Hook Type Tableaux and Partition Identities

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HOOK TYPE TABLEAUX AND PARTITION IDENTITIES

KOUSTAV BANERJEE AND MANOSIJ GHOSH DASTIDAR

Abstract. In this paper we exhibit the box-stacking principle (BSP) in conjunction with Young diagrams to prove generalizations of the Stanley's and Elder's theorem without the use of partition statistics in general. We explain how the principle can be used to prove another interesting theorem on partitions with parts separated by parity, a special case of which is George Andrews's result in [2].

Mathematics Subject Classifications. 05A19, 11P84, 11P81.
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1. Introduction

The field of hypergeometric series and partitions have been closely connected ever since Euler's primarily work on the subject. Since then it became a standard method to use results in q-series in order to aid the proofs of numerous partition identities. However, in some cases, as for the box-stacking principle (BSP) demonstrated subsequently, we will see how one can prove a host of partition identities in an elementary and elegant manner, find newer identities and also prove a infinite series identity by purely discrete geometrical construction and combinatorial arguments.

2. Preliminaries

Partitions. A partition of $n \geq 0$ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ of positive integers whose sum is n. We express this by writing $\lambda \vdash n$. Here $l = l(\lambda)$ is the number of parts in λ and $n = |\lambda|$ is the sum of parts of λ.
The partition function $p(n)$ is the number of partitions of n and $P(n)$ is the set of all partitions of n. For example, $p(4) = 5$ as we can write 4 into 5 ways explicitly into 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.

The empty partition () is the unique partition of 0; i.e., $p(0) = 1$.

Due to Euler, we have the following generating function for partitions of n,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Conjugate of a Partition. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, we may define a new partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_m) \vdash n$ (where m is the largest part of λ) by choosing λ'_i as the number of parts of λ that are $\geq i$. The partition of λ' is called the conjugate of λ.

Notice that the graphical representation of the conjugate is obtained by reflecting the diagram in the main diagonal.

For example, if $\lambda = (6, 3, 3, 2, 1)$, then conjugate of λ is $\lambda' = (5, 4, 3, 1, 1, 1)$.

Color Partitions. For a positive integer $l \geq 2$ we define the l-color partitions of n as follows:
$P^{(l)}(n)$ is the set of all partitions of n where multiples of l can occur with 2 colors and $P^{(l)}(0) := \{()\}$; moreover, $p^{(l)}(n) := \#P^{(l)}(n)$.

For example, for $l = 3$ and $n = 4$, $p^{(3)}(4) = 7$ as we can express 4 into 4_1, $3_1 + 1_1$, $3_2 + 1_1$, $2_1 + 2_1$, $2_1 + 1_1 + 1_1$, $1_1 + 1_1 + 1_1 + 1_1$. Note that those parts in the color partitions of 4 which are not multiples of 3 are indexed by 1 which is interpreted by having white color; whereas parts being multiples of 3 are indexed by both 1 and 2, where 2 is interpreted by having green color. The generating function of $p^{(l)}(n)$ is

$$\sum_{n=0}^{\infty} p^{(l)}(n) q^n = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)(1-q^{lj})}.$$

For $l = 2$, $p^{(l)}(n)$ is $a(n)$, the number of cubic partitions [5], which enjoy many arithmetic properties analogous to the classical partition function.

Young Diagrams. To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ. In this context, we prefer the representation to be 'upside down' (sometimes called as 'right side up'). For $\lambda = (8, 6, 6, 5, 1) \vdash 26$, Y_λ is given by

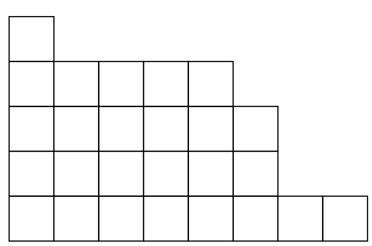


Figure 1

In the same fashion, for $\lambda = (5_2, 6_1, 10_2, 15_1) \in P^{(5)}(36)$, the associated colored Young diagram Y_λ is;

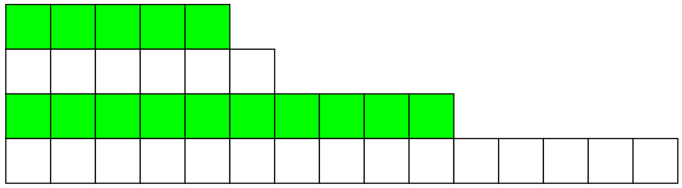


Figure 2

Hook Length Tableaux and Hook Type Tableaux. For each box v in Y_λ, define the hook length of v, denoted by $h_v(\lambda)$, to be the number of boxes u such that $u = v$ or u lies in the same column as v and above v or in the same row as v and to the right of v. The hook length multiset of λ, denoted by $\mathcal{H}_\lambda$, is the multiset of all hook lengths of λ. Each hook length h can be split into $h = a + l + 1$, where a is the arm length (the no. of boxes to the right in the same row) and l the leg length (the no. of boxes on above in the same column). The ordered pair (a, l) is called hook type of the chosen box in the Young tableau.

The boxes will be colored in accordance with the partition given (cf. Figure 2). E.g., for $\lambda = (6, 3, 3, 2) \vdash 14$, the hook length multiset $\mathcal{H}_\lambda = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$ according to

2	1				
4	3	1			
5	4	2			
9	8	6	3	2	1

Figure 3

For the hook type (of the boxes) of Y_λ for $\lambda = (6, 3, 3, 2) \vdash 14$ we have,

$(1,0)$	$(0,0)$				
$(2,1)$	$(1,1)$	$(0,0)$			
$(2,2)$	$(1,2)$	$(0,1)$			
$(5,3)$	$(4,3)$	$(3,2)$	$(2,0)$	$(1,0)$	$(0,0)$

Figure 4

In color perspective, $\lambda = (6_1, 6_2, 7_1, 12_1) \in P^{(6)}(31)$ possesses the following hook length tableau,

6	5	4	3	2	1						
7	6	5	4	3	2						
9	8	7	6	5	4	1					
15	14	13	12	11	10	7	5	4	3	2	1

Figure 5

The hook type tableau of λ is

$(5,0)$	$(4,0)$	$(3,0)$	$(2,0)$	$(1,0)$	$(0,0)$						
$(5,1)$	$(4,1)$	$(3,1)$	$(2,1)$	$(1,1)$	$(0,1)$						
$(6,2)$	$(5,2)$	$(4,2)$	$(3,2)$	$(2,2)$	$(1,2)$	$(0,0)$					
$(11,3)$	$(10,3)$	$(9,3)$	$(8,3)$	$(7,3)$	$(6,3)$	$(5,1)$	$(4,0)$	$(3,0)$	$(2,0)$	$(1,0)$	$(0,0)$

Figure 6

Notation. Basic notations used: for n, k and $l \geq 2$ positive integers,

$Q_k(n) :=$ Number of occurences of part k in $P(n)$.
$V_k(n) :=$ The number of parts occurring k or more times in the partitions of n.
$S(n) := \sum_{\lambda \vdash n} \text{dist}(\lambda)$, where $\text{dist}(\lambda)$ denotes the number of distinct parts in λ.
$Q_k^{(l)}(n) :=$ Number of occurences of parts k_1 and k_2 in $P^{(l)}(n)$ when k is a multiple of l; otherwise the number of occurences of the part k_1 in $P^{(l)}(n)$.
In short we will say, $Q_k^{(l)}(n)$ is the number of occurences of part k.

Theorem 1 (Bessenrodt [3]**, Bacher - Manivel** [4]**):** Let $1 \leq k \leq n$ be two integers. Then, for every positive $j < k$, the total number of occurrences of the part k among all partitions of n $(= Q_k(n))$ is equal to the number of boxes whose hook type is $(j, k-j-1)$.

For $k = 1$, j has to be 0, but for $k > 1$ without loss of generality, one can choose particularly $j = k-1$. This is because the total number of occurences of the part k among all partitions of n does not depend on j according to Theorem 1, and so for any two $j_1, j_2 > 0$, and $j_1 \neq j_2$, the number of boxes whose hook type is $(j_1, k-j_1-1)$ and $(j_2, k-j_2-1)$, respectively, and both of them exactly enumerate the total number of occurences of the part k among all partitions of n.
For $n = 5$, $k = 3$, $Q_3(5) = 2$ $(3+2,\ 3+1+1)$ and the number of boxes with hook type $(2,0)$ in the corresponding Young tableau is also 2:

$$\begin{array}{|c|c|c|c|c|}\hline (4,0) & (3,0) & (2,0) & (1,0) & (0,0) \\ \hline \end{array}\ ,\quad \begin{array}{|c|cccc}\hline (0,0) & \multicolumn{1}{|c}{} & & & \\ \hline (3,1) & \multicolumn{1}{|c|}{(2,0)} & \multicolumn{1}{c|}{(1,0)} & \multicolumn{1}{c|}{(0,0)} \\ \hline \end{array}$$

Figure 7

3. Partition Identities and Proofs

In this section we exploit the Theorem 1 in a pointwise sense so as to provide proofs of Stanley's theorem and subsequently Elder's theorem.

Stanley's Theorem [7]**.** The total number of 1's in all partitions of a positive integer n is equal to the sum of the numbers of distinct parts of those partitions of n. In our notation, $S(n) = Q_1(n)$.

For $n = 4$,

$P(4)$		
$\lambda \vdash 4$	no. of 1's in λ	no. of distinct parts in λ
4	0	1
$3+1$	1	2
$2+2$	0	1
$2+1+1$	2	2
$1+1+1+1$	4	1
Total	$Q_1(4) = 7$	$S(4) = 7$

Figure 8

Proof. It is enough to show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0,0)$. From the definition of hook-type of a box in Young tableau Y_λ, it is clear that the number of boxes with hook-type $(0,0)$ is same as the number of boxes in Y_λ with no boxes on the right and above of the chosen box in the tableau. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and suppose $\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_k}$ are all the distinct parts of λ with respective occurences $m_1, m_2, \ldots, m_k$ where $1 \leq a_i \leq r$ and $a_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Without loss of generality assume $\lambda_{a_1} > \lambda_{a_2} > \cdots > \lambda_{a_k}$. Obviously, $n = \lambda_{a_1} + \cdots + \lambda_{a_1} + \lambda_{a_2} + \cdots + \lambda_{a_2} + \cdots + \lambda_{a_k} + \cdots + \lambda_{a_k}$. Note that, the boxes with hook-type $(0,0)$ appear exactly once in Y_λ corresponding to the part λ_{a_m} subject to the condition that the immediate next part λ_{a_n} with $m \neq n$. Explicitly, if we look at the right most box for the part λ_{a_m} in Y_λ, can observe that there is no box above or right to it in the representation of Y_λ. For the immediate next part of λ_{a_m} is λ_{a_n}; i.e., the total number of boxes in the row (correspond to λ_{a_n}) are at least one less than that of λ_{a_m} in Y_λ. Therefore, it is clear that the number of boxes with hook-type $(0,0)$ equals the number of distinct parts of λ. Now, summing over all $\lambda \vdash n$ we get the Stanley's theorem.

For example, for $n = 4$,

$P(4)$		
$\lambda \vdash 4$	no. of boxes in Y_λ with hook-type $(0, 0)$	no. of distinct parts in λ
4	1	1
$3+1$	2	2
$2+2$	1	1
$2+1+1$	2	2
$1+1+1+1$	1	1
Total	7	$S(4) = 7$

Figure 9

Elder's Theorem [7]**.** The total number of occurences of an integer k among all partitions of n is equal to the number of occasions that a part occurs greater or equal k times in $P(n)$; i.e., $Q_k(n) = V_k(n)$.

For $n = 5$ and $k = 2$,

$P(5)$		
$\lambda \vdash 5$	no. of 2's in λ	no. of parts occurring ≥ 2 times in λ
5	0	0
$4+1$	0	0
$3+2$	1	0
$3+1+1$	0	1
$2+2+1$	2	1
$2+1+1+1$	1	1
1+1+1+1+1	0	1
Total	$Q_2(5) = 4$	$V_2(5) = 4$

Figure 10

Proof. We need only to show that the number of boxes with hook type $(k-1, 0)$, $k > 1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ. Now, a box with hook-type $(k-1, 0)$ in Y_λ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ precisely describes that there are $k-1$ boxes on the right to it but having no box above. When transforming λ to it's conjugate $\lambda^{'}$ it is clear that after conjugation, the box with hook-type $(k-1, 0)$ transforms into the box with hook-type $(0, k-1)$. This shows that there are total at least k verticals stacks of boxes (including the box itself); i.e., there exists a part that occurs at least k times in that conjugate partition. Therefore, corresponding to each box with hook-type $(k-1, 0)$ there exists a part that occurs at least k times. Now, summing over all partitions of n we have Elder's statement since counting over λ is same as the counting over the conjugate $\lambda^{'}$.

For $n = 5$ and $k = 2$,

$P(5)$			
$\lambda \vdash 5$	no. of boxes with hook-type $(1,0)$	$\lambda^{'} \vdash 5$	no. of parts occurring ≥ 2 times in $\lambda^{'}$
5	1	1+1+1+1+1	1
$4+1$	1	$2+1+1+1$	1
$3+2$	1	$2+2+1$	1
$3+1+1$	1	$3+1+1$	1
$2+2+1$	0	$3+2$	0
$2+1+1+1$	0	$4+1$	0
1+1+1+1+1	0	5	0
Total	4	Total	$V_2(5)=4$

Figure 11

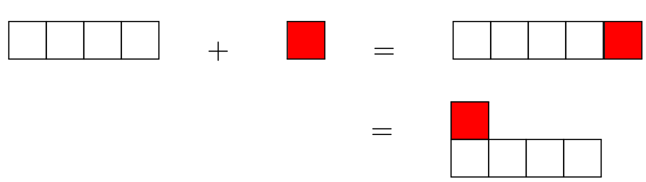
4. Box Stacking Principle

In this subsection, we shall introduce a specific type of combinatorial construction which we call the "Box Stacking Principle"(BSP).

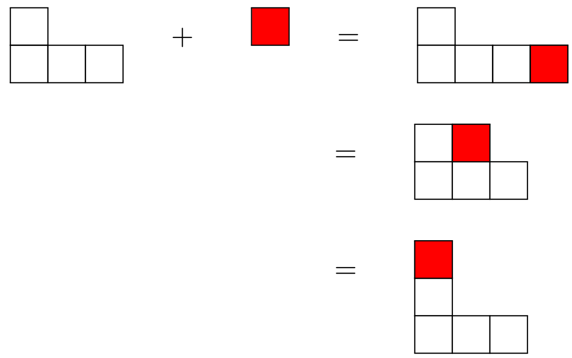
The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n+k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:
1. For $k=1$: We add one box to all permissible places in Y_λ. One can trivially add one box in two ways: (i) Add to the bottom row of Y_λ. (ii) Stack the box on the above of the top row of Y_λ. Also, one can add one box to a row in Y_λ if and only if the difference between the number of boxes in the chosen row and its immediate next is at least 1. In other words, for $\lambda := (\lambda_1, \ldots, \lambda_r) \vdash n$, following rule (i) the trivial addition of one box corresponds to $\mu := ((\lambda_1+1), \ldots, \lambda_r) \vdash n+1$ whereas by rule (ii) we have $\mu := (\lambda_1, \ldots, \lambda_r, 1) \vdash n+1$. Nontrivial addition of one box can be done if and only if for any two consecutive part say, λ_i and λ_j $(\lambda_i \geq \lambda_j)$, we have $\lambda_i - \lambda_j \geq 1$.

For example, to all partitions of 4 and applying the stacking principle for adding one box to the Young diagram gives:

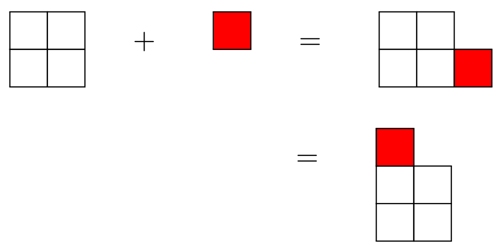
I. $\lambda = 4$:



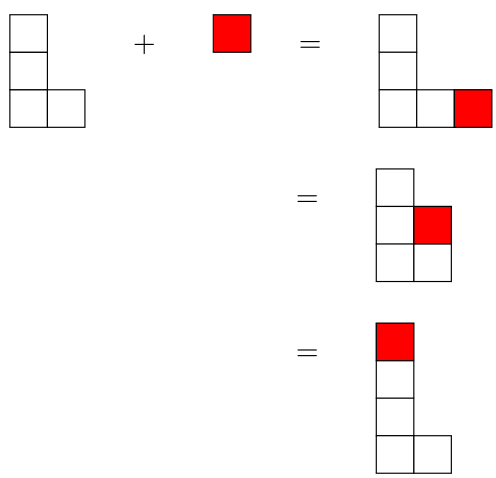
II. $\lambda = 3 + 1$:



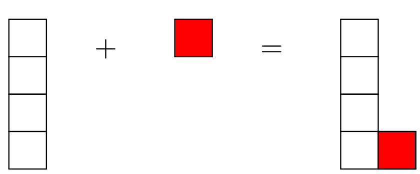
III. $\lambda = 2 + 2$:



IV. $\lambda = 2 + 1 + 1$:



V. $\lambda = 1 + 1 + 1 + 1$:



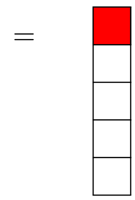


Figure 12

2. $k > 1$: Here we consider the addition of k boxes as a 'packet of k boxes', instead of adding 'k single boxes'. Again one can trivially add a 'packet of k boxes' to the bottom row of Y_λ with $\lambda := (\lambda_1, \ldots, \lambda_r)$. In this context, by adding 'packet of k boxes', we mean that adding k to λ_1 so that the resulting partition $\mu := ((\lambda_1 + k), \ldots, \lambda_r) \vdash n+k$. Now a nontrivial addition of a packet of k boxes to Y_λ can be done if and only if for any two consecutive part say, λ_i and λ_j ($\lambda_i \geq \lambda_j$), we have $\lambda_i - \lambda_j \geq k$. We do not consider the addition of 'k single boxes' which means that we do not allow the cases $\mu_1 := (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \vdash n+k$ and $\mu_2 := (\lambda_1, \ldots, (\lambda_{j_1} + 1), \ldots, (\lambda_{j_2} + 1), \ldots, , (\lambda_{j_k} + 1), \ldots, \lambda_r) \vdash n+k$.

This specific example will show what we do not allow. For stacking of $k = 2$ boxes with $\lambda = (3, 1) \vdash 4$, the following situations will be regarded as violating our rules:

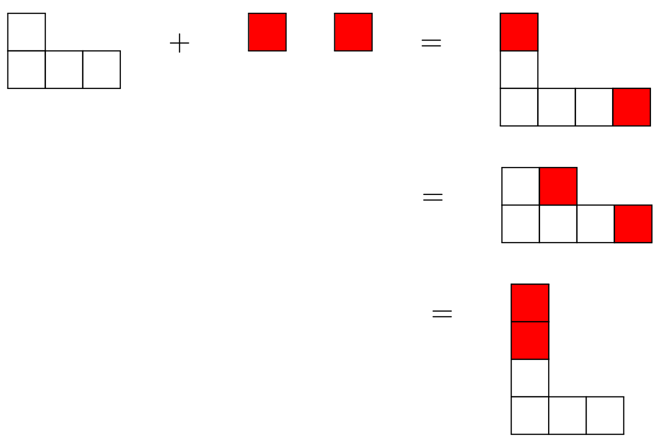


Figure 13

The correct addition of 'packet of 2 boxes' following BSP gives:

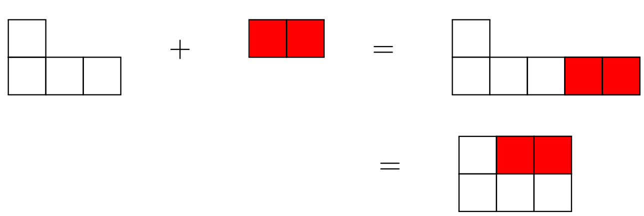


Figure 14

4.1. **Extension of Partition Identities in Recursive Format:** This section prepares for our generalization of Stanley's and Elder's theorem. To this end we exploit a recursive argument due to Dastidar and Sengupta [6]. We will provide an alternative proof of their observations using the BSP.

Theorem 2 (Dastidar, Sengupta [6]**):** For positive integers n and k,

$$S(n) = Q_k(n) + Q_k(n+1) + Q_k(n+2) + \cdots + Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

For example, for $n = 5$ and $k = 3$; we have $S(5) = 12$, $Q_3(5) = 2$, $Q_3(6) = 4$, $Q_3(7) = 6$. So, $S(5) = Q_3(5) + Q_3(5+1) + Q_3(5+(3-1))$.

Theorem 3 (Dastidar, Sengupta [6]**):** For positive integers n, r and k,

$$V_k(n) = Q_{rk}(n) + Q_{rk}(n+k) + Q_{rk}(n+2k) + \cdots + Q_{rk}(n+(r-1)k) = \sum_{l=0}^{r-1} Q_{rk}(n+lk).$$

For example, for $n = 5$, $k = 2$ and $r = 3$; we have $V_2(5) = 4$, $Q_6(5) = 0$, $Q_6(7) = 1$, $Q_6(9) = 3$. So, $V_2(5) = Q_6(5) + Q_6(5+2) + Q_6(5+(3-1)2)$.

Lemma 1: Stacking k boxes to the Young diagrams corresponding to all partitions of n following the BSP generates as many new partitions as there are occurences of k in all partitions of $n+k$.

Proof: The proof is divided into two cases as follows:

I. (The Trivial Stacking): We can obviously add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ (as discussed in the principle of construction) and immediately observe that the total number of generated new partition is $p(n)$.

II. (Non-trivial Stacking): We can also add a packet of k boxes to some part other than the largest one in a partition $\lambda \vdash n$ (cf. Figure 14) but not always. So, adding k-boxes to a Young diagram Y_λ following BSP is possible if and only if there exists a box in Y_λ with hook-type $(k-1, 0)$ (Because, having a box with hook-type $(k-1, 0)$ implies that above this box there are k-consecutive empty places where we can place the packet of k boxes. On the other hand, to place a packet of k boxes in the diagram without violating the BSP and structure of Y_λ there must exist a k-consecutive empty places; i.e., a box with hook-type $(k-1, 0)$). This explicitly shows the one to one correspondence between the number of permissible ways of non-trivial addition of packet of k boxes and the number of boxes with hook-type $(k-1, 0)$ in Y_λ. Summing over all partitions of n gives the number of occurrences of part k in $P(n)$.

Altogether I and II give that the total of new generated partition is $p(n) + Q_k(n)$ and it is immediate that $p(n) + Q_k(n) = Q_k(n+k)$.

Proof of Theorem 2 and 3: The proofs of Theorem 2 and Theorem 3 are immediate from Lemma 1 since it is enough to prove that $p(n) + Q_k(n) = Q_k(n+k)$.

5. Generalization of Stanley's Theorem in Color Context

In the previous section, we presented an extension of Stanley's theorem due to Dastidar, Sengupta [6] in the context of the classical partition function. Now we will provide a theorem in the context of color partitions which is analogous to the Theorem 2.

Theorem 4: For positive integers $k, l \geq 2$, and $n \in \mathbb{N}$,

$$Q_1^{(l)}(n) = \begin{cases} (Q_k^{(l)}(n) + Q_k^{(l)}(n+1) + \cdots + Q_k^{(l)}(n+k-1))/2, & \text{if } l \,|\, k, \\ Q_k^{(l)}(n) + Q_k^{(l)}(n+1) + \cdots + Q_k^{(l)}(n+k-1), & \text{otherwise} \end{cases}$$

Remark: In order to prove the above theorem, it is enough to prove the following recursions: If $l \geq 2, k, n \in \mathbb{N}$, then

$$\frac{Q_k^{(l)}(n+k)}{2} = p^{(l)}(n) + \frac{Q_k^{(l)}(n)}{2}, \quad \text{if} \quad l \mid k.$$
$$Q_k^{(l)}(n+k) = p^{(l)}(n) + Q_k^{(l)}(n), \quad \text{otherwise.}$$

5.1. **Box Stacking Principle in Color Partitions context.** The BSP in color partitions context consists of a set of rules to produce from all color partitions λ of n a new set of color partitions of $n+k$. This is done by adding k boxes in a particular way. Here 'packet of k boxes' has the same meaning as in the case of the BSP for the classical partition function. But in the color context we have to take care about the color of a 'packet of k boxes' (because following notations in Preliminary section, one can observe that for a $\lambda \in P^{(l)}(n)$ if k is multiple of l then k appears twice k_1, k_2 in λ). If k is not a mutiple of l, without loss of generality, we always add a 'packet of k boxes' prescribed by white color (one may also add a 'packet of k boxes' prescribed by green color). The set of rules are as follows:

Whenever, we say adding a 'packet of k boxes' it will mean that 'packet of k boxes' is colored by white color. Let $\lambda := (\lambda_{1_{i_1}}, \lambda_{2_{i_2}}, \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n)$ with $i_k \in \{1, 2\}$ and $1 \leq k \leq r$, $k \in \mathbb{N}$. So when we say $\lambda_{1_{i_1}}$ is the largest part of λ, it means that $\lambda_1 \geq \cdots \geq \lambda_r$ (independent of the indices). First, we will look at the index of the largest part $\lambda_{1_{i_1}}$. If $i_1 = 1$, then trivially we add the packet of k boxes to $\lambda_{1_{i_1}}$; i.e., to bottom row of Y_λ so that the resulting partition $\mu := ((\lambda_{1_{i_1}} + k_1), \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n+k)$. If $i_1 = 2$, then two cases will arise:

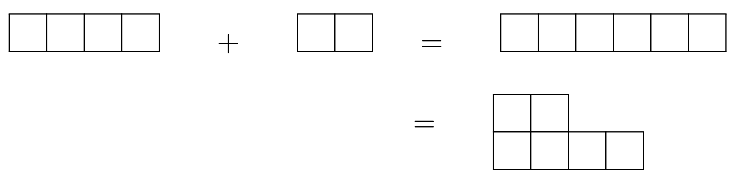
A. If $\lambda_1 \geq k$, then we consider following two cases: (i) If there exist any two consecutive parts say $\lambda_{s_{i_s}}$ and $\lambda_{t_{i_t}}$ ($\lambda_s \geq \lambda_t$) with $i_t = 1$ and $\lambda_s - \lambda_t \geq k$, then we add a packet of k boxes to the row corresponding to the part $\lambda_{t_{i_t}}$ in Y_λ. (ii) If there does not exists any two consecutive parts with the condition given in (i), then we simply insert the packet of k-boxes as a new row into Y_λ (cf. Figure 17).

B. If $\lambda_1 < k$, then we adjoin the packet of k boxes to the below of the bottom row of Y_λ so that resulting partition is $\mu := (k_1, \lambda_{1_{i_1}}, \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n+k)$ (cf. Figure 16).

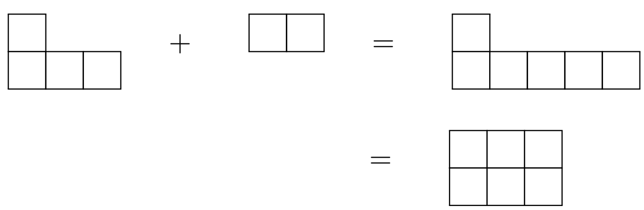
C. We have already stated all the rules of adding a packet of k boxes. Now, we state an exclusion rule; i.e, a case in which we will not allow for addition of k boxes. Here index of parts in the partition $\lambda \in P^{(l)}(n)$ is important. For any part, say $\lambda_{m_{i_m}}$ with $i_m = 2$, we do not allow the addition of a packet of k boxes to the row corresponding to the part $\lambda_{m_{i_m}}$ in Y_λ. In short, if the color of the row corresponding to the part with index 2 is green, we do not allow the addition of a packet of k boxes to it (cf. Figure 18).

An example to illustrate these rules, consider all 3 color partitions of 4; i.e,$P^{(3)}(4)$ and applying the color BSP for adding a packet of 2 boxes to the Young diagram gives:

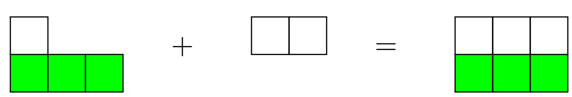
I. 4_1 :



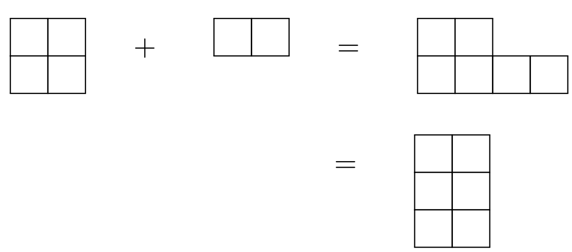
II. $3_1 + 1_1$:



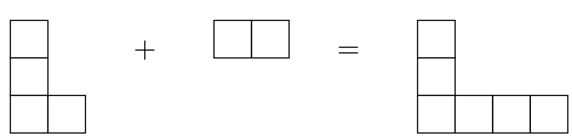
III. $3_2 + 1_1$:



IV. $2_1 + 2_1$:



V. $2_1 + 1_1 + 1_1$:



VI. $1_1 + 1_1 + 1_1 + 1_1$:

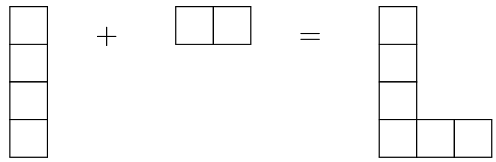


Figure 15

Now if we consider the addition of a packet of 5 boxes and 3 boxes, respectively, to the partition $(3_2, 1_1)$, then rules **B** and **A**(ii) imply the following:

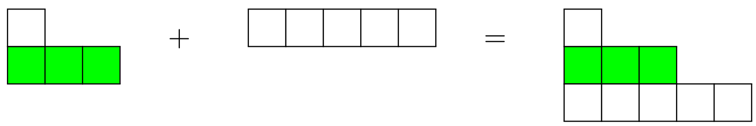


Figure 16

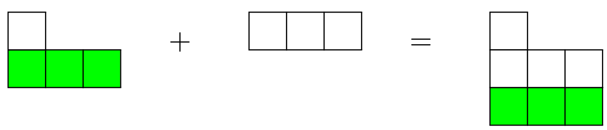


Figure 17

Next, we give an example for the exclusion following rule **C**. For instance, for $n = 11$, $l = 3$, $k = 2$ and $\lambda = (6_2, 3_2, 2_1) \in p^{(3)}(11)$:

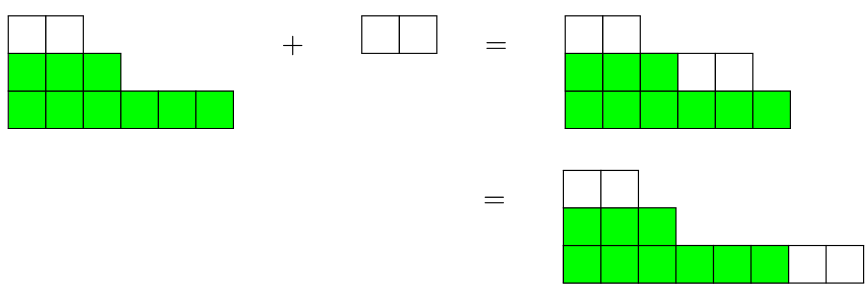


Figure 18

In the above figure, one can observe that we did not allow the addition of a packet of 2 white boxes to the parts which are colored by green color; i.e., by exclusion we mean that we excluded all possible addition of a packet of 2 boxes being white colored to the parts colored by green color following the rule **C**.

5.2. **Recursion and Proof of Theorem 4.** It is enough to prove the recursion for the above stated Theorem 4.

Lemma 2: Adding a packet of k boxes to the Young diagrams of $\lambda \in P^{(l)}(n)$ following the color BSP generates as many new color partitions as there are occurences of a part k in $P^{(l)}(n+k)$ subject to the condition that k is not a multiple of l. But if k is a multiple of l, then adding a packet of k boxes generates as many new color partitions which equals to half of the total number of occurences of the part k in $P^{(l)}(n+k)$.

Proof: Following the rules defined in the subsection 5.1, one can immediately observe that the trivial addition of a packet of k boxes generates the number of color partitions which equals to $p^{(l)}(n)$.

Now, if $l \nmid k$, then following rule **A**(ii), we conclude that the number of nontrivial addition of a packet of k boxes to Young diagrams is $Q_k^{(l)}(n)$. Therefore, total number of new generated color partitions is $p^{(l)}(n) + Q_k^{(l)}(n)$ and it is immediate that $p^{(l)}(n) + Q_k^{(l)}(n) = Q_k^{(l)}(n+k)$.

On the other hand, for $l \mid k$, the part k in $\lambda \in P^{(l)}(n)$ appears with two colors. Now, adding a packet of k boxes to Young Diagrams enumerate half of the total number of occurrences of k in $P^{(l)}(n)$ because we add only a white colored packet of k boxes. Whereas in this situation, we have to count the total number of occurrences of parts k_1 and k_2. In short, we have chosen only one representative of k_1 and k_2 in terms of adding only a white colored packet of k boxes. Hence one can observe that the total number of generated color partition is $p^{(l)}(n) + \frac{Q_k^{(l)}(n)}{2}$ and consequently, $p^{(l)}(n) + \frac{Q_k^{(l)}(n)}{2} = \frac{Q_k^{(l)}(n+k)}{2}$.

Proof of Theorem 4: The proof of Theorem 4 is immediate from Lemma 2.

6. Application

Recently, Andrews proved a beautiful result.

Theorem 5 (Andrews [2]): Let $\mathcal{O}_d(n)$ denote the number of partitions of n in which the odd parts are distinct and each positive odd integer smaller than the largest odd part must appear as a part. Then

$$p_{eu}^{od}(n) = \mathcal{O}_d(n),$$

where $p_{eu}^{od}(n)$ denotes the number of partitions of n in which each even part is less than each odd part and odd parts are distinct.
As an application of the BSP, we will prove a theorem on partitions with parts separated by parity. Adding the restriction of distinctness, we immediately obtain George Andrews's theorem.

Definition 1:

$$P_{eu}^{ou}(n) := \left\{ \lambda \vdash n : \begin{array}{c} (1) \text{ all the odd parts of } \lambda \text{ are unrestricted,} \\ (2) \text{ each even part of } \lambda \text{ is less than each odd part of } \lambda \end{array} \right\},$$

$p_{eu}^{ou}(n) := \#\{\lambda \vdash n : \lambda \in P_{eu}^{ou}(n)\}$.

For example, $p_{eu}^{ou}(9) = 12$ $(9, 7+2, 7+1+1, 5+4, 5+3+1, 5+2+2, 5+1+1+1+1, 3+3+3, 3+3+1+1+1, 3+2+2+2, 3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$.

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,

$$\text{OMax}(\lambda) := \text{greatest odd part of } \lambda,$$

$$\text{EMax}(\lambda) := \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{OEMaxSum}(\lambda) := \text{OMax}(\lambda) + \text{EMax}(\lambda),$$

and

$$\text{OEMaxDiff}(\lambda) := |\text{OMax}(\lambda) \text{ - } \text{EMax}(\lambda)|.$$

$$O_{\overline{u}}(n) := \left\{ \lambda \vdash n : \begin{array}{l} (1) \text{ for any odd k such that k} < \text{OMax}(\lambda); \text{ k must appear as a part of } \lambda, \\ (2) \text{ if some odd part of } \lambda \text{ occurs repeatedly then OEMaxSum } (\lambda) \leq n \end{array} \right\},$$

$$O_{\overline{u}}^{*}(n) := \{\lambda \in O_{\overline{u}}(n) : \text{OEMaxDiff } (\lambda) = \min \{\text{OEMaxDiff } (\lambda^{'}) : \lambda^{'} \in O_{\overline{u}}(n)\}\},$$

$$o_{\overline{u}}^{*}(n) := \#\{\lambda \vdash n : \lambda \in O_{\overline{u}}^{*}(n)\}.$$

For example, $o_{\overline{u}}^{*}(9) = 12$ $(8+1, 6+2+1, 5+3+1, 4+4+1, 4+3+1+1, 4+2+2+1, 3+2+1+1+1+1, 2+2+2+2+1, 2+2+2+1+1+1, 2+2+1+1+1+1+1, 2+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$; we see that according to our definition, the partition $\lambda = (6, 1, 1, 1) \notin O_{\overline{u}}^{*}(9)$ but the partition $(4, 3, 1, 1) \in O_{\overline{u}}^{*}(9)$

Note: In our diagrams, odd parts and even parts are interpreted by having blue and brown color, respectively.

Theorem 6: $o_{\overline{u}}^{*}(n) = p_{eu}^{ou}(n)$

Proof: Consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in O_{\overline{u}}^{*}(n)$. We separate λ into $\lambda^{'} = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq l$ and $\lambda^{''} = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq o_j \leq l$ according to the odd and even parts, respectively. Let $Y_{\lambda^{'}}$ and $Y_{\lambda^{''}}$ be the corresponding Young diagrams of $\lambda^{'}$ and $\lambda^{''}$. Next, we join $Y_{\lambda^{'}}$ and $Y_{\lambda^{''}}$ by successively adjoining their rows with respect to the ordering of the parts in $\lambda^{'}, \lambda^{''}$, respectively, starting with the largest one and end with the smallest one. Call the restricting Young diagram $Y_{\lambda^{'''}}$. Now, we consider the following three cases:

1. If the number of odd parts is equal to the number of even parts in a partition $\lambda \in O_{\overline{u}}^{*}(n)$, then $Y_{\lambda^{'''}}$ is with $\lambda^{'''} \in P_{eu}^{ou}(n)$. Since $\lambda^{'} = (\lambda_{o_1}, \ldots, \lambda_{o_r})$ and $\lambda^{''} = (\lambda_{e_1}, \ldots, \lambda_{e_r})$ have equal number of parts, no part remains left neither in $\lambda^{'}$ nor in $\lambda^{''}$ after adjoining because the resulting partition $\lambda^{'''} = (\lambda_{o_1} + \lambda_{e_1}, \ldots, \lambda_{o_r} + \lambda_{e_r})$. Correspondingly no row remains left neither in $Y_{\lambda^{'}}$ nor in $Y_{\lambda^{''}}$ after adjoining.

2. Suppose the number of odd parts is greater than the number of even parts in a partition $\lambda \in O_{\overline{u}}^{*}(n)$; let the difference be t. Then a similar argument shows that the t rows in $Y_{\lambda^{'}}$

remain left after adjoining of rows of $Y_{\lambda'}$ and $Y_{\lambda''}$. Therefore, in the resulting $Y_{\lambda'''}$ with $\lambda''' \in P_{eu}^{ou}(n)$, t rows will be positioned in the same order as in $Y_{\lambda'}$.
3. Suppose the number of even parts is greater than the number of odd parts in a partition $\lambda \in O_{\overline{u}}^{*}(n)$; let the difference be u. Similar to the argument given in (1) we see that u rows in $Y_{\lambda''}$ remain left after adjoining the rows of $Y_{\lambda'}$ and $Y_{\lambda''}$. Here u rows will be inserted into $Y_{\lambda'}$ so that the resulting $Y_{\lambda'''}$ with $\lambda''' \in P_{eu}^{ou}(n)$ does not violate the structure of the Young diagram.

For example, given Y_λ with the partition $\lambda = (5, 4, 3, 2, 1, 1) \in O_{\overline{u}}^{*}(16)$:

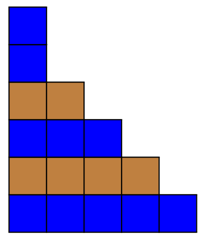


Figure 19

Separating Y_λ into the odd and even parts; i.e., into $Y_{\lambda'}$ with $\lambda' = (5, 3, 1, 1)$ and $Y_{\lambda''}$ with $\lambda'' = (4, 2)$ yields:

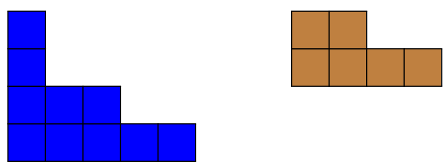


Figure 20

Adjoining the rows of $Y_{\lambda'}$ and $Y_{\lambda''}$ gives $Y_{\lambda'''}$ with the partition $\lambda''' = (9, 5, 1, 1) \in P_{eu}^{ou}(16)$:

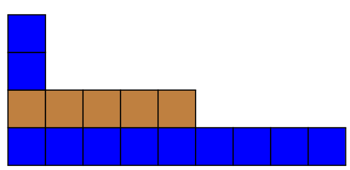


Figure 21

Let $\mu = (\mu_1, \ldots, \mu_s) \in P_{eu}^{ou}(n)$. Separate μ into $\mu' = (\mu_{o_1}, \ldots, \mu_{o_i})$ with the odd parts, $\mu_{o_i} \leq \mu_{o_{i-1}} \leq \cdots \leq \mu_{o_1}$ where $\mu_{o_i} \geq \mu_s$, $\mu_{o_1} \leq \mu_1$ and into μ'' with the even parts. We keep aside the even component $Y_{\mu''}$ of Y_μ. First, we assume that all odd parts of μ are distinct; i.e., there are i distinct odd values with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$. It is clear that there are μ_{o_i} boxes in the first row and $\mu_{o_{i-1}}$ boxes in the second row and continuing this, one can observe that there are μ_{o_1} boxes in the bottom most ith row of $Y_{\mu'}$. Now, for all j $(1 \leq j \leq i)$, we extract $2j - 1$ boxes from the jth row of $Y_{\mu'}$ and attach $2j - 1$ boxes to $Y_{\mu'}$ without violating the structure of the Young diagram $Y_{\mu'}$. Explicitly, we break an odd part μ_{o_t} of the partition $\mu^{'}$ into $(\mu_{o_t} - (2v-1), 2v-1)$ where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu'}$. The Young diagram $Y_{\mu'''}$ obtained from $Y_{\mu'}$ by the above construction and adjoining $Y_{\mu''}$ with it to get the unique resulting Young diagram, say Y_π with $\pi \in O^*_{\overline{u}}(n)$. This is because all the odd parts are distinct and their corresponding position in Y_μ is unique, and hence the resulting partition $\pi \in O^*_{\overline{u}}(n)$ is the unique pre-image of the partition $\mu \in P^{ou}_{eu}(n)$. Next, we consider $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$ with the assumption that $\mu_{o_1}, \ldots, \mu_{o_i}$ occurs with multiplicity $k_1, k_2, \ldots, k_i$, respectively; i.e., a part μ_{o_t} $(1 \le t \le i)$ occurs with multiplicity k_t. Then we break the k_t tuple $(\mu_{o_t}, \ldots, \mu_{o_t})$ into $((\mu_{o_t} - (2v-1), 2v-1), \ldots, (\mu_{o_t} - (2v-1), 2v-1))$ where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu'}$. Similar argument shows that the resulting partition, say $\pi \in O^*_{\overline{u}}(n)$. Therefore, the BSP provides a bijection to conclude the proof of the theorem 6.

For example, Y_μ with $\mu = (9, 7, 4, 2) \in P^{ou}_{eu}(22)$ breaks into $Y_{\mu'}$ with $\mu^{'} = (9, 7)$ and $Y_{\mu''}$ with $\mu^{''} = (4, 2)$:

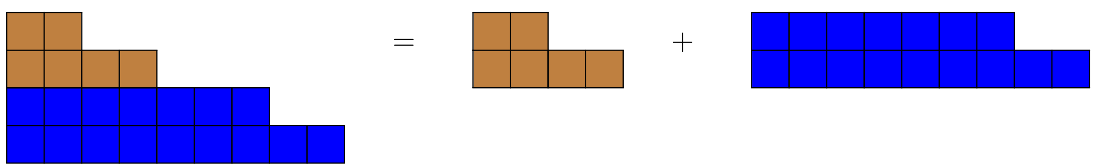


Figure 22

Following the above construction, $Y_{\mu'}$ yields $Y_{\mu'''}$ with $\mu^{'''} = (1, 3, 6, 6)$:

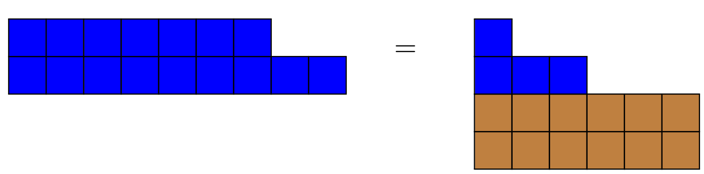


Figure 23

Then the resulting diagram Y_π with $\pi = (6, 6, 4, 3, 2, 1) \in O^*_{\overline{u}}(22)$ is the unique pre-image of μ:

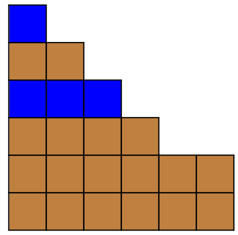


Figure 24

An example if an odd part repeats; e.g., the pre-image of $\mu = (7,7,5,1,1,1)$ is $\pi = (5,5,3,2,2,2,1,1,1) \in O^{*}_{\overline{u}}(22)$:

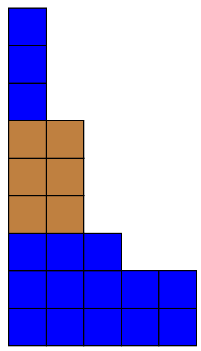


Figure 25

Remark: From the above theorem, it is clear that if we restrict ourselves to the distinct odd parts, then Andrews's result [2] follows as a corollary.

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