ÖAW Austrian Academy of Sciences

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

JKU JOHANNES KEPLER UNIVERSITY LINZ

A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero

Jiayue Qi

DK-Report No. 2021-01 01 2021

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by

Austrian Science Fund (FWF)

FWF Der Wissenschaftsfonds.

Upper Austria

LAND OBERÖSTERREICH

DK sponsors:

- **Johannes Kepler University Linz** (JKU)
- **Austrian Science Fund** (FWF)
- **Upper Austria**

A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero

Jiayue Qi *†

Abstract

The Chow group of zero cycles in the moduli space of stable pointed curves of genus zero is isomorphic to the integer additive group. With the help of a tree representation of the monomial, we are able to provide two different methods for computing this integer: one is algebraic, the other pictorial — the latter can be seen as an equivalent characterization of the former. Also, this tree representation enables us to further expedite the algorithm, via considering an extra property of the tree. With our result, we figure out the value of a specific type of monomials. In addition, we prove the equivalence between a multinomial coefficient identity and the theorem for the value of that specific type of monomials, using this graphical algorithm.

1 Introduction

Chow ring is a vital tool in intersection theory. In the Chow ring of some ambient variety, each subvariety is assigned to a class in this ring, which generalizes the degree of a curve in projective space of dimension two. For a standard reference in intersection theory, see [1]. See [2, Chapter 1] for more detailed explanations on Chow rings.

Multiplication of elements in the Chow ring corresponds to the intersection of cycles — formal sums of subvarieties — in the ambient variety. For a projective variety X of dimension k, the Chow ring $A^*(X)$ is the direct sum of $k+1$ groups, each of them is composed of cycles of a fixed dimension. The group consists of cycles of codimension r is usually denoted by $A^r(X)$, while $A^r(X)$ is always the zero group when r is bigger than the dimension of X. In particular, the Chow group $A^k(X)$ is the group of cycles of dimension zero and, it is isomorphic to the integer additive group.

In this paper, we work in the Chow ring of a specific variety, namely the moduli space of stable n-pointed curves of genus zero, denoted by

*Doctoral Program "Computational Mathematics" W1214, Johannes Kepler University Linz.

†Research Institute for Symbolic Computation, Johannes Kepler University Linz.

$\mathcal{M}_n$. It is a smooth irriducible projective compactification of stable n-pointed genus zero curves. This moduli space was originally introduced by Knudsen and Mumford in their series of papers [3], [4] and [5]. We denote the Chow ring of it by $A^*(n)$ instead of $A^*(\mathcal{M}_n)$, for simplicity. The paper [6] is a very good reference on this Chow ring and can be helpful for a better understanding of the background of the tools that we will mainly use in our paper. We will explain soon the relation between their work and ours. The Chow rings of moduli spaces of curves were first defined by Mumford in [7]. In that paper, they defined the Chow ring of moduli space of curves of a general degree. Also, they worked out completely the case when the genus of stable curves is two. Faber Carel worked in the Chow rings of moduli space of stable curves of genus 3 and 4 in [8] and [9], respectively. They determined completely the situation for genus three, while results were gained partially for genus four. Izadi Elham continued on this path the investigation in [10] for the genus five case. In [11], the author presented the Chow ring of the moduli space of curves of genus zero. They provide a specific basis for the Chow groups and the intersection pairings between the basis. They consider the same ambient ring as we do, their paper may help if one wants to know more properties of our ambient ring, in addition to those considered by us.

From the general property of a Chow ring, we can directly get the following properties of our ambient ring. The moduli space $\mathcal{M}_n$ has dimension $n-3$. Hence we have

$$A^*(n) = \bigoplus_{r=0}^{n-3} A^r(n),$$

where $A^r(n)$ is the Chow group of codimension r. When $r > n-3$, we have $A^r(n) = \{0\}$. To be more precise, we work in the Chow group of codimension $n-3$ — which is the group of cycles of dimension zero — and consider the isomorphism from this group to the integer additive group. We denote by $\int : A^{n-3}(n) \to \mathbb{Z}$ this isomorphism. In this paper, we focus on the following problem: Compute the corresponding integer under this isomorphism for any monomial in this Chow group. This corresponding integer is called the *integral value* or *value* of the given monomial.

This is a subproblem came onto the surface when we studied the realization-counting problem for Laman graphs (minimally rigid graphs) on a sphere, see [12]. Originally, we tried to find another algorithm, hopefully more efficient than the one given in their paper, for the realization-counting of Laman graphs on a sphere. During the study, we figured out that computing the value of any monomial in Chow group $A^{n-3}(n)$ is a subproblem of the target problem. Later on, we actually invented two algorithms for this subproblem — one will be presented in this paper, and another one, actually more efficient, can be checked in [13]. One thing we should confess is that our new algorithms do not seem faster than what they provided in [12]. Nevertheless, we find this subproblem and the corresponding algorithms interesting on their own, which also may have further applications. Hence, we consider this subproblem on its own and try to write down what we discovered in this winding but exciting journey.

As is mentioned in the last paragraph, the algorithm that we will introduce in this paper is not more efficient than the algorithm — which is called *forest algorithm* — presented in [13], while both solve the same problem. However, we value the idea of this algorithm, not only because it is the bridge for us to reach the forest algorithm, but we see more potential of it as well. Especially we praise the idea of vertex-splitting, which is an essential step of the algorithm and will be explained more into details in Section 4.2. When we tried to prove the base case of the forest algorithm, the idea of vertex-splitting was essential. What is worthwhile to mention is that, in the proof of that base case, an identity about multinomial coefficient is needed — we manage to prove it as well. Hence, the algorithm led to another work of us, namely the discovery of this identity. We find the identity as well as its proof fascinating. There is a connection between our algorithm and this identity on the multinomial coefficient. Furthermore, we prove the equivalence between this base case and the identity, which gives the identity an equivalent characterization. Or maybe we should say that the identity hides some structural information of our main algorithm, algebraically. For detailed story on it, see Section 7.

In addition but not the least, the idea of viewing algebraic reduction as tree generating process stands on its own. Starting from the equivalent tree-representation of the given monomial, we complete the characterization of the monomial reduction process within the tree representations and we prove that these two reductions are theoretically equivalent. However, the tree-version reduction is more efficient in many situations. Moreover, it allows us to introduce a necessary condition for the monomial to have value zero, which we call *balancing condition* in the paper. This condition sufficiently improve the efficiency of the algorithm. Pure algebraic reduction is not capable of doing this. This equivalent characterization, namely viewing the algebraic reduction as some operations on the trees, is also the main spirit that we would like to convey.

In order to explain the basic idea of our characterization, we need to introduce some specific properties of the ambient ring. Let $n \geq 3$ be an integer and denote by N the set $\{1, \ldots, n\}$. A bipartition $\{I, J\}$ of N is called a *cut* if both I and J have cardinality bigger than one — then, we say that I and J are *parts* of this cut. For every cut $\{I, J\}$, there is a hypersurface $D_{I,J}$ in the ambient variety $\mathcal{M}_n$. We denote by $\delta_{I,J}$ the corresponding element of $D_{I,J}$ in the Chow ring $A^*(n)$ — note that $\delta_{I,J} = \delta_{J,I}$ and $D_{I,J} = D_{J,I}$. In [6], Keel Sean introduced a linear relation ([6, Section 4, Theorem 1.(2)]) and a quadratic relation ([6, Section 4, Theorem 1.(3)]) between the generators $\delta_{I,J}$. This two relations are the main tools used by us and are called *Keel's linear relation* and *Keel's quadratic relation* in our paper, respectively.

These elements $\delta_{I,J}$ generate the Chow group of codimension one, i.e., $A^1(n)$. They also generate the whole ring, of course, when used as ring generators. Since we are working in a graded ring, it is not hard to see that the monomial $\prod_{i=1}^{n-3} \delta_{I_i,J_i}$ is an element in the Chow group of codimension $n-3$ and any monomial in this group should look like so — the product of $n-3$ generators. Recall that the problem we want to solve is to compute the (integral) value of such monomials.

We solve the problem via the following steps. First, we transfer the monomial to its equivalent tree-representation. See [14, Section 2.2] and [13, Theorem 0.2.] for the one-to-one correspondence between a monomial in $A^*(n)$ and its tree representation. Second, we apply a vertex-splitting operation on this tree, collect all possible outputs. This gives us a set of new trees, each of them corresponds to some degree-reduced-by-one monomial. These monomials are exactly the summand monomials we obtain on the right hand side of the equation when we do the reduction algebraically. Next, we check if each of these new trees fulfills the balancing condition, we exclude those do not — since they have value zero. For more detailed explanation of the balancing condition, see Section 5.3. Then, we apply the vertex splitting process recursively to these new trees. We have a specific principle how the vertex splitting operation should be done; when this is obeyed, it is guaranteed that after finitely many steps, all the obtained monomials are squarefree. Another theorem (Theorem 3.4) tells us that squarefree monomial always has value one. Thence, we are able to directly "count" the absolute value of the given monomial accordingly. The sign depends on the number of recursive steps: being odd leads to the negative sign, while even gives the positive.

The structure of this paper is as follows. In Section 2, we introduce the Keel's quadratic relation between the generators of the ambient ring. Also we introduce the tree representation of the monomials in the Chow group of cycles of dimension zero. In Section 3, we illustrate the algebraic reduction of any monomial in the ambient group with several examples. In the next section, we focus on one step of our algorithm, namely the vertex-splitting process. Then, in Section 5, we introduce the complete graphical algorithm, or maybe better called tree-based algorithm, which characterize the algebraic reduction in a graphical view. In Section 6, we complete the missing proofs from earlier context. In the end, we introduce an application of the main result in Section 7, which may lead to a deeper understanding of our algorithm.

2 Loaded trees

There are linear and quadratic relations between the generators of the ring $A^*(n)$. We say that the two generators δ_{I_1,J_1}, δ_{I_2,J_2} (or the pair $(\delta_{I_1,J_1}, \delta_{I_2,J_2})$) fulfill *Keel's quadratic relation* if the following four conditions hold:

1. $I_1 \cap I_2 \neq \emptyset$;
2. $I_1 \cap J_2 \neq \emptyset$;
3. $J_1 \cap I_2 \neq \emptyset$;
4. $J_1 \cap J_2 \neq \emptyset$.

And in this case, we have $\delta_{I_1,J_1} \cdot \delta_{I_2,J_2} = 0$. For example, when $n = 5$, $(\delta_{12,345}, \delta_{13,245})$ fulfills Keel's quadratic relation but $(\delta_{12,345}, \delta_{123,45})$ does not, also then we have $\delta_{12,345} \cdot \delta_{13,245} = 0$. This fact comes directly from [6, Section 4, Theorem 1.(3)], we just use a different notation. Note that the index are abbreviated a bit: $\delta_{12,345}$ refers to $\delta_{\{1,2\},\{3,4,5\}}$ for instance. We will also use this abbreviation in the later context.

Inspired by this property, we know that if any two factors of the monomial fulfills this relation, the integral value of the given monomial will then be zero. So, the problem is solved in this case. Now we only need to focus on those monomials where no two factors fulfill this quadratic relation. We call those monomials *tree monomials*, since there is a one-to-one correspondence between these monomials and a type of tree (see Theorem 2.2), which we call *loaded tree*.

Definition 2.1 ([13], Definition 0.1.)**.** A *loaded tree with n labels and k edges* is a tree (V, E) together with a labeling function $h : V \to 2^N$ and an edge multiplicity function $m : E \to \mathbb{N}^+$ such that the following three conditions hold:

1. $\{h(v)\}_{v\in V, h(v)\neq\emptyset}$ form a partition of N — elements in N are called the *labels* of T;
2. $\sum_{e\in E} m(e) = k$;
3. For every $v \in V$, $\deg(v) + |h(v)| \geq 3$, note that here multiple edges are only counted once for the degree of its incident vertex.

We say that two loaded trees are *of the same type* if they have the same number of labels and the same number of edges as well.

Note that when we want to verify if some given tree is a loaded tree (of any type) or not, we only need to verify the third condition in Definition 2.1, since the other two are just there to specify the type of the tree.

We define the *monomial of a given loaded tree* as follows. Since it is a tree, we will obtain two components when we remove any edge e. We collect the labels in each component and form the sets I and J, respectively. We say that $\{I, J\}$ is the corresponding cut for edge e, and that $\delta_{I,J}$ is the corresponding factor for edge e. Because of the third condition in Definition 2.1, one can check that no two edges can have the same corresponding cut. Therefore, given a loaded tree, there is a one-to-one correspondence between the edge set and the set of their corresponding cuts (or, the set of their corresponding factors). Hence, we can represent an edge by its corresponding cut. The monomial of a given loaded tree is $\prod_{i=1}^{m} \delta_{I_i,J_i}$, where δ_{I_i,J_i}s are the corresponding factors for the edges, and m refers to the number of edges of the given tree, where edges are coounted with multiplicities. We see that once the loaded tree is given, its monomial is then uniquely determined. For multiple edges, its corresponding factor in the monomial has the power same as its multiplicity. We see in Figure 1 and Figure 2 for two examples of loaded trees and their monomials.

In [14], the authors introduce the one-to-one correspondence between the set of cuts and the phylogenetic tree. In the next theorem, we refer to the same result, while the only difference is that we allow some cuts to appear more than once; that is to say, we consider a multi-set of cuts. Consequently, the corresponding phylogenetic tree differs from that in [14] in the sence that we allow multiple edges; and the multiplicities of edges equal to the occurrences of cuts. In addition, the tree is called differently, in our paper *loaded trees*, while in [14] *phylogenetic trees*.

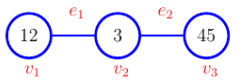


Figure 1: This is a loaded tree with 5 labels and 2 edges. $LT = (V, E, h, m)$, where $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2\}$, $h(v_1) = \{1, 2\}$, $h(v_2) = \{3\}$, $h(v_3) = \{4, 5\}$, $m(e_1) = m(e_2) = 1$. Its monomial is $\delta_{12,345} \cdot \delta_{123,45}$.

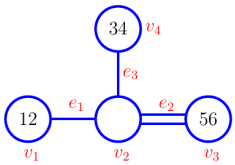


Figure 2: This is a loaded tree with 6 labels and 4 edges. $LT = (V, E, h, m)$, where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3\}$, $h(v_1) = \{1, 2\}$, $h(v_2) = \emptyset$, $h(v_3) = \{5, 6\}$, $h(v_4) = \{3, 4\}$, $m(e_1) = m(e_3) = 1$, $m(e_2) = 2$. Its monomial is $\delta_{12,3456} \cdot \delta_{34,1256} \cdot \delta^2_{56,1234}$.

Theorem 2.2. *[14, Section 2.2] There is a one-to-one correspondence between tree monomials $M = \prod_{i=1}^{k} \delta_{I_i,J_i}$ and loaded trees with n labels and k edges, where $I_i \cup J_i = N$ for all $1 \leq i \leq k$.*

Remark 2.3. Note that there is an extremal case for each $n \geq 3$, namely the tree containing only one vertex with labeling set $N = \{1, \cdots, n\}$. Such a loaded tree has no edges, is an element in the Chow group $A^0(n)$. We call its corresponding monomial the *empty monomial*, denote it by $\emptyset$; it is also a tree monomial. We see that a loaded tree with n labels and k edges represents a tree monomial in the Chow group $A^k(n)$. We often denote the corresponding loaded tree of a given tree monomial M as T_M; and the corresponding tree monomial of a given loaded tree T as M_T.

We have an algorithm realizing this correspondence, namely converting from the monomial to loaded tree, which we call *tree algorithm* in [13]. But we will not go into details on this algorithm here. Recall that our goal is to calculate $\int(M)$ for any tree monomial $M \in A^{n-3}(n)$, where $\int$ represents the isomorphism from $A^{n-3}(n)$ to $\mathbb{Z}$. We say that a monomial is *proper* if it is a tree monomial in $A^{n-3}(n)$ for some $n \geq 3$. Because of this one-to-one correspondence, we can define *value of a loaded tree* — denote it as $\int(T)$ — as $\int(M_T)$, where M_T is the corresponding monomial of the given loaded tree T. We say that a loaded tree is *proper* if its number of edges is three less than the number of labels. In the algebraic language, a proper loaded tree represents some tree monomial in $A^{n-3}(n)$ for some $n \in \mathbb{N}^+$. Given a proper loaded tree, we want to calculate its value. This is another way of expressing our goal.

In the sequel, we introduce the weight function of a loaded tree, which is usually denoted by w. This concept will be helpful in stating our al-

gorithm later on. The weight function $w : V \cup E \to \mathbb{N}$ is defined as $w(v) := deg(v) + |h(v)| - 3$ for all $v \in V$ and $w(e) := m(e) - 1$ for all $e \in E$. One additional remark is that: it is not hard to verify the following identity about the weight function for a proper loaded tree — we call it *weight identity*: $\sum_{v \in V} w(v) = \sum_{e \in E} w(e)$.

$$\begin{aligned}
\sum_{v \in V} w(v) &= \sum_{v \in V} (\deg(v) + |h(v)| - 3) \\
&= \sum_{v \in V} \deg(v) + \sum_{v \in V} |h(v)| - 3 \cdot |V| \\
&= 2 \cdot |E| + n - 3 \cdot |V| \\
&= 2 \cdot |E| + n - 3 \cdot |E| - 3 \\
&= n - 3 - |E| \\
\sum_{e \in E} w(e) &= \sum_{e \in E} (m(e) - 1) \\
&= \sum_{e \in E} m(e) - |E| \\
&= n - 3 - |E|
\end{aligned}$$

Up to here one may wonder: how indeed can we calculate the value of a proper monomial? We are still missing one important tool. Now is the stage time for *Keel's linear relation*, to help us out of the sludge.

3 Linear reduction

In this section, we describe how to figure out the value of a given proper monomial, using Keel's linear relation and Keel's quadratic relation, via algebraic reductions. Keel's linear relation was originally proved in [6, Theorem 1.(2)]. We state exactly the same content as follows, but in different notations.

Fact 3.1 (Keel's linear relation, [6] Theorem 1.(2))**.** Denote by $\epsilon_{ij|kl} := \sum_{i,j \in I, k,l \in J} \delta_{I,J}$. Then we have the equality relations $\epsilon_{ij|kl} = \epsilon_{il|kj} = \epsilon_{ik|jl}$. We call it *Keel's linear relation*.

Let us see a concrete example on it.

Example 3.2. When $n = 6$, we have $\epsilon_{12|35} = \epsilon_{13|25} = \epsilon_{15|23}$, i.e.,

$$\begin{aligned}
&\delta_{12,3456} + \delta_{124,356} + \delta_{126,345} + \delta_{1246,35} \\
&= \delta_{13,2456} + \delta_{134,256} + \delta_{136,245} + \delta_{1346,25} \\
&= \delta_{15,2346} + \delta_{145,236} + \delta_{156,234} + \delta_{1456,23}
\end{aligned}$$

Remark 3.3. From the example above we see that we can substitute some $\delta_{I,J}$, say $\delta_{12,3456}$, by $\epsilon_{13|25} - (\epsilon_{12|35} - \delta_{12,3456})$. Basically we can replace $\delta_{I,J}$ by a sum of $(2^{n-3} - 1)$ many $(\pm)\delta_{I',J'}$s.

It is possible that the follow theorem is known, but because of the lack of proper reference, we will prove it for the sake of completeness. We postpone the proof to Section 6, since our proof method will need some

more tools that will be introduced later on — and of course, those tools are independent from this theorem. For now, we continue our story by accepting this result.

Theorem 3.4. *If all factors are distinct in the tree monomial* $M := \Pi_{i=1}^{n-3}\delta_{I_i,J_i}$, *where* $I_i \cup J_i = N$ *for all* $1 \leq i \leq n-3$. *Then* $\int(M) = 1$ *and we call this type of tree monomial* ***clever monomial*** *and its corresponding loaded tree* ***clever tree****.*

Remark 3.5. Notice that clever trees are by definition proper. For clever trees, we know that they have value one. What about non-clever trees? We would like to use Keel's linear relation to reduce them, hoping to get a sum of several clever trees, then we can directly "count" the value. Let us see a concrete example for a better understanding of this idea.

Example 3.6. Given $\delta_{12,3456}^2 \cdot \delta_{1234,56}$, which is a tree monomial. We want to calculate its value. Now, apply Keel's linear relation, and replace one occurrence of $\delta_{12,3456}$, by $\epsilon_{13|25} - (\epsilon_{12|35} - \delta_{12,3456})$:

$$\delta_{12,3456}^2 \cdot \delta_{1234,56} = \delta_{12,3456} \cdot \delta_{1234,56} \cdot (\epsilon_{13|25} - \delta_{124,356} - \delta_{126,345} - \delta_{1246,35}).$$

Since any summand $\delta_{I,J}$ of $\epsilon_{13|25}$ has $1,3 \in I$ and $2,5 \in J$. Therefore, it together with $\delta_{12,3456}$ fulfills Keel's quadratic relation and we obtain that $\delta_{12,3456} \cdot \delta_{I,J} = 0$. Consequently, we have $\delta_{12,3456} \cdot \epsilon_{13|25} = 0$. Hence we have:

$$\delta_{12,3456}^2 \cdot \delta_{1234,56} = \delta_{12,3456} \cdot \delta_{1234,56} \cdot (-\delta_{124,356} - \delta_{126,345} - \delta_{1246,35}).$$

One can check that the two pairs $(\delta_{1234,56}, \delta_{126,345})$ and $(\delta_{1234,56}, \delta_{1246,35})$ both fulfill Keel's quadratic relation. Hence both products are zero. Therefore, we have

$$\delta_{12,3456}^2 \cdot \delta_{1234,56} = -\delta_{12,3456} \cdot \delta_{1234,56} \cdot \delta_{124,356}.$$

We get one clever monomial with a negative sign, so the tree value (monomial value) is -1.

The process is to replace one factor in the given monomial via some Keel's linear relation and then expand it into a sum of monomials. Then cancel all the terms that will lead to zero because of Keel's quadratic relation. This process is called a *linear reduction*. In the sequel, we see an example where the linear reduction is needed more than once, in order to finally get a sum of clever monomials (maybe with a negative sign).

Example 3.7. Let $M := \delta_{123,4567}^3 \cdot \delta_{12345,67} \in A^4(7)$ be the given monomial. We use Keel's linear relation for $n = 7$, replacing $\delta_{123,4567}$ via $\epsilon_{12|46} = \epsilon_{14|26}$. Then, we obtain

$$\delta_{123,4567}^3 \cdot \delta_{12345,67} = \delta_{123,4567}^2 \cdot \delta_{12345,67} \cdot (\epsilon_{14|26} - (\epsilon_{12|46} - \delta_{123,4567})).$$

Then we see that each summand of $\epsilon_{14|26}$ fulfills Keel's quadratic relation together with $\delta_{123,4567}$, hence we have $\epsilon_{14|26} \cdot \delta_{123,4567} = 0$. Hence we have:

$$\begin{aligned}\delta_{123,4567}^3 \cdot \delta_{12345,67} &= \delta_{123,4567}^2 \cdot \delta_{12345,67} \cdot (-(\epsilon_{12|46} - \delta_{123,4567})) \\ &= \delta_{123,4567}^2 \cdot \delta_{12345,67} \cdot (-\delta_{12,34567} - \delta_{125,3467} \\ &\quad - \delta_{127,3456} - \delta_{1235,467} - \delta_{1237,456} - \delta_{1257,346} \\ &\quad - \delta_{12357,46})\end{aligned} \tag{1}$$

Then we need to exclude those summands of $\epsilon_{12|46}$ which fulfill Keel's quadratic relation with any factor(s) of M – here it refers to $\delta_{12345,67}$ and $\delta_{123,4567}$. After the exclusion, we obtain that

$$\begin{aligned}\delta^3_{123,4567} \cdot \delta_{12345,67} &= \delta^2_{123,4567} \cdot \delta_{12345,67} \cdot (-\delta_{12,34567} - \delta_{1235,467}) \\ &= -\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \\ &\quad - \delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467}\end{aligned} \tag{2}$$

Then for the two tree monomials on right hand side, we continue with the linear reduction. For the first monomial, we use the relation $\epsilon_{13|46} = \epsilon_{14|36}$. We obtain:

$$\begin{aligned}&\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot (\epsilon_{14|36} - (\epsilon_{13|46} - \delta_{123,4567}))\end{aligned} \tag{3}$$

Then, for the same reason, we can ommit $\epsilon_{14|36}$. Hence we have

$$\begin{aligned}&\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot (-(\epsilon_{13|46} - \delta_{123,4567})) \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot (-\delta_{13,24567} - \delta_{135,2467} - \delta_{137,2456} \\ &\quad - \delta_{1235,467} - \delta_{1237,456} - \delta_{1357,246} - \delta_{12357,46})\end{aligned} \tag{4}$$

Now we need to detect those factors in the summands of $\epsilon_{13|46}$ which fulfills Keel's quadratic relation together with any factor among $\delta_{123,4567}$, $\delta_{12345,67}$ and $\delta_{12,34567}$. After the cancellations, only one of them is left.

$$\begin{aligned}&\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot (-(\epsilon_{13|46} - \delta_{123,4567})) \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot (-\delta_{13,24567} - \delta_{135,2467} - \delta_{137,2456} \\ &\quad - \delta_{1235,467} - \delta_{1237,456} - \delta_{1357,246} - \delta_{12357,46}) \\ &= -\delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{12,34567} \cdot \delta_{1235,467}\end{aligned} \tag{5}$$

We can see that we obtain a clever monomial with a negative sign, the value of which is -1. For the other monomial on right hand side of Equation 2 — $\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467}$ — we use $\epsilon_{12|45} = \epsilon_{14|25}$ to reduce the powered factor. Then we get

$$\begin{aligned}&\delta^2_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467} \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467} \cdot (\epsilon_{14|25} - (\epsilon_{12|45} - \delta_{123,4567})) \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467} \cdot (-(\epsilon_{12|45} - \delta_{123,4567})) \\ &= \delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467} \cdot (-\delta_{12,34567} - \delta_{126,3457} - \delta_{127,3456} \\ &\quad - \delta_{1236,457} - \delta_{1237,456} - \delta_{1267,345} - \delta_{12367,45}) \\ &= -\delta_{123,4567} \cdot \delta_{12345,67} \cdot \delta_{1235,467} \cdot \delta_{12,34567}\end{aligned} \tag{6}$$

Here we obtain a clever monomial with negative sign, the value of which is -1. Substituting these two values back to Equation 2, we obtain that the value of the given monomial $M := \delta^3_{123,4567} \cdot \delta_{12345,67}$ is 2. □

Remark 3.8. From the above two examples, we see clearly at least one thing: whenever we replace one occurrence of $\delta_{I,J}$ by some $\epsilon_{ik|jl}-(\epsilon_{ij|kl}-\delta_{I,J})$, we can directly ommit $\epsilon_{ik|jl}$, since any summand of it fulfills the Keel's quadratic relation with $\delta_{I,J}$ and there is at least one occurrence of $\delta_{I,J}$ still left in the remaining part of the monomial. Hence from now on we will only say that we replace $\delta_{I,J}$ by $-(\epsilon_{ij|kl}-\delta_{I,J})$.

Also, because of the same reason, whenever we do one-time linear reduction, we obtain a negative sign on the right hand side. Therefore, how many times of linear reduction we use decides the sign for the value of the given monomial — odd times gives a negative sign while even times leads to a positive sign. In the sequel, we only need to worry about how to obtain the absolute value of the given monomial.

4 Tree-version linear reduction

Non-squarefree tree monomials are in one-to-one correspondence with non-clever trees. After we do linear reductions to a given tree monomial, the obtained monomials in the expanded form on the right hand side of the equation are all tree monomials — because all monomials containing a pair of factors fulfilling Keel's quadratic relation have been cancelled out during the reduction process. Because of the one-to-one correspondence between tree monomials and loaded trees, this process can parallelly be viewed as a tree transforming/generating process. This is an equivalent process of the (algebraic) linear reduction. We call it the *tree-version linear reduction*.

In this section, we will describe it into details. Then, in Section 6 we prove that this tree-based algorithm is indeed an equivalent characterization of the algebraic linear reduction described in Section 3. We view the algebraic reduction as some operations on the loaded tree corresponding to the given monomial. Now we will try to explain the tree-version linear reduction step by step — translating the algebraic language to graphical language.

- Non-squarefree tree monomials corresponds precisely to loaded trees with multiple edge(s).
- In the linear reduction process, first we pick some $\delta_{I,J}$ that has power higher than one in the given monomial. In the graphical language, this operation means that we pick a multiple edge e whose corresponding factor is $\delta_{I,J}$, i.e., whose corresponding cut is $\{I,J\}$.
- Then we decide on which Keel's linear relation $\epsilon_{ij|kl}=\epsilon_{ik|jl}$ to use, in order to replace the factor $\delta_{I,J}$. In graphical language, it says that we pick $i,j\in I$ and $k,l\in J$.

We need to pause here, so as to explain this quadruple choosing step. In the algebraic reduction, it seems that we are allowed to choose whatever quadruple for Keel's linear relation as long as we can replace the factor — we will see later on that it is not true. We have a specific requirement in the pictorial algorithm — we require a specific way of choosing the quadruple, namely *a proper choice*.

4.1 Proper choice of the Keel's linear quadruple

Actually we can pick any quadruple (i, j, k, l) such that $i, j \in I$ and $k, l \in J$ for Keel's linear relation; the correctness is guaranteed in theory. We call such a quadruple *Keel's linear quadruple*. However, we want to make sure that no summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$ divides M. We call it *summand distinction property* of the quadruple (i, j, k, l) or of the set $\{i, j, k, l\}$, simply because it means any summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$ is distinct from all factors of M. Why do we need this property?

Suppose that we want to replace one occurrence of δ_{I_1,J_1} in $M = \delta^p_{I_1,J_1} \cdots \delta_{I_q,J_q}$ by $-(\epsilon_{ij|kl} - \delta_{I_1,J_1})$ and (i, j, k, l) fulfills the summand distinction property. W.l.o.g., assume that $-(\epsilon_{ij|kl} - \delta_{I_1,J_1}) = \sum_{s=1}^{t} \delta_s$. Then we obtain the following equation:

$$\begin{aligned} \delta^p_{I_1,J_1} \cdots \delta_{I_q,J_q} &= \delta^{p-1}_{I_1,J_1} \cdots \delta_{I_q,J_q} \cdot (-(\epsilon_{ij|kl} - \delta_{I_1,J_1})) \\ &= \delta^{p-1}_{I_1,J_1} \cdots \delta_{I_q,J_q} \cdot \sum_{s=1}^{t} \delta_s \\ &= \delta^{p-1}_{I_1,J_1} \cdots \delta_{I_q,J_q} \cdot \delta_1 + \cdots + \delta^{p-1}_{I_1,J_1} \cdots \delta_{I_q,J_q} \cdot \delta_t \end{aligned} \tag{7}$$

Now we focus on any monomial on right hand side of the equation, say $M_r := \delta^{p-1}_{I_1,J_1} \cdots \delta_{I_q,J_q} \cdot \delta_r$. Since δ_r does not divide M, δ_r is distinct from all factors of M. Therefore, no factor can have a higher power after the replacement. Then, we can really guarantee that after finitely many times linear reduction, we can indeed obtain a (maybe negative) sum of clever monomials, which, in value, is equal to the given monomial. The analysis above also tells us that we should also obey this rule when doing the algebraic reduction, otherwise we may not finish the reduction (until a sum of only clever monomials) after finitely many steps.

How can we guarantee that the chosen quadruple satisfies summand distinction property? By a *proper choice* of the Keel's linear quadruple. In order to explain what is this "proper choice", we need the concept of *cluster* first.

Definition 4.1. We say that $cl \subset N$ is a *cluster of vertex* v if and only if one of the following two conditions holds:

1. cl is a one-element-subset of $h(v)$.
2. cl is the collection of labels in one component when we remove vertex v and all its incident edges.

Note that here h denotes the labeling function of the loaded tree to which v belongs. If a cluster has cardinality one, we say that it is a *singleton*; otherwise, we say that it is a *proper cluster*.

Remark 4.2. We observe that collecting all parts of cuts of a loaded tree gives us exactly the collection of proper clusters of all vertices.

Remark 4.3. It is not hard to check that the above two cases are disjoint for any loaded tree. When a cluster of vertex v fulfills the first condition, it contributes one to the cardinality of $h(v)$. When a cluster of vertex v fulfills the second condition, it contributes one to the degree of v. Recall the expression $\deg(v) + |h(v)|$ in the third item of Definition 2.1: For a

vertex, each adjacent edge corresponds to a cluster of it, and each of its labels corresponds to a cluster of it as well.

For a better idea of this definition, let us see what are the clusters for vertices of the loaded tree in Figure 2.

- Clusters for v_1: $\{1\}$, $\{2\}$, $\{3,4,5,6\}$.
  Singletons for v_1: $\{1\}$, $\{2\}$.
  Proper clusters for v_1: $\{3,4,5,6\}$.
- Clusters for v_2: $\{1,2\}$, $\{3,4\}$, $\{5,6\}$.
  Singletons for v_2: none.
  Proper clusters for v_2: $\{1,2\}$, $\{3,4\}$, $\{5,6\}$.
- Clusters for v_3: $\{1,2,3,4\}$, $\{5\}$, $\{6\}$.
  Singletons for v_3: $\{5\}$, $\{6\}$.
  Proper clusters for v_3: $\{1,2,3,4\}$.
- Clusters for v_4: $\{1,2,5,6\}$, $\{3\}$, $\{4\}$.
  Singletons for v_4: $\{3\}$, $\{4\}$.
  Proper clusters for v_4: $\{1,2,5,6\}$.

Now we are prepared for the concept of a "proper choice" of Keel's linear quadruple. Assume w.l.o.g. that when we remove edge $e = \{v_1, v_2\}$, vertex v_1 is in the component where all labels collected to be I and v_2 is in the component where all labels collected to be J. We call the corresponding components *Component-I* and *Component-J*, respectively. We choose the quadruple (i,j,k,l) such that $i,j \in I$ are from two distinct clusters of v_1 and $k,l \in J$ are from two distinct clusters of v_2. We call this way of choosing i,j,k,l a *proper choice*. And we call the corresponding quadruple (i,j,k,l) a *proper quadruple* of the edge e, or of the cut $\{I,J\}$; we call $\{i,j,k,l\}$ a *proper quadruple set* of e or of $\{I,J\}$. Note that we are always able to choose a proper quadruple for any edge of some loaded tree, since apart from the cluster connected by edge e to v_2 (or v_1), v_1 (or v_2) has at least two more clusters, by the third condition of Definition 2.1.

Let us continue with focusing on the loaded tree in Figure 2. For this loaded tree, suppose that we want to replace one occurrence of edge e_2 (i.e., the cut $\{\{1,2,3,4\},\{5,6\}\}$). We should choose i,j from $\{1,2,3,4\}$ and k,l from $\{5,6\}$ for Keel's linear reduction. We see that $(1,3,5,6)$ is a proper choice, but neither $(1,2,5,6)$ nor $(3,4,5,6)$ is a proper choice. We claim that any proper quadruple fulfills the summand distinction property. We leave the proof of this to Section 6. For now, we continue by accepting this statement.

So when we have a loaded tree, we should decide on which multiple edge to reduce, then we should decide on a proper quadruple to do the reduction. In the sequel, we introduce the next step: how to directly tell which loaded trees or tree monomials will be generated on the right hand side of the equation in the algebraic linear reduction (see Section 3) but using a graphical method. For this, we need to introduce the concept of *vertex splitting*.

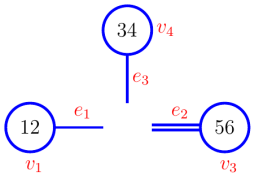


Figure 3: These are the (simplified) branches of vertex v_2 of the loaded tree in Figure 2. Vertex v_2 has three branches.

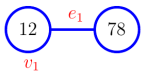


Figure 4: This is the loaded tree that is obtained from attaching the branch of v_2 containing labels 1 and 2 (in the loaded tree in Figure 2) to a single vertex with labeling set $\{7, 8\}$.

4.2 Vertex-splitting

Before we can explain this concept, we need to define *branches of a vertex* first. Let v be a vertex in loaded tree T. Removing vertex v but not its adjacent edges gives us $\deg(v)$ many parts. Each part is a structure of a tree but lacking a vertex. We make $\deg(v)$ many copies of vertex v, and concatenate it to each of these parts at the place where a vertex is missing. Then we obtain $\deg(v)$ many trees, we call them *branches of* v. We call the copy of v the *special vertex* in each of these trees. We say that a branch with the special vertex v is *attached to* some vertex v' (of some tree T') if we add an extra edge e' between v and v', and then contract it (two vertices merged into a new vertex v_1), and then set the labeling set of v_1 to be the labeling set of v'.

Note that each branch corresponds to a cluster of v. Each cluster of v corresponds either to a branch of v or to a label of v. The above statement is just to make things precise in a formal sense. However, in our concrete operation, we can think of the branch simply as the structure of a tree but lacking that special vertex. And when we attach a branch to some vertex v' of another tree T', we can just view it as to concatenate T' to the branch at vertex v', to the endmost of that branch where a vertex is missing. From now on, for convenience, we will use this simplified concept of branch and branch-attaching.

In order to better understand the concept, we see an example. Let us turn our focus back to vertex v_2 in loaded tree in Figure 2. See Figure 3 for its branches. See Figure 4 for an example of branch-attaching. Now we are prepared for the vertex splitting process.

We explain this process as a series of operations on a loaded tree. Recall that the weight function is defined as $\deg(v) + |h(v)| - 3$ for each vertex v.

1. Input: a loaded tree $T_1 = (V_1, E_1, h_1, m_1)$; a proper quadruple set

$Q := \{i, j, k, l\}$ of some multiple edge $e = \{v_1, v_2\}$ with the corresponding cut $\{I, J\}$.

2. Output: NULL or a loaded tree $T_2 = (V_2, E_2, h_2, m_2)$.
3. If both v_1 and v_2 have zero-weight, return NULL. Otherwise, pick one of the incident vertices of e with non-zero weight — assume w.l.o.g. that vertex v_1 is chosen. Let I be so that v_1 is in Component-I of T_1. Naturally, v_2 is in Component-J.
4. Replace v_1 by two vertices v_1' and v_1'' with a single edge $e' = \{v_1', v_1''\}$ connecting them.
5. Set the labeling sets of v_1' and that of v_1'' so that $h_2(v_1') \cup h_2(v_1'') = h_1(v_1)$ and $h_1(v_1) \cap (Q \cap I) \subset h_2(v_1')$;

   ($\star$) However, note that if v_1 has no other branches except for the ones that contain any label in $Q \cap I$ or edge e, then $|h_2(v_1'')| \geq 1$ must hold.
6. Arrange the branches of v_1 in the following way:

   Among the branches of v_1, those containing any label in $Q \cap I$ are attached to v_1' and the one that contains labels in $Q \cap J$ is attched to v_1'' — it is not hard to check that the two labels in $Q \cap J$ are in the same branch of v_1. The branch containing e should be modified slightly: the multiplicity of edge e in this branch gets reduced by one, then gets attached to v_1''.

   For any other branches of v_1 in T_1, it can be either attached to v_1' or v_1''.

   ($\star\star$) However, note that if $h_2(v_1'') = \emptyset$, then we must attach at least one branch to v_1''.

Remark 4.4. The requirements ($\star$) and ($\star\star$) are there to guarantee that vertex v_1'' gets at least one label or at least one branch (other than the one contains edge e). In this way, we guarantee that v_1'' fulfills the third condition in Definition 2.1. Since we require that the cluster of each of the two labels in $Q \cap I$ is either attached to v_1' (as a branch), or put into the labeling set of v_1' (as a singleton). Also we have $\{v_1', v_1''\} \in E(T_2)$. Hence vertex v_1' has at least three clusters, it also fulfills the third condition in Definition 2.1. Therefore, T_2 is a loaded tree.

Remark 4.5. Let w_1 be the weight function for T_1 and w_2 be that for T_2. Then we have

$$\begin{aligned} w_2(v_1') + w_2(v_1'') &= (\deg(v_1') + |h_2(v_1')| - 3) + (\deg(v_1'') + |h_2(v_1'')| - 3) \\ &= (\deg(v_1') + \deg(v_1'')) + (|h_2(v_1')| + |h_2(v_1'')|) - 6 \\ &= (\deg(v_1) + 2) + |h_1(v_1)| - 6 \\ &= \deg(v_1) + |h_1(v_1)| - 4 \\ &= (\deg(v_1) + |h_1(v_1)| - 3) - 1 \\ &= w_1(v_1) - 1 \end{aligned}$$

We see from the above reasoning that the vertex-splitting process indeed require the weight of vertex v_1 to be non-zero; otherwise, it cannot be splitted. Actually, if there is a multiple edge $e = \{v_1, v_2\}$ in some proper

loaded tree, both v_1 and v_2 being weight-zero leads to the value of this tree being zero. Therefore, if none of the weights is non-zero, we can immediately output zero as the wanted value. We will prove this conclusion in Section 6. This is the reason why we assume that v_1 has non-zero weight in the input statement. If v_1 has zero weight, while v_2 has non-zero weight, we can simply rename the vertices.

Remark 4.6. It is not hard to see that T_1 and T_2 have the same set of labels and the same number of edges. Hence, T_2 is of the same type with T_1. Thence when T_1 is proper, T_2 is also proper. After these analysis, we see that the above stated process is an algorithm — it terminates, and returns a loaded tree or NULL. We say that in this algorithm, vertex v_1 *is splitted into* vertices v_1' and v_1''.

Because of the summand distinction property, the weight sum of edges of T_2 is always one less than that of T_1. Therefore, if we recursively apply this process, then after finitely many steps, we will obtain only clever tree. This observation provides us an idea on calculating the value of a given tree monomial.

In the above algorithm, output is just one loaded tree. We observe that we actually have some freedom at several steps:

1. If both v_1 and v_2 have non-zero weight, then we can split any of them.
2. We could also have some freedom on how to set up the labeling function for v_1' and v_1'', respectively — as long as condition $(\star)$ holds.
3. Also, we have some freedom on the arrangements of branches of v_1' and those of v_1'' — as long as condition $(\star\star)$ holds.

When we consider all these freedom, and collect all the possibly generated loaded trees, we obtain the tree-version linear reduction algorithm. This algorithm does the same thing and should give us the same result with the algebraic reasoning which is introduced in Section 3 — after a slight modification. We postpone the correctness proof of it to Section 6.

5 Reduction algorithm

5.1 Tree-version linear reduction algorithm

In this subsection, we explain the tree-version linear reduction algorithm, see Algorithm 1.

Remark 5.1. Note that we could also obtain NULL in some steps, in that case, we simply do not add anything to ST. We modify the input loaded tree into a proper tree monomial M, the output into the negative formal sum of all corresponding monomials of the loaded trees in ST. And add one step at the very beginning — using tree algorithm, transferring M to T_M, one step at the very end — transferring loaded trees in ST into their corresponding monomials. Then the above algorithm does the same thing and should give us the same result with the algebraic reasoning/algorithm which is introduced in Section 3. By Theorem 2.2, the function of Algorithm 1 is equivalent to the modified version stated above.

Algorithm 1: tree-version linear reduction

input : a proper loaded tree T_M (the corresponding loaded tree of proper monomial M); a multiple edge $e = \{v_1, v_2\}$ (with corresponding cut $\{I, J\}$) of T_M; a proper quadruple set $\{i, j, k, l\}$ of e, or equivalently, of $\{I, J\}$ such that $i, j \in I$ and $k, l \in J$.

output: loaded trees whose corresponding monomials are the ones that survive after the (algebraic) linear reduction on M which uses Keel's linear reduction on the relation $\epsilon_{ij|kl} = \epsilon_{ik|jl}$.

$w_1 \leftarrow$ weight of v_1;
$w_2 \leftarrow$ weight of v_2;
if $w_1 = 0$ *and* $w_2 = 0$ **then**
  return $\emptyset$
end if
$ST \leftarrow$ the set of all loaded trees that can be obtained from T_M with edge e and the proper quadruple set $\{i, j, k, l\}$, after applied to the vertex splitting algorithm — either by splitting vertex v_1, or by splitting vertex v_2 — in the set ST;
return ST

Remark 5.2. Note that our algorithm is applicable also to non-proper monomials (trees). However, in our problem, the input is always a proper monomial (tree), and we always obtain one or more proper monomials (loaded trees) after each linear reduction.

Now we see an example for a better understanding of the above algorithm, as well of the vertex splitting algorithm stated earlier.

Example 5.3. Given a proper loaded tree T_1 as in Figure 5. Let us follow the above vertex-splitting process, see what we will obtain. We pick a multiple edge e_2 and a proper quadruple set $\{1, 7, 5, 6\}$. We calculate the weight of its incident vertices, find out that the weight of v_3 is zero while that of v_2 is 2 (non-zero). Hence we can only choose v_2 to split, and note that v_2 is in Component-$\{1, 2, 3, 4, 7, 8\}$ of T_1.

We should bipartition the labeling set of v_2 such that

$$\{7, 8\} \cap (\{1, 2, 3, 4, 7, 8\} \cap \{1, 7, 5, 6\}) = \{7\} \subset h_2(v_2')$$

holds. So if we follow the instructions in Algorithm 1, we have two options in this step:

1. $h_2(v_2') = \{7\}$ and $h_2(v_2'') = \{8\}$;
2. $h_2(v_2') = \{7, 8\}$ and $h_2(v_2'') = \emptyset$.

Then, based on the above two choices, we distribute the branches of v_2 to be the branches of v_2' or those of v_2''. From the principle we know that the branch containing 1 should be attached to v_2' and the branch containing 5 or 6 should be attached to v_2''; edge multiplicity of e_2 should be reduced by one. The remaining branch — the branch containing labels

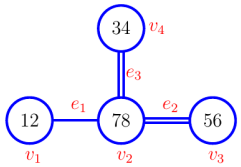


Figure 5: This a proper loaded tree T_1 with 8 labels and 5 edges. We want to reduce edge e_2 with the proper quadruple set $\{1, 7, 5, 6\}$, via splitting the vertex v_2.

3 and 4 can be attached to either v_2' or v_2'' in the first label-distribution option; but it must be attached to v_2'' in the second label-distribution option because of the $(\star\star)$ requirement. Hence, we obtain in total three loaded trees, after applying Algorithm 1.

Figure 6 shows these loaded trees. It is not hard to check that each of them is still a loaded tree, and is of the same type with T_1. Note that vertices in T_1 and those in the new trees do not necessarily have the same names/symbols. We keep most of them the same, just to make it easilier to compare and understand the vertex-splitting process. This also applies to some later context or examples.

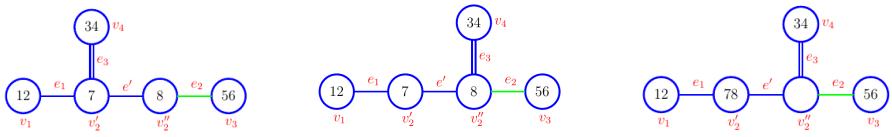


Figure 6: The set of these three loaded trees is the output of Algorithm 1 - applied to the loaded tree T_1 with multiple edge e_2 (in Figure 5) and the proper quadruple set $\{1, 7, 5, 6\}$. The edge which gets reduced is marked in green. Vertex v_2 is splitted into v_2' and v_2''. The new edge is denoted as e'. Each of them is a loaded tree with 8 labels and 5 edges.

In the sequel, we look back on our examples in Section 3. We will apply the tree-version linear reduction algorithm and see if we will obtain the same result as if we apply the algebraic method.

Example 5.4. See Figure 7 for the corresponding loaded tree T of monomial $\delta_{12,3456}^2 \cdot \delta_{1234,56}$ in Example 3.6. Now we want to apply the tree-version linear reduction algorithm to it. We pick the multiple edge e_1 to reduce. Its corresponding cut is $\{\{3, 4, 5, 6\}, \{1, 2\}\}$. The proper linear reduction quadruple set we pick here is $\{3, 5, 1, 2\}$. By an easy calculation we know that the weight of v_1 is zero and that of v_2 is 1. Therefore, we can only split vertex v_2 — no freedom of choice here. And v_2 is in Component-$\{3, 4, 5, 6\}$ of T.

First we split vertex v_2 into v_2' and v_2'' such that $\{3, 4\} \cap (\{3, 4, 5, 6\} \cap \{3, 5, 1, 2\}) = \{3\} \subset h_2(v_2')$. And observe that v_2 has only two branches,

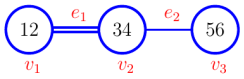


Figure 7: This is the corresponding loaded tree of monomial $M = \delta^2_{12,3456} \cdot \delta_{1234,56}$. We want to reduce edge e_1 with the proper quadruple set $\{3, 5, 1, 2\}$.

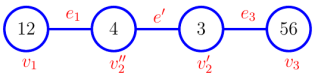


Figure 8: This is the only loaded tree that can be obtained after applying the tree-version linear reduction algorithm to the loaded tree in Figure 7 with edge e_1 and the quadruple set $\{3, 5, 1, 2\}$. The new edge is denoted by e'.

one contains 5, the other contains edge e_1. Therefore, by condition $(\star)$, $h_2(v_2'')$ should contain at least one label. Hence we have that $h_2(v_2'') = \{4\}$. We see that there is also no freedom of choice in this step.

The next step is to arrange the branches of v_2 to be the branches of v_2' or those of v_2''. The branch containing e_1 should be modified — multiplicity of edge e_1 should get reduced by one — and then gets attached to v_2''. The branch containing label 5 should be attached to v_2'. We see that there is also no freedom of different options in this step. We obtain only one loaded tree, which is also a clever tree.

Hence we only need one time linear reduction for calculating the value of T, the sign for the result is then $(-1)^1 = -1$. In Figure 8 we see the output tree. We can easily obtain that the corresponding monomial of this new tree is $\delta_{12,3456} \cdot \delta_{124,356} \cdot \delta_{1234,56}$, which coincides with the result we obtain in Example 3.6, with the algebraic reasoning.

In the sequel, we apply the tree-version linear reduction algorithm to the monomial in Example 3.7. Let us see if we can save some labor, comparing to the algebraic method in Section 3.

Example 5.5. See Figure 9 for the corresponding loaded tree T of monomial $M := \delta^3_{123,4567} \cdot \delta_{12345,67}$ in Example 3.7. This is a proper tree. The edge $e_1 = \{v_1, v_2\}$ to be reduced is the corresponding edge of cut $\{\{1, 2, 3\}, \{4, 5, 6, 7\}\}$. We pick $1, 2$ from two distinct clusters of v_1 and $4, 6$ from two distinct clusters of v_2.

We see that we can either split v_1 or v_2 since they both have non-zero weight. When we split vertex v_1, there is only one loaded tree T_1 that can be obtained, see the loaded tree in Figure 10a. When we split vertex v_2, there is only one loaded tree T_2 that can be obtained, see the loaded tree in Figure 10b. Hence after we apply one time tree-version linear reduction algorithm to T, we obtain in total two loaded trees: T_1 and T_2. Their corresponding monomials are $\delta_{12,34567} \cdot \delta^2_{123,4567} \cdot \delta_{12345,67}$ and $\delta^2_{123,4567} \cdot \delta_{1235,467} \cdot \delta_{12345,67}$, respectively. This coincides with the resulting monomials in Equation 2.

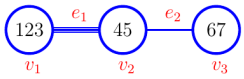
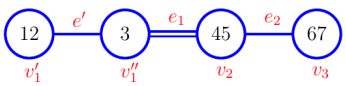
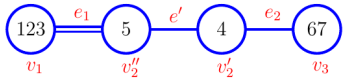


Figure 9: This is the corresponding loaded tree T of monomial $\delta^3_{123,4567}\cdot\delta_{12345,67}$. We want to reduce edge e_1 with the proper quadruple set $\{1,2,4,6\}$.



(a) loaded tree T_1



(b) loaded tree T_2

Figure 10: The set of these two loaded trees is the output set of Algorithm 1 applied to tree T in Figure 9 with edge e_1 and quadruple set $\{1,2,4,6\}$. When we choose to split vertex v_1, we obtain the loaded tree in Figure 10a. When we choose to split vertex v_2, we obtain the loaded tree in Figure 10b.

We see that in the above two examples, we obtain exactly the same result as when we did it via algebraic approach; however, much labor saved. Note that we should choose the Keel's quadruple set coinciding with the one used in the algebraic reasoning, each time; then we will always get the same reasult with the algebric algorithm. This is an equivalent characterization of the algebraic linear reduction, in a more visualized way. This characterization naturally also leads to a complete algorithm that is equivalent to the whole process of the algebraic linear reduction until only clever monomials are in the expansion on right hand side of the equation. This is called *reduction chain algorithm* and will be introduced in the upcoming section.

5.2 Reduction chain algorithm

Tree-version linear reduction algorithm tells us what monomials are on the right hand side of the equation after one-time linear reduction, using the Keel's linear relation. However, as mentioned earlier, we want to know the value of any proper tree monomial (loaded tree).

A very natural idea is that we can apply Algorithm 1 recursively on the obtained set of loaded trees, after finitely many steps, we should obtain a set of clever trees. The cardinality of the obtained set should give us the absolute value of the given loaded tree. And from Remark 3.8 we know that the sign is just -1 to the power of number of recursions. Correctness of this algorithm is guaranteed by the correctness of tree-version linear reduction algorithm. We call this process *reduction chain algorithm*. This is an equivalent characterization of the algebraic algorithm for computing the value of the given monomial described in Section 3, using graphical language. We state this process as follows.

1. Input: a proper loaded tree T_M (of which the corresponding mono-

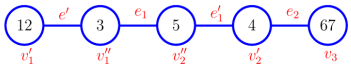


Figure 11: This is the only loaded tree T_1' one can obtain after applying Algorithm 1 to tree T_1 in Figure 10a with edge e_1 and quadruple set $\{4,6,1,3\}$. Edge e_1 gets reduced, vertex v_2 gets splitted into vertices v_2'' and v_2'. And e_1' denotes the newly-generated edge.

mial is M).

2. Output: a set of clever trees that are the clever monomials we obtain in the algebraic reasoning, given that in each reduction we use the same quadruple set (for algebraic-version reduction and for tree-version reduction).
3. Let A denote the output set of Algorithm 1 applied to T_M with any multiple edge and any corresponding proper quadruple.
4. Then, apply Algorithm 1 on each element in A. Let A_1 denote the union of the output sets of these results.
5. We recursively apply Algorithm 1 on each element in A_1 and union the resulting sets, define the union as A_2.
6. From Remark 4.6 we know that this process must end at some step; w.l.o.g., assume that all loaded trees in A_k are already clever trees. Return A_k.

Now, let us see an example for a better idea. We continue with Example 5.5.

Example 5.6. We want to apply the above stated algorithm to T in Figure 9, we already obtain A_1 in Example 5.5. The set A_1 is the set of T_1 and T_2, when we choose e_1 and $\{3,5,1,2\}$ to do the reduction. Then we continue with loaded tree T_1 in Figure 10a, the edge to be reduced is evidently (the corresponding edge of cut) $\{\{1,2,3\},\{4,5,6,7\}\}$. We pick a proper linear reduction quadruple set $\{4,6,1,3\}$. We can only split v_2 from the weight information. Then we carry out the vertex-splitting process, there is only one loaded tree that can be obtained, see Figure 11. We see that its monomial coincides with the right hand side of Equation 5.

Now we continue with loaded tree T_2, the edge to be reduced is evidently (the corresponding edge of cut) $\{\{1,2,3\},\{4,5,6,7\}\}$. We pick a proper linear reduction quadruple set $\{1,2,4,5\}$. We can only split v_1 from the weight information. Then we carry out the vertex-splitting process, there is only one loaded tree that can be obtained, see Figure 12. We see that its monomial coincides with the right hand side of Equation 5. We obtain exactly the same result as in Example 3.7!

Therefore, we only need two times recursions, in order to obtain a set of clever loaded trees. And the output is $A_2 = \{T_1', T_2'\}$. Therefore, the value of the given monomial $M := \delta^3_{123,4567} \cdot \delta_{12345,67}$ is $(-1)^2 \cdot 2 = 2$. This coincides with the result in Example 3.7.

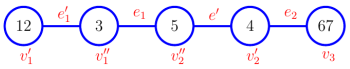


Figure 12: This is the only loaded tree T_2' one can obtain after applying Algorithm 1 to tree T_2 in Figure 10b with edge e_1 and quadruple set $\{1, 2, 4, 5\}$. Edge e_1 gets reduced, vertex v_1 gets splitted into vertices v_1' and v_1''. And e_1' denotes the newly-generated edge.

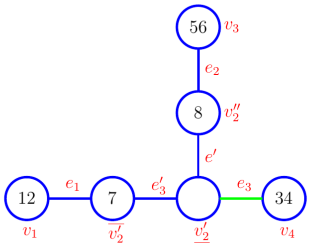


Figure 13: This is the only loaded tree T_1'' one can obtain after applying Algorithm 1 to tree T_1' on the leftmost loaded tree of Figure 6 with edge e_3 and quadruple set $\{1, 7, 3, 4\}$. Edge e_3 gets reduced and is marked in green, vertex v_2' gets splitted into vertices $\overline{v_2'}$ and $\underline{v_2'}$. And e_3' denotes the newly-generated edge. Notice that this is a clever tree.

Next, let us continue with Example 5.3. We will apply the reduction chain algorithm to the loaded tree T_1 in Figure 5 and see how many clever trees can we obtain.

Example 5.7. In Example 5.3, we already obtain all the loaded trees after one time recursion. These loaded trees are shown in Figure 6. Now we apply Algorithm 1 on each of them. However, we see that both incident vertices of the only multiple edge are zero-weighted in the rightmost loaded tree. From Remark 4.4, we know that this tree has value zero. Therefore, we can omit it. We only need to consider the first and the second loaded trees — denote them as $\overline{T_1}$ and $\underline{T_1}$, respectively.

First, consider $\overline{T_1}$. We pick e_3 to reduce and the proper quadruple set $\{3, 4, 1, 7\}$. Manifestly, we can only use vertex v_2' to split. After Algorithm 1, we obtain only one loaded tree T_1'' in the output set. This loaded tree is shown in Figure 13. It is a clever tree. Then let us consider $\underline{T_1}$. We pick edge e_3 to reduce and a proper quadruple set $\{3, 4, 5, 8\}$. We leave it to the reader to check the result — we should obtain only one clever tree as well in the output set. Then we union these two sets and obtain the output set for the reduction chain algorithm. Since we did two times recursion, the sign should be positive. Therefore, the value for tree T_1 in Figure 5 is 2.

In the above example, we already see that after the linear reduction, some loaded trees in the output set have value zero anyway. We can

(a) (b)

Figure 14: These are the two loaded trees that are obtained after an edge-cutting operation on loaded tree T in Figure 9. The loaded tree in Figure 14a has 6 labels and 3 edges. The labeling sets of the two vertices are $\{1, 2, 3\}$ and $\{4, 5, \star\}$, respectively. The loaded tree in Figure 14b has 3 labels and 0 edges. The labeling set of the only vertex is $\{\diamond, 6, 7\}$.

directly omit those trees since they do not contribute for the value we want to count. In this way, we can also gain some efficiency. *Balancing condition* is a necessary condition for a loaded tree to have value zero. In the next subsection, we introduce this condition. Thanks to the tree representation of the monomials, we can well-define this concept.

5.3 Balancing condition

Although those monomials/trees obtained after tree-version linear reduction algorithm are already precisely the ones that are on right hand side of the equation in algebraic expressions, we want to further improve the algorithm, so that there is more efficiency. We will introduce an edge-cutting lemma, which offers one more filter to judge whether a monomial has value zero — in addition to Keel's quadratic relation. And it is the first filter for the zeroness property of a loaded tree — though only a necessary one. Before this, we need to introduce the concept of *edge-cutting*.

Let T be a proper loaded tree with some weight-zero/single edge $e = \{v_1, v_2\}$. Then we can construct two new loaded trees T_1, T_2 by cutting off this weight-zero edge and adding one more label to the two incident vertices of this edge, respectively. We call this operation *edge-cutting operation* on edge e. Let us see an example on it.

Example 5.8. Consider the loaded tree T in Figure 9, we see that e_2 is a single edge. We can cut it off, add one extra label, say $\star$, to vertex v_2, and one extra label, say $\diamond$, to vertex v_3. Then we obtain two new loaded trees, see Figure 14.

In the above example, we see that after applying edge-cutting operation on a given proper loaded tree, we obtain two new loaded trees. And they are even proper. Naturally one may wonder, if this is always the case. So, yes, this operation keeps the property of being a loaded tree; but it does not keep the property of being proper.

Lemma 5.9. *Let T be a loaded tree and T_1 and T_2 are the two trees obtained from an edge-cutting operation executed on the edge e (whose corresponding cut is $\{I, J\}$) of T. Then, both T_1 and T_2 are loaded trees.*

Proof. T_1 is a loaded tree with labeling set $I \cup \{\star\}$, T_2 is a loaded tree with labeling set $J \cup \{\diamond\}$. The only thing that we need to verify is the third condition of Definition 2.1 for the two incident vertices of e in T_1 and

T_2, respectively. Actually, since we add one more label to v_1 (v_2) after removing edge e, the sum of degree and labeling set cardinality stays unchanged for v_1 (v_2). □

As for the properness, we can see a counter-example. Consider the rightmost tree in Figure 6. It is a loaded tree with 8 labels and 5 edges; evidently proper. We can cut the edge e'. Then it is not hard to check that we obtain a loaded tree with 5 labels and 1 edge; and another loaded tree with 5 labels and 3 edges. Both of them are not proper. Later on, with the help of this being "improper" property, we can derive the filter that is claimed at the beginning of this section.

Next, we introduce the main statement of the section. We call it "edge-cutting lemma", although it is stated as a proposition — since the author views it interesting on its own.

Proposition 5.10 (edge-cutting lemma)**.** *Let T be a loaded tree and T_1 and T_2 are the two trees obtained from an edge-cutting operation executed on edge $e = \{v_1, v_2\}$ of T. Then, we have that $|\int(T)| = |\int(T_1)| \cdot |\int(T_2)|$.*

Proof. Let $\{I, J\}$ be the corresponding cut of e and w.l.o.g. assume that v_1 is in Component-I of T and v_2 is in Component-J. Also, assume that the extra label added to v_1 is $\star$ and the one added to v_2 is $\diamond$.

Let $M = M_1 \cdot \delta_{I,J} \cdot M_2$ be the corresponding monomial of T, where M_1 contains all corresponding factors of edges in Component-I; M_2 contains all corresponding factors of edges in Component-J and $\delta_{I,J}$ is the corresponding factor for edge e. From the reduction chain algorithm, we know that after finitely many times (algebraic) linear reduction (applied to M), M_1 can be represented as a sum of, say m_1 many, clever monomials — maybe with a negative sign; so does M_2, say it is a sum of m_2 many clever monomials — maybe with a negative sign. Then, after an elementary expansion of the polynomial, M can be represented as a sum of $m_1 \cdot m_2$ many clever monomials (maybe with a negative sign). Formally, and algebraically, we will have

$$\begin{aligned} M &= (M_1 + M_2 + \cdots + M_{m_1}) \cdot \delta_{I,J} \cdot (M'_1 + M'_2 + \cdots + M'_{m_2}) \\ &= \sum_{1 \le i \le m_1, 1 \le j \le m_2} M_i \cdot \delta_{I,J} \cdot M'_j, \end{aligned}$$

after these finitely many reductions. From the reduction chain algorithm, we know that each summand in this equation is a clever monomial. And the absolute value of M (or T) should be $m_1 \cdot m_2$, i.e., $|\int(T)| = m_1 \cdot m_2$. Then, it is sufficient to show that $|\int(T_1)| = m_1$ and $|\int(T_2)| = m_2$.

Let $M = \overline{M_1} \cdot \delta_{I,J} \cdot \overline{M_2}$ be the monomial representation of any loaded tree $\overline{T}$ that is generated at some recursion round. Let $\overline{\delta}_{I_1,J_1}$ be any factor of $\overline{M_1}$ — let $\overline{e_1} = \{\overline{v_1}, \overline{v_2}\}$ be its corresponding edge. W.l.o.g., assume that $\overline{v_2}$ is in Component-J_1 of $\overline{T}$ and the unique path from $\overline{v_1}$ to edge e (corresponding edge of cut $\{I, J\}$, or factor $\delta_{I,J}$) is incident with $\overline{v_2}$. Then, from Keel's quadratic relation, we obtain that $J \subsetneq J_1$. Hence, all labels in J always appear together (as a subset of one part of the corresponding cut) in any factor in $\overline{M_1}$-part. Consequently, in each intermediate tree, for any vertex v in Component-I, all labels in J belong to the same cluster of v. Therefore, whenever we split some vertex in Component-I, there exists at

most one label from J in any proper quadruple. And this property remains true no matter which recursion step we are at during the reduction chain algorithm.

We replace each occurrence of J (as a subset of some cut, say J_1 of factor δ_{I_1,J_1}) in M_1 by $\star$, and denote the new monomial by $M_1^\star$. One can see that $M_1^\star$ is actually the monomial of T_1. Whenever a quadruple containing any label in J is chosen for the linear reduction, we only need to replace that label by $\star$; if the chosen quadruple does not contain any label in J, then we leave it unchanged. By this replacement, we see that the whole excution process of reduction chain algorithm induces an excution of the reduction chain algorithm on T_1. Parallelly, in algebraic view (language), it is the following process: We do this replacement for all factors through all those linear reductions, until the representation of M_1 is a sum of clever monomials (maybe with a negative sign). Then, we actually obtain a linear reduction series for $M_1^\star$. And we simultaneously obtain the representation of M_1' by a sum of clever monomials (maybe with a negative sign)! Therefore, we have that $|\int(T_1)| = m_1$ and $|\int(T_2)| = \ m_2$. □

Corollary 5.11. *Let T be a proper tree with a single edge e and T_1, T_2 be two loaded trees obtained from T via an edge-cutting operation on edge e. If T_1 is not proper, then $\int(T) = 0$.*

Proof. Assume that T has n labels and $n-3$ edges, T_1 has n_1 labels and k_1 edges, T_2 has n_2 labels and k_2 edges. It is not hard to obtain that $n_1 + n_2 = n + 2$ and $k_1 + k_2 = n - 4$. Therefore, we obtain that $n_1 + n_2 = k_1 + k_2 - 6$. Hence T_1 is proper if and only if T_2 is proper. Suppose w.l.o.g. that $k_1 > n_1 - 3$. Then we know that T_1 corresponds to a monomial in the Chow group $A^{k_1}(n_1)$. From the fact stated in Section 1 we have that $A^{k_1}(n_1) = \{0\}$ and hence $\int(T_1) = 0$. By Proposition 5.10, we obtain that

$$|\int(T)| = |\int(T_1)| \cdot |\int(T_2)| = 0 \cdot |\int(T_2)| = 0.$$

□

If both T_1 and T_2 in above stated process are proper, we say that T *is balanced or fulfills the balancing condition with respect to edge e*. A loaded tree is *balanced* if and only if it is balanced with respect to any single edge.

Actually, the edge-cutting operation is defined locally. If we do this operation on each single edge of loaded tree T, then no matter in which sequence we choose to cut those edges, we always obtain a same set of loaded trees. With the similar reasoning, edge-cutting lemma has a generalized version; so does Corollary 5.11. We conclude them in the following lemma, and we omit the proof.

Lemma 5.12. *Let T be a loaded tree and $T_1, \cdots, T_n$ are the trees obtained from edge-cutting operations on all single edges of T. Then, we have that $|\int(T)| = |\int(T_1)| \cdot |\int(T_2)| \cdots |\int(T_n)|$. And if T_i is not proper for any $1 \leq i \leq n$, $|\int(T)| = 0$.*

The next result holds consequenctly.

Theorem 5.13. *Unbalanced loaded trees have value zero.*

Proof. By Lemma 5.12, straightforward. □

Now let us look back on the vertex-splitting process. Whenever we split a vertex, one single edge e' is generated. Therefore, in the tree-version linear reduction algorithm, we can further filter the loaded trees in the output set — those that are not balanced w.r.t. edge e' can be directly removed from the set, since zero does not contribute anything to the value calculation. In this sence, we can always check the local balanceness (w.r.t. edge e') of the generated loaded tree. Furthermore, we can check the (overall) balanceness of the generated trees and omit the unbalanced ones. In this way, more efficiency is brought into the algorithm. See Algorithm 2 for an optimized version of Algorithm 1.

Algorithm 2: optimized tree-version linear reduction

input : a proper loaded tree T_M (corresponding loaded tree of the proper monomial M); a multiple edge $e = \{v_1, v_2\}$ (with corresponding cut $\{I, J\}$) of T_M; a proper quadruple set $\{i, j, k, l\}$ of e, or equivalently, of $\{I, J\}$.

output: *balanced* loaded trees whose corresponding monomials are the ones that survive after the linear reduction on M which uses Keel's linear reduction on the relation $\epsilon_{ij|kl} = \epsilon_{ik|jl}$

$w_1 \leftarrow$ weight of v_1;
$w_2 \leftarrow$ weight of v_2;
if $w_1 = 0$ *and* $w_2 = 0$ **then**
  return $\emptyset$
end if
$ST \leftarrow$ the set of all *balanced* loaded trees that can be obtained from T_M with edge e and the proper quadruple set $\{i, j, k, l\}$, after applying the vertex splitting algorithm — either by splitting vertex v_1, or by splitting vertex v_2 — in the set ST;
return ST

As an example, we consider the loaded tree T_1 in Figure 5. After the first recursion round of Algorithm 1, we obtain three loaded trees as shown in Figure 6. However, if we check the balanceness of them, we see that the rightmost tree is not balanced with respect to edge e'. Hence it is not balanced, has value zero. The output of optimized tree-version linear reduction algorithm will just contain the two loaded trees on the right for this round.

In some other cases, this condition can help filter out a lot of loaded trees. One may realize already that this condition can be much more complicated if we consider it for the monomials. But of course, the corresponding monomial of an unbalanced tree also has value zero. Hence this also shows the strength of the tree representation. Only when we consider the monomial reductions in the view point of loaded trees, can we make use of the balancing condition, and gain much more efficiency for the computing.

To integrate the balancing checking to reduction chain algorithm, we only need to substitute the occurrences of tree-version linear reduction algorithm by the *optimized* tree-version linear reduction algorithm. With this, we conclude this section. In the next section, all the undone proofs will be settled.

6 The missing proofs

In this section, we will settle down all the missing proofs that are postponed in the earlier sections. After going through the previous context, we collect in total four missing proofs:

1. Theorem 3.4.
2. Any proper quadruple fulfills the summand distinction property.
3. If a proper loaded tree has a multiple edge $e = \{v_1, v_2\}$ where the weights of v_1, v_2 are both zero, then the tree has value zero.
4. Correctness of the tree-version linear reduction algorithm.

We will first prove Theorem 3.4. The theorem can be restated as follows:

Theorem 6.1. *A clever tree has value one.*

Proof. First, we consider the situation in the Chow ring $A^*(3)$. This ring is exactly the Chow group $A^0(3)$, which is isomorphic to the integer additive group. There is only loaded tree T in this ring, which consists of a single vertex with three labels 1, 2 and 3. Hence the group $A^0(3)$ is generated by this single clever tree (or by the empty monomial). Combining with the fact that $A^0(3)$ is isomorphic to $\mathbb{Z}$, we obtain that the value of T is either 1 or -1. We choose it to be one. And note that T is the only clever tree that has no edges.

Let T_1 be any other clever tree — then it has at least one edge. Any edge of T_1 is a single edge. By the edge-cutting lemma, we can apply the edge-cutting operation on each of its edge, at the end obtaining $|V(T)|$ many clever trees with three labels, where $V(T)$ denotes the vertex set of T. Since renaming or permuting the labels does not influence the integral value of a loaded tree, we know that the value of T_1 is the product of $|V(T)|$ many 1, hence it has value one. □

Remark 6.2. Note that we can also choose -1 as the value of T. Then the value monomial in $A^{n-3}(n)$ is just the product of -1 and the value of it when we choose 1 for the value of T — since there are in total two automorphisms on the integer additive group.

Then let us focus on the algebraic reduction, try to prove the second item above. We will need some help from the tree representation of the (tree) monomial for the proof.

Theorem 6.3. *Let M be a proper non-clever tree monomial and $\delta_{I,J}$ the (power-higher-than-one) factor to be reduced, let $Q = \{i, j, k, l\}$ be a proper quadruple set such that $i, j \in I$, $k, l \in J$ hold. Then, no summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$ divides M.*

Proof. Assume that after the linear reduction, we obtain the representation of M as the negative sum (see Remark 3.8) of several tree monomials: $M = -\sum_{i=1}^{k} M_r$, where any M_r $(1 \leq r \leq k)$ is a monomial. Let M_r be any summand on the right hand side. Let $e = \{v_1, v_2\}$ be the corresponding multiple edge of cut $\{I, J\}$ — w.l.o.g. assume that v_1 is in Component-I — and $T_M = (V, E, h, m)$ the corresponding proper loaded tree for M.

It is obvious from the algebraic point of view that after the linear reduction, all the other factors except for $\delta_{I,J}$ stay unchanged, we only need to argue that the new factor $\delta_{I',J'}$ (corresponding to edge e') in M_r is distinct from all the factors in M. Since $\delta_{I',J'}$ is a summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$, we see that it is also distinct from $\delta_{I,J}$, by Fact 3.1. In the graphical language: we only need to argue that it cannot be the same edge with any other edges in tree T_M. Note that $\delta_{I',J'}$ is a summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$, hence we have w.l.o.g. $i, j \in I'$ and $k, l \in J'$.

When we do a proper choice of the quadruple set $\{i, j, k, l\}$, there are in total six different cases. Denote by $h(v) := h(v_1) \cup h(v_2)$.

1. $i, j \notin h(v)$, $k, l \notin h(v)$.

   Since i, j are from two distinct clusters, say I_1 and I_2, of v_1, we see that $\delta_{I',J'}$ cannot coincide with any edge in Component-I. Because of the symmetry of i, j and k, l in this case, $\delta_{I',J'}$ also cannot coincide with any edge in Component-J. Also, we see that $I_1, I_2 \subset I'$ and $I_1, I_2 \subset I$.

   Suppose that e' is not incident with e (in T_{M_r}), then there exists another edge e'' (with corresponding cut is $\{I'', J''\}$) on the path from e' to e. Note that e'' should also be an edge in T_M. Then we see that $I' \subsetneq I'' \subsetneq I$. Since there is no proper cluster containing both I_1, I_2 which is a proper subset of I in T_M, we obtained a contradiction. Therefore, e' is incident with e.

2. $i, j \notin h(v)$, $k \notin h(v)$.

   The same reasoning as in item 1. tells us that e' cannot coincide with any edge in Component-I. Denote by J_1 the cluster of k in T_M. We know that $J_1 \subset J'$ since $k, l \in J'$. Hence e' cannot be coincident with any edge in Component-J. A similar reasoning with item 1. proves that e' is incident with e.

3. $i, j \notin h(v)$, $k, l \in h(v)$. Since $k, l \in J'$ but not in any other cluster of v_2 (except for J), e' is distinct from all edges in T_M. A similar reasoning with item 1. proves that e' is incident with e.

4. $i, k \notin h(v)$, $j, l \in h(v)$.

5. $i \notin h(v)$, $j, k, l \in h(v)$.

6. $i, j, k, l \in h(v)$.

Proof for case 4., 5., 6. can be obtained analogously via the analysis of the first three items. We conclude that Q fulfills the summand distinction property. □

From the above proof, we actually gained more than needed. We can see that the set of labels (can be seen from the index of factors in M,

or M_r) stay unchanged. Also the number of factors in M is the same as that in M_r: only one occurrence of $\delta_{I,J}$ is replaced by $\delta_{I',J'}$. Hence T_M and T_{M_r} are loaded trees of the same type. The proper loaded tree T_{M_r} differs from T_M only in the following aspects:

1. Multiplicity of edge e is reduced by one.
2. A new edge e' (incident to e) with multiplicity one is generated.
3. All other edges stay unchanged: both the corresponding cut and the multiplicity stay unchanged.

In the sequel, we prove the correctness of tree-version linear reduction algorithm. We will keep using the notations from the above analysis.

Theorem 6.4. *Tree-version linear reduction algorithm is correct.*

Proof. First, we will show that any loaded tree T_{M_r} of some summand M_r (with respect to the Keel's linear relation $\epsilon_{ij|kl} = \epsilon_{il|kj} = \epsilon_{ik|jl}$) can be obtained by a vertex splitting process on T_M with respect to the proper quadruple set $\{i, j, k, l\}$.

In order to keep all the other edges (except for e) unchanged, it is necessary to have step 6. in vertex splitting process; otherwise, once the structure of any branch (of v_1) is changed, some edge (and corresponding cut) in this branch will change accordingly. In order to generate the new edge e' incident with e, it is necessary to have step 4. And in order to make sure that we obtain a loaded tree as well, it is necessary to have step 5. Since $i, j \in I \cap I'$ and $k, l \in J \cap J'$, the specific requirements in Step 5. and Step 6. are also necessary. From Remark 4.5 we already see that Step 3. is necessary. Hence, the vertex splitting process is necessary for obtaining a monomial on right hand side after one time linear reduction.

In the sequel, we prove that any loaded tree T_2 obtained by a vertex splitting process on T_1 with respect to the proper quadruple set $\{i, j, k, l\}$ corresponds to some monomial M_r in $M = -\sum_{i=1}^{k} M_r$ after an algebraic linear reduction using the Keel's linear relation $\epsilon_{ij|kl} = \epsilon_{il|kj} = \epsilon_{ik|jl}$, where $M = M_{T_1}$ is the corresponding monomial of input (proper) loaded tree T_1.

First, we see that all edges of T_2 — compare to T_1, the input of vertex splitting process — stay unchanged except for e; but a new edge e' incident to e is generated. In algebraic language: all factors stay unchanged (compare M_{T_1} with M_{T_2}) except for one occurrence of $\delta_{I,J}$, which is replaced by $\delta_{I',J'} \neq \delta_{I,J}$; we know that $i, j \in I \cap I'$ and $k, l \in J \cap J'$. Hence we see that $\delta_{I',J'}$ is a summand in $-(\epsilon_{ij|kl} - \delta_{I,J})$. Therefore, $M_{T_2} = M_r$ for some $1 \leq r \leq k$. □

From the correctness of tree-version linear reduction algorithm, we naturally obtain the following proposition.

Proposition 6.5. *If a proper loaded tree has a multiple edge $e = \{v_1, v_2\}$ where the weights of v_1, v_2 are both zero, then the tree has value zero.*

Proof. In this case, when we want to replace one occurence of the corresponding factor of e to do the linear reduction, we see that it is not possible to split any adjacent vertices of e via the vertex splitting process. Hence we obtain NULL via the vertex splitting process. By Theorem 6.4,

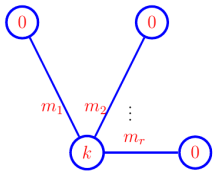


Figure 15: This is the weighted tree of the above described sun-like proper loaded tree, where the weights for vertices and edges are marked in red. Note that $m_i > 0$ for all $1 \leq i \leq r$ and that $k = \sum_{i=1}^{r} m_i$.

we know no momomial can survive the Keel's quadratic relation, i.e., all of them contain some pair of generators fulfilling Keel's quadratic relation. Hence all the monomials we obtain on right hand side after linear reduction have integral value zero. Therefore, the value of the given loaded tree is zero. □

7 An application: sun-like trees

In this section, we introduce a specific type of proper trees, and compute their values. This computation can also serve as the base case for the proof of forest algorithm correctness ([13, Theorem 0.5.]).

7.1 Problem

We say a proper loaded tree is *sun-like* if it has domination number equal to one — there exists a vertex v such that all other vertices are neighbors of it — and all adjacent vertices of v have weights zero, all edges have positive weights. We call this vertex v the *middle vertex*. Let k be the weight for the middle vertex and $m_1, \cdots, m_r$ the weights for its incident edges, respectively. By the weight identity for proper loaded trees, we know that $k = \sum_{i=1}^{r} m_i$. See Figure 15 for a visualization.

The **specific problem** that we want to handle in this section is: prove that the absolute value of a sun-like proper loaded tree — using the above notations — is $\binom{k}{m_1,\cdots,m_r}$. It can be formulated as follows:

Theorem 7.1. *Let $T = (V, E, h, m)$ be a sun-like proper loaded tree with v its middle vertex. Let $e_1, \cdots, e_r$ be the edges of T, where $e_i = \{v, v_i\}$ for $1 \leq i \leq r$. Denote by w the weight function for T, assume that $w(v) = k$ and $w(e_i) = m_i \geq 1$ for $1 \leq i \leq r$. Then we have $|\int(T)| = \binom{k}{m_1,\cdots,m_r}$.*

Before the proof, we need to introduce an identity on multinomial coefficient, which will help us in the proof later on.

7.2 An identity on multinomial coeffient

For any r-many positive-integer parameters $m_1, m_2, \cdots, m_r$, define $s := \sum_{i=1}^{r} m_i$. Denote a set of r-many indeterminates as

$$X := \{x_1, x_2, \cdots, x_r\}.$$

Define $T := \{B \mid B \subset X, x_1 \in B\}$ and

$$\mathcal{B} := \{(B_1, B_2) \mid B_1 \in T, B_2 = X \setminus B_1\}.$$

Define $g : X \to \{m_1 - 1, m_2, \cdots, m_r\}$ by $g(x_i) := m_1 - 1$ if $i = 1$ and $g(x_i) := m_i$ otherwise. Here note that the set $\{m_1 - 1, m_2, \cdots, m_r\}$ may be a multiset in form but we just consider it as a normal set, in practice. For convenience in the later writing, we introduce the following notation. Define for $B \subset X$,

$$S(B) := \sum_{x \in B} g(x), \quad \binom{S(B)}{B} := \frac{S(B)!}{\prod_{x \in B} (g(x)!)}.$$

Based on the above preparation, the identity we want to introduce can be formulated as follows.

Theorem 7.2.

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1, B_2) \in \mathcal{B}} \binom{s - r + 1}{S(B_2) - |B_2|} \cdot \binom{S(B_1)}{B_1} \cdot \binom{S(B_2)}{B_2},$$

where $|B_2|$ refers to the cardinality of B_2.

The proof for this identity is postponed to Section 7.5, in order not to distract our main rhythm. In the next section, we provide the proof of Theorem 7.1 using the above identity.

7.3 Proof for the value of sun-like tree

- **Base case:** We prove by induction on k. When $k = 0$, since it is proper and all weights of the edges are positive, we know that in this case the tree has no edge. By Theorem 6.1, we know that it has value 1, which coincides with the wanted formula since $0! = 1$.
- **Proof idea for the general case — apply Algorithm 2:** When $k \geq 1$, by Remark 3.8 and Algorithm 2, we know that the absolute vaue of T equals to the sum of the values of balanced loaded trees obtained in the output of Algorithm 2, when applied to T with a multiple edge and a proper quadruple. This indicates our proof idea: we choose a multiple edge and a proper quadruple, then consider what balanced loaded trees can be generated after the vertex-splitting process. Since in this process, one new weight-zero edge is generated and an old edge's weight is reduced by one, hence the edge weight sum, or equivalently the vertex weight sum, is reduced by one. For any tree in the output, we can cut-off the zero-weighted edge generated in the vertex-splitting process. Then for the obtained two loaded trees, we can apply the induction hypothesis.
- **Get prepared for the input data:** We choose edge e_1 to reduce. In order to choose a proper quadruple, we should first figure out how many labels does the middle vertex v have. By Definition 2.1 third condition, we have $k = \deg(v) + |h(v)| - 3$. Clearly $\deg(v) = r$ and we also have $k = \sum_{i=1}^{r} m_i \geq r$. Hence

$$|h(v)| = k - \deg(v) + 3 = k - r + 3 \geq 3.$$

Hence v has at least three labels, denote by i, j two of them. Since vertex v_1 has degree one and weight zero, it is not hard to reason that it has two labels, denote them by k, l. We see that $\{i, j, k, l\}$ is a proper quadruple set for the multiple edge e_1. And obviously we can only split vertex v, since v_1 has value zero. We will use as input these data for Algorithm 2.

- **Arranging the labels — part one:** Suppose that the statement is true for all cases when the middle vertex has weight less or equal to $k-1$, where $k \geq 1$, now let us consider the case when the middle vertex has weight k. Let $T_1 = (V_1, E_1, h_1, m_1)$ be any loaded tree in the output. We split vertex v into v' and v'', denote by e_1' the edge $\{v', v''\}$. By Step 5. of vertex-splitting, we know that $i, j \in h_1(v')$. Hence, we have the freedom on the arrangements of all other labels of v to be the labels of either v' or v''. How many choices do we have? The answer will be revealed a bit later.

- **Arranging the branches:** Also, we have the freedom for arranging the branches. Denote by B_1 the set of branches for v'' and B_2 for that of v'. Denote by $x_1, \cdots, x_r$ the branches corresponding to $e_1, \cdots, e_r$, respectively. Let $B := \{x_1, \cdots, x_r\}$, we see that $B = B_1 \cup B_2$. By Step 6. of vertex-splitting, we know that $x_1 \in B_1$, since this branch contains labels k and l. For each bipartition of B into B_1 and B_2 such that $x_1 \in B_1$, we need to consider the distribution of labels so that the obtained tree is balanced. For this, we need to introduce several notations, so as to express the arrangements of branches.

- **A formal model for the arrangements of branches:** Define a function $g : B \to \{m_1 - 1, m_2, \cdots, m_r\}$ by $x_1 \mapsto m_1 - 1$, $x_i \mapsto m_i$ for $2 \leq i \leq r$. Define $S(A) := \sum_{x \in A} g(x)$ and

$$\binom{S(A)}{A} := \frac{S(A)!}{\prod_{x \in A} (g(x)!)}.$$

  Let B_1 be any arrangement of branches for vertex v''. Then $B_2 = B \setminus B_1$.

- **Balancing condition considered:** Since we require loaded tree T_1 to be balanced, it is also balanced with respect to edge e_1'. After we cut-off the edge e_1', the tree containing v' should be proper, donote it by T_1'; so does the tree containing v'', denote it by T_1'' — the weight identity should hold for both trees. From this, we obtain that the weight of v' in T_1' is $S(B_2)$ and that the weight of v'' in T_1'' is $S(B_1)$. Since the edge-cutting operation does not change the weight of vertex, we know that $w_1(v') = S(B_2)$ and $w_1(v'') = S(B_1)$. And obviously, degree of v' is $|B_2| + 1$, while that of v'' is $|B_1| + 1$, since we need to count edge e_1' once also for their degree, respectively.

- **Arranging the labels — part two:**
  Now we can figure our how many labels should we distribute to v'. In T_1:

$$w_1(v') = \deg(v') + |h_1(v')| - 3.$$

Therefore, we have:

$$\begin{aligned}|h_1(v')| &= w_1(v') - \deg(v') + 3 \\ &= S(B_2) - (|B_2| + 1) + 3 \\ &= S(B_2) - |B_2| + 2\end{aligned}$$

However, we already know that $i, j \in h_1(v')$. Therefore, we should distribute $S(B_2) - |B_2|$ many labels of v to vertex v', in the vertex-splitting process. Then naturally the remaining labels went to the labeling set of vertex v''.

- **The induction step:** Then we cut off the single edge e_1' in T_1, obtaining two smaller proper trees T_1' and T_1'', since T_1 is balanced. By Remark 4.5, we have $w_1(v') + w_1(v'') = w(v) - 1$. Hence $0 \leq w_1(v') < w(v) = k$ and $0 \leq w_1(v'') < w(v) = k$. Therefore, we can use the induction hypothesis on T_1' and T_1'' if $m_1 - 1 \neq 0$.
- **Special case for the induction step:** When $m_1 - 1 = 0$, we can cut off edge e_1 in T_1'', then apply the induction hypothesis for the same weight for vertex v'' — since edge-cutting does not influence the weights of the two adjacent vertices of the cut-off edge. After this edge-cutting, we obtain a single vertex with three labels (the two of which are k and l) — this loaded tree has value one. Then by the edge-cutting lemma (Proposition 5.10), we see that the value of T_1'' equals to the value of the tree containing v'' after cutting off e_1. Then we can use the induction hypothesis on this smaller tree. However, since

$$\binom{S}{s_1, \cdots, s_p} = \binom{S}{0, s_1, \cdots, s_p}$$

for $S = \sum_{i=1}^{p} s_i$ and $s_i \in \mathbb{N}^+$, we see that our induction hypothesis can also apply in this case — the value of T_1'' is not influenced by whether $m_1 - 1$ is zero or not.

- **Express the value of each loaded tree in the output:** For each given B_1, among all the $k - r + 3$ labels of v, i, j are doomed to belong to v'. We should choose $S(B_2) - |B_2|$ many labels for v', from $k - r + 1$ many labels of v'. Then by the edge-cutting lemma, we know that for this arrangement, the value of the obtained balanced proper loaded tree is the product of the values of two smaller trees, after cutting off the edge e_1' — which is exactly $\binom{S(B_2)}{B_2} \cdot \binom{S(B_1)}{B_1}$.
- **Summing over these values:** That is to say, whenever the arrangement of the branches is fixed, because of the balancing condition requirement, the label distribution is also fixed. There are $\binom{k-r+1}{S(B_2)-|B_2|}$ many distributions of labels, up to the permutations of labels. Since permutation or renaming the labels does not influence the value of the loaded tree, we know that for any two such permutations, the two trees have the same value: $\binom{S(B_2)}{B_2} \cdot \binom{S(B_1)}{B_1}$ — the product of the values of the two smaller trees. Summing over the different arrangements of the branches, we obtain that the sum of

the values of all the trees in the output is precisely:

$$\sum_{(B_1,B_2)\in B} \binom{k-r+1}{S(B_2)-|B_2|} \cdot \binom{S(B_1)}{B_1} \cdot \binom{S(B_2)}{B_2}.$$

- **Show-time for the identity:** Hence the only thing that is needed for our proof is the following identity:

$$\sum_{(B_1,B_2)\in B} \binom{k-r+1}{S(B_2)-|B_2|} \cdot \binom{S(B_1)}{B_1} \cdot \binom{S(B_2)}{B_2} = \binom{k}{m_1,\cdots,m_r}.$$

  With Theorem 7.2, we conclude the proof.

7.4 Equivalent characterization

In the last section, we see that we can prove Theorem 7.1 using Theorem 7.2. In this section, we show that we can also prove the identity given that Theorem 7.1 holds.

Let us reconsider the proof steps in Section 7.3. All the steps until the last do not depend on the identity, therefore we can still use those analysis for the proof in this section. We keep using the notation from last section, suppose that Theorem 7.1 holds. Then we know that if we apply Algorithm 2 to T with multiple edge e_1 and corresponding quadruple set $\{i,j,k,l\}$ same as in the last section, the sum of the values of output loaded trees can be expressed as:

$$\sum_{(B_1,B_2)\in B} \binom{k-r+1}{S(B_2)-|B_2|} \cdot \binom{S(B_1)}{B_1} \cdot \binom{S(B_2)}{B_2}.$$

Correctness of Algorithm 2 tells us that the above sum equals to $\binom{k}{m_1,\cdots,m_r}$. That is to say, we have

$$\binom{k}{m_1,m_2,\cdots,m_r} = \sum_{(B_1,B_2)\in B} \binom{k-r+1}{S(B_2)-|B_2|} \cdot \binom{S(B_1)}{B_1} \cdot \binom{S(B_2)}{B_2}.$$

Hence, Theorem 7.1 is an equivalent characterization of the identity. However, in order to show their equivalence, Algorithm 2, or the vertex-splitting process plays the essential role. We believe that the identity indicates some complicated structural information of the vertex-splitting process, in an algebraic way. In the next section, we prove this identity.

7.5 Proof of the identity

Continuing with the notations in Section 7.2, we give a combinatorics proof for Theorem 7.2 in the sequel — which then also leads to the correctness of both Theorem 7.2 and Theorem 7.1.

Let us first recall the notations, note that we slightly modify the notations, so that they serve well for our proof — namely we add an index r for many of them, indicating that we are considering r many sums for

the multinomial coefficient. We will see later on that this index is helpful. We will also add some comments on why these notations were used in the identity, to show some original idea behind.

- $m_1, m_2, \cdots, m_r$: r-many positive-integer parameters.
- $s_r := \sum_{i=1}^{r} m_i$.
- $X_r := \{x_1, x_2, \cdots, x_r\}$: a set of r-many indeterminates. This set is introduced so that we can consider all the bipartitions of the values $\{m_1-1, m_2, \cdots, m_r\}$. In this way, we are able to formally go through all combinations.
- $T_r := \{B \mid B \subset X_r, x_1 \in B\}$. The elements in T_r indicates one part of the bipartition and we always put x_1 in it, so as to avoid repetition.
- $\mathcal{B}_r := \{(B_1, B_2) \mid B_1 \in T_r, B_2 = X_r \setminus B_1\}$. This set is exactly the collection of all the bipartition of X_r.
- $g_r : X_r \to \{m_1 - 1, m_2, \cdots, m_r\}$, $x_1 \mapsto m_1 - 1$, $x_i \mapsto m_i$ for $i \neq 1$. This function is introduced for the sake of the next two notations, mainly because the value for m_1 is reduced by one.
- $S(B) := \sum_{x \in B} g_r(x)$, for $B \subset X_r$. This is just the normal sum of m_i for $1 \leq i \leq r$, except that m_1 is replaced by $m_1 - 1$ as a summand — this is also why we need the function g_r.
- $$\binom{S(B)}{B} := \frac{S(B)!}{\prod_{x \in B} (g_r(x)!)},$$
  for $B \subset X_r$. Note that this notation is just a generalized definition of multinomial coefficient.

Besides these notations from earlier, we still need several more, so as to present our proof properly.

- Define
  $$S_r := \{(P_1, P_2, \cdots, P_r) \mid \cup_{i=1}^{r} P_i = \{1, 2, \cdots, s_r\},\ |P_i| = m_i\}.$$
  With this set, we collect all partitions of the set $\{1, 2, \cdots, s_r\}$ into r parts such that the i-th part has cardinality m_i.
- Let $L_r := \{2, 3, \cdots, r\}$. These elements are *special* elements in $\{1, 2, \cdots, s_r\}$. Later on we will see why or how they are special, in the definition of function φ_r. The next two notations are also there to serve the definition of function φ_r.
- For $A \subset \{1, 2, \cdots, r\}$, define $P_A := \cup_{i \in A} P_i$. P_A is the union of the part which has index in A.
- For $A \subset \{1, 2, \cdots, r\}$, define $X_A := \{x_i \mid i \in A\}$. X_A collect the indeterminate that has index in A.

Let us see an example, so that we do not get lost among the ocean of notations.

Example 7.3. Given $r = 3$, the following facts are already clear:

- $X_3 = \{x_1, x_2, x_3\}$.
- $T_3 = \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}\}$. This is the collection of one part of the bipartition of X_3 that contains x_1.
- $\mathcal{B}_3 = \{(\{x_1\}, \{x_2, x_3\}), (\{x_1, x_2\}, \{x_3\}), (\{x_1, x_3\}, \{x_2\}), (\{x_1, x_2, x_3\}, \emptyset)\}$. This is the collection of all bipartitions of X_3.
- $L_3 = \{2, 3\}$. The elements 2 and 3 are special.
- Take $A = \{1, 2\} \subset \{1, 2, 3\}$ for instance, then $X_A = \{x_1, x_2\}$ — the collection of indeterminate with index in A.

However, in order to figure out those remaining notations, we should know the values of m_i for $1 \leq i \leq r$. Let $m_1 = 2$, $m_2 = 2$ and $m_3 = 1$ for instance, then we also obtain the following facts:

- $s_3 = \sum_{i=1}^{3} m_i = 2+2+1 = 5$. Now we know that $2, 3$ are considered special among $1, 2, 3, 4, 5$.
- $g_3 : X_3 \to \{1, 2\}$ is defined as $g_3(x_1) = m_1 - 1 = 2 - 1 = 1$, $g_3(x_2) = m_2 = 2$ and $g_3(x_3) = m_3 = 1$.
- Take $B = \{x_2, x_3\} \subset X_3$ for instance, then

  $$S(B) = g_3(x_2) + g_3(x_3) = m_2 + m_3 = 2 + 1 = 3.$$

  Note that in this case $S(B)$ is just the sum of m_2 and m_3, since $x_1 \notin B$.
- Take $B = \{x_2, x_3\} \subset X_3$ for instance, then

  $$\binom{S(B)}{B} = \frac{S(B)!}{\prod_{x \in B} (g_3(x)!)} = \frac{3!}{g_3(x_2) \cdot g_3(x_3)} = \frac{6}{m_2 \cdot m_3} = \frac{6}{2 \cdot 1} = 3.$$

  Note that in this case, $\binom{S(B)}{B}$ is just the normal multinomial coeffienct $\binom{m_2+m_3}{m_2, m_3}$, since $x_1 \notin B$.
- S_3 is the set of all partitions (P_1, P_2, P_3) of the set $\{1, 2, 3, 4, 5\}$ into three parts P_1, P_2, P_3 such that $|P_1| = m_1 = 2$, $|P_2| = m_2 = 2$ and $|P_3| = m_3 = 1$.
- Take $A = \{1, 2\} \subset \{1, 2, 3\}$ for instance, then $P_A = P_1 \cup P_2$ for some $(P_1, P_2, P_3) \in S_3$.

We leave it to the readers to check that the identity holds in this example, which may help to have a taste of the identity.

Now we define a function $\varphi_r : S_r \to T_r$, $(P_1, \cdots, P_r) \mapsto B$ by Algorithm 3. We will proof that it is indeed an algorithm later on.

For a better understanding, let us see how is this function defined in our running example.

Example 7.4. $\varphi_3 : S_3 \to T_3$, $(P_1, P_2, P_3) \mapsto B \in T_3$. Let us go through Algorithm 3 with the input $(P_1, P_2, P_3) = (\{1, 3\}, \{4, 5\}, \{2\})$.

1. Input: $(P_1, P_2, P_3) = (\{1, 3\}, \{4, 5\}, \{2\})$.
2. Initial values: $B = \{x_1\}$, $A_0 = \{1\}$, $i = 1$.

Algorithm 3: function φ_r

input : $(P_1, \cdots, P_r) \in S_r$.
output: $B \in T_r$.
$B \leftarrow \{x_1\}$;
$A_0 \leftarrow \{1\}$;
$i \leftarrow 1$;
for $i = 1$ **do**
  $A_i := L_r \cap P_{A_{i-1}}$;
  if $A_i = \emptyset$ **then**
    return B
  end if
  else
    $B = B \cup X_{A_i}$;
    $i = i + 1$
  end if
end for

3. Recall that $L_3 = \{2, 3\}$. First loop:

$$A_1 = L_3 \cap P_{A_0} = L_3 \cap P_1 = \{2, 3\} \cap \{1, 3\} = \{3\} \neq \emptyset.$$

Hence

$$B = \{x_1\} \cup X_{A_1} = \{x_1\} \cup \{x_3\} = \{x_1, x_3\},$$
$$i = 1 + 1 = 2.$$

4. Second loop:

$$A_2 = L_3 \cap P_{A_1} = \{2, 3\} \cap P_3 = \{2, 3\} \cap \{2\} = \{2\} \neq \emptyset.$$

Hence

$$B = \{x_1, x_3\} \cup \{x_2\} = \{x_1, x_2, x_3\},$$
$$i = 2 + 1 = 3.$$

5. Third loop:

$$A_3 = \{2, 3\} \cap P_{A_2} = \{2, 3\} \cap P_2 = \{2, 3\} \cap \{4, 5\} = \emptyset.$$

Return $B = \{x_1, x_2, x_3\}$.

6. Output: $B = \{x_1, x_2, x_3\}$.

Proposition 7.5. *In the above defined process (Algorithm 3), $A_i \cap A_j = \emptyset$ for all $i \neq j$.*

Proof. When $i = 0$, $j \neq 0$, we have $A_0 \cap A_j = \emptyset$ since $A_j \subset L_r$, $A_0 = \{1\}$ and $1 \notin L_r$. Suppose $A_i \cap A_j \neq \emptyset$ when $i, j > 0$. W.l.o.g., assume $i < j$, since $A_i := L_r \cap P_{A_{i-1}}$, we obtain that $L_r \cap P_{A_{i-1}} \cap P_{A_{j-1}} \neq \emptyset$, hence $A_{i-1} \cap A_{j-1} \neq \emptyset$. Repeating the similar process, after finite steps we reach a situation where $A_0 \cap A_{j-i} \neq \emptyset$. This is a contradiction and we see this from the beginning part of this proof. □

Remark 7.6. Since $|L_r| < \infty$, $A_i \subset L_r$ for all $1 \leq i \leq r$, and $A_i \cap A_j = \emptyset$ for all $i \neq j$, there must exist $i \in \mathbb{N}^+$ such that $A_i = \emptyset$. Therefore, this process terminates. And it is well-defined — once the input is given, the output is uniquely determined via the process and clearly $B \in T_r$ — thence it is indeed an algorithm.

Proposition 7.7. *Function $\varphi_r : S_r \to T_r$ is a surjection.*

Proof. For any $B \in T_r$, define $(P_1, \cdots, P_r)$ as follows:

- Define $Q := \{i \mid i \in L_r,\ x_i \in B\}$. Since $|Q| < \infty$, we can list its elements as: $q_1, \cdots, q_t$, assuming that $|Q| = t$.
- Let $P_1 := \{q_1\}$, $P_{q_j} = \{q_{j+1}\}$ for $1 \leq j \leq t-1$.
- Now, we already defined all P_i when $i \in L_r$ and $x_i \in B$ except for P_t, and we have defined P_1 as well.
- Let $P_i = \{i\}$ if $i \in L_r$ but $x_i \notin B$. Let $P_t := \{1, \cdots, s_r\} \setminus (\cup_{i \neq t} P_i)$.

It is clear that $1 \in P_t$, hence all parts defined above are non-empty. Also, it is not hard to see that they are indeed a partition of $\{1, \cdots, s_r\}$ into r parts. Hence the input is well-defined. Furthermore, one can check that $\varphi_r(P_1, \cdots, P_r) = B$. □

We see that φ_r is a well-defined surjective function. Then, using a basic property of any mapping, we obtain

$$\cup_{B \in T_r} \varphi_r^{-1}(B) = S_r$$

and

$$|S_r| = \sum_{B \in T_r} |\varphi_r^{-1}(B)| = \sum_{(B, X \setminus B) \in \mathcal{B}_r} |\varphi_r^{-1}(B)|.$$

In order to prove the identity, we only need to show one thing and it is formulated as the following lemma.

Lemma 7.8. *For any $B_1 \in T_r$, define $B_2 := X_r \setminus B_1$, then*

$$|\{x \in S_r \mid \varphi_r(x) = B_1\}| = \binom{s_r - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2}.$$

In order to prove the above lemma, we need to introduce the following proposition.

Proposition 7.9. *If $\varphi_r(P_1, \cdots, P_r) = B_1$ for some $(P_1, \cdots, P_r) \in S_r$ and $B_1 \in T_r$; denote $B_2 := X_r \setminus B_1$. Then $P_{F_{B_1}} \cap L_r = F_{B_1} \setminus \{1\}$, where $F_B := \{i \mid x_i \in B\}$. Consequently, we have $P_{F_{B_2}} \cap L_r = F_{B_2}$ and $|P_{F_{B_2}} \cap L_r| = |B_2|$.*

Proof. From Remark 7.6 we know that there exists $k \in \mathbb{N}^+$ such that $A_k = \emptyset$. Let $t := k - 1$, we claim that $\cup_{i=0}^t A_i = F_{B_1}$. It is clear from Algorithm 3 that $B_1 = \cup_{i=0}^t X_{A_i}$, which is equivalent to $F_{B_1} = \cup_{i=0}^t A_i$. Hence we know that $F_{B_1} \setminus \{1\} = \cup_{i=1}^t A_i$. We only need to show $P_{F_{B_1}} \cap L_r = \cup_{i=1}^t A_i$.

For any $m \in \cup_{i=1}^t A_i$, there exists $1 \leq j \leq t$ such that $m \in A_j$. From Algorithm 3 we know that $A_j := L_r \cap P_{A_{j-1}}$. Hence $m \in L_r$. And $X_{A_{j-1}} \subset B_1$ is equivalent to $A_{j-1} \subset F_{B_1}$, which implies $P_{A_{j-1}} \subset P_{F_{B_1}}$.

Hence $m \in P_{A_{j-1}} \subset P_{F_{B_1}}$. We obtain that $m \in P_{F_{B_1}} \cap L_r$. For any $m \in P_{F_{B_1}} \cap L_r$, equivalently we have $m \in \cup_{i=0}^{t} A_i \cap L_r = \cup_{i=1}^{t} A_i \cap L_r$. Since $A_i \subset L_r$ for any $i \neq 1$. We obtain that $m \in \cup_{i=1}^{t} A_i$.

Since $P_{F_{B_1}} \cup P_{F_{B_2}} = \{1, \cdots, s_r\}$ and $L_r \subset \{1, \cdots, s_r\}$, we know that $(P_{F_{B_1}} \cap L_r) \cup (P_{F_{B_1}} \cap L_r) = L_r$. Therefore

$$P_{F_{B_2}} \cap L_r = L_r \setminus (P_{F_{B_1}} \cap L_r) = L_r \setminus (F_{B_1} \setminus \{1\}) = F_{B_2}.$$

Then, $|P_{F_{B_2}} \cap L_r| = |B_2|$ follows naturally, since $|F_{B_2}| = |B_2|$. □

Given $B_1 \in T_r$, by Proposition 7.9, we know that $P_{F_{B_1}} \cap L_r = F_{B_1} \setminus \{1\}$, hence the special elements in $P_{F_{B_1}}$ are fixed — so do the special elements in $P_{F_{B_2}}$ since $P_{F_{B_2}} \cap L_r = F_{B_2}$, where $B_2 := X_r \setminus B_1$. Let $K_r := \{1, \cdots, s_r\}$. Also, it is evident that $P_{F_{B_1}} \cup P_{F_{B_2}} = K_r$. Inspired by Proposition 7.9, in order to figure out what configurations in S_r have value B_1 under φ_r for some given $B_1 \in T_r$, we view the problem in the following way.

Given $B_1 \in T_r$, we know that the special elements in $P_{F_{B_1}}$ are fixed. We only need to choose a proper amount of elements in $K_r \setminus L_r$ and put them into $P_{F_{B_1}}$. We call the elements in $K_r \setminus L_r$ *non-special*. We need to choose $|P_{F_{B_1}}| - (|B_1| - 1) = S(B_1) - |B_1| + 1$ many elements from

$$|K_r \setminus L_r| = s_r - |L_r| = s_r - r + 1$$

many elements, and put them in the group of $P_{F_{B_1}}$. There are

$$\binom{s_r - r + 1}{S(B_1) - |B_1| + 1}$$

many ways to do so. Since $(S(B_1) - |B_1| + 1) + (S(B_2) - |B_2|) = (S(B_1) + S(B_2)) - (|B_1| + |B_2|) + 1 = s_r - r + 1$, we can also say that there are

$$\binom{s_r - r + 1}{S(B_2) - |B_2|}$$

many ways to arrange the non-special elements. Considering the definition of φ_r, we see that no matter how we arrange the elements in $P_{F_{B_2}}$, the value of φ_r is not influenced. Therefore, there are $\binom{S(B_2)}{B_2}$ many configurations for the elements in $P_{F_{B_2}}$. As for the arrangements in $P_{F_{B_1}}$, they need to obey certain rules in order to guarantee that the value of φ_r is B_1.

From the analysis above, the first and third coefficients in the equation in Lemma 7.8 are both explained well in a combinatorics way. In order to prove this lemma, we only need to prove that given $B_1 \in T_r$, the number of configurations for the elements in $P_{F_{B_1}}$ is exactly $\binom{S(B_1)}{B_1}$. Recalling the definition of $\binom{S(B_1)}{B_1}$, one can see that it is equivalent to proving the following proposition.

Proposition 7.10. *$f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}$, $k \in \mathbb{N}^+$, $m_i \in \mathbb{N}^+$, where $f_k : (\mathbb{N}^+)^k \to \mathbb{N}$,*

$$(m_1, m_2, \cdots, m_k) \mapsto |\{(P_1, P_2, \cdots, P_k) \in S_k \mid \varphi_k(P_1, P_2, \cdots, P_k) = X_k\}|.$$

Before we approach the proof, we need to introduce a known identity on multinomial coefficients and we will need it in later.

Lemma 7.11. *For all $n, m, k_1, \cdots, k_m \in \mathbb{N}$ with $k_1 + \cdots + k_m = n$, $n \geq 1$ and $m \geq 2$, we have*

$$\binom{n}{k_1, \cdots, k_m} = \sum_{i=1}^{m} \binom{n-1}{k_1, \cdots, k_i - 1, \cdots, k_m}.$$

Proof of Proposition 7.10. Prove by induction. When $k = 1$, $L_1 = \emptyset$, for any $m_1 \in \mathbb{N}^+$, we have

$$|\{(P_1) \in S_1 \mid \varphi_1((P_1)) = \{x_1\}\}| = 1 = \binom{s_1 - 1}{m_1 - 1}$$

since $s_1 = m_1$ in this case. Assume that the proposition holds whenever the number of parameters is less or equal to $k - 1$, where $k \geq 2$.

When the number of parameters is k, we start the inner induction on s_k. Obviously $s_k \geq k$. When $s_k = k$, we know that

$$m_1 = m_2 = \cdots = m_k = 1.$$

In the configurations that is mapped to X_k under φ_k, we can choose any element in L_k for P_1, say i_1; there are $|L_k| = k - 1$ many possibilities. Then we can choose an element in $L_k \setminus \{i_1\}$ for P_{i_1}, and so on. Until we choose the element $i_{k-1} \in L_k$ for $P_{i_{k-2}}$. Then we already arranged $k - 2$ many parts, then the only remaining part $P_{i_{k-1}}$ can only be $\{1\}$. In total there are $(k - 1)!$ many configurations. Hence we have

$$f_k(m_1, m_2, \cdots, m_k) = (k-1)! = \binom{k}{1, \cdots, 1} = \binom{1}{0, 1, \cdots, 1},$$

which equals to

$$\binom{s_k}{m_1 - 1, m_2, \cdots, m_k}.$$

Assume that the proposition holds whenever the sum of these parameters is less or equal to $s_k - 1$, where we can assume $s_k - 1 \geq k$, i.e., $s_k \geq k + 1$. When the sum of these parameters equals s_k, left hand side of the proposition is $f_k(m_1, m_2, \cdots, m_k)$, by definition we want to count the number of configuration that is mapped to X_k under φ_k. We focus on the distribution of the element $1 \in K_k$. Since $1 \notin L_k$, it does not influence the value of φ_k on any configuration. So in the case when $m_i \geq 2$ for all $1 \leq i \leq k$, there are k-many cases for the distribution of 1, i.e., it can belong to any part P_i for $1 \leq i \leq k$. Hence in this case we obtain the following identity:

$$\begin{aligned} f_k(m_1, m_2, \cdots, m_k) = {} & f_k(m_1 - 1, m_2, \cdots, m_k) + f_k(m_1, m_2 - 1, \cdots, m_k) \\ & + \cdots + f_k(m_1, m_2, \cdots, m_k - 1). \end{aligned}$$

Now we can apply the induction hypothesis on the sum of the parameters. Then we obtain the following equation:

$$f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 2}{m_1 - 2, m_2, \cdots, m_k}$$
$$+ \binom{s_k - 2}{m_1 - 1, m_2 - 1, \cdots, m_k} + \cdots + \binom{s_k - 2}{m_1 - 1, m_2, \cdots, m_k - 1}.$$

Then by Lemma 7.11, we obtain

$$f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}.$$

If $m_i = 1$ for some $i \neq 1$. When we put 1 into P_i, the problem can be reduced to counting the number of corresponding configurations of P_j for $j \neq i$, since $1 \notin L_k$. Therefore, in this case, considering the distribution of the element 1 gives us the following identity:

$$\begin{aligned} f_k(m_1, \cdots, m_k) = f_k(&m_1 - 1, \cdots, m_k) + \\ &\cdots + f_k(m_1, \cdots, m_{i-1} - 1, m_i, \cdots, m_k) \\ &+ f_{k-1}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k) \\ &+ f_k(m_1, \cdots, m_i, m_{i+1} - 1, \cdots, m_k) + \\ &\cdots + f_k(m_1, \cdots, m_{i-1}, m_i, m_{i+1}, \cdots, m_k - 1). \end{aligned}$$

By induction hypothesis on k we obtain

$$\begin{aligned} &f_{k-1}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k) \\ &\quad = \binom{(s_k - m_i) - 1}{m_1 - 1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k} \\ &\quad = \binom{(s_k - 1) - 1}{m_1 - 1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k} \\ &\quad = \binom{s_k - 2}{m_1 - 1, \cdots, m_{i-1}, 0, m_{i+1}, \cdots, m_k} \\ &\quad = \binom{s_k - 2}{m_1 - 1, \cdots, m_{i-1}, m_i - 1, m_{i+1}, \cdots, m_k}. \end{aligned}$$

Substituting back this term, we get the same recurrence for $f_k(m_1, \cdots, m_k)$ as in the case where $m_i \geq 2$ for $1 \leq i \leq k$. By Lemma 7.11 we as well obtain

$$f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}.$$

With the same idea, it is not hard to prove that the statement holds however many parameters except for m_1 equals one.

If $m_1 = 1$, from the definition of function f_k and φ_k, we know that $1 \notin P_1$. Hence considering the distribution of the element 1, the recurrence

formula becomes

$$\begin{aligned} f_r(m_1, m_2, \cdots, m_r) =& f_r(m_1, m_2 - 1, \cdots, m_r) + \\ & \cdots + f_r(m_1, m_2, \cdots, m_r - 1). \end{aligned}$$

Then by induction hypothesis on the sum of the parameters, we obtain

$$\begin{aligned} & f_k(m_1, m_2, \cdots, m_k) \\ &= \binom{s_k - 2}{m_1 - 1, m_2 - 1, \cdots, m_k} + \cdots + \binom{s_k - 2}{m_1 - 1, m_2, \cdots, m_k - 1} \\ &= \binom{s_k - 2}{0, m_2 - 1, \cdots, m_k} + \cdots + \binom{s_k - 2}{0, m_2, \cdots, m_k - 1} \\ &= \binom{s_k - 2}{m_2 - 1, \cdots, m_k} + \cdots + \binom{s_k - 2}{m_2, \cdots, m_k - 1}. \end{aligned}$$

Now we can apply Lemma 7.11 and then obtain

$$\begin{aligned} f_k(m_1, m_2, \cdots, m_k) &= \binom{s_k - 1}{m_2, \cdots, m_k} \\ &= \binom{s_k - 1}{0, m_2, \cdots, m_k} \\ &= \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}. \end{aligned}$$

By induction, the proposition holds. □

With this, we finished proving Theorem 7.2.

8 Conclusion

In this paper, we introduce an equivalent characterization of the algebraic reduction algorithm for computing the integral value of monomials in the Chow group of zero cycles in the moduli space of stable curves, in a graphical view. This characterization naturally can serve as an algorithm, which is also more efficient in many cases, compare to the algebraic one. Furthermore, making use of this tree-representation, we introduce the balancing condition, which brings much more efficiency to the algorithm and leads to an optimized version of the calculus.

Then, we introduce the application of our algorithm on a specific type of monomials. With the help of our main result, the equivalence between an identity on multinomial coefficient and the value of that specific type of monomials is shown, which further reveals the potential strength of our result. Also, the identity indicates some structrual information of the tree-based algorithm, in an algebraic way. In the end, we give the proof of that identity, which completes the story.

9 Acknowledgement

The research was funded by the Austrian Science Fund (FWF): W1214-N15, project DK9.

I sincerely thank Josef Schicho for suggesting me to consider the Laman graph realization counting problem, which then leads to the whole journey on this path. Also, I am truly grateful to Josef Schicho for introducing to me the background knowledge of the problem and the basic properties of the ambient ring etc., providing me with the result of Theorem 3.4, and inspiring me with the idea of adding two extra labels to the adjacent vertices of the edge during the edge-cutting operation — which is the essential point for passing on the properness property to the generated trees.

I am genuinely grateful to Cristian-Silviu Radu for providing me with the rather detailed suggestions on how to formulate the whole algorithm properly, step by step — which to a great extend leads to the completion of this manuscript. Also, I thank Cristian-Silviu Radu for the suggestion on adding on the reduction chain algorithm, so as to give a complete algorithm; this makes the write-up more elegant and integral. Last but not the least, I genuinely thank Cristian-Silviu Radu for helping me formulate the identity and as well its proof in a mathematically proper way.

I am very much grateful to Matteo Gallet for inspiring me with the viewpoint on how to formulate a good introduction — which to a great extend improve the quality of the first section. Also, I sincerely thank Matteo Gallet for answering to me the general questions in intersection theory or specifically about the ambient ring, offering me quite some valuable and detailed suggestions for general mathematical writings.

The birth of Section 7.4 — which not only enhances our comprehension of the main result of the paper but also makes the last section more complete and elegant — was inspired by another discussion with Dongsheng Wu, I thank him a lot for it. Also, I am sincerely thankful to Dongsheng Wu for the instructive discussions on the proof of the identity, specifically for providing me with the main idea of the proof of Proposition 7.10.

References

[1] Fulton William. Intersection theory. *Springer Science & Business Media*, Volume 2, 2013.

[2] Eisenbud David and Harris Joe. 3264 and all that: A second course in algebraic geometry. *Cambridge University Press*, 2016.

[3] Knudsen Finn, and Mumford David. The Projectivity of the moduli space of stable curves I: Preliminaries on "det" and "Div". *Mathematica Scandinavica*, 39.1 (1977), 19-55.

[4] Knudsen Finn F. The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$. *Mathematica Scandinavica*, 52.2 (1983), 161-199.

[5] Knudsen Finn F. The Projectivity of the Moduli Space of Stable Curves, III: The Line Bundles on $M_{g,n}$, and a proof of the Projectivity of $M_{g,n}$ in Characteristic 0. *Mathematica Scandinavica* (1983), 200-212.

[6] Keel Sean. Intersection theory of moduli space of stable n-pointed vurves of genus zero. *Transaction of the American Mathematical Society*, **330** (1992), no. 2, 545-574.

[7] Mumford David. Towards an enumerative geometry of the moduli space of curves. *Arithmetic and geometry*, Birkhuser, Boston, MA, 1983. 271-328.

[8] Faber Carel. Chow Rings of Moduli Spaces of Curves I: The Chow Ring of $\overline{\mathcal{M}_3}$. *Annals of Mathematics* (1990): 331-419.

[9] Faber Carel. Chow rings of moduli spaces of curves II: Some results on the Chow ring of $\overline{\mathcal{M}_4}$. *Annals of Mathematics* (1990): 421-449.

[10] Izadi Elham. The Chow ring of the moduli space of curves of genus 5. *The moduli space of curves*, pp. 267-303. Birkhuser Boston, 1995.

[11] Tavakol Mehdi. The Chow ring of the moduli space of curves of genus zero. *Journal of Pure and Applied Algebra*, Volume 221, Issue 4, April 2017, Page 757-772.

[12] Gallet Matteo, Grassegger Georg, Schicho Josef. Counting realizations of Laman graphs on the sphere. *The Electronic Journal of Combinatorics*, Volume 27, Issue 2 (2020).

[13] Qi Jiayue. A calculus for monomials in Chow group of zero cycles in the moduli space of stable curves. *DK-Report*, No. 2020-11, September 2020.

[14] Qi Jiayue, Schicho Josef. Five Equivalent Ways to Describe Phylogenetic Trees. *arXiv:2011.11774*, preprint.

Technical Reports of the Doctoral Program "Computational Mathematics"

2021

2021-01 J. Qi: *A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero* Jan 2021. Eds.: S. Radu, J. Schicho

2020

2020-01 N. Smoot: *A Single-Variable Proof of the Omega SPT Congruence Family Over Powers of 5* Feb 2020. Eds.: P. Paule, S. Radu

2020-02 A. Schafelner, P.S. Vassilevski: *Numerical Results for Adaptive (Negative Norm) Constrained First Order System Least Squares Formulations* March 2020. Eds.: U. Langer, V. Pillwein

2020-03 U. Langer, A. Schafelner: *Adaptive space-time finite element methods for non-autonomous parabolic problems with distributional sources* March 2020. Eds.: B. Jüttler, V. Pillwein

2020-04 A. Giust, B. Jüttler, A. Mantzaflaris: *Local (T)HB-spline projectors via restricted hierarchical spline fitting* March 2020. Eds.: U. Langer, V. Pillwein

2020-05 K. Banerjee, M. Ghosh Dastidar: *Hook Type Tableaux and Partition Identities* June 2020. Eds.: P. Paule, S. Radu

2020-06 A. Bostan, F. Chyzak, A. Jiménez-Pastor, P. Lairez: *The Sage Package* `comb_walks` *for Walks in the Quarter Plane* June 2020. Eds.: M. Kauers, V. Pillwein

2020-07 A. Meddah: *A stochastic multiscale mathematical model for low grade Glioma spread* June 2020. Eds.: E. Buckwar, V. Pillwein

2020-08 M. Ouafoudi: *A Mathematical Description for Taste Perception Using Stochastic Leaky Integrate-and-Fire Model* June 2020. Eds.: E. Buckwar, V. Pillwein

2020-09 A. Bostan, A. Jiménez-Pastor: *On the exponential generating function of labelled trees* July 2020. Eds.: M. Kauers, V. Pillwein

2020-10 J, Forcan, J. Qi: *How fast can Dominator win in the Maker-Breaker domination game?* July 2020. Eds.: V. Pillwein, J. Schicho

2020-11 J. Qi: *A calculus for monomials in Chow group of zero cycles in the moduli space of stable curves* Sept 2020. Eds.: P. Paule, J. Schicho

2020-12 J. Qi, J. Schicho: *Five Equivalent Ways to Describe a Phylogenetic Tree* November 2020. Eds.: M. Kauers, S. Radu

The complete list since 2009 can be found at
`https://www.dk-compmath.jku.at/publications/`

Doctoral Program
"Computational Mathematics"

Director:

Assoc. Prof. Dr. Veronika Pillwein
Research Institute for Symbolic Computation

Deputy Director:

Prof. Dr. Bert Jüttler
Institute of Applied Geometry

Address:

Johannes Kepler University Linz
Doctoral Program "Computational Mathematics"
Altenbergerstr. 69
A-4040 Linz
Austria
Tel.: ++43 732-2468-6840

E-Mail:

office@dk-compmath.jku.at

Homepage:

http://www.dk-compmath.jku.at

Submissions to the DK-Report Series are sent to two members of the Editorial Board who communicate their decision to the Managing Editor.