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Jovana Forcan Jiayue Qi

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How fast can Dominator win in the Maker–Breaker domination game?

Jovana Forcan*† and Jiayue Qi‡§

Abstract

We study the Maker–Breaker domination games played by two players, Dominator and Staller. We give a structural characterization for graphs with Maker-Breaker domination number equal to the domination number. Specifically, we show how fast Dominator can win in the game on $P_2 \square P_n$, for $n \geq 1$.
Keywords: domination number, Maker–Braker domination number, positional game, grid, winning strategy.

1 Introduction

In this paper we study the Maker-Breaker domination games, first introduced in literature by Duchêne, Gledel, Parreau and Renault in [5]. The games combine two following research directions. In the original domination game, introduced by Brešar, Klavžar, and Rall in [2], two players, *Dominator* and *Staller*, alternately take a turn in claiming vertices from the finite graph G, which were not yet chosen in the course of the game. Dominator has a goal to dominate the graph in as few moves as possible while Staller tries to prolong the game as much as possible.

The Maker-Breaker games, introduced by Erdős and Selfridge in [6], are played on a finite hypergraph $(X, \mathcal{F})$ with the vertex set X and a set $\mathcal{F} \subseteq 2^X$ of hyperedges. The set X is called the *board* of the game, and $\mathcal{F}$ the *family of winning sets*. Two players, *Maker* and *Breaker* take turns in claiming previously unclaimed elements of X. Maker wins the game if, by the end of the game, claims all elements of some $F \in \mathcal{F}$. Otherwise, Breaker wins. For a deeper and more comprehensive analysis of Maker-Breaker games see the

*Department of Mathematics and Informatics, Faculty of Sciences, Univeristy of Novi Sad, Serbia. Email: dmi.jovana.jankovic@student.pmf.uns.ac.rs

†Department of Mathematics, Informatics and Physics, Faculty of Philosophy, University of East Sarajevo, Bosnia and Herzegovina

‡Doctoral Program "Computational Mathematics" W1214, Johannes Kepler University, Linz, Austria. The research was funded by the Austrian Science Fund (FWF): W1214-N15, project DK9. Email: jiayue.qi@dk-compmath.jku.at

§Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria.

book of Beck [1], and the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [9].

The Maker–Breaker domination game (MBD for short) is played on graph $G = (V, E)$ by two players Dominator and Staller. The board of the game is the set V, and family of winning sets consist of all dominating sets of G. The aim of Dominator is to build a dominating set of the graph, that is a set T such that every vertex not in T has a neighbour in T. The aim of Staller is to claim a vertex from the graph G and all its neighbours.

When it is not hard to determine the identity of the winner in some Maker–Breaker game, then the more interesting question to ask is how fast player with the winning strategy can win. Fast winning strategies for Maker in the Maker–Breaker games have received a lot of attention in recent years (see e.g. [3, 4, 8]).
Specifically, for the Maker–Breaker domination game the smallest number of moves for Dominator is studied in [7], where Gledel, Iršič, and Klavžar introduced the Maker–Breaker domination number $\gamma_{MB}(G)$ of a graph G, as the minimum number of moves of Dominator to win in the game on G where he is the first player. If Dominator is the second player, then the corresponding invariant authors denoted by $\gamma'_{MB}(G)$.

In [7], the authors proved that $\gamma_{MB}(G) = \gamma(G) = 2$ if and only if G has a vertex that lies in at least two γ-sets of G, where $\gamma(G)$ is the *domination number* of G, that is the order of a smallest dominating set of G and γ-set is a dominating set of size $\gamma(G)$.
In this paper, we want to find a structural characterization of the graphs G with domination number $\gamma(G) = k$, where $k \geq 2$ is a fixed integer, for which $\gamma_{MB}(G) = \gamma(G) = k$ holds, answering a related question from [7]. So, in Section 2, we provide a graph $\mathcal{G}$ with the corresponding structural characterization and prove the following theorem.

Theorem 1.1. *Let G be a graph with $\gamma(G) = k$, $k \geq 2$. Then $\gamma_{MB}(G) = \gamma(G) = k$ for all $k \geq 2$ if and only if $G \supseteq \mathcal{G}$.*

In the same paper [7], the authors proposed finding the minimum number of moves for Dominator in the MBD game on the Cartesian product of two graphs. Motivated by a given problem, we focus on estimating invariants $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ for the Cartesian product of two graphs and prove the following theorems in Section 3.

Theorem 1.2. *Let G and H be two arbitrary graphs on n and m vertices, respectively. Suppose that Maker has a winning strategy in MBD game on at least one of these two graphs as the first and as the second player. Then*

$$\gamma_{MB}(G\Box H) \leq \min\{\gamma_{MB}(G) + (m-1)\gamma'_{MB}(G), \gamma_{MB}(H) + (n-1)\gamma'_{MB}(H)\}$$

and

$$\gamma'_{MB}(G\Box H) \leq \min\{m \cdot \gamma_{MB}(G), n \cdot \gamma'_{MB}(H)\}.$$

Theorem 1.3. *Let G be a graph on n vertices. Then Dominator can win the game on $G\square K_2$ in at most n moves. If Dominator has a wining strategy as the first and as the second player in the game on G, then $\gamma_{MB}(G\square K_2) \leq \min\{\gamma_{MB}(G) + \gamma'_{MB}(G), n\}$ and $\gamma'_{MB}(G\square K_2) \leq \min\{2\gamma'_{MB}(G), n\}$.*

Especially, we focus on determining how long does it take Dominator to win on $P_2\square P_n$, for $n \geq 1$. So, in Section 3, we also prove the following two theorems.

Theorem 1.4. *$\gamma'_{MB}(P_2\square P_n) = n$ for $n \geq 1$.*

Theorem 1.5. *$\gamma_{MB}(P_2\square P_n) = n - 2$, for $n \geq 13$.*

1.1 Preliminaries

For given graph G by $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. The order of graph G is denoted by $v(G) = |V(G)|$, and the size of the graph by $e(G) = |E(G)|$.
Assume that the MBD game is in progress. We denote by $d_1, d_2, ...$ the sequence of vertices chosen by Dominator and by $s_1, s_2, ...$ the sequence chosen by Staller. At any given moment during this game, we denote the set of vertices claimed by Dominator by $\mathfrak{D}$ and the set of vertices claimed by Staller by $\mathfrak{S}$. As in [7], we say that the game is the D-game if Dominator is the first to play, i.e. one *round* consists of a move by Dominator followed by a move of Staller. In the S-game, one round consists of a move by Staller followed by a move of Dominator. We say that the vertex v is *isolated* by Staller if v and all its neighbours are claimed by Staller.

2 Relation between γ and γ_{MB}

Let $\mathcal{G}$ be a graph with $\gamma(\mathcal{G}) = k$, where $k \geq 2$ is an integer. Let $U = \{a, b_2, c_2, ..., b_k, c_k\} \subseteq V(\mathcal{G})$ be a set of all vertices, which appear in γ-sets. Divide the set U into following subsets: $\{a\}$ and $\{b_i, c_i\}$, for all $i \in \{2, ..., k\}$. Suppose that

- all vertices from $V(\mathcal{G}) \setminus U$ can be divided into k pairwise disjoint sets $A_1, A_2, ...A_{k-1}, A_k$ such that all vertices from some A_i are adjacent to $\{b_i, c_i\}$, for $i = 2..., k$ and $N_U(A_i) \cap N_U(A_j) = \emptyset$, for all $i \neq j$.
- vertices from A_1 are the leaves of the star with the center in the vertex $a \in U$ and these vertices do not have other neighbours in U.

At least one of the next four cases must hold

1. $b_ic_i \in E(\mathcal{G})$,
2. $b_ia \in E(\mathcal{G})$ and $c_ia \in E(\mathcal{G})$

3. $b_i a \in E(\mathcal{G})$ and there exist $j \neq i$ such that $c_i b_j, c_i c_j$, or $c_i a \in E(\mathcal{G})$ and there exist $j \neq i$ such that $b_i b_j, b_i c_j \in E(\mathcal{G})$,
4. there exist $j, k \neq i$ such that $b_i b_j, b_i c_j, c_i b_k, c_i c_k \in E(\mathcal{G})$ (note that k and j could be equal).

One example of the graph $\mathcal{G}$ is illustrated on Figure 1.

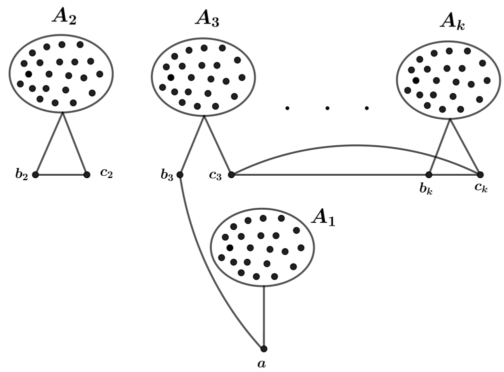


Figure 1: An example of graph $\mathcal{G}$.

Lemma 2.1. *The number of γ-sets in graph $\mathcal{G}$ is 2^{k-1}. In particular, the vertex a lies in every γ-set, the vertex b_i lies in exactly 2^{k-2} γ-sets which do not contain vertex c_i and the vertex c_i lies in other 2^{k-2} γ-sets which do not contain vertex b_i.*

Proof. Denote by $\mathcal{F}$ a family of all γ-sets of graph G and let $N = |\mathcal{F}|$. In every γ-set from the family $\mathcal{F}$ for each vertex define positions in the corresponding γ-set. Since every γ-set is of order k, denote positions in sets by $1, 2..., k$ and place vertices $a, b_2, b_3, ..., b_k, c_2, c_3, ..., c_k$ on the corresponding positions in γ-sets in the following way.
Since vertices from A_1 have only one neighbour from U, a vertex a, it follows that each set from $\mathcal{F}$ must contain this vertex a. Its position in each γ-set we denote by 1.
Also, since vertices from some set A_i, $i = 2, ..., k$ have two common neighbours from U, b_i and c_i, then b_i or c_i will be placed at the position i, $i = 2, 3, ..., k$. More precisely, the vertex b_i will appear in $N/2$ γ-sets and c_i will appear in other $N/2$ γ-sets which do not contain vertex b_i.
It follows that for each position i in some γ-set there are two possibilities, b_i or c_i, $i = 2, 3, ..., k$. So, we obtain that the total number of γ-sets is $N = 2^{k-1}$. □

Proof of Theorem 1.1. First, suppose that $\mathcal{G} \subseteq G$ and prove that $\gamma_{MB}(G) = k$. It is enough to prove that $\gamma_{MB}(\mathcal{G}) = k$.
In his first move Dominator plays $d_1 = a$. In every other round $2 \leq r \leq k$, Dominator plays in the following way. If Staller in her $(r-1)^{\text{st}}$ move plays $s_{r-1} = b_i$ (or $s_{r-1} = c_i$) then Dominator responses with $d_r = c_i$ (or $d_r = b_i$), for each $i = 2, 3, ..., k$. So, $\gamma_{MB}(\mathcal{G}) = k$.

Suppose, now, that $\gamma_{MB}(G) = k$ and prove that $G \supseteq \mathcal{G}$.
After Dominator's first move d_1, it is Staller's turn to make a move. If she claims s_1 such that d_1 and s_1 are part of some γ-set, then there exists at least one more vertex, say d_2, such that d_1 and d_2 are part of some other γ-set. Otherwise, this is a contradiction with the statement that Dominator wins the game. So, this gives at least two γ-sets: $\{d_1, d_2, ...\}$ and $\{d_1, s_1, ...\}$.
Since Staller plays according to her optimal strategy, the vertex she claims in each round is the best choice for her. So, for her first move she had at least two best choices, s_1 and d_2. We consider separately the cases when Staller claims s_1 and when she claims d_2 in the first round.

Case 1. Suppose that Staller claimed s_1 in her first move and Dominator claimed d_2 in his second move. Then Staller in her second move can claim s_2 such that d_1, d_2 and s_2 are part of some γ-set. Then there exists at least one more vertex, say d_3, such that d_1, d_2 are d_3 are part of some other γ-set.

Case 2. Suppose that Staller claimed d_2 in her first move and Dominator claimed s_1 in his second move. Then, Staller in her second move can claim some s_2 such that d_1, s_1 are s_2 are part of some γ-set. Then, there exists at least one more vertex, say d_3, such that d_1, s_1 are d_3 are part of some other γ-set. Dominator claims d_3.

After Dominator's third move, above analyses gives at least $4 = 2^2$ γ-sets: $\{d_1, d_2, d_3...\}$, $\{d_1, d_2, s_2...\}$, $\{d_1, s_1, d_3...\}$ and $\{d_1, s_1, s_2...\}$.

Suppose that after Dominator's i^{th} move we obtain that there are 2^{i-1} γ-sets. Assume that after Dominator's move in round i, he owns vertices: $d_1, d_2, d_3, ..., d_i$.
If in round i Staller claims some s_i such that $d_1, d_2, ..., d_i$ are s_i are part of some γ-set, then according to the statement of theorem that Dominator wins in the game, there exists a vertex d_{i+1}, such that $d_1, d_2, ..., d_i$ and d_{i+1} are part of some other γ-set. So, s_i or d_{i+1} is the vertex on the $(i+1)^{\text{st}}$ position of previously found 2^{i-1} sets. So, this gives at least 2^{i-1} new sets which is, in total, at least $2 \cdot 2^{i-1} = 2^i$ γ-sets.
Since Dominator in each round i, $i = 2, 3, .., k$ can find the corresponding vertex, as the response to Staller's $(i-1)^{\text{st}}$ move, it follows that for each position in every γ-set there are at least two possible choices. This gives at least 2^{k-1} γ-sets. The vertex d_i (or s_{i-1}), for every $i = 2, 3, ..., k$, appears in at least 2^{k-2} γ-sets which do not contain s_{i-1} (or d_i). The vertex d_1 must appear in all γ-sets. Otherwise, after some number of rounds Dominator will lose the game which would be a contradiction. Also, at least one of the next four cases must hold for each $i \in \{1, ..., k-1\}$.

1. $s_i d_{i+1} \in E(G)$,
2. $d_1 d_{i+1}, d_1 s_i \in E(G)$,

3. $d_1d_{i+1} \in E(G)$ and there exist $j \neq i$ such that $s_is_j, s_id_{j+1} \in E(G)$, or $d_1s_i \in E(G)$ and there exist $j \neq i$ such that $d_{i+1}s_j, d_{i+1}d_{j+1} \in E(G)$,

4. there exist $j, k \neq i$ such that $s_is_j, s_id_{j+1}, d_{i+1}s_k, d_{i+1}d_{k+1} \in E(G)$ (where k and j could be equal).

So, $G \supseteq \mathcal{G}$. □

3 MBD game on $G\Box H$

First, we consider the MBD game on $G\Box K_2$ and prove Theorem 1.3.

Proof of Theorem 1.3. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Let H be a copy of the graph G and let $V(H) = \{v'_1, v'_2, ..., v'_n\}$, where $v'_i = v_i$ for each $i \in \{1, 2, ..., n\}$. Then $V(G\Box K_2) = V(G) \cup V(H) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ and $E(G\Box K_2) = E(G) \cup E(H) \cup \{v_1v'_1, v_2v'_2, ..., v_nv'_n\}$.
In order to win, Dominator can always use the pairing strategy. That is, when Staller claims v_i (or v'_i), for some $i \in \{1, 2, ..., n\}$, Dominator responses by claiming vertex v'_i (or v_i). So, Dominator wins in at most n moves. To see that bound is tight consider G as the disjoint union of K_1s.
Next, suppose that Dominator can win in the game on the graph G as the first and as the second player. Assume that Dominator starts the game. Note that $\gamma_{MB}(G) = \gamma_{MB}(H)$ and $\gamma'_{MB}(G) = \gamma'_{MB}(H)$.
By S_D and S'_D denote Dominator's winning strategy on G (and also on H) in the D-game and the S-game, respectively.
If $\gamma_{MB}(G) + \gamma'_{MB}(G) \geq n$, Dominator will use the pairing strategy. So, suppose that $\gamma_{MB}(G) + \gamma'_{MB}(G) < n$.
For his first move Dominator chooses a vertex from $V(G)$ according to his winning strategy S_D. In this way he starts the D-game on G.
In every other round $r \geq 2$, Dominator looks on the $(r-1)^{\text{st}}$ move of Staller. If Staller claims a vertex from $V(G)$, Dominator responses by claiming a vertex from $V(G)$ and if Staller claims a vertex from $V(H)$, Dominator also claims a vertex from $V(H)$.
If Staller was first to claim a vertex from $V(H)$, then the S-game was played on H.
So, in the game on $G\Box K_2$, Dominator can win in at most $\gamma_{MB}(G) + \gamma'_{MB}(G)$ moves.

Next, assume that Staller starts the game on $G\Box K_2$. If $2\gamma'_{MB}(G) \geq n$, Dominator will use the pairing strategy. So, let $2\gamma'_{MB}(G) < n$. Since in this case, Staller can make the first move on G and after, also, on H, Dominator will need to play according to the strategy S'_D on both graphs G and H. So, to win in the game on $G\Box K_2$, he needs to play at most $2\gamma'_{MB}(G)$ moves. □

Remark 3.1. *The domination number of the $r \times l$ rook's graph $K_r\Box K_l$ is $\gamma = \min(r, l)$. It is not hard to see that Dominator can win in γ moves. Note that the graph $\mathcal{G}$, described in Section 2, is the subgraph of $K_r\Box K_l$.*

Proof of Theorem 1.2. The proof for the first part of theorem is similar to the proof of Theorem 1.3. Consider, first, the D-game on $G\Box H$. Suppose that Dominator has a winning strategy as the second player on G. Let $\gamma_{MB}(G)+(m-1)\gamma'_{MB}(G)\leq\gamma_{MB}(H)+(n-1)\gamma'_{MB}(H)$.
By $G^{(1)},G^{(2)},...,G^{(m)}$ denote copies of the graph G. By S_D and S'_D denote Dominator's winning strategy on G in the D-game and the S-game, respectively.
His first move Dominator will play on one copy of G according to his winning strategy S_D. In every other round $i\geq 2$, he looks on the $(i-1)^{\text{st}}$ move of Staller. If Staller in his $(i-1)^{\text{st}}$ move claimed vertex from some $V(G^j)$, Dominator responds by claiming a vertex from the same set $V(G^j)$ according to the corresponding winning strategy S_D or S'_D. Since Staller can be the first player on at most $m-1$ copies of the graph G, the statement holds.
If $\gamma_{MB}(G)+(m-1)\gamma'_{MB}(G)>\gamma_{MB}(H)+(n-1)\gamma'_{MB}(H)$, then we consider n copies of graph H and the proof is the same. $\Box$

3.1 MBD game on $P_2\Box P_n$

Definition 3.2. *For $1\leq m\leq n$, let $V=\{u_1,...,u_m,v_1,...,v_m\}$ and $E=\{u_iu_{i+1}:i=1,2,...,m-1\}\cup\{v_iv_{i+1}:i=1,2,...,m-1\}\cup\{u_iv_i:i=1,2,...,m\}$. Suppose that Maker–Breaker domination game on $P_2\Box P_n$ is in progress, where $n\geq 5$.*

1. *By X_m $(1\leq m\leq n)$ denote a subgraph of $P_2\Box P_n$, where $V(X_m)=V$ and $E(X_m)=E$, such that u_1 is a free vertex which is dominated by Dominator with its neighbour $u_0\in V(P_2\Box P_n)\setminus V(X_m)$ (Figure 2(a)).*

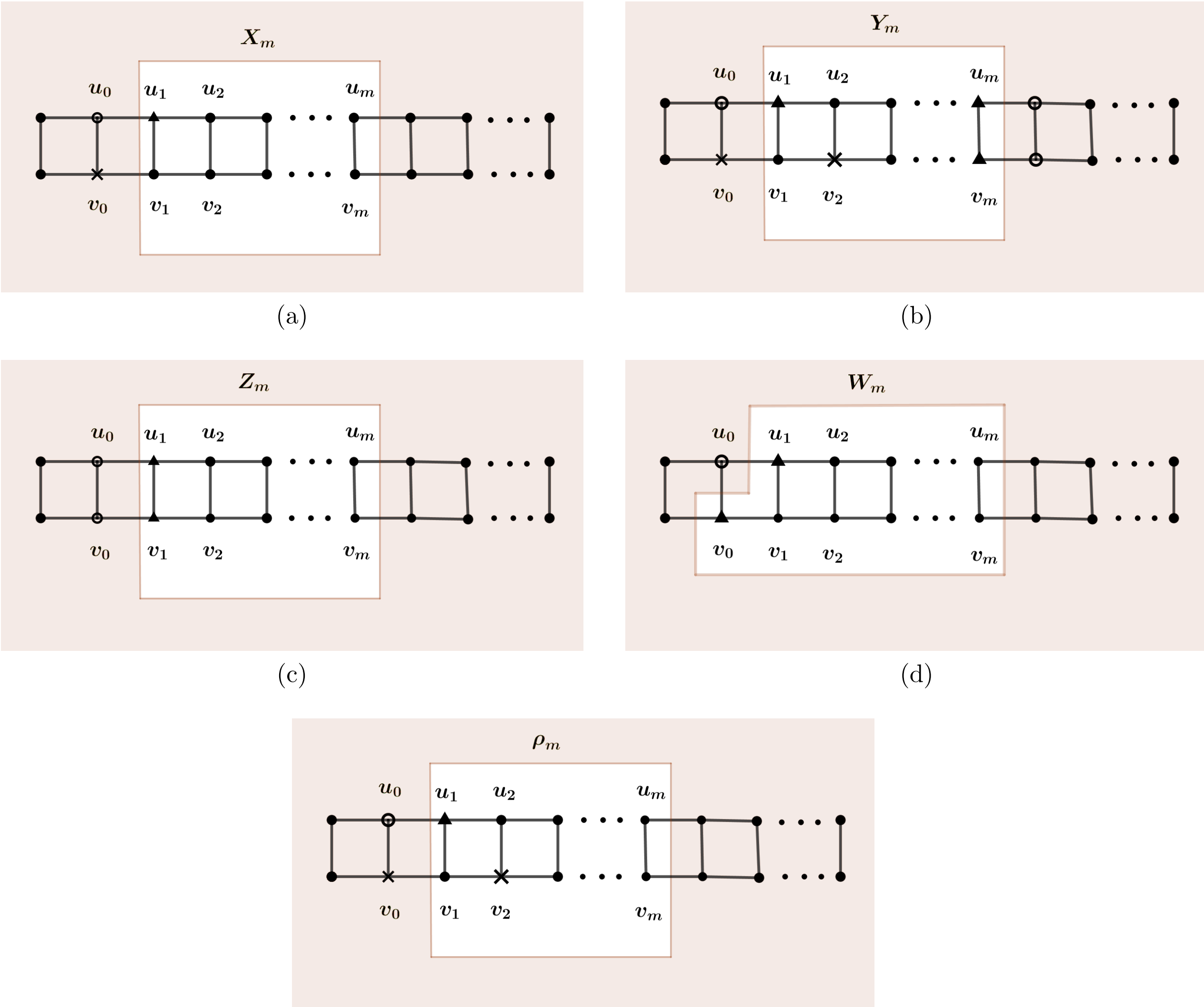
2. *By Y_m $(3\leq m\leq n)$ denote a subgraph of $P_2\Box P_n$, where $V(Y_m)=V$ and $E(Y_m)=E$, such that v_2 is claimed by Staller and u_1,u_m and v_m are free vertices which are dominated by Dominator with their corresponding neighbours from the set $V(P_2\Box P_n)\setminus V(Y_m)$ (Figure 2(b)).*
   When consider the D-game on Y_m, we set $s_0=v_2$.

3. *By Z_m $(1\leq m\leq n)$ denote a subgraph of $P_2\Box P_n$, where $V(Z_m)=V$ and $E(Z_m)=E$, such that u_1 and v_1 are free vertices which are dominated by Dominator with their corresponding neighbours from the set $V(P_2\Box P_n)\setminus V(Z_m)$ (Figure 2(c)).*

4. *By W_m $(1\leq m\leq n)$ denote a subgraph of $P_2\Box P_n$, where $V(W_m)=V\cup\{v_0\}$ and $E(W_m)=E\cup\{v_0v_1\}$, such that u_1 and v_0 are free vertices which are dominated by Dominator with their corresponding neighbour $u_0\in V(P_2\Box P_n)\setminus V(W_m)$ (Figure 2(d)).*

5. *By ρ_m $(2\leq m\leq n)$ denote a subgraph of $P_2\Box P_n$, where $V(\rho_m)=V$ and $E(\rho_m)=E$, such that v_2 is claimed by Staller and u_1 is a free vertex which is dominated by Dominator with its neighbour $u_0\in V(P_2\Box P_n)\setminus V(\rho_m)$ (Figure 2(e)).*
   When consider the D-game on ρ_m, we set $s_0=v_2$.

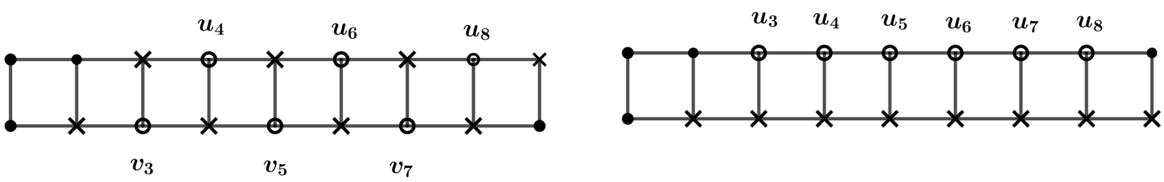
We define two types of traps Staller can create in the MBD game on $P_2\Box P_n$ for $n\geq 3$.



(e)

Figure 2: Subgraph (a) X_m (b) Y_m (c) Z_m (d) W_m (e) ρ_m
Vertices claimed by Dominator are denoted by cycles and vertices claimed by Staller by crosses. Triangle vertices are free vertices dominated by Dominator.

Trap 1 - triangle trap. We say that Staller created a *triangle trap* if after her move Dominator is forced to claim a vertex v_i in order to dominate v_i, where $2 \leq i \leq n-1$, because all its neighbours v_{i-1}, v_{i+1} and u_i are claimed by Staller and Staller can isolate v_i by claiming it in her next move. Similarly, Staller created the triangle trap if Dominator is forced to claim u_i in order to dominate u_i, where $2 \leq i \leq n-1$, because all its neighbours u_{i-1}, u_{i+1} and v_i are claimed by Staller.
We say that Staller creates a *sequence of triangle traps* $v_i - v_j$ (or $v_i - u_j$), where $2 \leq i \leq n-2$ and $i+1 \leq j \leq n-1$, if Dominator is consecutively forced to claimed vertices $v_i, u_{i+1}, v_{i+2}, u_{i+3}, ..., v_j$ (or $v_i, u_{i+1}, v_{i+2}, u_{i+3}, ..., u_j$). In this sequence of triangle traps the



(a) (b)

Figure 3: The example of the sequence of (a) triangle traps (b) line traps
Vertices claimed by Dominator are denoted by cycles and vertices claimed by Staller by crosses.

triple of vertices claimed by Staller which form the first trap is v_{i-1}, u_i, v_{i+1}, and the triple of vertices which form the last trap in this sequence is v_{j-1}, u_j, v_{j+1} (or u_{j-1}, v_j, u_{j+1}), and v_{j+1} (or u_{j+1}) is the vertex which is claimed last by Staller in the sequence of traps. The sequence of triangle traps $v_3 - u_8$ is illustrated on Figure 3(a).
Similarly, we say that Staller creates a sequence of triangle traps $u_i - v_j$ (or $u_i - u_j$), where $2 \leq i \leq n-2$ and $i+1 \leq j \leq n-1$, if Dominator is consecutively forced to claimed vertices $u_i, v_{i+1}, u_{i+2}, v_{i+3}, ..., v_j$ (or $u_i, v_{i+1}, u_{i+2}, v_{i+3}, ..., u_j$). In this sequence of triangle traps the triple of vertices claimed by Staller which form the first trap is u_{i-1}, v_i, u_{i+1}, and the triple of vertices which form the last trap in this sequence is v_{j-1}, u_j, v_{j+1} (or u_{j-1}, v_j, u_{j+1}), and v_{j+1} (or u_{j+1}) is the vertex which claimed last by Staller in the sequence of triangle traps.

Trap 2 - line trap. We say that Staller created a *line trap* if after her move Dominator is forced to claim a vertex v_i, $2 \leq i \leq n-1$, in order to dominate u_i because vertices u_{i-1}, u_i and u_{i+1} are claimed by Staller and Staller can isolate u_i by claiming v_i in her next move. Similarly, Staller created a line trap if Dominator is forced to claim u_i in order to dominate v_i, $2 \leq i \leq n-1$, because vertices v_{i-1}, v_i and v_{i+1} are claimed by Staller.
We say that Staller creates a *sequence of line traps* $v_i - v_j$ (or $u_i - u_j$), where $2 \leq i \leq n-2$ and $i+1 \leq j \leq n-1$, if Dominator is consecutively forced to claimed vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, ..., v_j$ (or $u_i, u_{i+1}, u_{i+2}, u_{i+3}, ..., u_j$) and where the last vertex claimed by Staller in this sequence is u_{j+1} (or v_{j+1}). The sequence of line traps $u_3 - u_8$ is illustrated on Figure 3(b).

Lemma 3.3. *Let $m \geq 2$. Then $\gamma_{MB}(\rho_m) = m$. Also, if Dominator skips his move in any round, he can not win.*

Proof. Let $s_0 = v_2$. To prove the upper and the lower bound we use induction on k, where $2 \leq k \leq m$. For $k = 2$, ρ_2 is a cycle C_4. To dominate v_1 and u_2, Dominator needs to play two moves. So, $\gamma_{MB}(\rho_2) = 2$. If Dominator skips his first move on ρ_2, which we denote by $d_1 = \emptyset$, then $s_1 = v_1$ and in her next round Staller can isolate either v_1 or v_2.
To prove that $\gamma_{MB}(\rho_3) = 3$, we analyse the following cases. It is not hard to see that if Dominator skips any move on ρ_3, Staller can isolate some vertex.

1. $d_1 = u_3$ (or $d_1 = v_1$)
   Then s_1 must be equal to v_1 (or $s_1 = u_3$), as otherwise Dominator will need exactly one more move to win. To dominate v_1 and v_2 (or u_2, u_3 and v_3), Dominator needs two more moves.

2. $d_1 = v_3$ (or $d_1 = u_1$).
   This case is symmetric to the previous case.

3. $d_1 = u_2$.
   Dominator needs two more moves to dominate v_1 and v_3.

So, $\gamma_{MB}(\rho_3) = 3$.
For $k \in \{2, 3\}$, statement holds. Suppose that $\gamma_{MB}(\rho_{k-1}) \leq k - 1$, for $4 \leq k \leq m$ and $m \geq 4$. Consider the D-game on ρ_k. Dominator's strategy is to split ρ_k into two parts, a graph ρ_{k-1} and an edge $u_k v_k$. By induction hypothesis, $\gamma_{MB}(\rho_{k-1}) \leq k - 1$. Also, when Staller claims u_k (or v_k), Dominator claims v_k (or u_k). So, it follows that $\gamma_{MB}(\rho_k) \leq k$.

Next, we prove that Staller has a strategy to postpone Dominator's winning by at least k moves and which ensures that Dominator can not skip any move on ρ_k.
Assume that $\gamma_{MB}(\rho_{k-1}) \geq k - 1$ and Dominator can not skip any move in the game on ρ_{k-1}, for $4 \leq k \leq m$. Consider the D-game on ρ_k and prove that $\gamma_{MB}(\rho_k) \geq k$ and Dominator is not able to skip any move on ρ_k.
If $d_1 = \emptyset$, we propose the following strategy for Staller: $s_1 = v_1$ which forces $d_2 = u_1$, as otherwise Staller can isolate v_1 in her next move. By playing $s_2 = v_3$ Staller starts the sequence of line traps $u_2 - u_{k-1}$. In her last move Staller claims u_k and isolates v_k. Next, we consider all possibilities for d_1 and propose Staller's strategy.

Case 1. $d_1 = u_i$, $(i \neq 1)$.
Then $s_1 = v_1$ which forces $d_2 = u_1$, as otherwise Staller can isolate v_1 by claiming u_1 in her third move.
If $i = 2$, that is, if $d_1 = u_2$, then $s_2 = v_4$. Consider the D-game on subgraph ρ_{k-2} on $V(\rho_{k-2}) = \{u_3, ..., u_k, v_3, .., v_k\}$, where $v_4 \in \mathfrak{S}$ and u_3 is a free vertex dominated by Dominator with u_2.
By induction hypothesis $\gamma_{MB}(\rho_{k-2}) \geq k - 2$ and Dominator can not skip any move. So, Dominator needs at least k moves to win on ρ_k without skipping any move.

If $i > 2$, then $s_2 = v_3$ and Staller starts the sequence of line traps $u_2 - u_{i-1}$.
In round i Staller claims $s_i = v_{i+2}$. Consider the D-game on subgraph ρ_{k-i} with the vertex set $V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, .., v_k\}$, where $v_{i+2} \in \mathfrak{S}$ and u_{i+1} is a free vertex dominated by Dominator with u_i. By induction hypothesis, $\gamma_{MB}(\rho_{k-i}) \geq k - i$ and Dominator can not skip any move. So, Dominator needs at least k moves to win on ρ_k without skipping any move.
If $i = k$, that is, if $d_1 = u_k$, then Dominator already played k moves since he was forced to claim all from $\{u_1, ..., u_k\}$.

If $i = k-1$, that is, if $d_1 = u_{k-1}$, then $s_i = s_{k-1} \in \{u_k, v_k\}$. So, Dominator needs to play one more move to dominate v_k. So, in total, he plays k moves.

Case 2. $d_1 = v_i$, $i \geq 3$.

Claim 3.4. *If $d_1 \notin \{v_3, v_4\}$, then Dominator can not win.*

Proof of Claim 3.4. Suppose that $d_1 \notin \{v_3, v_4\}$.
Then $s_1 = u_2$.
If $d_2 = u_3$, Staller claims $s_2 = v_1$ and forces $d_3 = u_1$ or $d_3 = v_3$. Since Dominator can not dominate vertices v_1, u_1 and v_2 at the same time, in her next move Staller will isolate v_1 and u_1 by claiming u_1, or v_2 by claiming v_3.
If $d_2 = v_3$, Staller claims $s_2 = u_1$. Since Dominator can not dominate u_1, v_1 and u_2 at the same time, he will lose the game after Staller next move.
If $d_2 = u_1$, then $s_2 = v_3$ which forces $d_3 = v_1$. Next, $s_3 = u_3$. Dominator can not dominate both u_3 and v_3 in his next move. In her next move Staller isolates u_3 or v_3.
If $d_2 = v_1$, then $s_2 = u_3$ which forces $d_3 = u_1$. Next, $s_3 = v_3$. Dominator can not dominate both u_3 and v_3. So, he will lose the game after Staller's next move.
Finally, if $d_2 \notin \{u_1, v_1, u_3, v_3\}$, then Staller claims $s_2 = u_1$. Since Dominator can not dominate u_1, v_1, u_2 and v_2 at the same time, he will lose the game after Staller's next move. □

So, $d_1 \in \{v_3, v_4\}$.

Case 2.1 $d_1 = v_3$.
Then, $s_1 = u_1$ which forces $d_2 = v_1$, and $s_2 = u_3$ which forces $d_3 = u_2$ (a triangle trap). Next, $s_3 = u_5$. Consider the D-game on the subgraph ρ_{k-3} with the vertex set $V(\rho_{k-3}) = \{u_4, ..., u_k, v_4, ..., v_k\}$. By induction hypothesis, it holds that $\gamma_{MB}(\rho_{k-3}) \geq k-3$ and he can not skip any move. So, Dominator needs at least k on ρ_k moves without skipping any move.

Case 2.2 $d_1 = v_4$.
Then $s_1 = u_2$.

Claim 3.5. *If $d_2 \notin \{u_1, v_1\}$, then Dominator can not win.*

Proof of Claim 3.5. The proof of this claim is very similar to the proof of Claim 3.4. □

Case 2.2.1 $d_2 = u_1$.
Then, $s_2 = v_3$ which forces $d_3 = v_1$ and $s_3 = u_4$ which forces u_3 (a triangle trap). Next, if $k \geq 6$, then $s_4 = u_6$. Consider the D-game on subgraph ρ_{k-4} with the vertex set $V(\rho_{k-4}) = \{u_5, ..., u_k, v_5, ..., v_k\}$ where v_5 is already dominated by Dominator with v_4, and the vertex u_6 is claimed by Staller. By induction hypothesis, it holds $\gamma_{MB}(\rho_{k-4}) \geq k-4$ and he can not skip

any move. So, Dominator needs at least k moves without skipping any move.
If $k = 5$, then no matter what Staller claims in her fourth move, Dominator will need one more move to dominate u_5.

Case 2.2.2 $d_2 = v_1$.
Then, $s_2 = u_3$ which forces $d_3 = u_1$ and $s_3 = u_4$ which forces v_3 (a line trap). Next, $s_4 = u_6$ and the rest of the proof is the same as in Case 2.2.1.

Case 3. $d_1 = u_1$.
Then $s_1 = v_3$.

Case 3.1. $d_2 = u_i$, $i > 2$.
Then, $s_2 = u_2$ which forces $d_3 = v_1$.
Next, $s_3 = v_4$ and Staller starts the sequence of line traps $u_3 - u_{i-1}$. In round i, Staller claims v_{i+2}. Consider the D-game on ρ_{k-i} with the vertex set $V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$ where u_{i+1} is already dominated by Dominator with u_i and v_{i+2} is claimed by Staller. The rest of the proof is the same as in Case 1. So, Dominator needs at least k moves on ρ_k without skipping any move.

Case 3.2. $d_2 = v_i$, where $i > 3$.
Then, $s_2 = u_2$ which forces $d_3 = v_1$.
If $i > 4$, that is, if $d_2 = v_i \neq v_4$, then $s_3 = u_3$. Dominator can not dominate both u_3 and v_3 at the same time. In her next move, Staller isolates u_3 or v_3 and Dominator loses the game.
If $i = 4$, then $s_3 = u_4$ which forces $d_4 = u_3$. Next, if $k \geq 6$, then $s_4 = u_6$ and the rest of the proof is the same as in Case 2.2.1.

Case 3.3. $d_2 \in \{v_1, u_2\}$.
Then, in round $2 \leq r \leq k-2$, Staller claims $s_r = v_{r+2}$ and forces Dominator to claim $d_{r+1} = u_{r+1}$, as otherwise Staller can isolate v_{r+1} by claiming u_{r+1} in the next round, that is Staller creates the sequence of line traps $u_3 - u_{k-1}$. In the last round $k-1$, Staller claims u_k and in this way she isolates v_k.

Case 4. $d_1 = v_1$.
Then, Staller claims $s_1 = u_3$.

Claim 3.6. *If $d_2 \notin \{v_3, u_4, v_4\}$, then Dominator can not win.*

Proof of Claim 3.6. Assume that $d_2 \notin \{v_3, u_4, v_4\}$.
Let $d_2 = u_1$ or $d_2 = u_2$.
Then, in round 2, by playing $s_2 = v_4$, Staller starts the sequence of triangle traps $v_3 - v_{k-1}$ (for even k) or $v_3 - u_{k-1}$ (for odd k). In the last move, if k is even, Staller claims u_k and isolates it. If k is odd, she claims v_k and isolates it.

Let $d_2 \notin \{u_1, u_2, v_3, u_4, v_4\}$.

Then, we have the following sequences of the moves: $s_2 = u_2 \Rightarrow d_3 = u_1$ and $s_3 = v_3$. Dominator can not dominate both u_3 and v_3 at the same time. □

From Claim 3.6, it follows that $d_2 \in \{v_3, u_4, v_4\}$. We have $s_0 = v_2$, $d_1 = v_1$ and $s_1 = u_3$. Next, we consider the following cases.

Case 4.1. $d_2 = v_3$.
Then, Staller claims $s_2 = u_1$ which forces Dominator to claim $d_3 = u_2$. In the next round Staller claims u_5. The rest of the proof is the same as in Case 2.1. So, Dominator needs at least k moves on ρ_k without skipping any move.

Case 4.2. $d_2 = u_4$.
Then, Staller claims $s_2 = u_1$ which forces Dominator to claim $d_3 = u_2$. In the next round Staller claims $s_3 = v_4$ and forces Dominator to play $d_4 = v_3$. If $k \geq 6$, then $s_4 = v_6$. Consider the D-game on ρ_{k-4} with the vertex set $V(\rho_{k-4}) = \{u_5, u_6, ..., u_k, v_5, v_6, ..., v_k\}$ where u_5 is already dominated by Dominator with u_4 and v_6 is claimed by Staller. The rest of the proof is the same as in Case 2.2.1.

Case 4.3. $d_2 = v_4$.
Then, Staller claims $s_2 = u_2$ which forces Dominator to claim $d_3 = u_1$. In the next round Staller claims $s_3 = u_4$ and forces Dominator to play $d_4 = v_3$. Next, $s_4 = u_6$ and the rest of the proof is similar to the proof from Case 4.2. So, Dominator needs to play at least k moves on ρ_k without skipping any move.

This concludes the proof of the lemma.

□

Remark 3.7. *Note that the D-game on graph ρ_m can be considered as the S-game on X_m where $s_1 = v_2$. This means that v_2 is one of the optimal choices for the first move for Staller in the S-game on X_m since by playing v_2 in her first move and then following her strategy for ρ_m Staller can force Dominator to play the maximum number of moves, which is m.*

Lemma 3.8. *Let $m \geq 3$. Then $\gamma_{MB}(Y_m) = m - 1$.*

Proof. Let $s_0 = v_2$.
The proof is very similar to the proof of Lemma 3.3. To prove the upper and the lower bound, we use induction on k where $2 \leq k \leq m$. In the proof for the lower bound we follow the same case analysis from Lemma 3.3. □

Lemma 3.9. *Let $m \geq 1$. Then $\gamma'_{MB}(Z_m) = m - 1$.*

Proof. To prove the upper bound we use induction on k, where $1 \leq k \leq m$. For $k \in \{1, 2, 3\}$ it is not hard to see that statement holds, that is, Dominator needs to play $k - 1$ moves in the S-game on Z_k. Suppose that $\gamma'_{MB}(Z_{k-1}) \leq k - 2$ for $4 \leq k \leq m$ and prove that $\gamma'_{MB}(Z_k) \leq k-1$. Dominator splits the graph into two parts, a graph Z_{k-1} and an edge u_kv_k. By induction hypothesis, $\gamma'_{MB}(Z_{k-1}) \leq k-2$. Also, when Staller claims u_k (or v_k), Dominator claims v_k (or u_k). So, $\gamma'_{MB}(Z_k) \leq k-1$.

To prove the lower bound we propose the following strategy for Staller:
$s_1 = u_m$, which forces $d_1 \in \{u_{m-1}, v_{m-1}, v_m\}$. Otherwise, in her second move Staller can choose v_m and in the third move she can isolate either u_m or v_m by claiming u_{m-1} or v_{m-1}, since Dominator will not be able to dominate both u_m and v_m in his second move.
If $d_1 = u_{m-1}$, Staller plays $s_2 = v_{m-1}$ which forces $d_2 = v_m$. Then, $s_3 = v_{m-3}$. In this way Staller creates Y_{m-2} with the vertex set $V(Y_{m-2}) = \{u_1, u_2, ..., u_{m-2}, v_1, v_2, ..., v_{m-2}\}$. From Lemma 3.8 we know that $\gamma_{MB}(Y_{m-2}) = m-3$, so Dominator needs to play $m-1$ moves on Z_m.
If $d_1 = v_{m-1}$, Staller plays $s_2 = u_{m-1}$ which forces $d_2 = v_m$. Then, $s_3 = u_{m-3}$. In this way Staller creates Y_{m-2}. According to Lemma 3.8, $\gamma_{MB}(Y_{m-2}) = m-3$, so Dominator needs to play $m-1$ moves on Z_m.
Finally, if $d_1 = v_m$ Staller plays $s_2 = u_{m-2}$ and creates Y_{m-1}. According to Lemma 3.8, $\gamma_{MB}(Y_{m-1}) = m-2$, so Dominator needs to play $m-1$ moves on Z_m.
It follows that, $\gamma'_{MB}(Z_m) = m-1$ for $m \geq 4$. □

Lemma 3.10. *Let $m \geq 4$. Then $\gamma'_{MB}(W_m) = m-1$. In particularly, if $m \in \{1,2,3\}$, then $\gamma'_{MB}(W_m) = m$.*

Proof. For $m \in \{1,2,3\}$ it is not hard to see that Dominator needs m moves to win in the S-game on W_m.
Let $m \geq 4$. Since W_m has one more undominated vertex than Z_m, Dominator needs to play at least as many moves as he needs to play on Z_m. So, it follows that $\gamma'_{MB}(W_m) \geq \gamma'_{MB}(Z_m)$ and the lower bound holds.
The proof for the upper bound follows by induction on k where $4 \leq k \leq m$. For $k = 4$, we consider the following cases and propose Dominator's strategy.

Case 1. $s_1 = v_2$.
Then, $d_1 = u_3$.
If $s_2 = v_1$, then $d_2 = v_3$. To dominate v_1 Dominator will claim a vertex from $\{v_0, u_1\}$. One of these two vertices must be free after Staller's third move. Otherwise, if $s_2 \neq v_1$, then $d_2 = v_1$. To dominate v_4 Dominator will claim a vertex from $\{v_3, v_4, u_4\}$. One of these three vertices must be free after Staller's third move.

Case 2. $s_1 \neq v_2$.

Case 2.1 $s_1 = u_4$ (or $s_1 = v_4$).
Then, $d_1 = u_3$. If $s_2 = v_3$, then $d_2 = v_4$ (or $d_2 = u_4$) and $d_3 \in \{v_1, v_2\}$. If $s_2 = v_4$ (or $s_2 = u_4$), then $d_2 = v_3$ and $d_3 \in \{v_1, v_2\}$.

Case 2.2 $s_1 \notin \{u_4, v_4\}$.
Then, $d_1 = v_2$. Dominator needs at most two more moves to dominate the remaining vertices.

Suppose that $\gamma'_{MB}(W_{k-1}) \le k-2$, for $5 \le k \le m-1$. Consider the S-game on W_k. Dominator divides W_k into two parts, W_{k-1} and an edge u_kv_k. Since $\gamma'_{MB}(W_{k-1}) \le k-2$ and since he needs at most one more move to dominate u_k and v_k, it follows that $\gamma'_{MB}(W_k) \le k-1$. □

Lemma 3.11. *Let $m \ge 6$. Then $\gamma_{MB}(X_m) = m-2$. In particularly, if $m = 1$ then $X_1 = 1$ and if $m \in \{2,3,4,5\}$, then $X_m = m-1$.*

Proof. For $m \in \{1,2,3\}$ it is not hard to see that the statement holds. For $m = 4$ and $m = 5$ simple case analysis gives the result.
Let $m \ge 6$. The proof for the upper bound goes by induction on k, where $6 \le k \le m$. First, we consider the D-game on X_6. In his first move Dominator plays $d_1 = v_2$ and he creates a subgraph W_4 with the vertex set $V(W_4) = \{u_2, u_3, u_4, u_5, u_6, v_3, v_4, v_5, v_6\}$. By Lemma 3.10, we have $\gamma'_{MB}(W_4) = 3$. So, $\gamma_{MB}(X_6) = 4$.
Suppose that $\gamma_{MB}(X_{k-1}) \le k-3$ for $7 \le k \le m$ and $m \ge 7$, and prove that $\gamma_{MB}(X_k) \le k-2$. Dominator divides X_k on two parts, the graph X_{k-1} and an edge u_kv_k. Since $\gamma_{MB}(X_{k-1}) \le k-3$ and since he needs at most one more move to dominate u_k and v_k, it follows that $\gamma_{MB}(X_k) \le k-2$.

To prove the lower bound, we also use induction on k and we do the case analysis. Suppose that $\gamma_{MB}(X_{k-1}) \ge k-3$, for $7 \le k \le m$ and $m \ge 7$, and prove that $\gamma_{MB}(X_k) \ge k-2$.
We analyse the following cases and propose the following strategy for Staller.

Case I $d_1 \in \{u_1, v_1, u_2, v_2\}$.
If $d_1 = u_1$ (or $d_1 = v_1$), then consider the S-game on W_{k-1} with the vertex set $V(W_{k-1}) = \{v_1, v_2, ..., v_k, u_2, ..., u_k\}$ (or $V(W_{k-1}) = \{v_2, ..., v_k, u_1, u_2, ..., u_k\}$). By Lemma 3.10, $\gamma'_{MB}(W_{k-1}) = k-2$. So, Dominator needs to play $k-1$ moves on X_k.
If $d_2 = u_2$ (or $d_2 = v_2$), then consider the S-game on W_{k-2} with the vertex set $V(W_{k-2}) = \{v_2, v_3, ..., v_k, u_3, ..., u_k\}$ (or $V(W_{k-2}) = \{v_3, ..., v_k, u_2, u_3, ..., u_k\}$). By Lemma 3.10, $\gamma'_{MB}(W_{k-2}) = k-3$. Also, if $d_1 = u_2$ Dominator needs to play one more move to dominate v_1. So, Dominator needs to play at least $k-2$ moves on X_k.

Case II $d_1 = u_i$, $i \ge 3$.
Then, $s_1 = v_2$.
The rest of Staller's strategy depends on Dominator's second move:

Case 1. $d_2 = u_1$.
If $i = 3$, that is $d_1 = u_3$, then consider the S-game on W_{k-3} with the vertex set $V(W_{k-3}) = \{u_4..., u_k, v_3, v_4, ..., v_k\}$. By Lemma 3.10, $\gamma'_{MB}(W_{k-3}) = k-4$. So, Dominator needs at least $k-2$ moves.

Let $i \ge 4$, then $s_2 = v_3$. Depending of Dominator's third move, we consider the following subcases.

Case 1.1. $d_3 = u_2$ or $d_3 = v_1$.

Then, by playing $s_3 = v_4$ Staller starts the sequence of line traps $u_3 - u_{i-1}$ where $s_{i-1} = v_i$ and $d_i = u_{i-1}$. Then, if $k - i \geq 2$ Staller plays $s_i = v_{i+2}$. Consider the D-game on the subgraph ρ_{k-i} with the vertex set $V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$. According to Lemma 3.3, $\gamma_{MB}(\rho_{k-i}) = k - i$, so Dominator needs to play k moves on X_k. If $k - i = 1$, then $s_i \in \{u_k, v_k\}$ and Dominator needs to play one more move to dominate v_k. If $k - i = 0$, then Dominator already played k moves.

Case 1.2. $d_3 = u_j$, $j \geq 3$ or $d_3 = v_j$, $j \geq 4$.

Claim 3.12. *If* $\min\{i, j\} \notin \{3, 4\}$*, then Dominator can not win.*

Proof of Claim 3.12. Assume $\min\{i, j\} \notin \{3, 4\}$. Then $s_3 = u_2$ which forces $d_4 = v_1$. Next, $s_4 = u_3$. Dominator can not dominate both u_3 and v_3 at the same time. □

Case 1.2.1. $d_3 = u_j$, where $j \geq 3$ and $j < i$. According to Claim 3.12, $j \in \{3, 4\}$.

1.2.1.a. $j = 3$, that is $d_3 = u_3$. Then, $s_3 = u_2$ which forces $d_4 = v_1$. Consider the subgraph X_{k-3} with the vertex set $V(X_{k-3}) = \{u_4, ..., u_k, v_4, ..., v_k\}$ where u_4 is already dominated with u_3 by Dominator. Also, $d_1 = u_i \in X_{k-3}$ and now it is Staller's turn to make her move on X_{k-3}. By induction hypothesis, if $k - 3 \geq 6$, then $\gamma_{MB}(X_{k-3}) \geq k - 5$, so Dominator needs at least $k - 2$ moves. If $4 \leq k - 3 \leq 5$, then, since $\gamma_{MB}(X_{k-3}) = k - 4$, Dominator needs $k - 1$ moves.

1.2.1.b. $j = 4$, that is, $d_3 = u_4$. Then, $s_3 = u_2$ which forces $d_4 = v_1$ and $s_4 = v_4$ which forces $d_5 = u_3$ (a line trap). Consider the subgraph X_{k-4} on $V(X_{k-4}) = \{u_5, ..., u_k, v_5, ..., v_k\}$ where u_5 is already dominated with u_4 by Dominator. Also, $d_1 = u_i \in X_{k-4}$ and now it is Staller's turn to make her move on X_{k-4}. By induction hypothesis, if $k - 4 \geq 6$, then $\gamma_{MB}(X_{k-4}) \geq k - 6$, so Dominator needs at least $k - 2$ moves. If $3 \leq k - 4 \leq 5$, then, since $\gamma_{MB}(X_{k-4}) = k - 5$, Dominator needs to play $k - 1$ moves on X_k.

Case 1.2.2. $d_3 = u_j$, where $j \geq 3$ and $j > i$. According to Claim 3.12, $i \in \{3, 4\}$, that is, $d_1 = u_3$ or $d_1 = u_4$. Staller's strategy is the same as in Case 1.2.1.

Case 1.2.3. $d_3 = v_j$, $j < i$. According to Claim 3.12 and since $s_2 = v_3$, it follows that $j = 4$, that is, $d_3 = v_4$.

Then, $s_3 = u_2$ which forces $d_4 = v_1$ and $s_4 = u_4$ which forces $d_5 = u_3$ (a triangle trap). Consider the subgraph X_{k-4}. Note that $d_1 = u_i \in X_{k-4}$. The rest of the proof is the same as in Case 1.2.1.b.

Case 1.2.4. $d_3 = v_j$, $j = i$. According to Claim 3.12, $d_1 = u_4$ and $d_3 = v_4$.

Then, $s_3 = u_2$ which forces $d_4 = v_1$. Consider the S-game on Z_{k-4} with the vertex set $V(Z_{k-4}) = \{u_5, ..., u_k, v_5, ..., v_k\}$ where u_5 and v_5 are dominated with u_4 and v_4. According to Lemma 3.9, $\gamma'_{MB}(Z_{k-4}) = k - 5$, so Dominator needs $k - 1$ moves.

Case 1.2.5. $d_3 = v_j$, $j > i$. According to Claim 3.12, $i \in \{3, 4\}$, that is, $d_1 \in \{u_3, u_4\}$. The proof of this case is similar to the proof of Case 1.2.1.

Case 2. $d_2 = v_1$.
If $i = 3$, that is, $d_1 = u_3$, then consider the S-game on the subgraph W_{k-3} with the vertex set $V(W_{k-3}) = \{u_4, ..., u_k, v_3, v_4, ..., v_k\}$ where v_3 and u_4 are dominated with u_3. According to Lemma 3.10, $\gamma'_{MB}(W_{k-3}) = k - 4$, so Dominator needs $k - 2$ moves.

Let $i \geq 4$. Then, $s_2 = u_3$.
Depending of Dominator's third move, we consider the following cases.

Case 2.1. $d_3 = u_1$ or $d_3 = u_2$.
Let i be an even number. Then, $s_4 = v_4$ and Staller starts the sequence of triangle traps $v_3 - v_{i-1}$, where $s_{i-1} = v_i$ and $d_i = v_{i-1}$. Next, if $k - i \geq 2$, then $s_i = v_{i+2}$ and we have the subgraph ρ_{k-i} with the vertex set $V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$ where u_{i+1} is dominated with u_i. Consider the D-game on ρ_{k-i}. According to Lemma 3.3, $\gamma_{MB}(\rho_{k-i}) = k - i$, so Dominator needs k moves. If $k - i = 1$, then $s_i = v_k$ which forces $d_{i+1} = u_k$, so Dominator needs k moves. If $k - i = 0$, then Dominator already played k moves.

Let i be an odd number. Then, $s_4 = v_4$ and Staller starts the sequence of triangle traps $v_3 - v_{i-2}$, where $s_{i-2} = v_{i-1}$ and $d_{i-1} = v_{i-2}$. Consider the subgraph W_{k-i} with the vertex set $V(W_{k-i}) = \{u_{i+1}, ..., u_k, v_i, v_{i+1}, ..., v_k\}$ where v_i and u_{i+1} are dominated with u_i. Consider the S-game on W_{k-i}. This means that $s_{i-1} \in V(W_{k-i})$. According to Lemma 3.10, if $k - i \geq 4$, then $\gamma'_{MB}(W_{k-i}) = k - i - 1$, so Dominator needs $k - 2$ moves. If $1 \leq k - i \leq 3$, then $\gamma'_{MB}(W_{k-i}) = k - i$, so Dominator needs $k - 1$ moves. If $d_1 = u_i = u_k$, then Dominator already played $k - 1$ moves.

Case 2.2. $d_3 = u_j$, $j \geq 4$ or $d_3 = v_j$, $j \geq 3$.
It is not hard to check that Claim 3.12 can be also applied on this case. So, $\min\{i, j\} \in \{3, 4\}$.

Case 2.2.1. $d_3 = u_j$, $j < i$. According to Claim 3.12 and since $s_2 = u_3$, it follows that $j = 4$, that is, $d_3 = u_4$. Then, $s_3 = u_1$ which forces $d_4 = u_2$ and $s_4 = v_4$ which forces $d_5 = v_3$ (a triangle trap). Consider X_{k-4} with the vertex set $V(X_{k-4}) = \{u_5, ..., u_k, v_5, ..., v_k\}$, where $d_1 = u_i \in X_{k-4}$ and now it is Staller's turn to make her move on X_{k-4}. Dominator needs at least $k - 2$ moves.

Case 2.2.2. $d_3 = u_j$, $j > i$. According to Claim 3.12, $i = 4$, that is, $d_1 = u_4$.
The proof is the same as the proof for Case 2.2.1.

Case 2.2.3. $d_3 = v_j$, $j > i$. According to Claim 3.12, $i = 4$, that is, $d_1 = u_4$.
The proof is the same as the proof for Case 2.2.1.

Case 2.2.4. $d_3 = v_j$, $j = i$. According to Claim 3.12, $i = j = 4$, that is, $d_1 = u_4$ and $d_3 = v_4$. Then, $s_3 = u_1$ which forces $d_4 = u_2$ (a triangle trap). We get the subgraph Z_{k-4} and the rest of the prof is the same as in Case 1.2.4. Dominator needs $k - 1$ moves.

Case 2.2.5. $d_3 = v_j$, $j < i$. According to Claim 3.12, $j \in \{3, 4\}$.
Let $j = 3$, that is, $d_3 = v_3$.
Then, $s_3 = u_1$ which forces $d_4 = u_2$. Consider X_{k-3} and the rest of the proof is the same as for Case 1.2.1.a. So, Dominator needs at least $k - 2$ moves.
Let $j = 4$, that is, $d_3 = v_4$.
Then, $s_3 = u_2$, which forces $d_4 = u_1$ and $s_4 = u_4$ which forces $d_5 = v_3$ (a line trap). We get the subgraph X_{k-4} and the rest of the prof is the same as in Case 1.2.1.b.

Case 3. $d_2 = u_j$, $j \geq 3$.
Then, $s_2 = v_1$. In his third move Dominator is forced to claims u_1, as otherwise Staller can isolate v_1 by claiming u_1 in her next move. So, $d_3 = u_1$.
Let $l = \min\{i, j\}$ and let $h = \max\{i, j\}$. Then, $s_3 = v_3$ and in this way Staller starts the sequence of line traps $u_2 - u_{l-1}$, where $s_l = v_l$ and $d_{l+1} = u_{l-1}$. Consider the subgraph X_{k-l} on $V(X_{k-l}) = \{u_{l+1}, ..., u_k, v_{l+1}, ..., v_k\}$ where u_{l+1} is a free vertex already dominated by Dominator with u_l. Also, $u_h \in X_{k-l}$ and it is already claimed by Dominator (in his first or the second move), and now it is Staller's turn to make a move on X_{k-l}. By induction hypothesis, if $k - l \geq 6$, then $\gamma_{MB}(X_{k-l}) \geq k - l - 2$, so Dominator needs at least $k - 2$ on X_k.
If $2 \leq k - l \leq 5$, then, since $\gamma_{MB}(X_{k-l}) \geq k - l - 1$, it follows that Dominator needs $k - 1$ moves. Finally, if $k - j = 1$, then Dominator needs k moves.

Case 4. $d_2 = v_j$, $i < j$.
Then, $s_2 = v_1$. In his third move Dominator is forced to claim u_1, so $d_3 = u_1$. Then, $s_3 = v_3$ and Staller starts the sequence of line traps $u_2 - u_{i-1}$, where the $s_i = v_i$ and $d_{i+1} = u_{i-1}$. Consider X_{k-i} with the vertex set $V(X_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$, where $d_2 = v_j \in X_{k-i}$. Dominator needs at least $k - 2$ moves.

Case 5. $d_2 = v_j$, $i = j$, where $j \geq 3$.
Staller plays $s_2 = v_1$ and Dominator is forced to play $d_3 = u_1$. Then, $s_3 = v_3$ and Staller starts the sequence of line traps $u_2 - u_{i-2}$, where the $s_{i-1} = v_{i-1}$ and $d_i = u_{i-2}$. Since $u_i, v_i \in \mathfrak{D}$, we have the subgraph Z_{k-i} with the vertex set $V(Z_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$. Next, $s_i \in V(Z_{k-i})$, so we consider the S-game on Z_{k-i}. By Lemma 3.9, $\gamma'_{MB}(Z_{k-i}) = k - i - 1$. This means that Dominator needs to play at least $k - 1$ moves on X_k.

Case 6. $d_2 = v_j$, $i > j \geq 2$ and j is even.
Then, $s_2 = u_2$.
We claim the following.

Claim 3.13. *If $d_3 \notin \{u_1, v_1\}$, Dominator can not win.*

Proof of Claim 3.13. Let $d_3 \notin \{u_1, v_1\}$. After Dominator's third move at least one of the vertices u_3, v_3 needs to be free.
Suppose that v_3 is a free vertex. Then, $s_3 = v_1$, so Dominator is not able to dominate u_1, v_1 and v_2 at the same time. In her next move Staller can isolate either u_1 and v_1, or v_2 by claiming either u_1 or v_3.
If u_3 is a free vertex, then $s_3 = u_1$ and Dominator is not able to dominate u_1, v_1 and u_2 at the same time. In her next move Staller can isolate either u_1 and v_1, or u_2 by claiming either v_1 or u_3. □

Case 6.1. $d_3 = u_1$.
Then, $s_3 = v_3$ which forces $d_4 = v_1$. By playing $s_4 = u_4$ Staller starts the sequence of triangle traps $u_3 - u_{j-1}$, where $s_j = u_j$. After Dominator's move in round $j + 1$, $d_{j+1} = u_{j-1}$, we have the subgraph X_{k-j} with the vertex set $V(X_{k-j}) = \{u_{j+1}, ..., u_k, v_{j+1}, ..., v_k\}$, where v_{j+1} is dominated by Dominator with v_j. Also, $d_1 = u_i \in X_{k-j}$ and now it is Staller's turn to make her move on X_{k-j}. By induction hypothesis, if $k - j \geq 6$, then $\gamma_{MB}(X_{k-j}) \geq k - j - 2$, so Dominator needs at least $k - 2$ moves.
If $2 \leq k - j \leq 5$, then, since $\gamma_{MB}(X_{k-j}) \geq k - j - 1$, it follows that Dominator needs at least $k - 1$ moves. Also, if $k - j = 1$, Dominator needs k moves.

Case 6.2. $d_3 = v_1$.
Then, $s_3 = u_3$ which forces $d_4 = u_1$. By playing $s_4 = u_4$ Staller starts the sequence of line traps $v_3 - v_{j-1}$, where $s_j = u_j$. After Dominator's move in round $j + 1$, where $d_{j+1} = v_{j-1}$, we have the subgraph X_{k-j}. The rest of the proof is the same as in Case 6.1. So, Dominator needs at least $k - 2$ moves. Also, if $k - j = 1$, Dominator needs k moves.

Case 7. $d_2 = v_j$, $i > j \geq 2$ and j is odd.
Staller's second move $s_2 = u_1$ forces $d_3 = v_1$. By claiming u_3 Staller starts the sequence of triangle traps $u_2 - u_{j-1}$ where $s_j = u_j$. After Dominator's move in round $j + 1$, that is, $d_{j+1} = u_{j-1}$, we have the subgraph X_{k-j} with the vertex set $V(X_j) = \{u_{j+1}, ..., u_k, v_{j+1}, ..., v_k\}$. The vertex $u_i \in X_{k-j}$ is already claimed by Dominator in his first move and now it is Staller's turn to make her move. After using induction hypothesis, we obtain that Dominator needs to play at least $k - 2$ moves on X_k.

Case III. $d_1 = v_i$, $i \geq 3$.
Then, $s_1 = v_2$. The rest of Staller's strategy depends on Dominator's second move:

Case i. $d_2 = u_1$.
If $i = 3$, that is, $d_1 = v_3$, then consider the S-game on the subgraph W_{k-3} with the vertex set $V(W_{k-3}) = \{u_3, u_4, ..., u_k, v_4, ..., v_k\}$, where u_3 and v_4 are dominated with v_3. Since $\gamma'_{MB}(W_{k-3}) = k - 4$, Dominator needs to play at least $k - 2$ moves on X_k.

Let $i \geq 4$. Then, $s_2 = v_3$.
Depending on Dominator's third move we consider the following cases.

Case i.1. $d_3 = v_1$.
If $d_1 = v_4$, consider W_{k-4} with the vertex set $V(W_{k-4}) = \{u_4, u_5, ..., u_k, v_5, ..., v_k\}$. According to Lemma 3.10, $\gamma'_{MB}(W_{k-4}) = k - 5$, so Dominator needs $k - 2$ moves. Otherwise, if $d_1 = v_i, i > 4$, then $s_3 = v_4$ and Staller starts the sequence of line traps $u_3 - u_{i-2}$. Consider the subgraph W_{k-i} with the vertex set $V(W_{k-i}) = \{u_i, u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$, where u_i and v_{i+1} is dominated with v_i. Next, $s_{i-1} \in V(W_{k-i})$. According to Lemma 3.10, if $k - i \geq 4$, $\gamma'_{MB}(W_{k-i}) = k - i - 1$, so Dominator needs $k - 2$ moves. If $1 \leq k - i \leq 3$, then $\gamma'_{MB}(W_{k-i}) = k - i$, so Dominator needs $k - 1$ moves. If $d_1 = v_k$, then Dominator already played $k - 1$ moves.

Case i.2. $d_3 = v_j$, $j \geq 4$, or $d_3 = u_j$, $j \geq 3$.
It is not hard to see that Claim 3.12 also holds in this case.

Case i.2.1. $d_3 = v_j$, $j \geq 4$. Let $l = \min\{i, j\}$.
According to Claim 3.12, $l = 4$.
Then, $s_3 = u_2$ which forces $d_4 = v_1$ and $s_4 = u_4$ which forces $d_5 = u_3$ (a triangle trap). Consider the subgraph X_{k-4}. It follows that Dominator needs at least $k - 2$ moves.

Case i.2.2. $d_3 = u_j$, $j > i$.
According to Claim 3.12, $i = 4$, that is, $d_1 = v_4$.
Then, Staller's strategy is the same as in Case i.2.1.

Case i.2.3. $d_3 = u_j$, $i = j$.
According to Claim 3.12, $i = j = 4$.
Consider the subgraph Z_{k-4}. It follows that Dominator needs at least $k - 2$ moves.

Case i.2.4. $d_3 = u_j$, $j < i$.
According to Claim 3.12, $j \in \{3, 4\}$.

i.2.4.a. Let $j = 3$, that is, $d_3 = u_3$. Then, $s_3 = v_1$ which forces $d_4 = u_2$ (a line trap). Consider X_{k-3}. It is Staller's turn to maker her move on X_{k-3}. It follows that Dominator needs at least $k - 2$ moves.

i.2.4.b. Let $j = 4$, that is, $d_3 = u_4$.
Then, $s_3 = u_2$ which forces $d_4 = v_1$ and $s_4 = v_4$ which forces $d_5 = u_3$ (a line trap). Consider the subgraph X_{k-4}. It follows that Dominator needs at least $k - 2$ moves.

Case ii. $d_2 = v_1$.
Then, $s_2 = u_3$. Depending of Dominator's third move we consider the following cases.

Case ii.1. $d_3 = u_1$ or $d_3 = u_2$.

ii.1.a. i is even.
Then, $s_3 = v_4$ and Staller starts the sequence of triangle traps $v_3 - u_{i-2}$, where $s_{i-2} = u_{i-1}$ and $d_{i-1} = u_{i-2}$. Consider the S-game on the subgraph W_{k-i} with the vertex set $V(W_{k-i}) = \{u_i, ..., u_k, v_{i+1}, ..., v_k\}$, where u_i and v_{i+1} are dominated with v_i. According to Lemma 3.10, if $k - i \geq$, $\gamma'_{MB}(W_{k-i}) = k - i - 1$, so Dominator needs $k - 2$ moves.
If $1 \leq k - i \leq 5$, then since $\gamma'_{MB}(W_{k-i}) = k - i$, Dominator needs $k - 1$ moves.

ii.1.b. i is odd.
Then, $s_3 = v_4$ and Staller starts the sequence of triangle traps $v_3 - u_{i-1}$, where $s_{i-1} = u_i$ and $d_i = u_{i-1}$. Next, if $k - i \geq 2$, $s_i = u_{i+2}$. Consider the subgraph ρ_{k-i} with the vertex set $V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$, where v_{i+1} is dominated with v_i. According to Lemma 3.3, $\gamma_{MB}(\rho_{k-i}) = k - i$, so Dominator needs k moves.
If $k - i = 1$, then $s_i = u_k$ which forces $d_{i+1} = v_k$, so Dominator again needs k moves.
If $k - i = 0$, then Dominator already played k moves.

Case ii.2. $d_3 = u_j$, $j \geq 4$, or $d_3 = v_j$, $j \geq 3$.
It is not hard to check that Claim 3.12 also holds in this case.

Case ii.2.1. $d_3 = u_j$, $j < i$. According to Claim 3.12, $j = 4$, that is, $d_3 = u_4$.
Staller's strategy is the same as in Case 2.2.1.

Case ii.2.2. $d_3 = u_j$, $j = i$. According to Claim 3.12, $i = j = 4$, that is, $d_1 = v_4$ and $d_3 = u_4$. Then, $s_3 = u_2$ which forces $d_4 = u_1$. Consider the subgraph Z_{k-4} and the rest of the proof is the same as for Case 1.2.4.

Case ii.2.3. $d_3 = u_j$, $j > i$. According to Claim 3.12, $i \in \{3, 4\}$.

ii.2.3.a. Let $i = 3$, that is, $d_1 = v_3$. Then, $s_3 = u_2$ which forces $d_4 = u_1$. We get the subgraph X_{k-3} with the vertex set $V(X_{k-3}) = \{u_4, ..., u_k, v_4, ..., v_k\}$ where v_4 is dominated with v_3. The rest of the proof is the same as in Case 1.2.1.a.

ii.2.3.b. Let $i = 4$, that is, $d_1 = v_4$. Then, $s_3 = u_2$ which forces $d_4 = u_1$ and $s_4 = u_4$ which forces $d_5 = v_3$. We get the subgraph X_{k-4} with the vertex set $V(X_{k-4}) = \{u_5, ..., u_k, v_5, ..., v_k\}$, where v_5 is dominated with v_4. The rest of the proof is the same as in Case 1.2.3.

Case ii.2.4. $d_3 = v_j$, $j < i$. According to Claim 3.12, $j = 4$, that is, $d_3 = v_4$.
Staller's strategy is the same as in Case ii.2.3.b.

Case ii.2.5. $d_3 = v_j$, $j > i$. According to Claim 3.12, $i \in \{3, 4\}$.

ii.2.5.a. Let $i = 3$, that is, $d_1 = v_3$. The proof is the same as in Case ii.2.3.a.

ii.2.5.b. Let $i = 4$, that is, $d_1 = v_4$. The proof is the same as in Case ii.2.3.b.

Case iii. $d_2 = u_j$, $i < j$ and i is even.
Then, $s_2 = u_2$. Staller's strategy from round 3 is the same as in Case 6.

Case iv. $d_2 = u_j$, $i < j$ and i is odd.
Then, $s_2 = u_1$. Staller's strategy from round 3 is the same as in Case 7.

Case v. $d_2 = u_j$, $i = j$.
Then, $s_2 = v_1$. Staller's strategy from round 3 is the same as in Case 5.

Case vi. $d_2 = u_j$, $i > j$.
Then, $s_2 = v_1$. Staller's strategy from round 3 is the same as in Case 4.

Case vii. $d_2 = v_j$, $\min\{i, j\}$ is odd.
Then, $s_2 = u_1$.
Staller's strategy from round 3 is the same as in Case 7.

Case viii. $d_2 = v_j$, $\min\{i, j\}$ is even.
Then, $s_2 = u_2$. Staller's strategy from round 3 is the same as in Case 6.

From this case analysis it follows that $\gamma_{MB}(X_k) \geq k - 2$, for $14 \leq k \leq m$. □

Proof of Theorem 1.4. Let $V(P_2 \square P_n) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and let $E(P_2 \square P_n) = \{u_i u_{i+1} : i = 1, 2, ..., n-1\} \cup \{v_i v_{i+1} : i = 1, 2, ..., n-1\} \cup \{u_i v_i : i = 1, 2, ..., n\}$.

To prove that $\gamma'_{MB}(P_2 \square P_n) \leq n$ we use the pairing strategy for Dominator. That is, when Staller claims u_i (or v_i) for some $i \in \{1, 2, ..., n\}$, Dominator responses by claiming v_i (or u_i). In this way Dominator can win in n moves in the S-game.
Next, we prove that Staller has a strategy to postpone Dominator's winning for at least n moves.
For her first move, Staller claims vertex v_2, that is, $s_1 = v_2$. Since it is harder to dominate the graph $P_2 \square P_n$ in the S-game, where $s_1 = v_2$ than the graph ρ_n in the D-game, and since $\gamma_{MB}(\rho_n) = n$, according to Lemma 3.3, it follows that $\gamma'_{MB}(P_2 \square P_n) \geq n$. □

To prove Theorem 1.5, we need the following lemma.

Lemma 3.14. $\gamma_{MB}(P_2 \square P_{13}) = 11$.

Proof. Let $V(P_2 \square P_{13}) = \{u_1, u_2, ..., u_{13}, v_1, v_2, ..., v_{13}\}$ and let $E(P_2 \square P_{13}) = \{u_i u_{i+1} : i = 1, 2, ..., 12\} \cup \{v_i v_{i+1} : i = 1, 2, ..., 12\} \cup \{u_i v_i : i = 1, 2, ..., 13\}$.
It is not hard to see that $\gamma_{MB}(P_2 \square P_{13}) \geq 11$. Indeed, since $P_2 \square P_{13}$ has one more undominated vertex than X_{13}, it follow that $\gamma_{MB}(P_2 \square P_{13}) \geq \gamma_{MB}(X_{13})$. So, by Lemma 3.11, $\gamma_{MB}(P_2 \square P_{13}) \geq 11$.

Next, we prove the upper bound. First, we give two claims.

Claim 3.15. *Consider the S-game on W_4, where $V(W_4) = \{v_0, v_1, ..., v_4, u_1, ..., u_4\}$ and $E(W_4) = \{u_i u_{i+1} : i = 1, 2, 3\} \cup \{v_i v_{i+1} : i = 1, 2, 3\} \cup \{u_i v_i : i = 1, 2, 3\} \cup \{v_0 v_1\}$, and suppose that Dominator skips the first move. If $s_1 \notin \{u_3, v_3, u_4, v_4\}$, then Dominator can win in at most 4 moves.*

Claim 3.16. *Consider the S-game on W_6, where $V(W_6) = \{v_0, v_1, ..., v_6, u_1, ..., u_6\}$ and $E(W_6) = \{u_i u_{i+1} : i = 1, 2, ..., 6\} \cup \{v_i v_{i+1} : i = 1, 2, ..., 6\} \cup \{u_i v_i : i = 1, 2, ..., 6\} \cup \{v_0 v_1\}$, and suppose that Dominator skips the first move. Let $s_1 = v_2$. Then Dominator can win in at most 6 moves.*

The proofs for these two claims can be obtained by simple case analysis, so we skip it.

Suppose that the game on $P_2 \square P_{13}$ is in progress. If in some point of the game we obtain a subgraph W_4 with the situation described in Claim 3.15, we denote this subgraph by W_4'. If we get a subgraph W_6 with the situation described in Claim 3.16, we denote this subgraph by W_6'.
Let L be a subgraph of $P_2 \square P_{13}$ induced by the set $\{u_1, ..., u_6, v_1, ..., v_6\}$ and let R be a subgraph of $P_2 \square P_{13}$ induced by the set $\{u_8, ..., u_{13}, v_8, ..., v_{13}\}$.
We propose the following strategy for Dominator.

Strategy $\mathcal{S}_D$. For his first move Dominator claims v_7. The rest of the Dominator's strategy depends on Staller's first move. It is enough to consider the case when $s_1 \in L \cup \{u_7\}$. The case when Staller for her first move claims a vertex from the R is symmetric to the case when Staller claims a vertex from the set L. We analyse the following cases.

Case 1. $s_1 = u_7$.
In his second move, Dominator plays $d_2 = u_9$. Consider the subgraph $W_4 \subset R$, where $V(W_4) = \{v_9, v_{10}, ..., v_{13}, u_{10}, ..., u_{13}\}$. When Staller plays on W_4 (or L), Dominator responds on W_4 (or L). According to Lemma 3.10, $\gamma'_{MB}(W_4) = 3$. On L he uses the pairing strategy where the pairing sets are $\{u_i, v_i\}$, for each $i \in \{1, ..., 6\}$. So, Dominator needs at most 11 moves.

Case 2. $s_1 = u_5$.
In his second move Dominator plays $d_2 = u_9$.
Consider $W_4 \subset R$, where $V(W_4) = \{v_9, v_{10}, ..., v_{13}, u_{10}, ..., u_{13}\}$ and consider W_6', where $V(W_6') = \{u_7, u_6, ..., u_1, v_6, ..., v_1\}$ (note $u_5 \in \mathfrak{S}$ and Dominator skipped to play his first move on W_6').
When Staller plays on W_4 (or W_6'), Dominator responds on W_4 (or W_6'). According to Lemma 3.10, $\gamma'_{MB}(W_4) = 3$. By Claim 3.16, Dominator needs at most 6 moves to play on W_6'. So, Dominator needs at most 11 moves.

Case 3. $s_1 \in \{u_3, v_3, u_4, v_4, v_5, u_6, v_6\}$.
In his second move Dominator plays $d_2 = u_5$.
If $s_1 \in \{u_6, v_6\}$, then we have $W_4 \subset L$ on $V(W_4) = \{v_5, v_4, ..., v_1, u_4, ..., u_1\}$ and according to Lemma 3.10, $\gamma'_{MB}(W_4) = 3$. Otherwise, if $s_1 \notin \{u_6, v_6\}$, we have $W_4' \subset L$

on $V(W_4') = \{v_5, v_4, ..., v_1, u_4, ..., u_1\}$ and according to Claim 3.15, $\gamma'_{MB}(W_4') \leq 4$. Also, consider the S-game on W_6 where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. By Lemma 3.10, $\gamma'_{MB}(W_6) = 5$. When Staller plays on W_4 or W_4', Dominator responds on W_4 or W_4', and when Staller plays on W_6, Dominator responds on W_6. So, Dominator needs at most 11 moves.

Case 4. $s_1 \in \{u_2, v_2\}$.
In his second move Dominator plays $d_2 = u_3$.
Consider the S-game on W_6, where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. When Staller plays on W_6 (or L), Dominator responds on W_6 (or L). By Lemma 3.10, $\gamma'_{MB}(W_6) = 5$.
On the L Dominator will use the pairing strategy where the pairing sets are $\{u_1, v_1\}, \{v_4, v_5\}, \{u_5, u_6\}$. Also, to dominate v_2 Dominator will need at most 1 more move. He will claim a free vertex from the set $\{u_2, v_2, v_3\}$. So, Dominator needs at most 11 moves.

Case 5. $s_1 \in \{u_1, v_1\}$.
Then, Dominator claims $d_2 = v_2$. Consider the subgraph W_6 with the vertex set $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. When Staller plays on W_6 (or L), Dominator responds on W_6 (or L). By Lemma 3.10, $\gamma'_{MB}(W_6) = 5$.
On the L Dominator will use the pairing strategy where the pairing sets are $\{u_3, u_4\}, \{v_4, v_5\}, \{u_5, u_6\}$. Also, to dominate u_1 Dominator will need at most 1 more move. He will claim a free vertex from the set $\{u_1, v_1, u_2\}$. So, Dominator needs at most 11 moves.

According to the considered cases, it follows that $\gamma_{MB}(P_2 \square P_{13}) \leq 11$. □

Proof of Theorem 1.5. Let $V(P_2 \square P_n) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and let $E(P_2 \square P_n) = \{u_i u_{i+1} : i = 1, 2, ..., n-1\} \cup \{v_i v_{i+1} : i = 1, 2, ..., n-1\} \cup \{u_i v_i : i = 1, 2, ..., n\}$.
First, prove that $\gamma_{MB}(P_2 \square P_n) \leq n - 2$. For $n = 13$ the statement holds, according to Lemma 3.14. Let $n \geq 14$. Dominator's strategy is to divide a graph $P_2 \square P_n$ into two graphs, $P_2 \square P_{13}$ and $P_2 \square P_{n-13}$. In his first move Dominator claims $v_7 \in V(P_2 \square P_{13})$. When Staller plays on $P_2 \square P_{13}$, Dominator also plays on $P_2 \square P_{13}$ by using his winning strategy $\mathcal{S}_D$ from Lemma 3.14. On graph $P_2 \square P_{n-13}$, Dominator uses the pairing strategy, that is, when Staller claim u_i (or v_i) from $P_2 \square P_{n-13}$, Dominator claims v_i (or u_i) from $P_2 \square P_{n-13}$. So, $\gamma_{MB}(P_2 \square P_n) \leq 11 + (n - 13) = n - 2$.
To prove the lower bound we use Lemma 3.11. Since $P_2 \square P_n$ has one more undominated vertex than X_n, it follow that $\gamma_{MB}(P_2 \square P_n) \geq \gamma_{MB}(X_n)$. So, $\gamma_{MB}(P_2 \square P_n) \geq n - 2$. □

Corollary 3.17. *Let* $3 \leq m \leq n$. *Then*

(i) If m *is even,* $\gamma_{MB}(P_m \square P_n) \leq \gamma_{MB}(P_2 \square P_n) + \left(\frac{m}{2} - 1\right) \gamma'_{MB}(P_2 \square P_n)$.

(ii) If m *and* n *are odd,* $\gamma_{MB}(P_m \square P_n) \leq \gamma_{MB}(P_n) + \left\lfloor \frac{m}{2} \right\rfloor \gamma'_{MB}(P_2 \square P_n)$.

(iii) If m is odd and n is even, $\gamma_{MB}(P_m \Box P_n) \leq \gamma_{MB}(P_2 \Box P_m) + \left(\frac{n}{2} - 1\right) \gamma'_{MB}(P_2 \Box P_m)$.

Sketch of the proof. Consider the D-game on the grid $P_m \Box P_n$.

(i) Divide the graph $P_m \Box P_n$ on $\frac{m}{2}$ grids $P_2 \Box P_n$. On one grid $P_2 \Box P_n$ Dominator is the first player. On the other $\frac{m}{2} - 1$ grids $P_2 \Box P_n$, Staller can be the first player. Applying the Theorem 1.5 and 1.4, we obtain the upper bound for $\gamma_{MB}(P_m \Box P_n)$.

(ii) Divide the graph $P_m \Box P_n$ on $\lfloor \frac{m}{2} \rfloor$ grids $P_2 \Box P_n$ and one path P_n. Dominator will start the game on the path.

The proof for case (iii) is similar to the proof of case (i). □

4 Concluding remarks

In this paper we gave the structural characterization for the graphs G with $\gamma(G) = k \geq 2$ for which $\gamma_{MB}(G) = \gamma(G)$ holds. We proved that Dominator needs exactly n moves to win in the S MBD game on $P_2 \Box P_n$ for every $n \geq 1$, while in the D-game he needs exactly $n - 2$ moves, for $n \geq 13$. Determining the exact values of the invariants $\gamma_{MB}(P_m \Box P_n)$ and $\gamma'_{MB}(P_m \Box P_n)$, where $m, n > 3$ it does not seem as an easy task. So, it would be interesting first to investigate $\gamma_{MB}(P_3 \Box P_n)$ and $\gamma'_{MB}(P_3 \Box P_n)$, for $n \geq 3$, and to see how this improves the upper bounds given in Corollary 3.17.

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Address:

Johannes Kepler University Linz
Doctoral Program "Computational Mathematics"
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-6840

E-Mail:

office@dk-compmath.jku.at

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