite family for $A(n)$, in the style of Ramanujan's classic congruences, modulo powers of 5 [17, Chapter 7]. This infinite family was given a precise formulation after careful investigation using our standard package, as well as a modified version [30] of the package designed to check large congruences. After substantial evidence was gathered, the conjecture was proved [34]. We will here consider the case $A(25n+24)$ by examining $a(25n+24)$.

Taking $M = 4, r = (-3, 5, -2), m = 25, j = 24$, and setting $N = 20$, we find that

In[24] = RKDelta$[20, 4, \{-3, 5, -2\}, 25, 24]$

Out[24] =

$$P_{m,r}(j) = \{24\}$$

$$f_1(q) = \frac{(q;q)_\infty^{35}(q^4;q^4)_\infty^{18}(q^{10};q^{10})_\infty^{30}}{q^{26}(q^2;q^2)_\infty^{27}(q^5;q^5)_\infty^{8}(q^{20};q^{20})_\infty^{48}}$$

$$t = \frac{(q^4;q^4)_\infty^{4}(q^{10};q^{10})_\infty^{2}}{q^2(q^2;q^2)_\infty^{2}(q^{20};q^{20})_\infty^{4}}$$

AB =

$$\{1, \frac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{q^3(q;q)_\infty(q^{20};q^{20})_\infty^5} - \frac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}\}$$

$$\{p_g(t) : g \in AB\} =$$

$$\{126953125 + 74218750t - 174609375t^2 + 25390625t^3$$
$$- 1237031250t^4 + 1542084375t^5 + 3798876250t^6$$
$$- 7568402750t^7 + 3755535625t^8 + 210440100t^9$$
$$- 754603995t^{10} + 190492925t^{11} + 10649860t^{12} + 5735t^{13},$$
$$- 78125000 + 62500000t - 46093750t^2 + 128906250t^3$$
$$+ 551875000t^4 - 1636475000t^5 + 430767500t^6 + 1615951500t^7$$
$$- 1247744000t^8 + 145803400t^9 + 72090170t^{10} + 543930t^{11}\}$$

Common Factor = 5

4 Accessibility

Our software package is freely available as RaduRK via the software page of the Computer Algebra for Combinatorics Group at RISC (https://risc.jku.at/research_topic/computer-algebra-for-combinatorics/). The implementation uses Mathematica, and requires installation of a Diophantine software package called 4ti2 [1]. In particular, we used the interface developed by Ralf Hemmecke and Silviu Radu. Unfortunately, because 4ti2 is a Linux program, some additional steps are necessary in order to properly install our program. We list the instructions for full installation below. Any difficulty in installation should be communicated immediately to the author's email, nsmoot@risc.uni-linz.ac.at.

Step 1:
Install the 64 bit Cygwin, which can be found at https://cygwin.com/setup-x86_64.exe. Be sure to install the latest non-test versions of the following packages: binutils, gcc-core, gcc-g++, gmp, and make.

Step 2:
Once installed, open the Cygwin terminal. This will establish a folder in the directory in `cygwin64\home`, which we will call `ME` (i.e., `cygwin64\home\ME`), but which may have a different name.

Step 3:
Download `4ti2`, version 1.6.7 or higher, which can be found online at [1]. Be sure to place the file `4ti2-1.6.7.tar.gz` into `cygwin64\home\ME`.

Step 4:
In the Cygwin terminal, extract and compile `4ti2` by typing the following:

```
cd /home/ME
tar xzf 4ti2-1.6.7.tar.gz
cd 4ti2.1.6.7
./configure --prefix=/home/ME/4ti2
make
make install-exec
```

This should properly define the `zsolve` command, which is used throughout our software package.

Step 5:
Open Mathematica and type `$UserBaseDirectory`. A possible directory would resemble
`C:\Users\USERNAME\AppData\Roaming\Mathematica`.

Step 6:
Install the package `math4ti2.m` by Ralf Hemmecke and Silviu Radu, which can be found online at [15]. Be sure to place the file `math4ti2.m` in the directory
`C:\Users\USERNAME\AppData\Roaming\Mathematica \Applications`.

Step 7:
We need to modify `math4ti2.m` so as to recognize the separately installed `zsolve` command (see Step 4). Open `math4ti2.m`, and go to the line
`zsolvecmd = "/usr/bin/4ti2-zsolve"`.
Replace `/usr/bin/4ti2-zsolve` with the path to `zsolve` *as defined through Windows*. For example,
`zsolvecmd = "C:\\cygwin64\\home\\ME\\4ti2\\bin";`.

Step 8:
Download the package `RaduRK` from `https://risc.jku.at/research_topic/computer-algebra-for-combinatorics/` [35].

The notebooks `RaduRKexamples.nb` and `OverpartitionExamples.nb` can also be found at [35].

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