Comparative asymptotics for discrete semiclassical orthogonal polynomials

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Abstract

We study the ratio \( \frac{P_n(x; z)}{\phi_n(x)} \) asymptotically as \( n \to \infty \), where the polynomials \( P_n(x; z) \) are orthogonal with respect to a discrete linear functional and \( \phi_n(x) \) denote the falling factorial polynomials.

We give recurrences that allow the computation of high order asymptotic expansions of \( P_n(x; z) \) and give examples for most discrete semiclassical polynomials of class \( s \leq 2 \).

We show several plots illustrating the accuracy of our results.

Keywords: Semiclassical orthogonal polynomials, asymptotic expansions, ordinary differential equations.

Subject Classification Codes: 41A60 (primary), 33C47, 34E05 (secondary).

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1 Introduction

Let \( \mathbb{K} \) be a commutative ring (for our purposes, we mostly think of \( \mathbb{K} \) as the set of complex numbers \( \mathbb{C} \)) and \( \mathbb{N}_0 \) be the set of nonnegative integers

\[
\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.
\]

We will denote by \( \delta_{k,n} \) the Kronecker delta, defined by

\[
\delta_{k,n} = \begin{cases} 
1, & k = n \\
0, & k \neq n
\end{cases}, \quad k, n \in \mathbb{N}_0,
\]

and let \( \mathbb{F} \) be the ring of \textit{formal power series} in the variable \( z \)

\[
\mathbb{F} = \mathbb{K}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : c_n \in \mathbb{K} \right\}.
\]

We consider the differential operator \( \vartheta : \mathbb{F} \to \mathbb{F} \) defined by [37, 16.8.2]

\[
\vartheta = z \partial_z,
\]

where \( \partial_z \) is the \textit{derivative operator}

\[
\partial_z = \frac{\partial}{\partial z}.
\]

The action of \( \vartheta \) on the monomials is given by

\[
\vartheta^k [z^x] = x^k z^x,
\]

where we always assume that \( x \) and \( z \) are \textit{independent variables}.

Suppose that \( L : \mathbb{F}[x] \to \mathbb{F} \) is a \textit{linear functional} (acting on the variable \( x \)), and \( \{\Lambda_n(x)\}_{n \geq 0} \subset \mathbb{K}[x] \) is a sequence of \textit{monic polynomials} with \( \deg(\Lambda_n) = n \). If the system of linear equations

\[
L[\Lambda_k \Lambda_n] + \sum_{i=0}^{n-1} L[\Lambda_k \Lambda_i] \xi_{n,i} = 0, \quad 0 \leq k \leq n - 1,
\]

has a \textit{unique solution} \( \{\xi_{n,i}(z)\}_{0 \leq i \leq n-1} \subset \mathbb{F} \), we can define \textit{monic polynomials} \( P_n(x;z) \) by \( P_0(x;z) = 1 \) and
\[ P_n (x; z) = \Lambda_n (x) + \sum_{i=0}^{n-1} \xi_{n,i} (z) \Lambda_i (x), \quad n \geq 1. \]  

(4)

We say that \( \{ P_n (x; z) \}_{n \geq 0} \) is a sequence of (monic) orthogonal polynomials with respect to the functional \( L \), \([2],[4],[21],[22],[27],[28],[46]\).

In this paper, we focus on linear functionals of the form

\[ L[u] = \sum_{x=0}^{\infty} u(x) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}, \quad u \in \mathbb{F}[x], \]  

(5)

and we use the notation

\[ (a)_n = \prod_{i=1}^{p} (a_i)_n, \quad (b)_n = \prod_{i=1}^{q} (b_i)_n, \quad n \in \mathbb{N}_0, \]

\[ c + r = (c_1 + r, c_2 + r, \ldots, c_m + r) \in \mathbb{K}^m, \quad r \in \mathbb{K}, \quad c \in \mathbb{K}^m, \]

where

\[ a = (a_1, \ldots, a_p) \in \mathbb{K}^p, \quad b = (b_1, \ldots, b_q) \in \mathbb{K}^q, \quad p, q \in \mathbb{N}_0, \]  

(6)

and the Pochhammer polynomial \((x)_n\) is defined by \((x)_0 = 1\) and \([37],[18:12]\)

\[ (x)_n = \prod_{j=0}^{n-1} (x+j), \quad n \in \mathbb{N}. \]  

(7)

If \( \mu_n (z) \in \mathbb{F} \) denote the standard moments of \( L \) on the monomial basis

\[ \mu_n (z) = L [x^n], \quad n \in \mathbb{N}_0, \]  

(8)

it follows from (2) and (5) that

\[ \mu_{n+1} = \vartheta [\mu_n] = \vartheta^n [\mu_0], \quad n \in \mathbb{N}_0. \]  

(9)

Moreover, using (5) we can see that \([15]\)

\[ L [\sigma (x) u (x)] = L [z \tau (x) u (x + 1)], \quad u \in \mathbb{K}[x], \]  

(10)

where

\[ \sigma (x) = x (x + b)_1, \quad \tau (x) = (x + a)_1. \]
Because of (9), we say that the functional $L$ is of \textit{Toda-type} \cite{3, 14, 38}, \cite{47}, and because of (10) we also call $L$ \textit{discrete semiclassical} \cite{1, 16, 18}, \cite{33, 36, 49}. The class of the functional $L$ is defined by

$$s = \max \{ \deg (\sigma) - 1, \deg (\tau) - 1 \} = \max \{ p - 1, q \},$$

and semiclassical functional of class $s = 0$ are called \textit{classical}.

Our objective is to obtain \textit{comparative asymptotics} (also called \textit{relative asymptotics}) \cite{5, 23, 24, 25, 29, 30, 31, 32, 34, 39, 40, 41, 42, 43, 44}, for the polynomials $P_n (x; z)$ with respect to the basis of \textit{falling factorial polynomials} defined by $\phi_0 (x) = 1$ and

$$\phi_n (x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \quad (11)$$

In other words, we want to study the limit

$$\lim_{n \to \infty} \frac{P_n (x; z)}{\phi_n (x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,$$

where $z$ is a fixed number, and $x$ belongs to a compact subset of the complex plane containing the origin. We already considered this type of limits in \cite{10}, \cite{12} (Charlier and Meixner polynomials), and in \cite{13} (Krawtchouk polynomials).

The organization of the paper is as follows: in Section 2 we review some of our results from \cite{14}. The polynomials $P_n (x; z)$ have different asymptotic approximations depending on the relation between the parameters $p$ and $q$ defined in (6). Thus, we will consider the cases $p = q$ (Section 4.1), $p = q - 1$ (Section 4.2), $p < q - 1$ (Section 4.3), and $p = q + 1$ (Section 4.4). Finally, in the conclusions’ section we will summarize the results and discuss future directions.

\section{Previous results}

In \cite{14}, we studied families of polynomials (that we said to be of \textit{Toda type}), orthogonal with respect to a linear functional $L : F[x] \to F$ satisfying

$$D_x L [u] = L [xu], \quad u \in F [x],$$

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$$D_x L [u] = L [xu], \quad u \in F [x],$$
where \( D_z : \mathbb{F} \rightarrow \mathbb{F} \) is a fixed derivation (on the variable \( z \)) associated to \( L \).

In this section, we review some of the results that we obtained, and apply them to the particular cases:

(i) \( D_z = \vartheta \), where the operator \( \vartheta \) was defined in (1).

(ii) The variable transformation

\[
D_w = w (1 - w) \partial_w, \quad w = \frac{z}{z - 1}.
\]

2.1 Toda-type orthogonal polynomials

The linear system (3) can be written as

\[
L [\Lambda_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,
\]

and we see that the sequence \( \{P_n(x; z)\}_{n \geq 0} \) satisfies the orthogonality conditions

\[
L [P_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \tag{12}
\]

where \( h_n(z) \in \mathbb{F} \setminus \{0\} \) is the norm of \( P_n(x; z) \).

From (12), we see that

\[
L [x P_k P_n] = 0, \quad k \neq n, n \pm 1,
\]

and therefore the polynomials \( P_n(x; z) \) satisfy the three term recurrence relation

\[
x P_n(x; z) = P_{n+1}(x; z) + \beta_n(z) P_n(x; z) + \gamma_n(z) P_{n-1}(x; z) \tag{13}
\]

with \( P_{-1} = 0, \ P_0 = 1 \). The coefficients \( \beta_n(z), \gamma_n(z) \in \mathbb{F} \) are given by [8]

\[
\beta_0 = \frac{L[x]}{L[1]}, \quad \gamma_0 = 0, \tag{14}
\]

and

\[
\beta_n = \frac{L[x P_n^2]}{h_n}, \quad \gamma_n = \frac{L[x P_n P_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}. \tag{15}
\]

If we define \( \sigma_n(z) \in \mathbb{F} \) by

\[
P_n(x; z) = x^n - \sigma_n(z) x^{n-1} + u_n(x; z), \quad \deg(u_n) \leq n - 2, \tag{16}
\]
we have $\sigma_0 = 0$, and using (13) we get
\[ x^{n+1} - \sigma_n x^n + xu_n = x^{n+1} - \sigma_{n+1} x^{n+1} + \beta_n \left( x^n - \sigma_n x^{n-1} + u_n \right) + \gamma_n P_{n-1}. \]
Comparing coefficients of $x^n$, we obtain $-\sigma_n = -\sigma_{n+1} + \beta_n$, or
\[ \beta_n = \sigma_{n+1} - \sigma_n. \tag{17} \]
Our next result relates $\sigma_n, h_n, \beta_n$ and $\gamma_n$.

**Proposition 1** Let $\vartheta$ be defined by (1), $h_n$ be defined by (12), $\beta_n, \gamma_n$ be defined by (15), and $\sigma_n$ be defined by (16). Then, we have
\[ \vartheta [\sigma_n] = \gamma_n \] (18)
and
\[ \vartheta [\ln h_n] = \beta_n. \tag{19} \]

**Proof.** See [14]. ■

As a direct consequence, we see that $(\beta_n, \gamma_n)$ are solutions of the *Toda equations* [47].

**Corollary 2** The coefficients of the 3-term recurrence relation (13) are solutions of the differential-difference equations
\[ \vartheta [\beta_n] = \Delta \gamma_n, \quad \vartheta [\ln \gamma_n] = \nabla \beta_n, \tag{20} \]
with initial conditions (14), where
\[ \Delta f (n) = f (n + 1) - f (n), \quad \nabla f (n) = f (n) - f (n - 1). \tag{21} \]

Essential for our work in this paper is the following theorem.

**Theorem 3** The polynomials $P_n (x; z)$ defined by (12) satisfy the recurrence
\[ \vartheta [P_n] = -\gamma_n P_{n-1}, \tag{22} \]
and the ODE
\[ [\vartheta^2 + (x - \beta_n) \vartheta + \gamma_n] [P_n] = 0. \tag{23} \]
Proof. See [14]. ■

Since $\vartheta = z\partial_z$, we have

$$z\partial_z P_n = -\gamma_n P_{n-1},$$

and

$$z \left( z\partial_z^2 P_n + \partial_z P_n \right) + (x - \beta_n) z\partial_z P_n + \gamma_n P_n = 0. \quad (24)$$

If we define $g_n (z) \in \mathbb{F}$ by

$$\gamma_n (z) = zg_n (z), \quad (25)$$

then

$$P'_n = -g_n P_{n-1}, \quad (26)$$

and (24) becomes

$$zP''_n + (x + 1 - \beta_n) P'_n + g_n P_n = 0, \quad (27)$$

where we will always use the notation

$$P'_n = \partial_z P_n.$$

2.2 The function $\sigma_n (z)$

A fundamental quantity in our studies is $\sigma_n (z)$ defined in (16).

Theorem 4 The coefficients in the power series expansion

$$\sigma_n (z) = \sum_{k=0}^{\infty} s_k (n) z^k, \quad (28)$$

are given by

$$s_0 (n) = \frac{n(n - 1)}{2}, \quad s_1 (n) = n \frac{(n - 1 + a)}{(n + b)}, \quad (29)$$

and

$$s_k (n) = \frac{1}{k(k - 1)} \sum_{j=1}^{k-1} (k - j) s_{k-j} (n) \Delta \nabla [s_j (n)], \quad k \geq 2, \quad (30)$$

$\Delta, \nabla$ are the finite difference operators (acting on $n$) defined in (21).
Proof. See [14]. ■

Using (17) and (18), we obtain the following result.

Corollary 5 The coefficients of the 3-term recurrence relation (13) admit the formal power series

$$
\beta_n(z) = \sum_{k=0}^{\infty} \Delta s_k(n) z^k, \quad \gamma_n(z) = \sum_{k=1}^{\infty} k s_k(n) z^k, \quad (31)
$$

where the coefficients $s_k(n)$ are defined by (28). In particular,

$$
\beta_n(0) = n, \quad \gamma_n(0) = 0. \quad (32)
$$

Remark 6 From (25) and (31), we have

$$
g_n(z) = \sum_{k=0}^{\infty} (k + 1) s_{k+1}(n) z^k. \quad (33)
$$

From (29), we see that

$$
s_1(n) = n^\theta \frac{(1 - n^{-1} + n^{-1} a)_1}{(1 + n^{-1} b)_1},
$$

where

$$
\theta = p + 1 - q. \quad (34)
$$

If we write

$$
s_1(n) = n^\theta \sum_{k=0}^{\infty} r_k n^{-k}, \quad (35)
$$

we get

$$
\sum_{j=0}^{k} e_{k-j}(b) r_j = e_k(a-1),
$$

where the elementary symmetric polynomials $e_n(c)$ are defined by the generating function [37, 19.19.4]

$$
\sum_{n=0}^{\infty} e_n(c) t^n = \prod_{i=1}^{m} \frac{1}{(1 + tc_i)}, \quad c \in \mathbb{K}^m. \quad (36)
$$
Since $e_0 = 1$, we obtain the recurrence
\[ r_k = e_k (a - 1) - \sum_{j=0}^{k-1} e_{k-j} (b) r_j, \quad r_0 = 1. \quad (37) \]

The first two coefficients $r_k$ are
\[ r_1 = e_1 (a - 1) - e_1 (b), \]
\[ r_2 = e_2 (a - 1) - e_2 (b) - e_1 (a - 1) e_1 (b) + e_1^2 (b). \]

**Theorem 7** Let
\[ \Theta_k = (\theta - 2) k + \eta (\theta), \]
with
\[ \eta (\theta) = \begin{cases} 
0, & \theta = 1 \\
1, & \theta = 0 \\
2, & \theta \neq 0,1 
\end{cases} \]

We have:
(i) If $\theta < 0$, then
\[ s_k (n) \sim A_k (\theta) n^{\Theta_k}, \quad n \to \infty, \quad (38) \]
where $A_1 = 1$ and for $k \geq 2$
\[ A_k = \frac{1}{k(k - 1)} \sum_{j=1}^{k-1} (k - j) \Theta_j (\Theta_j - 1) A_j A_{k-j}. \quad (39) \]

(ii) If $\theta = 0$, then as $n \to \infty$,
\[ s_1 (n) \sim 1, \quad s_k (n) \sim r_1 C (k - 1) n^{-2k+1}, \quad k \geq 2, \]
where $C (k)$ is the $k^{th}$ Catalan number [37, 26.5(i)]
\[ C (k) = \frac{1}{k + 1} \binom{2k}{k}. \]

(iii) If $\theta = 1$, then as $n \to \infty$,
\[ s_1 (n) \sim n, \quad s_k (n) \sim r_2 n^{-k}, \quad k \geq 2. \]
Proof. See [14].

Remark 8 Using induction, we can see that the solution of (39) is given by

\[ A_k(\theta) = -\theta \frac{(1-\theta)^k}{(k-1)!} (1 + k - \theta k)_{k-3}. \]

As a direct application of (30), we can illustrate the results of Theorem 7 for some particular cases.

Example 9 Let \( \theta = 1 \). As \( n \to \infty \), we have

\[
\begin{align*}
  s_2 &= r_2 n^{-2} + (r_1 r_2 + 3r_3)n^{-3} + O\left(n^{-4}\right), \\
  s_3 &= r_2 n^{-3} + 3(r_1 r_2 + 2r_3)n^{-4} + O\left(n^{-5}\right),
\end{align*}
\]

and we see that \( s_k(n) \sim r_2 n^{-k}, n \geq 2 \), as expected. Also,

\[
\begin{align*}
  \sigma_n(z) &= \frac{n^2}{2} + \left( z - \frac{1}{2} \right) n + r_1 z + r_2 z n^{-1} + (r_3 + r_2 z) z n^{-2} \\
  &\quad + \left[ r_4 + (r_1 r_2 + 3r_3) z + r_2 z^2 \right] z n^{-3} + O\left(n^{-4}\right), \\
  \beta_n(z) &= n + z - r_2 z n^{-2} + \left[ (1 - 2z) r_2 - 2r_3 \right] z n^{-3} + O\left(n^{-4}\right), \quad (40)
\end{align*}
\]

and

\[
\begin{align*}
  g_n(z) &= n + r_1 + r_2 n^{-1} + (2 z r_2 + r_3) n^{-2} + O\left(n^{-3}\right). \quad (41)
\end{align*}
\]

Example 10 Let \( \theta = 0 \). As \( n \to \infty \), we have

\[
\begin{align*}
  s_2 &= r_1 n^{-3} + (r_1^2 + 3r_2)n^{-4} + O\left(n^{-5}\right), \\
  s_3 &= 2r_1 n^{-5} + 2(3r_1^2 + 5r_2)n^{-6} + O\left(n^{-7}\right),
\end{align*}
\]

and we see that \( s_k(n) \sim C (k - 1) r_1 n^{-2k+1}, n \geq 2 \), as expected. Also,

\[
\begin{align*}
  \sigma_n(z) &= \frac{n^2}{2} - \frac{1}{2} n + z + r_1 z n^{-1} + r_2 z n^{-2} + (r_1 z + r_3) z n^{-3} + O\left(n^{-4}\right), \\
  \beta_n(z) &= n - r_1 z n^{-2} + (r_1 - 2r_2) z n^{-3} \\
  &\quad - [r_1 (3z + 1) - 3 (r_2 - r_3)] z n^{-4} + O\left(n^{-5}\right), \quad (42)
\end{align*}
\]

and

\[
\begin{align*}
  g_n(z) &= 1 + r_1 n^{-1} + r_2 n^{-2} + (2 z r_1 + r_3) n^{-3} + O\left(n^{-4}\right). \quad (43)
\end{align*}
\]
Example 11 Let $\theta = -1$. As $n \to \infty$, we have

\[
\begin{align*}
s_2 &= n^{-4} + 4r_1n^{-5} + (1 + 3r_1^2 + 7r_2) n^{-6} + O \left( n^{-7} \right), \\
s_3 &= 4n^{-7} + 28r_1n^{-8} + (20 + 51r_1^2 + 61r_2) n^{-9} + O \left( n^{-10} \right),
\end{align*}
\]

and we see that $s_k (n) \sim A (k) \ r_1 n^{-3k+2}$, $n \geq 2$, as expected. Also,

\[
\begin{align*}
\sigma_n (z) &= \frac{n^2}{2} - \frac{1}{2} n + zn^{-1} + r_1 zn^{-2} + r_2 zn^{-3} + (z + r_3) zn^{-4} + O \left( n^{-5} \right), \\
\beta_n (z) &= n - zn^{-2} + (1 - 2r_1) zn^{-3} - [1 + 3 (r_2 - r_1)] zn^{-4} + O \left( n^{-5} \right), \quad (44) \\
and \\
g_n (z) &= n^{-1} + r_1 n^{-2} + r_2 n^{-3} + (2 + r_3) n^{-4} + O \left( n^{-5} \right). \quad (45)
\end{align*}
\]

2.3 The function $\Phi_n (z; x)$

Sometimes, the falling factorial polynomials $\phi_n (x)$ defined in (11), are called binomial polynomials, since we have

\[
\frac{\phi_n (x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \quad (46)
\]

From the definition (11), we see that

\[
\phi_{n+1} (x) = (x - n) \phi_n (x) = x \phi_n (x - 1), \quad n \geq 0, \quad (47)
\]

and from (7) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

\[
\phi_n (x) = (-1)^n (-x)_n = (x + 1 - n)_n.
\]

Using (32) in (13), we obtain

\[
P_{n+1} (x; 0) = (x - n) P_n (x; 0), \quad P_0 (x; 0) = 1,
\]

and comparing with the recurrence satisfied by the falling factorial polynomials (47), we conclude that

\[
P_n (x; 0) = \phi_n (x). \quad (48)
\]
Note that from (26) and (48), we see that
\[ P'_n(x; 0) = -g_n(0) \phi_{n-1}(x). \]  
(49)

If we define \( \Phi_n(z; x) \) by
\[ P_n(x; z) = \phi_n(x) \Phi_n(z; x), \]  
(50)
then (47) and (49) give the recurrence
\[ \Phi'_n(z; x) = -\frac{g_n(z)}{x + 1 - n} \Phi_{n-1}(z; x). \]  
(51)

It also follows from (27) and (48) that \( \Phi_n(z; x) \) is the solution of the ODE
\[ z \Phi''_n + (x + 1 - \beta_n) \Phi'_n + g_n \Phi_n = 0, \]  
(52)
with initial condition
\[ \Phi_n(0; x) = 1. \]  
(53)
Note that setting \( z = 0 \) in (52) and using (32) gives
\[ \Phi'_n(0; x) = -\frac{g_n(0)}{x + 1 - n} \]
in agreement with (51).

**Proposition 12** Suppose that
\[ \Phi_n(z; x) = \sum_{k=0}^{\infty} \frac{\alpha_k(n)}{(x + 1 - n)_k} \frac{z^k}{k!}, \quad \alpha_0(n) = 1. \]  
(54)

Then, the coefficients \( \alpha_k(n) \) satisfy the recurrence
\[ \alpha_{k+1}(n) = -\sum_{j=0}^{k} s_{j+1}(n) \alpha_{k-j}(n-1)(x + 2 - n + k - j) \]  
(55)
In particular,
\[ \alpha_1(n) = -s_1(n). \]  
(56)
Proof. Taking a derivative in (54), we have

\[ \Phi_n'(z; x) = \sum_{k=0}^{\infty} \frac{k\alpha_k(n)}{(x+1-n)_k} \frac{z^{k-1}}{k!} = \frac{1}{x + 1 - n} \sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!}, \]

since from (7) we see that

\[ (x)_{k+1} = x (x+1)_k. \]

From (51), we conclude that

\[ \sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!} = -g_n(z) \sum_{k=0}^{\infty} \frac{\alpha_k(n-1)}{(x+2-n)_k} \frac{z^k}{k!}, \]

and using (33), we get

\[ \frac{\alpha_{k+1}(n)}{(x+2-n)_k} = -\sum_{j=0}^{k} s_{j+1}(n) \frac{\alpha_{k-j}(n-1)}{(x+2-n)_{k-j}}. \tag{57} \]

The result follows after using the identity

\[ \frac{(x)_n}{(x)_m} = (x+m)_{n-m}, \quad m \leq n. \]

Remark 13 Suppose that \( \theta < 2. \) It follows from (57) that to find the leading term in the asymptotic expansion of \( \alpha_k(n) \) as \( n \to \infty, \) one needs to consider only the term with \( j = 0. \) Thus,

\[ \alpha_{k+1}(n) \sim -s_1(n) \alpha_k(n-1), \quad n \to \infty \]

and we conclude that

\[ \alpha_k(n) \sim (-1)^k \prod_{j=0}^{k-1} s_1(n-j), \quad n \to \infty. \]

Using (35), we get

\[ \alpha_k(n) = (-1)^k n^{k\theta} \left[ 1 + k \left( r_1 - \frac{k-1}{2} \theta \right) n^{-1} + O \left( n^{-2} \right) \right], \quad n \to \infty. \]
Example 14 Let $\theta = 1$. As $n \to \infty$, we have
\[
\frac{\alpha_k(n)}{(x + 1 - n)_k} = 1 + \frac{x + 1 + r_1}{n} k + O(n^{-2}),
\]
and therefore
\[
\Phi_n(z; x) = e^z \left[ 1 + \frac{x + 1 + r_1}{n} z + O(n^{-2}) \right], \quad n \to \infty. \tag{58}
\]

2.4 The variable $w$

If we use (30) with $\theta = 2$, we get
\[
s_1 = n^2 + r_1 n + r_2 + r_3 n^{-1} + O(n^{-2}),
\]
\[
s_2 = n^2 + r_1 n + r_2 + 2 r_3 n^{-1} + O(n^{-2}),
\]
\[
s_3 = n^2 + r_1 n + r_2 + 3 r_3 n^{-1} + O(n^{-2}),
\]
and this is clearly not an asymptotic sequence. As we showed in [14], what we need is to change variables from $z$ to
\[
w = \frac{z}{z - 1}. \tag{59}
\]

Theorem 15 Let $\sigma_n(z)$ defined by (16). If we write
\[
\sigma_n(w) = \sum_{k=0}^{\infty} \xi_k(n) w^k,
\]
we have
\[
\xi_0(n) = \frac{n(n-1)}{2}, \quad \xi_1(n) = -n \frac{(n-1 + a)_1}{(n + b)_1}, \tag{60}
\]
and
\[
\xi_k = \xi_{k-1} + \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \xi_{k-j} \Delta \xi_j, \quad k \geq 2. \tag{61}
\]

Proof. See [14].
Remark 16 If we use (35) in (60), we get

\[ \xi_1(n) = -n^2 \sum_{k=0}^{\infty} r_k n^{-k}, \]  

(62)

where the coefficients \( r_k \) can be computed using (37).

The asymptotic behavior of the coefficients \( \xi_k(n) \) is given in the following result.

Theorem 17 For all \( k \geq 2 \), we have

\[ \xi_k(n) = O\left(n^{-k+1}\right), \quad n \to \infty. \]  

(63)

Proof. See [14]. \[ \square \]

Remark 18 For the first few \( \xi_k(n) \), we can use (61) and (62), and obtain

\[ \begin{align*}
\xi_2(n) &= \frac{r_3}{n} + \frac{r_1 r_3 + 3 r_4}{n^2} + O\left(n^{-3}\right), \\
\xi_3(n) &= -\frac{r_1 r_3 + 2 r_4}{n^2} + O\left(n^{-3}\right), \\
\xi_4(n) &= \frac{(1 + r_1^2 + r_2) r_3 + 5(r_1 r_4 + r_5)}{n^3} + O\left(n^{-4}\right),
\end{align*} \]

(64)

as \( n \to \infty \), in agreement with (63).

Note that we have

\[ \gamma_n = z \sigma'_n(z) = w (1 - w) \hat{\sigma}_n(w), \]

where we will always use the notation

\[ \hat{\Phi}_n = \partial_w \Phi_n. \]

Therefore, in this case we define

\[ \gamma_n(w) = w (1 - w) g_n(w), \]  

(65)

with

\[ g_n(w) = \sum_{k=0}^{\infty} (k + 1) \xi_{n,k+1} w^k. \]
Example 19 Using (62) and (64), we can compute the first terms in the asymptotic expansions of $\sigma_n(w)$, $\beta_n(w)$, and $g_n(w)$:

$\sigma_n(w) = \left(\frac{1}{2} - w\right)n^2 - \left(\frac{1}{2} + r_1w\right)n - r_2w + r_3(w - 1)wn^{-1} + O(n^{-2})$,

$\beta_n(w) = (1 - 2w)n - (1 + r_1)w - r_3(w - 1)wn^{-2} + O(n^{-3})$, (66)

and

$g_n(w) = -n^2 - r_1n - r_2 + r_3(2w - 1)n^{-1} + O(n^{-2})$, (67)

as $n \to \infty$.

3 Numerical results

Since we can write the falling factorial polynomials in terms of factorials (46), we can use the reflection formula for the Gamma function [37, 5.5.3]

$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$,

and obtain

$\phi_n(x) = \frac{x!}{\Gamma(x + 1 - n)} = \frac{x! \sin[\pi (n-x)]}{\pi} \Gamma(n-x)$.

But

$\sin(\pi (n-x)) = -\cos(\pi n) \sin(\pi x) = (-1)^{n+1} \sin(\pi x)$,

and therefore

$\phi_n(x) = (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \Gamma(n-x)$.

Thus, in order to plot the different asymptotic approximations for $P_n(x; z)$, we will consider two cases:

i) On the negative real axis, we shall graph

$\frac{P_n(x; z)}{\Gamma(n-x)}$ and $(-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \Phi_n(z; x)$, (68)

since both functions are analytic, nonzero, and bounded in this region.
ii) On the positive real axis (with \( x < n \)), we shall graph
\[
\frac{P_n(x; z)}{x! \Gamma(n - x)} \quad \text{and} \quad (-1)^{n+1} \frac{\sin(\pi x)}{\pi} \Phi_n(z; x),
\] (69)
since both functions are analytic and bounded in this region.

To compute the polynomials \( P_n(x; z) \), we first compute the moments of \( L \) on the monomial basis (8) to a very high order of accuracy (with error less than \( \varepsilon = 10^{-100} \)), solve the system of equations (3)
\[
\mu_{n+k} + \sum_{i=0}^{n-1} \mu_{k+i} \xi_{n,i} = 0, \quad 0 \leq k \leq n - 1,
\]
and construct the polynomials using (4),
\[
P_n(x; z) = x^n + \sum_{i=0}^{n-1} \xi_{n,i}(z) x^i.
\]
After that, we double-check that
\[
|L[x^k P_n]| < \varepsilon, \quad 0 \leq k \leq n - 1, \quad |L[x^n P_n]| > \varepsilon.
\]

We have tried other methods (using Hankel determinants, recurrences, or the Toda equations and the 3-term recurrence relation), but found them unsatisfactory from a numerical point of view.

4 Asymptotic analysis

In this section, we will obtain asymptotic approximations for \( P_n(x; z) \) as \( n \to \infty \), with \( x = O(1) \) and all other parameters fixed. Because of the moments’ recurrence (9), the analyticity of all the moments \( \mu_n(z) \) (and in consequence the polynomials \( P_n \) themselves) as functions of \( z \) will agree with that of the first moment \( \mu_0(z) \).

But since \( \mu_0(z) \) is a hypergeometric function,
\[
\mu_0(z) = _pF_q \left( \begin{array}{c} a \\ b \end{array} ; z \right) = \sum_{x=0}^{\infty} \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}, \quad a \in \mathbb{K}^p, b \in \mathbb{K}^q,
\]
its domain of analyticity depends on the parameters \( p, q \). We have three cases to consider:

(i) If \( p < q + 1 \), then \( \mu_0(z) \) is an entire function of \( z \). From (34), we see that this corresponds to the case \( \theta < 2 \).

(ii) If \( p = q + 1 \) \((\theta = 2)\), then \( \mu_0(z) \) is analytic inside the unit circle, \(|z| < 1\), and can be extended by analytic continuation to the cut plane \( \mathbb{C} \setminus (1, \infty) \).

(iii) If \( p > q + 1 \) \((\theta > 2)\), then \( \mu_0(z) \) diverges for all \( z \neq 0 \), except when one of the numerator parameters is a negative integer, and \( \mu_0(z) \) becomes a polynomial \((\text{in } z)\) of degree \( N \). We will not study this situation in this paper, since in this case we need to scale \( n \) in terms of \( N \) and consider the limit as \( N \to \infty \) (see [13] for the Krawtchouk polynomials).

We will divide the first case (i) in 3 subcases:

(a) When \( p = q \) \((\theta = 1)\), \( \mu_0(z) \) is entire (but barely!) and the asymptotic expansion of \( P_n(x; z) \) will contain an exponential multiple \( e^z \).

(b) When \( p = q - 1 \) \((\theta = 0)\), \( P_n(x; z) \) will have a regular asymptotic expansion.

(c) When \( p < q - 1 \) \((\theta < 0)\), some of the first terms in the asymptotic expansion of \( P_n(x; z) \) will be missing.

If \( p = q + 1 \) \((\theta = 2)\), then \( \mu_0(z) \) will have a logarithmic singularity at \( z = 1 \). Thus, we expect that the asymptotic expansion of \( P_n(x; z) \) will have a factor of the form \((1 - z)^{\varsigma} \), where the power could depend on \( n \) (and \( x \)). In this case, it is better to perform a change of variables and work with \( w \) defined in (59).

**Notation 20** We say that a family of polynomials is of type \((p, q)\), if it’s orthogonal with respect to the functional (5) with \( a \in \mathbb{K}^p \) and \( b \in \mathbb{K}^q \).

### 4.1 Case \( p = q \) \((\theta = 1)\)

From (58), we see that in this case we should ”peel off” an exponential term from \( \Phi_n(z; x) \). Thus, if

\[
\Phi_n(z; x) = e^z \Lambda_n(z; x),
\]

we have

\[
\Phi'_n = e^z (\Lambda_n + \Lambda'_n), \quad \Phi''_n = e^z (\Lambda_n + 2\Lambda'_n + \Lambda''_n).
\]
and (52) becomes
\[ z\Lambda''_n + (2z + x + 1 - \beta_n)\Lambda'_n + (z + x + 1 - \beta_n + g_n)\Lambda_n = 0. \]  
(71)

From (40) and (41), we see that
\[ \beta_n = n + \tilde{\beta}_n, \quad g_n = n + \tilde{g}_n, \quad \tilde{\beta}_n = O(1), \quad \tilde{g}_n = O(1), \quad n \to \infty, \]
and hence
\[ z\Lambda''_n + \left( 2z + x + 1 - n - \tilde{\beta}_n \right)\Lambda'_n + \left( z + x + 1 + \tilde{g}_n - \tilde{\beta}_n \right)\Lambda_n = 0. \]  
(72)

Thus, we shall have \( \Lambda_n = O(1), \quad n \to \infty. \)
Replacing
\[ \tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad \tilde{g}_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k}, \]
and
\[ \Lambda_n(z; x) = \sum_{k=0}^{\infty} \lambda_k(z; x) n^{-k}, \]
in (72) and comparing coefficients of \( n^{-k} \), we obtain the recurrence
\[ \lambda'_{k+1} = z\lambda''_k + (2z + x + 1)\lambda'_k + (z + x + 1)\lambda_k + \sum_{j=0}^{k} [(u_{k-j} - v_{k-j})\lambda_j - v_{k-j}\lambda'_j]. \]  
(73)

From (53) and (70) we have \( \Lambda_n(0; x) = \Phi_n(0; x) = 1, \) and therefore
\[ \lambda_k(0; x) = \delta_{0,k}, \quad k \geq 0. \]  
(74)

Note that from (40) and (41) we see that
\[ u_0 = r_1, \quad u_1 = r_2, \quad u_2 = 2zr_2 + r_3, \]
\[ v_0 = z, \quad v_1 = 0, \quad v_2 = -r_2z. \]

When \( k = -1, \) (73) and (74) give
\[ \lambda'_0 = 0, \quad \lambda_0(0; x) = 1, \]
and thus
\[ \lambda_0(z; x) = 1. \] (75)

Using (75) in (73), we get
\[ \lambda'_1 = z + x + 1 + u_0 - v_0 = x + 1 + r_1, \]
and since \( \lambda_1(0; x) = 0 \), we obtain
\[ \lambda_1(z; x) = (x + 1 + r_1) z. \] (76)

Similarly, using (75) and (76) in (73), we get after some simplification
\[ \lambda'_2 = \lambda'_1 (x + 1 + z) + \lambda_1 \lambda'_1 + r_2, \]
and since \( \lambda_2(0; x) = 0 \), we conclude that
\[ \lambda_2 = \lambda'_1 \left(x + \frac{z}{2} + 1 \right) z + \frac{1}{2} (\lambda_1)^2 + r_2 z, \]
or
\[ \lambda_2(z; x) = [(x + 1) (x + 1 + r_1) + r_2] z + (x + 1 + r_1) (x + 2 + r_1) \frac{z^2}{2}. \] (77)

### 4.1.1 Polynomials of type \((0, 0)\) (Charlier polynomials).

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [7] and have the hypergeometric representation
\[ P_n(x; z) = (-z)^n \binom{-n, -x}{-} \frac{-1}{z}. \]

For this family, we have \( r_k = 0, \ k \geq 1 \), and therefore
\[ \beta_n = n + z, \quad g_n = n. \]

Replacing in (71), we get
\[ z \Lambda''_n + (z + x + 1 - n) \Lambda'_n + (x + 1) \Lambda_n = 0. \] (78)

Therefore, the recurrence (73) becomes
\[ \lambda'_{k+1} = z \lambda''_k + (z + x + 1) \lambda'_k + (x + 1) \lambda_k, \]
\[
\lambda_{k+1}(z) = z(\lambda_k' + \lambda_k) + x \left[ \lambda_k(z) - \lambda_k(0) \right] + x \int_0^z \lambda_k(t) \, dt.
\]

Starting with \( \lambda_0(z) = 1 \), we obtain
\[
\begin{align*}
\lambda_1(z) &= (x+1)z, \\
\lambda_2(z) &= (x+1)^2 z + (x+1) z^2, \\
\lambda_3(z) &= (x+1)^3 z + (x+1)^2 (2x+3) \frac{z^2}{2} + (x+1) \frac{z^3}{6}.
\end{align*}
\]

However, in this case the ODE satisfied by \( \Lambda_n(z; x) \) (78) has the exact solution [12]
\[
\Lambda_n(z; x) = \frac{1}{F_1 \left( \frac{x+1}{x+1-n}; -z \right)},
\]
where we have used the initial value \( \Lambda_n(0; x) = 1 \). Therefore,
\[
\Lambda_n(z; x) = \sum_{k=0}^{\infty} \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} (80)
\]
and using the first few terms we obtain
\[
\sum_{k=0}^{3} \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} = 1 + \frac{(x+1)z}{n} + \left[ (x+1)^2 z + (x+1)^2 \frac{z^2}{2} \right] n^{-2} + \left[ (x+1)^3 z + (x+1)^2 (2x+3) \frac{z^2}{2} + (x+1) \frac{z^3}{6} \right] n^{-3} + O(n^{-4})
\]
as \( n \to \infty \), in agreement with (79).

### 4.1.2 Polynomials of type \((1, 1)\) (generalized Meixner)

For this family, we have
\[
\frac{s_1(n)}{n} = \frac{n+a-1}{n+b} = 1 + \frac{a-b-1}{n+b} = 1 + (a-b-1) \sum_{k=1}^{\infty} \frac{(-b)^{k-1}}{n^k},
\]
and therefore
\[
r_k = (a-b-1)(-b)^{k-1}, \quad k \geq 1.
\]

Using (81) in (75)–(77), we get \( \lambda_0(z; x) = 1 \),
\[
\lambda_1(z; x) = (x+a-b)z,
\]

and
\[
\lambda_2(z; x) = (x+1)z,
\]

and so on.
Figure 1: A plot of the scaled generalized Meixner polynomial $P_{10}(x; z)$ and its approximation.

and

$$\lambda_2(z; x) = \left[ (x + a)(x + 1 - b) + b^2 \right] z + (x + a - b + 1)(x + a - b) \frac{z^2}{2}.$$ 

In Figures 1 and 2, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = e^{z} \left[ 1 + \frac{\lambda_1(z; x)}{n} + \frac{\lambda_2(z; x)}{n^2} \right],$$

$n = 10, a = 0.2479357, b = 0.7146983$, and $z = 0.3974126$.

For additional information on these polynomials, see [6], [9], [15], [16], [17], [19].
Figure 2: A plot of the scaled generalized Meixner polynomial $P_{10}(x; z)$ and its approximation.
4.1.3 Polynomials of type $(2, 2)$

For this family, we have

$$s_1(n) = \frac{(n + a_1 - 1)(n + a_2 - 1)}{(n + b_1)(n + b_2)} =$$

$$1 + \frac{(a_1 - b_2 - 1)(a_2 - b_1 - 1)}{(b_1 - b_2)(n + b_2)} - \frac{(a_1 - b_1 - 1)(a_2 - b_1 - 1)}{(b_1 - b_2)(n + b_1)}$$

and therefore

$$r_k = \frac{\tau_k^{(1)}(b_2) - \tau_k^{(1)}(b_1)}{b_1 - b_2}, \quad k \geq 1,$$

with

$$\tau_k^{(1)}(b) = (b - a_1 + 1)(b - a_2 + 1)(-b)^{k-1}.$$ 

In particular,

$$r_1 = a_1 + a_2 - b_1 - b_2 - 2,$$

$$r_2 = 1 - a_1 - a_2 - (a_1 + a_2 - 2)(b_1 + b_2) + b_1^2 + b_2^2 + b_1b_2 + a_1a_2.$$ 

Using (82) in (75)–(77), we get $\lambda_0(z; x) = 1$,

$$\lambda_1(z; x) = (x + a_1 + a_2 - b_1 - b_2 - 1)z,$$

and

$$\lambda_2(z; x) = [(x + 1)(x + a_1 + a_2 - b_1 - b_2 - 1) + r_2]z$$

$$+ (x + a_1 + a_2 - b_1 - b_2 - 1)(x + a_1 + a_2 - b_1 - b_2)\frac{z^2}{2}.$$ 

In Figures 3 and 4, we plot the functions (68)–(69) with

$$\Phi_n(z; x) = e^z \left[1 + \frac{\lambda_1(z; x)}{n} + \frac{\lambda_2(z; x)}{n^2}\right],$$

$n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

For additional information on these polynomials, see [15] and [17].
Figure 3: A plot of the scaled polynomial of type (2,2) $P_{10}(x;z)$ and its approximation.
Figure 4: A plot of the scaled polynomial of type $(2,2)\ P_{10}(x; z)$ and its approximation.
4.2 Case $p = q - 1$ ($\theta = 0$)

From (42) and (43), we see that

$$\beta_n = n + n^{-2}\tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = O(1), \quad n \to \infty,$$

and replacing in (52), we get

$$z\Phi_n'' + \left(x + 1 - n - n^{-2}\tilde{\beta}_n\right)\Phi_n' + g_n\Phi_n = 0.$$  \hfill (83)

Thus, we shall have $\Phi_n = O(1), n \to \infty$ with $\Phi_n(0; x) = 1$. Replacing

$$\tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k},$$

and

$$\Phi_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = \delta_{0,k}, \quad k \geq 0,$$

in (83) and comparing coefficients of $n^{-k}$, we obtain the recurrence

$$\varphi_{k+1}' = z\varphi_k'' + (x + 1)\varphi_k' + \sum_{j=0}^{k} \varphi_j u_{k-j} - \sum_{j=0}^{k-2} \varphi_j' v_{k-2-j}.$$  \hfill (84)

Replacing $\varphi_0 = 1$ in (84) with $k = 0$, we have

$$\varphi_1' = u_0 = 1,$$

and therefore

$$\varphi_1(z; x) = z.$$  \hfill (85)

Using $\varphi_0 = 1, \varphi_1 = z$ in (84) with $k = 1$, we get

$$\varphi_2' = x + 1 + u_1 + z u_0 = x + 1 + r_1 + z,$$

and hence

$$\varphi_2(z; x) = (x + 1 + r_1) z + \frac{z^2}{2}.$$  \hfill (86)

Similarly, we have

$$\varphi_3' = z + (x + 1)\varphi_2' + \varphi_0 u_2 + \varphi_1 u_1 + \varphi_2 u_0 - \varphi_0' v_0$$

$$= z + (x + 1)\varphi_2' + r_2 + r_1 z + \varphi_2,$$

and we conclude that

$$\varphi_3(z; x) = [(x + 1)(x + 1 + r_1) + r_2] z + [2(x + 1 + r_1) + 1] \frac{z^2}{2} + \frac{z^3}{6}.$$  \hfill (87)
4.2.1 Polynomials of type \((0, 1)\) (generalized Charlier)

For this family, we have

\[
s_1(n) = \frac{n + b}{n + b_1 \cdot n + b_2} = \sum_{k=0}^{\infty} \frac{(-b)^k}{n^k},
\]

and therefore

\[
r_k = (-b)^k, \quad k \geq 0.
\]

Using (88) in (85)–(87), we get

\[
\Phi_n(z; x) \sim 1 + \frac{z}{n} + \frac{(x + 1 - b) \cdot z + \frac{z^2}{2}}{n^2} + \frac{[(x + 1) (x + 1 - b) + b^2] \cdot z + [2 (x + 1 - b) + 1] \cdot \frac{z^2}{2} + \frac{z^3}{6}}{n^3}
\]
as \(n \to \infty\).

In Figures 5 and 6, we plot the functions (68)–(69) with

\[
\Phi_n(z; x) = 1 + \frac{\varphi_1(z; x)}{n} + \frac{\varphi_2(z; x)}{n^2},
\]

\(n = 10, b = 0.7146983,\) and \(z = 0.3974126.\)

For additional information on these polynomials, see [9], [15], [16], [17], [26], [45], [48].

4.2.2 Polynomials of type \((1, 2)\)

For this family, we have

\[
s_1(n) = \frac{n (n + a - 1)}{(n + b_1) \cdot (n + b_2)} = 1 + \frac{(a - 1 - b_1) \cdot b_1}{(b_1 - b_2) \cdot (n + b_1)} - \frac{(a - 1 - b_2) \cdot b_2}{(b_1 - b_2) \cdot (n + b_2)},
\]

and therefore

\[
r_k = \frac{(b_1 + 1 - a) \cdot (-b_1)^k + (a - 1 - b_2) \cdot (-b_2)^k}{b_1 - b_2}, \quad k \geq 0.
\]

In particular,

\[
r_0 = 1, \quad r_1 = a - b_1 - b_2 - 1, \quad r_2 = (1 - a) \cdot (b_1 + b_2) + b_1^2 + b_2^2 + b_1 b_2.
\]
Figure 5: A plot of the scaled generalized Charlier polynomial $P_{10}(x; z)$ and its approximation.
Figure 6: A plot of the scaled generalized Charlier polynomial $P_{10}(x; z)$ and its approximation.
Using (89) in (85)–(87), we get
\[ \Phi_n(z; x) = 1 + zn^{-1} + \left[ (x + a - b_1 - b_2) z + \frac{z^2}{2} \right] n^{-2} + \left[ (x + 1) (x + a - b_1 - b_2) + r_2 \right] zn^{-3} + \left[ (x + a - b_1 - b_2 + \frac{1}{2}) z^2 + \frac{z^3}{6} \right] n^{-3} + O(n^{-4}) \]
as \( n \to \infty \).

In Figures 7 and 8, we plot the functions (68)–(69) with
\[ \Phi_n(z; x) = 1 + \frac{\varphi_1(z; x)}{n} + \frac{\varphi_2(z; x)}{n^2}, \]
n = 10, \( a = 0.2479357 \), \( b_1 = 0.7146983 \), \( b_2 = 0.5712349 \), and \( z = 0.3974126 \).

For additional information on these polynomials, see [15] and [17].

4.3 Case \( p < q - 1 \) (\( \theta < 0 \))
Looking at (44) and (45), suggests that as \( n \to \infty \),
\[ \beta_n = n + n^{\theta-1} \tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = n^\theta \tilde{g}_n, \quad \tilde{g}_n = O(1), \]
and replacing in (52), we get
\[ z \Phi''_n + \left( x + 1 - n - n^{\theta-1} \tilde{\beta}_n \right) \Phi'_n + n^\theta \tilde{g}_n \Phi_n = 0. \tag{90} \]
Thus, we expect that
\[ \Phi_n(z; x) = 1 + n^{\theta-1} \tilde{\Phi}_n(z; x), \quad \tilde{\Phi}_n = O(1), \quad n \to \infty \]
with \( \tilde{\Phi}_n(0; x) = 0 \), and therefore the ODE (90) becomes
\[ zn^{\theta-1} \tilde{\Phi}''_n + \left( x + 1 - n - n^{\theta-1} \tilde{\beta}_n \right) n^{\theta-1} \tilde{\Phi}'_n + n^\theta \tilde{g}_n + n^{2\theta-1} \tilde{g}_n \tilde{\Phi}_n = 0, \]
or
\[ z \tilde{\Phi}''_n + \left( x + 1 - n - n^{\theta-1} \tilde{\beta}_n \right) \tilde{\Phi}'_n + n \tilde{g}_n + n^\theta \tilde{g}_n \tilde{\Phi}_n = 0. \tag{91} \]
Replacing
\[ \tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k}, \]
Figure 7: A plot of the scaled polynomial of type \((1,2)\) \(P_{10}(x; z)\) and its approximation.
Figure 8: A plot of the scaled polynomial of type (1,2) $P_{10}(x;\tau)$ and its approximation.
and
\[ \tilde{\Phi}_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = 0, \quad k \geq 0 \]
in (91) and comparing coefficients of $n^{-k}$, we obtain the recurrence
\[ \varphi'_k = u_k + z\varphi''_{k-1} + (x + 1) \varphi'_{k-1} + \sum_{j=0}^{k-1+\theta} \varphi_j u_{k-1+\theta-j} - \sum_{j=0}^{k+\theta-2} \varphi'_j v_{k+\theta-2-j}. \quad (92) \]

Setting $k = 0$ in (92), we get
\[ \varphi'_0 = u_0 = 1, \]
and therefore
\[ \varphi_0(z; x) = z. \quad (93) \]

For $k = 1$, we have
\[ \varphi'_1 = u_1 + z\varphi''_0 + (x + 1) \varphi'_0 + \sum_{j=0}^{\theta} \varphi_j u_{\theta-j} - \sum_{j=0}^{\theta-1} \varphi'_j v_{\theta-1-j}, \]
but since $\theta < 0$ and $\varphi_0 = z$,
\[ \varphi'_1 = u_1 + x + 1 \]
and hence
\[ \varphi_1(z; x) = (x + 1 + r_1) z. \quad (94) \]

Continuing this way, we see that
\[ \varphi'_k = u_k + z\varphi''_{k-1} + (x + 1) \varphi'_{k-1}, \quad 1 \leq k < 1 - \theta, \]
and for $k = 1 - \theta$
\[ \varphi'_{1-\theta} = u_{1-\theta} + z\varphi''_{-\theta} + (x + 1) \varphi'_{-\theta} + \varphi_0 u_0. \]

Thus,
\[ \varphi_k(z; x) = \int_0^z u_k(t) \, dt + z\varphi'_{k-1}(z; x) + x\varphi_{k-1}(z; x), \quad 1 \leq k < 1 - \theta, \quad (95) \]
and
\[ \varphi_{1-\theta}(z; x) = \int_0^z u_{1-\theta}(t) \, dt + z\varphi'_{-\theta}(z; x) + x\varphi_{-\theta}(z; x) + \frac{z^2}{2}. \quad (96) \]
4.3.1 Polynomials of type $(0, 2)$

For this family, we have

\[
\frac{s_1(n)}{n^{-1}} = \frac{n^2}{(n + b_1)(n + b_2)} = 1 + \frac{b_2^2}{(b_1 - b_2)(n + b_2)} - \frac{b_1^2}{(b_1 - b_2)(n + b_1)},
\]

and therefore

\[
r_k = \frac{(-b_2)^{k+1} - (-b_1)^{k+1}}{b_1 - b_2}, \quad k \geq 0.
\]

In particular,

\[
r_0 = 1, \quad r_1 = -(b_1 + b_2), \quad r_2 = b_1 b_2 + b_1^2 + b_2^2.
\]

Using (97) in (94) and (96), we get

\[
\varphi_1(z; x) = (x + 1 - b_1 - b_2) z,
\]

\[
\varphi_2 = \int_0^z u_2(t) \, dt + z \varphi'_1 + x \varphi_1 + \frac{z^2}{2} = \int_0^z r_2 dt + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2},
\]

and hence

\[
\varphi_2(z; x) = (b_1 b_2 + b_1^2 + b_2^2) z + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2}.
\]

Combining the results above and recalling that $\varphi_0 = z$, we obtain

\[
\Phi_n(z; x) = 1 + \frac{z}{n^2} + (x + 1 - b_1 - b_2) zn^{-3}
\]

\[
+ \left[ (b_1 b_2 + b_1^2 + b_2^2) z + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2} \right] n^{-4} + O \left( n^{-5} \right).
\]

In Figures 9 and 10, we plot the functions (68)–(69) with

\[
\Phi_n(z; x) = 1 + n^{-2} \left[ \varphi_0(z; x) + \frac{\varphi_1(z; x)}{n} \right],
\]

$n = 10, b_1 = 0.7146983, b_2 = 0.5712349, \text{ and } z = 0.3974126.$

For additional information on these polynomials, see [15] and [17].
Figure 9: A plot of the scaled polynomial of type (0,2) $P_{10}(x;z)$ and its approximation.
Figure 10: A plot of the scaled polynomial of type $(0,2)\ P_{10}(x; z)$ and its approximation.
4.4 Case $p = q + 1$ ($\theta = 2$)

Let $w$ be defined by (59). Using

$$\frac{\partial z}{\partial z} = -(w - 1)^2 \partial w, \quad \frac{\partial^2 z}{\partial w^2} = (w - 1)^4 \partial_w^2 + 2(w - 1)^3 \partial w,$$

in (24), we get

$$w^2 (1-w)^2 \partial_w^2 \Phi_n + (x + 1 - \beta_n - 2w) w (1-w) \partial_w \Phi_n + \gamma_n \Phi_n = 0,$$

and from (65) we have

$$w (1-w) \ddot{\Phi}_n + (x + 1 - \beta_n - 2w) \dot{\Phi}_n + g_n \Phi_n = 0. \quad (98)$$

Based on the case $\theta = 1$ (Section 4.1), we expect that $\Phi_n (w; x)$ will contain an exponential term. Replacing

$$\Phi_n (w; x) = \exp \left[ \Upsilon_n (w; x) \right], \quad \Upsilon_n (0; x) = 0,$$

in (98), we obtain

$$w (1-w) \left[ \ddot{\Upsilon}_n + \left( \dot{\Upsilon}_n \right)^2 \right] + (x + 1 - \beta_n - 2w) \dot{\Upsilon}_n + g_n = 0. \quad (99)$$

From (66)–(67), we have

$$\beta_n = (1 - 2w) n - (1 + r_1) w + \bar{\beta}_n, \quad \bar{\beta}_n = O(n^{-2}), \quad n \to \infty,$$

$$g_n = -n^2 - r_1 n + \tilde{g}_n, \quad \tilde{g}_n = O(1), \quad n \to \infty, \quad (100)$$

and replacing in (99) gives, to leading order,

$$w (1-w) \left( \dot{\Upsilon}_n \right)^2 \sim (1-2w) n \dot{\Upsilon}_n + n^2, \quad n \to \infty$$

and therefore

$$\dot{\Upsilon}_n \sim \frac{n}{w}, \quad \text{or} \quad \dot{\Upsilon}_n \sim \frac{n}{w-1}, \quad n \to \infty.$$

Since we want $\Upsilon_n (w; x)$ to be analytic in a neighborhood of $w = 0$, we choose

$$\Upsilon_n (w; x) \sim \ln (1-w) n, \quad n \to \infty,$$

and set

$$\Upsilon_n (w; x) = \ln (1-w) n + \sum_{k=0}^{\infty} \epsilon_k (w; x) n^{-k}, \quad \epsilon_k (0; x) = 0, \quad k \geq 0, \quad (101)$$
\[ \tilde{\beta}_n (w) = \sum_{k=2}^{\infty} v_k (w; x) n^{-k}, \quad \tilde{g}_n (w) = \sum_{k=0}^{\infty} u_k (w; x) n^{-k}, \quad (102) \]

where from (66)–(67) we see that
\[ v_2 = r_3 (1 - w) w, \quad u_0 = -r_2, \quad u_1 = r_3 (2w - 1). \quad (103) \]

Using (101)–(102) in (99) and comparing powers of \( n \), we get
\[ \epsilon_0 = \frac{x + 1 + r_1}{w - 1}. \]

Thus, since \( \epsilon_0 (0; x) = 0 \),
\[ \epsilon_0 (w; x) = (x + 1 + r_1) \ln (1 - w). \]

We could proceed in this manner, but instead we consider \( \Psi_n (w; x) \) defined by
\[ \Phi_n (w; x) = (1 - w)^{n+x+1+r_1} \Psi_n (w; x), \quad (104) \]
so that
\[ \Psi_n (w; x) = \exp \left[ \sum_{k=1}^{\infty} \epsilon_k (w; x) n^{-k} \right] = O (1), \quad n \to \infty. \]

Using (100) and (104) in (98), we get
\[ w (1 - w)^2 \ddot{\Psi}_n + (1 - w) \left[ x + 1 - w(r_1 + 2x + 3) - \tilde{\beta}_n - n \right] \dot{\Psi}_n \]
\[ + \left[ (n + x + 1 + r_1) \tilde{\beta}_n + (1 - w) \tilde{g}_n - (x + 1)(x + 1 + r_1) \right] \Psi_n = 0. \quad (105) \]

Replacing (102) and
\[ \Psi_n (w; x) = \sum_{k=0}^{\infty} \psi_k (w; x) n^{-k}, \quad \psi_k (0; x) = \delta_{0,k}, \quad k \geq 0 \]
in (105), we obtain the recurrence
\[ (1 - w) \dot{\psi}_{k+1} = w (1 - w)^2 \ddot{\psi}_k + (1 - w) \left[ x + 1 - (r_1 + 2x + 3)w \right] \dot{\psi}_k \]
\[ + (x + 1) (x + 1 + r_1) (w - 1) \psi_k + (1 - w) \sum_{j=0}^{k} \psi_j u_{k-j} \]
\[ + \sum_{j=0}^{k-1} \psi_j v_{k+1-j} + \sum_{j=0}^{k-2} \left[ (x + 1 + r_1) \psi_j - \dot{\psi}_j \right] v_{k-j} = 0. \quad (106) \]
Setting \( k = 0 \) and \( \psi_0 = 1 \) in (106), we obtain

\[
\dot{\psi}_1 = - (x + 1) (x + 1 + r_1) + u_0,
\]

and since \( u_0 = -r_2 \) and \( \psi_1 (0; x) = 0 \), we conclude that

\[
\psi_1 (w; x) = - [(x + 1) (x + 1 + r_1) + r_2] w.
\]  \hspace{1cm} (107)

Replacing \( k = 1 \) and \( \psi_0 = 1 \) in (106), we have

\[
(1 - w) \dot{\psi}_2 = (1 - w) [x + 1 - (r_1 + 2x + 3)w] \dot{\psi}_1
\]

\[
+ (x + 1) (x + 1 + r_1) (w - 1) \psi_1 + (1 - w) (u_1 + \psi_1 u_0) + v_2,
\]

and using (103) and \( \psi_1 = w \dot{\psi}_1 \), we get

\[
(1 - w) \dot{\psi}_2 = (1 - w) (x + 1 - (r_1 + 2x + 3)w) \dot{\psi}_1
\]

\[
+ (x + 1) (x + 1 + r_1) (w - 1) w \dot{\psi}_1
\]

\[
+ (1 - w) \left( r_3 (2w - 1) - r_2 w \dot{\psi}_1 \right) + r_3 (1 - w) w,
\]

or

\[
\dot{\psi}_2 = [x + 1 - ((x + 2) (x + 2 + r_1) + r_2) w] \dot{\psi}_1 + r_3 (3w - 1).
\]

Since \( \psi_2 (0; x) = 0 \), we conclude that

\[
\psi_2 (w; x) = \left[ (x + 1) w - ((x + 2) (x + 2 + r_1) + r_2) \frac{w^2}{2} \right] \dot{\psi}_1 + \frac{r_3}{2} w (3w - 2),
\]

and noting from (107) that

\[
- [(x + 2) (x + 2 + r_1) + r_2] w = \psi_1 (w; x + 1),
\]

we can write

\[
\psi_2 (w; x) = \left[ x + 1 + \frac{1}{2} \psi_1 (w; x + 1) \right] \psi_1 (w; x) + \frac{r_3}{2} w (3w - 2). \]  \hspace{1cm} (108)

### 4.4.1 Polynomials of type \((1, 0)\) (Meixner polynomials)

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [35] and have the representation

\[
P_n (x; z) = (a)_n \left( 1 - \frac{1}{z} \right)^{-n} _2F_1 \left[ \begin{array}{c} -n, -x \\ a \end{array}; 1 - \frac{1}{z} \right], \quad z \in \mathbb{C} \setminus [1, \infty).
\]
For this family, we have
\[-\frac{\xi_1(n)}{n^2} = \frac{n + a - 1}{n},\]
and therefore
\[r_0 = 1, \quad r_1 = a - 1, \quad r_k = 0, \quad k \geq 2, \tag{109}\]
and
\[
\beta_n (w) = (1 - 2w) n - aw, \quad g_n (w) = -n^2 - (a - 1) n. \tag{110}
\]
Thus, in this case \( \tilde{\beta}_n = \tilde{g}_n = 0 \), and using (109) in (105), we obtain
\[
w (1 - w) \ddot{\Psi}_n + [x + 1 - (2x + 2 + a)w - n] \Psi_n - (x + 1) (x + a) \Psi_n = 0, \tag{111}
\]
while the recurrence (106) becomes
\[
\dot{\psi}_{k+1} = w (1 - w) \ddot{\psi}_k + [x + 1 - (2x + 2 + a)w] \dot{\psi}_k - (x + 1) (x + a) \psi_k.
\]
It follows that, as \( n \to \infty \),
\[
\Psi_n (w; x) \sim 1 - (x + 1) (x + a) wn^{-1} - [x + 1 - \frac{1}{2} (x + 2) (x + 1 + a) w] (x + 1) (x + a) wn^{-2}. \tag{112}
\]
However, the ODE (111) can be solved exactly, and we have [12]
\[
\Psi_n (w; x) = _2F_1 \left( \begin{array}{c} x + 1, x + a \\ x + 1 - n \end{array} ; w \right),
\]
and using the first couple of terms, we get
\[
\Psi_n (w; x) \sim \sum_{k=0}^{2} \frac{(x + 1)_k (x + a)_k}{(x + 1 - n)_k} \frac{w^k}{k!} \sim - (x + 1) (x + a) wn^{-1} - (x + 1) (x + a) w \left[ x + 1 - \frac{1}{2} (x + 2) (x + 1 + a) w \right] n^{-2}, \quad n \to \infty,
\]
in agreement with (112).
4.4.2 Polynomials of type $(2, 1)$ (generalized Hahn polynomials of type I)

For this family, we have

\[
\frac{\xi_1 (n)}{n^2} = \frac{(n + a_1 - 1)(n + a_2 - 1)}{n(n + b)}
= 1 + \frac{(a_1 - 1)(a_2 - 1)}{bn} - \frac{(b + 1 - a_1)(b + 1 - a_2)}{b(n + b)},
\]

and therefore

\[
\begin{align*}
    r_0 &= 1, \quad r_1 = a_1 + a_2 - 2 - b, \\
    r_k &= (b + 1 - a_1)(b + 1 - a_2)(-b)^{k-2}, \quad k \geq 2.
\end{align*}
\]  

(113)

Using (113) in (107)–(108), we get

\[
\begin{align*}
    \psi_1 (w; x) &= -\left[ (x + 1)(x + a_1 + a_2 - 1 - b) + (b - a_1 + 1)(b - a_2 + 1) \right] w \\
    \psi_2 (w; x) &= \left[ x + 1 + \frac{1}{2} \psi_1 (w; x + 1) \right] \psi_1 (w; x) \\
    &\quad - \frac{1}{2} (b - a_1 + 1)(b - a_2 + 1)(3w - 2).
\end{align*}
\]

In Figures 11 and 12, we plot the functions (68)–(69) with

\[
\Phi_n (w; x) = (1 - w)^{n + x + 1 + r_1} \left[ 1 + \frac{\psi_1 (w; x)}{n} + \frac{\psi_2 (w; x)}{n^2} \right],
\]

\[
n = 10, \quad a_1 = 0.2479357, \quad a_2 = 0.1963478, \quad b = 0.7146983, \quad z = -0.01574126,
\]

and \( w = 0.0154973 \).

For additional information on these polynomials, see [11], [15], [16], [17], [20].
Figure 11: A plot of the scaled generalized Hahn polynomial $P_{10}(x; z)$ and its approximation.
Figure 12: A plot of the scaled generalized Hahn polynomial $P_{10}(x; z)$ and its approximation.
4.4.3 Polynomials of type $(3, 2)$

For this family, we have

\[
\frac{-\xi_1(n)}{n^2} = \frac{(n + a_1 - 1)(n + a_2 - 1)(n + a_3 - 1)}{n(n + b_1)(n + b_2)}
\]

\[
= 1 + \frac{(a_1 - 1)(a_2 - 1)(a_3 - 1)}{b_1 b_2 n} + \frac{(a_1 - b_1 - 1)(a_2 - b_1 - 1)(a_3 - b_1 - 1)}{(b_1 - b_2)b_1(n + b_1)}
\]

\[
- \frac{(a_1 - b_2 - 1)(a_2 - b_2 - 1)(a_3 - b_2 - 1)}{(b_1 - b_2)b_2(n + b_2)},
\]

and therefore

\[
r_0 = 1, \quad r_1 = a_1 + a_2 + a_3 - 3 - b_1 - b_2,
\]

\[
r_k = \frac{\tau_k^{(2)}(b_1) - \tau_k^{(2)}(b_2)}{b_1 - b_2}, \quad k \geq 2,
\]

where

\[
\tau_k^{(2)}(b) = (b - a_1 + 1)(b - a_2 + 1)(b - a_3 + 1)(-b)^{k-2}.
\]

At this point, we truly reach the limit of being able to type expressions in a compact way. For the first terms in the asymptotic expansion of these polynomials, we refer to the general formulas (107)–(108) with \( r_1, r_2 \) given by (114).

In Figures 13 and 14, we plot the functions (68)–(69) with

\[
\Phi_n(w; x) = (1 - w)^{n+x+1+r_1} \left[ 1 + \frac{\psi_1(w; x)}{n} + \frac{\psi_2(w; x)}{n^2} \right],
\]

\( n = 10, a_1 = 0.2479357, a_2 = 0.1963478, a_3 = 0.3614782, b_1 = 0.7146983, b_2 = 0.5712349, z = -0.01574126, \) and \( w = 0.0154973. \)

For additional information on these polynomials, see [15] and [17].

5 Conclusions

We have given asymptotic expansions for the ratio

\[
\frac{P_n(x; z)}{\phi_n(x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,
\]
Figure 13: A plot of the scaled polynomial of type (3,2) $P_{10}(x; z)$ and its approximation.
Figure 14: A plot of the scaled polynomial of type (3,2) $P_{10}(x;z)$ and its approximation.
as $n \to \infty$, where $z$ (and any other parameters) is fixed. The polynomials $P_n(x; z)$ are orthogonal with respect to the linear functional

$$L[u] = \sum_{x=0}^{\infty} u(x) \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}, \quad a \in \mathbb{K}^p, \ b \in \mathbb{K}^q,$$

and depending on the value of the parameter $\theta = p + 1 - q$, we have the following cases:

(i) If $\theta < 1$, then

$$\frac{P_n(x; z)}{\phi_n(x)} = 1 + zn^{\theta-1} \left[ 1 + \frac{x + 1 + r_1}{n} + O\left(n^{-2}\right) \right], \quad n \to \infty,$$

where

$$\frac{(1-n^{-1} + an^{-1})}{(1+bn^{-1})} = \sum_{k=0}^{\infty} r_k n^{-k}.$$ 

(ii) If $\theta = 1$, then as $n \to \infty$

$$\frac{P_n(x; z)}{\phi_n(x)} = e^z \left[ 1 + \frac{x + 1 + r_1}{n} z + O\left(n^{-2}\right) \right].$$

This result extends our previous work on the Charlier polynomials, [10], [12].

(iii) If $\theta = 2$, then as $n \to \infty$

$$\frac{P_n(x; w)}{\phi_n(x)} = (1-w)^{n+x+1+r_1} \left[ 1 - \frac{(x + 1)(x + 1 + r_1) + r_2}{n} w + O\left(n^{-2}\right) \right],$$

where $w = \frac{z}{z-1}$. This result extends our previous work on the Meixner polynomials, [10], [12].

(iv) If $\theta > 2$, then the polynomials $P_n(x; w)$ depend on a parameter $N$, with $-N \in \mathbb{N}$. We have not analyzed this case, since it will require scaling $N$ in terms of $n$. For some related work on the Krawtchouk polynomials, see [13]. We plan to study this case in a forthcoming paper.

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