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Zusammenfassung

Wir untersuchen das folgende Hauptproblem: "*Für ein gegebenes trivariates Polynomial* $F(x, y, z)$ *entscheide man die Existenz einer rationalen allgemeinen Lösung der algebraischen gewöhnlichen Differentialgleichung (ODE) erster Ordnung* $F(x, y, y') = 0$*; ist das der Fall, so berechne man explizit eine rationale allgemeine Lösung*".

Zunächst vernachlässigen wir den differentiellen Aspekt des Problems, und betrachten die durch $F(x, y, z) = 0$ definierte algebraische Lösungsfläche. Das Problem besteht nun darin, auf der algebraischen Fläche $F(x, y, z) = 0$ eine rationale Lösungskurve zu finden. Hat die gegebene ODE eine allgemeine rationale Lösung, dann muss die Lösungsfläche rational parametrisierbar sein. Deshalb untersuchen wir algebraische ODEs der Ordnung 1, deren Lösungsflächen rational parametrisierbar sind. Mittels der Parametrisierungsabbildung wird die Struktur der rationalen Lösungskurven auf der Lösungsfläche $F(x, y, z) = 0$ durch ein sogenanntes assoziiertes System autonomer ODEs dargestellt. Dieses assoziierte System ist bedeutend einfacher als die gegebene Differentialgleichung, denn es ist von der Ordnung 1 und vom Grad 1 in den Ableitungen.

Wir untersuchen verschiedene Aspekte eines solchen assoziierten Systems. Seine rationalen allgemeinen Lösungen stehen in ein-eindeutiger Beziehung zu den rationalen allgemeinen Lösungen der gegebenen Differentialgleichung. Darüberhinaus ist es relativ leicht lösbar durch Auffinden einer irreduziblen invarianten algebraischen Kurve. Dieses Problem kann im generischen Fall gelöst werden mittels einer Gradschranke und Ansatz für die unbestimmten Koeffizienten. Das assoziierte System ist von zentraler Bedeutung, ist es doch invariant unter bestimmten birationalen Transformationen der gegebenen Differentialgleichung. Die Gruppenaktion dieser Transformationen zerlegt die Menge der algebraischen ODEs in Äquivalenzklassen, deren rationale Lösbarkeit invariant ist. Man kann das Problem also auch verstehen als die Untersuchung von Normalformen dieser Klassen bezüglich der Transformationsgruppe. Wir beschreiben einige Klassen, deren zugehöriges assoziiertes System sich einfach rational lösen lässt. Alle diese Klassen haben interessante geometrische Eigenschaften. Solche Transformationen stellen eine neue und erfolgreiche Methode zur Lösung algebraischer ODEs dar.

Wir formulieren den Begriff einer allgemeinen Lösung einer algebraischen gewöhnlichen Differentialgleichung vom Standpunkt der differentiellen Algebra aus. Parametrisierungsmethoden zur Lösung von ODEs sind zwar bekannt, eine derartige Kombination konstruktiver algebraischer Geometrie und differentieller Algebra ist jedoch neu.

Abstract

We consider the following main problem: "*Given a trivariate polynomial $F(x, y, z)$, decide the existence of a rational general solution of the algebraic ODE $F(x, y, y') = 0$ of order 1; in the affirmative case, compute a rational general solution explicitly*".

First we neglect the differential aspect of the problem, and we consider the algebraic solution surface defined by $F(x, y, z) = 0$. Then, geometrically, the problem is equivalent to looking for rational solution curves on the algebraic surface $F(x, y, z) = 0$. Furthermore, the surface must admit a rational parametrization if the differential equation has a rational general solution. Therefore, we consider algebraic ODEs of order 1, whose solution surfaces admit a rational parametrization. The structure of the rational solution curves on the algebraic surface $F(x, y, z) = 0$ is encoded into a so-called associated system of autonomous ODEs via the parametrization map. This associated system is much simpler than the given differential equation because it is an autonomous system of order 1 and of degree 1 in the derivatives.

We investigate several aspects of such an associated system. Its rational general solutions are in one-to-one correspondence with the rational general solutions of the given differential equation. Moreover, it is rather simple to solve by just looking for an irreducible general invariant algebraic curve of the associated system. This problem is solvable in the generic case using a degree bound and the undetermined coefficients method. The associated system is the core of the problem since it is invariant under a certain birational transformation of the given differential equation. The group action of these transformations partitions the set of algebraic ODEs into equivalence classes, whose rational solvability is invariant. Therefore, the problem can be seen as studying a normal form for each class w.r.t. the group of transformations. We describe some classes where the associated systems are simple in terms of rational solvability and all of them have their own interesting geometric properties. Such transformations provide a new and powerful method for solving algebraic ODEs.

We formulate the notion of a general solution of an algebraic ODE from the point of view of differential algebra. Although parametrization methods for solving ODEs are known, such a combination of constructive algebraic geometry and differential algebra is new.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Publication

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Contents

Chapter 1

Motivation and introduction

There are two natural questions in the theory of differential equations: find a differential equation of a given function and solve a differential equation for its solutions. The first question is usually easier than the second one since one can apply the elimination theory to find a relation between the function itself and its derivatives. While the second question is more difficult and hence one might restrict oneself on either the classes of differential equations or the classes of solutions to solve them.

Ordinary differential equations (ODEs) come in two varieties: *linear* and *non-linear*. Of course, linear ODEs are just a special case of non-linear ones. However, the solutions of linear ODEs have a linear structure and they can be studied by linear algebra tools. Solutions of non-linear ODEs are more difficult to study and there is no general method for all those differential equations. While one can deal with linear ODEs of any order, the order of a non-linear ODE plays an essential role.

In the literature, e.g. Piaggio (1933), Murphy (1960), ODEs of order 1 are usually studied in two classes: *differential equations of order 1 and of degree 1*, and *differential equations of order 1 and of degree higher than 1*. Typically, an ODE of the second class is manipulated in order to reduce it to an ODE of the first class, which one might know how to solve. Still, it is a challenge in symbolic computation to determine explicit solutions of these differential equations, unless the equations are of special types.

In Ince (1926), there is already a geometrical treatment to ODEs of order 1, where the differential equation is viewed as a surface—we later call the *solution surface*—and its solutions are corresponding to the integral curves on the solution surface. If the solution surface admits a parametrization, then it can be derived to a new differential equation of order 1 and of degree 1 and one has to solve it. However, it is shortly mentioned in the book and there are no details how one can go further in this direction.

We have investigated further on the idea of this method for the class of *first-order rational parametrizable ODEs*. There are several reasons motivating this aim. First of all, it is a natural generalization of the recent works by Feng and Gao (2004, 2006) on the rational general solutions of algebraic autonomous ODEs of order 1. Essentially, the works used the exact degree bound of a proper parametrization of a rational algebraic curve by Sendra and Winkler (2001). Moreover, the generalization covers several interesting classes of first-order ODEs in literature, i.e., we can re-interpret some of the classical results and might have a different outlook.

Second, the method should be developed as a differential counterpart of the theory of rational parametrization of rational curves (Sendra et al. (2008)) and rational surfaces (Schicho (1997)). We know that a rational parametrization of a curve or a surface is a rational generic point on the corresponding geometrical object. A rational general solution of an algebraic ODE should do a similar task but now for a differential equation.

Finally, the rational solutions of a non-linear differential equation is meaningful, for instance, when one studies the exponential solutions of a linear differential equation Bronstein (1992), (see also Section 4.2.4).

The thesis consists of three main chapters 2, 3 and 4. In Chapter 2, we present a geometric method to decide the existence of a rational general solution of a parametrizable algebraic ODE of order 1. In the affirmative case this decision method can be turned into an algorithm for actually computing such a rational general solution. More precisely, a proper rational parametrization of the solution surface allows us to reduce the given differential equation to a system of autonomous algebraic ODEs of order 1 and of degree 1 in the derivatives, called the *associated system* w.r.t the chosen parametrization. This often turns out to be an advantage because the original differential equation is typically of higher degree in the derivative. The main result in this chapter is the one-to-one correspondence between a rational general solution of the associated system and that of the given differential equation. We prove this result in differential algebra formulation. Moreover, in this context, we give a criterion for the existence of a rational general solution of the associated system via the differential pseudo reduction of Feng-Gao's differential polynomials. As an application of this criterion, we given a complete description for linear polynomial systems having a rational general solution.

In Chapter 3, we continue the aim of Chapter 2 by focusing on a method to decide the existence of a rational general solution of the associated system. It is based on the notion of an *invariant algebraic curve* of a planar polynomial system, which is well-developed in Darboux's theory on the integrability of the system. Our main contribution in this chapter is the method to determine a rational solution of a planar rational system by rational parametrizations of a rational invariant algebraic curve. Moreover, the method is adapted to a general invariant algebraic curve in order to determine a rational general solution of the system. We also present a closed relation between a rational general solution and a rational first integral of the system. Hence, we have another characterization of the rational solvability of the system via the existence of a rational first integral.

In Chapter 4, we propose a method of classifying of first-order parametrizable algebraic ODEs w.r.t rational solvability. Precisely, we define a group of linear affine transformations, that the group action yields a partition on the set of all first-order parametrizable algebraic ODEs. The main idea in the construction of the group is to preserve the rational solvability of the differential equations under the group action. This is the first step in constructing a group of birational transformations of first-order algebraic ODEs. We give a description for some representatives of some classes of first-order parametrizable ODEs that are special either in their parametrizations or in their associated systems. In addition, Section 4.2.4 gives an observation on a rational solution of the differential equation $y' = R(x, y)$. By looking at infinity we transform the differential equation into the system at infinity and give the form of a possible rational invariant algebraic curve of the system at infinity. This makes a simplification in computing a rational invariant algebraic curve of this system, especially when we use the undetermined coefficients method.

Finally, we summarize this thesis with the list of main contributions and some open problem areas where one could try to generalize our work.

Chapter 2

Rational general solutions of first-order algebraic ODEs

We recall the notion of a general solution of a first-order algebraic ordinary differential equation (ODE) from the point of view of differential algebra, i.e., it is defined as a generic zero of a prime differential ideal in a differential ring. We refer to the appendix section (B) on differential algebra for most of preliminary notions that need for this chapter. The main development of this chapter is the algebraic geometric method for determining a rational general solution of a first-order algebraic ODE. We observe that the solution surface of an algebraic ODE of order 1 having a rational general solution must be a unirational surface. By Castelnuovo's theorem, every unirational surface over an algebraically closed field of characteristic 0 (e.g. the field of complex numbers) is a rational surface. Therefore, we only consider the class of all first-order parametrizable algebraic ODEs, i.e., the differential equation $F(x, y, y') = 0$ such that $F(x, y, z) = 0$ defines a rational surface. This class naturally extends the class of first-order autonomous ODEs in Feng and Gao (2004, 2006). In this class, we derive an associated system from a proper rational parametrization of the solution surface of the given differential equation. Then we prove that there is a one-to-one correspondence between a rational general solution of the given first-order parametrizable algebraic ODE and that of its associated system. In the last section, we give a criterion, based on Ritt's reduction of Feng-Gao's differential polynomials, for the existence of a rational general solution of the associated system. As an application, we use the criterion for determining the linear systems with rational general solutions.

Throughout this chapter, we consider $\mathbb{K}$ to be an algebraically closed field of characteristic zero, i.e., $\mathbb{K}$ contains the field of rational numbers $\mathbb{Q}$. The content of this chapter is essentially based on Ngô and Winkler (2010).

2.1 Definition of (rational) general solutions

Given an algebraic ODE $F(x, y', y'', \cdots, y^{(n)}) = 0$ of order n, where F is a polynomial over $\mathbb{K}$. Classically, a solution of $F(x, y', y'', \cdots, y^{(n)}) = 0$ is a function depending on x, $y = f(x)$, such that $F(x, f(x), f'(x), \cdots, f^{(n)}(x)) = 0$. From the point of view of algebra, the polynomial F can be formally seen as a differential polynomial in the differential ring $(\mathbb{K}(x)\{y\}, \delta)$, where y is a differential indeterminate and δ is the unique derivation extended from the usual derivation $\frac{d}{dx}$ of the differential field $\mathbb{K}(x)$ (see B). Let us write the above differential equation in the form $F(y) = 0$ to simplify the notation when we do not want to stress on the order of the equation.

Let $(\mathcal{K}, \delta)$ be a differential field extension of $(\mathbb{K}(x), \frac{d}{dx})$. A solution of $F(y) = 0$ in $\mathcal{K}$ is an element $\eta \in \mathcal{K}$ such that $F(x, \eta, \delta\eta, \cdots, \delta^n\eta) = 0$. Observe that, if η is a solution of $F(y) = 0$, then η is also a solution of all $\delta^m(F)(y) = 0$ for any natural number $m \geq 1$. In fact, η is also a solution of the differential ideal generated by F, denoted by $[F]$. Furthermore, according to the theorem of zeros, Ritt (1950), II, §7 (also known as the differential Nullstellensatz), the collection of all differential polynomials in $\mathbb{K}(x)\{y\}$ vanishing on the solutions of F is the *radical differential ideal* generated by F, denoted by $\{F\}$ the set

$$\{F\} = \{A \in \mathbb{K}(x)\{y\} \mid \exists m \in \mathbb{N},\ A^m \in [F]\}.$$

It is known from Ritt (1950), II, §14, that we can decompose $\{F\}$ as

$$\{F\} = (\{F\} : S) \cap \{F, S\}, \tag{2.1}$$

where S is the separant of F and $\{F\} : S = \{A \in \mathbb{K}(x)\{y\} \mid SA \in \{F\}\}$. Note that $\{F\} : S$ is a radical differential ideal and $\{F\} : S = \{F\} : S^\infty$, defined by

$$\{F\} : S^\infty = \{A \in \mathbb{K}(x)\{y\} \mid \exists m \in \mathbb{N},\ \ S^m A \in \{F\}\}.$$

The ideal $\{F\} : S^\infty$ is called the saturation ideal of $\{F\}$ by S. Moreover, if F is an irreducible polynomial in the polynomial ring $\mathbb{K}[x, y_1, y_2, \ldots, y_n]$, which we assume from now on, then $\{F\} : S$ is a prime differential ideal (by Ritt (1950), II, §12). One can further decompose $\{F, S\}$ as the intersection of finite number of prime differential ideals because $\mathbb{K}(x)\{y\}$ is a radical Noetherian ring, i.e., the ring in which every radical differential ideal is finitely generated. In the end, if we exclude all redundant prime differential ideals, then we obtain a unique minimal decomposition of $\{F\}$ into an intersection of irredundant prime differential ideals, which are called *essential components* of $\{F\}$. Furthermore, $\{F\} : S$ is the unique essential component of $\{F\}$ that does not contain the separant S because if $\{F\} : S$ would contain S, then $S^2 \in \{F\}$. Hence, $S \in \{F\}$, a contradiction to the fact that $\deg_{y^{(n)}} S < \deg_{y^{(n)}} F$.

Definition 2.1.1. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of $F(y) = 0$. By a *generic zero* η of $\{F\} : S$ we mean for all $G \in \mathbb{K}(x)\{y\}$, $G(\eta) = 0 \Longleftrightarrow G \in \{F\} : S$.

Definition 2.1.2. A zero of $\{F, S\}$ is called a *singular solution* of $F(y) = 0$.

Of course, when we decompose $\{F, S\}$ into prime differential ideals, there might be some components that contain $\{F\} : S$. These components will be corresponding to the *particular solutions* of $F(y) = 0$ in the classical sense. We will demonstrate this in an example later.

In the quotient ring $\mathbb{K}(x)\{y\}/(\{F\} : S)$, which is an integral domain*, the class of y is a generic zero of the prime differential ideal $\{F\} : S$. However, this is still an implicit description of a general solution of $F(y) = 0$. Another way of describing a general solution of $F(y) = 0$ is computing a basis of the differential ideal $\{F\} : S$. In her paper Hubert (1996), Hubert presents an algorithmic method to determine a Gröbner basis of the differential ideal $\{F\} : S$ in the case $\mathrm{ord}(F) = 1$. Our question is:

> **Q 1.** *How to construct a generic zero of* $\{F\} : S$*, i.e., a general solution of* $F(y) = 0$ *explicitly?*
>
> Our goal, in this chapter and in the next chapter(s), is to develop a method to construct explicitly a rational general solution of $F(y) = 0$ in the case of *first-order parametrizable algebraic ODEs*, $F(x, y, y') = 0$, where $F(x, y, z) = 0$ defines a rational surface.

Definition 2.1.3. A *rational general solution* of $F(y) = 0$ is defined as a general solution of $F(y) = 0$ of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}, \tag{2.2}$$

where a_i, b_j are constants in a differential field extension of $\mathbb{K}(x)$.

From the definition of a general solution of $F(y) = 0$, it is important to know when a differential polynomial belongs to $\{F\} : S$. This ideal membership problem is solved by using Ritt's reduction. Precisely, consider the differential ring $\mathbb{K}(x)\{y\}$ with the orderly ranking (see B.2). Then for any $G \in \mathbb{K}(x)\{y\}$, we have

$$S^{s_F} I^{i_F} G = \sum_{i \geq 0} Q_i \delta^i(F) + R, \tag{2.3}$$

*In Ritt (1950), II, §6, one can define a natural derivation on the quotient field of this integral domain so that it becomes a differential field.

where $\delta^i(F)$ is the i-th derivative of F and the order of $\delta^i(F)$ is less than or equal the order of G; S is the *separant* of F, I is the *initial* of F, $s_F, i_F \in \mathbb{N}$, $Q_i \in \mathbb{K}(x)\{y\}$ and $R \in \mathbb{K}(x)\{y\}$ is *reduced* with respect to F, i.e., if n is the order of F, then the order of R is at most n and $\deg_{y^{(n)}}(R) < \deg_{y^{(n)}}(F)$. Moreover, if m is the order of G, then $0 \le i \le m-n$. By convention, the sum is empty if $m < n$; in this case, $R = G$.

Definition 2.1.4. The differential polynomial R in (2.3) is called the *differential pseudo remainder* of G with respect to F, denoted by $\mathrm{prem}(G, F)$.

The following theorem, whose proof can be found in Ritt (1950), II, §13, gives an algorithmic method to solve the ideal membership problem of $\{F\} : S$.

Theorem 2.1.1. *For every $G \in \mathbb{K}(x)\{y\}$, $G \in \{F\} : S \Longleftrightarrow \mathrm{prem}(G, F) = 0$.*

Corollary 2.1.2. *Suppose that η is a general solution of $F(y) = 0$. Then for every $G \in \mathbb{K}(x)\{y\}$, $G(\eta) = 0 \Longleftrightarrow \mathrm{prem}(G, F) = 0$.*

Observe that $S \notin \{F\} : S$ because $\mathrm{prem}(S, F) = S \neq 0$. Therefore, a general solution of $F(y) = 0$ is not annulled by S.

It is known from the point of view of analysis[†] that the most general solution of $F(x, y, y')$ contains one arbitrary constant (e.g. Ince (1926); Piaggio (1933))[‡]. The conclusion applies for higher order ODEs accordingly. In differential algebra context, one have to make the meaning of the term "arbitrary constant" precisely. By Ritt (1950), III, §5, an *arbitrary constant* w.r.t a given field $\mathbb{K}$ is a quantity c which can be adjoined to the field $\mathbb{K}$—to obtain an extension field $\mathbb{K}(c)$—which is transcendental over $\mathbb{K}$ and the derivative of c in the extension field $\mathbb{K}(c)$ is zero.

Let us see the above fact in the case of rational general solutions. Suppose that

$$y^* = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0}$$

is a rational general solution of $F(y) = 0$. Then there exists a coefficient of y^* does not belong to $\mathbb{K}$. Otherwise, the differential polynomial (of order 0)

$$G = (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0)y - (a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0) \in \mathbb{K}(x)\{y\}$$

vanishes on y^*, but $\mathrm{prem}(G, F) = G \neq 0$. Therefore, y^* must contain a constant which is not in $\mathbb{K}$ and hence it is transcendental over $\mathbb{K}$ because $\mathbb{K}$ is algebraically closed.

[†]A solution is expanded into a Taylor series, one looks for its coefficients in the expansion.

[‡]In Piaggio (1933): Chapter I, §7.

Remark 2.1.1. Suppose that $y = f(x, c)$ is a rational general solution of $F(x, y, y') = 0$, where c is an arbitrary constant, i.e., we have

$$F(x, f(x, c), f_x(x, c)) = 0,$$

where f_x is the partial derivative of f w.r.t. x. Let us view x and c as parameters of the rational map

$$\mathcal{P}(x, c) = (x, f(x, c), f_x(x, c)). \tag{2.4}$$

The Jacobian matrix of $\mathcal{P}(x, c)$ is

$$J_{\mathcal{P}} = \begin{pmatrix} 1 & f_x(x, c) & f_{xx}(x, c) \\ 0 & f_c(x, c) & f_{xc}(x, c) \end{pmatrix}. \tag{2.5}$$

Since f effectively depends on c, we have $f_c(x, c) \neq 0$. Therefore, the generic rank of $J_{\mathcal{P}}$ is 2. Hence, $\mathcal{P}(x, c)$ is a rational parametrization of the surface $F(x, y, z) = 0$. This is the reason why we restrict the consideration to the class of rational parametrizable algebraic ODEs of order 1.

2.2 The associated system of first-order parametrizable algebraic ODEs

2.2.1 Determination of the associated system

In this section, we present a method to determine a rational general solution of the first-order algebraic ODE

$$F(x, y, y') = 0, \tag{2.6}$$

where $F(x, y, z)$, an irreducible polynomial in $\mathbb{K}[x, y, z]$, defines a rational surface.

First of all, suppose that $y = f(x)$ is a rational solution of $F(x, y, y') = 0$. Then it generates a rational space curve parametrized by $(x, f(x), f'(x))$, here x is viewed as a parameter, and the curve belongs to the algebraic surface defined by $F(x, y, z)$. Therefore, solving the differential equation $F(x, y, y') = 0$ amounts to look for all such parametric curves on the algebraic surface $F(x, y, z) = 0$. By Remark 2.1.1, it is natural to consider those algebraic surfaces possessing rational parametrizations.

Definition 2.2.1. The algebraic surface $F(x, y, z) = 0$ is called a *unirational surface* iff there exists a rational map

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)) \tag{2.7}$$

such that $F(\mathcal{P}(s,t)) = 0$, where χ_1, χ_2 and χ_3 are rational functions in s and t, and the Jacobian matrix of $\mathcal{P}(s,t)$ has a generic rank 2. Then $\mathcal{P}(s,t)$ is called a *rational parametrization* of $F(x,y,z) = 0$.

Since the Jacobian matrix $J_{\mathcal{P}}$ of $\mathcal{P}(s,t)$ has a generic rank 2, at least one of the 2×2 minors of $J_{\mathcal{P}}$ is non-zero. We can assume w.l.o.g that

$$\chi_{1s}\chi_{2t} - \chi_{1t}\chi_{2s} \neq 0, \tag{2.8}$$

where χ_{is}, χ_{it} are the partial derivatives of χ_i w.r.t s and t. Because if it is not the case, then $\chi_1(s,t)$ and $\chi_2(s,t)$ must be related by the equation $\chi_1(s,t) = \phi(-\chi_2(s,t))$, where $\phi(y)$ is an arbitrary function of one variable. Since χ_1 and χ_2 are rational functions, it follows that $\phi(y)$ is a rational function. Then the surface $x - \phi(-y) = 0$ is a component of $F(x,y,z) = 0$, hence by irreducibility, $F(x,y,z)$ is the numerator of $x - \phi(-y)$. In this case, the surface defined by F is not corresponding to any first-order algebraic ODE.

Definition 2.2.2. A rational parametrization $\mathcal{P}(s,t)$ of $F(x,y,z) = 0$ is called *proper* iff it has an inverse and its inverse is also rational, i.e., there is a rational map

$$\mathcal{Q}(x,y,z) = (\psi_1(x,y,z), \psi_2(x,y,z))$$

such that $(\mathcal{Q} \circ \mathcal{P})(s,t) = (s,t)$ for almost all s, t and $(\mathcal{P} \circ \mathcal{Q})(x,y,z) = (x,y,z)$ for almost all (x,y,z) on the surface $F(x,y,z) = 0$ or, equivalently, $\mathbb{K}(\mathcal{P}(s,t)) = \mathbb{K}(s,t)$. Such a $\mathcal{P}(s,t)$ is called a birational map. The surface defined by $F(x,y,z) = 0$ is called *rational* iff it has a proper rational parametrization.

Note that, the parametrization (2.4) in Remark 2.1.1 may be not proper.

Definition 2.2.3. The *solution surface* of $F(x,y,y') = 0$, denoted by $\mathcal{S}$, is the surface $F(x,y,z) = 0$ when we view x, y, z as independent variables.

Definition 2.2.4. An algebraic ODE $F(x,y,y') = 0$ is called a *parametrizable algebraic ODE* if its solution surface is rational, i.e., it admits a rational parametrization of the form (2.7).

In the sequel, we denote by $\mathcal{AODE}$ the set $\mathcal{AODE} = \{F(x,y,y') = 0 \mid F \in \mathbb{K}[x,y,z]\}$ and by $\mathcal{PODE}$ the set

$$\mathcal{PODE} = \{F \in \mathcal{AODE} \mid \text{the solution surface } F = 0 \text{ is rationally parametrizable}\}.$$

In $\mathcal{AODE}$, if F is not involving x, then the differential equation (2.6) is called *autonomous*. In general, F is possibly involving x, the differential equation (2.6) is called *non-autonomous*. We view an autonomous ODE as a special case of the non-autonomous one.

Definition 2.2.5. Let $y = f(x)$ be a rational solution of $F(x, y, y') = 0$. The space curve parametrized by $\mathcal{C}(x) = \{(x, f(x), f'(x)) \mid x \in \mathbb{K}\}$ is called the *solution curve* of f w.r.t. $F(x, y, y') = 0$ or simply a solution curve of f when the differential equation is clear from the context.

In some textbooks, the curve $\mathcal{C}(x) = (x, f(x), f'(x))$ is called an *integral curve*. We will use this terminology when we consider the curve $\mathcal{C}(x)$ without taking into account an algebraic differential equation $F(x, y, y') = 0$ having $f(x)$ as a solution. If $f(x)$ is a rational function in x, then one can easily generate an algebraic differential equation $F(x, y, y') = 0$ having $f(x)$ as a solution. Hence, $\mathcal{C}(x)$ becomes a solution curve of f w.r.t. $F(x, y, y') = 0$.

From now on, we always consider F in $\mathcal{PODE}$ and $\mathcal{P}(s, t)$ to be a proper rational parametrization of $F(x, y, z) = 0$. The inverse map of $\mathcal{P}$, denoted by $\mathcal{P}^{-1}$, defines on the surface $\mathcal{S}$, except for finitely many curves or points on $\mathcal{S}$.

Definition 2.2.6. Let $f(x)$ be a rational solution of the equation $F(x, y, y') = 0$. Let $\mathcal{S}$ be the solution surface of $F(x, y, y') = 0$ and $\mathcal{C}(x)$ be the solution curve of f. Let $\mathcal{P}$ be a proper rational parametrization of $F(x, y, z) = 0$. The solution curve $\mathcal{C}(x)$ is *parametrizable* by $\mathcal{P}$ iff $\mathcal{C}(x)$ is almost contained in $\mathrm{im}(\mathcal{P}) \cap \mathrm{dom}(\mathcal{P}^{-1})$, i.e., except for finitely many points on $\mathcal{C}(x)$. Here $\mathrm{im}(\mathcal{P})$ and $\mathrm{dom}(\mathcal{P}^{-1})$ are the image and the domain of the corresponding maps.

Proposition 2.2.1. *Let $F \in \mathcal{PODE}$ with a proper parametrization $\mathcal{P}(s, t)$. The differential equation $F(x, y, y') = 0$ has a rational solution whose solution curve is parametrizable by $\mathcal{P}$ if and only if the system*

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x))' = \chi_3(s(x), t(x)) \end{cases} \tag{2.9}$$

has a rational solution $(s(x), t(x))$. In that case, $y = \chi_2(s(x), t(x))$ is a rational solution of $F(x, y, y') = 0$.

Proof. Assume that $y = f(x)$ is a rational solution of $F(x, y, y') = 0$ and the solution curve of $f(x)$ is parametrizable by $\mathcal{P}$. Let $(s(x), t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$. Then $s(x)$ and $t(x)$ are rational functions because $f(x)$ is a rational function and $\mathcal{P}^{-1}$ is a rational map. We have

$$\mathcal{P}(s(x), t(x)) = \mathcal{P}(\mathcal{P}^{-1}(x, f(x), f'(x))) = (x, f(x), f'(x)).$$

In other words, $(s(x), t(x))$ is a rational solution of the system

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x)) = f(x) \\ \chi_3(s(x), t(x)) = f'(x). \end{cases} \tag{2.10}$$

Therefore, $\chi_1(s(x), t(x)) = x$ and $\chi_2(s(x), t(x))' = \chi_3(s(x), t(x))$. Conversely, if two rational functions $s = s(x)$ and $t = t(x)$ satisfy the system (2.9), then $y = \chi_2(s(x), t(x))$ is a rational solution of $F(x, y, y') = 0$ because $F(\mathcal{P}(s(x), t(x))) = 0$. $\square$

Note that most of solution curves of $F(x, y, y') = 0$ will be parametrizable by $\mathcal{P}$ because $\mathcal{P}$ covers almost all the solution surface $\mathcal{S}$ and $\mathcal{P}^{-1}$ is defined at almost everywhere on the solution surface $\mathcal{S}$ except for finitely many curves or points on $\mathcal{S}$.

Q 2. *What are rational solutions of the system* (2.9)*?*

Let us have a closed looking at the system (2.9). We see that it can be decomposed as a differential system and an algebraic system. Indeed, differentiating the first equation of (2.9) and expanding the last equation of (2.9), we obtain a linear system of equations in $s'(x)$ and $t'(x)$

$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \cdot t'(x) = 1 \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x), t(x)). \end{cases} \tag{2.11}$$

If

$$\det \begin{pmatrix} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \end{pmatrix} \neq 0, \tag{2.12}$$

then $(s(x), t(x))$ is a rational solution of the autonomous system of differential equations

$$\left\{ s' = \frac{f_1(s,t)}{g(s,t)}, \;\; t' = \frac{f_2(s,t)}{g(s,t)} \right\} \tag{2.13}$$

where $f_1(s,t), f_2(s,t), g(s,t) \in \mathbb{K}(s,t)$ are defined by

$$\begin{aligned} f_1(s,t) =& \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t}, \quad f_2(s,t) = \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s} - \frac{\partial \chi_2(s,t)}{\partial s}, \\ g(s,t) =& \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}. \end{aligned} \tag{2.14}$$

If the determinant (2.12) is equal to 0, then $(s(x), t(x))$ is a solution of the algebraic system

$$\left\{\bar{g}(s,t) = 0, \quad \bar{f}_1(s,t) = 0\right\} \tag{2.15}$$

where $\bar{g}(s,t)$ and $\bar{f}_1(s,t)$ are the numerators of $g(s,t)$ and $f_1(s,t)$, respectively. In the latter case, $(s(x), t(x))$ defines a curve if and only if $\gcd(\bar{g}(s,t), \bar{f}_1(s,t))$ is a non-constant polynomial in s, t. Otherwise, $(s(x), t(x))$ is just an intersection point of the two algebraic curves $\bar{g}(s,t) = 0$ and $\bar{f}_1(s,t) = 0$, which does not satisfy the relation (2.9).

Therefore, the rational solutions of the system (2.9) is the union of the rational solutions of (2.13) and the non-trivial rational solutions of (2.15).

Definition 2.2.7. The autonomous system (2.13) is called the *associated system* of the differential equation $F(x, y, y') = 0$ with respect to $\mathcal{P}(s,t)$.

The main features of the associated system are autonomous, of order 1 and of degree 1 with respect to s' and t'. Later, these features turn out to be the advantages of the approach.

Claim 1. *A rational general solution of the system* (2.13) *completely determines a rational general solution of the differential equation* $F(x, y, y') = 0$.

At this point we define, from the point of view of differential algebra, what we mean by a rational general solution of the system (2.13). For this purpose we need some preparations.

2.2.2 Rational general solutions of the associated system

Consider the new differential ring $\mathbb{K}(x)\{s,t\}$ with the usual derivation δ extended from the derivation $\dfrac{d}{dx}$ of $\mathbb{K}(x)$, where s, t are two differential indeterminates. We denote by s_i and t_i the i-th derivatives of s and t, respectively.

Definition 2.2.8. Let $\mathcal{V} = \{s_i \mid i \in \mathbb{N}\} \cup \{t_i \mid i \in \mathbb{N}\}$. The *ord-lex ranking* on $\mathcal{V}$ is the total order defined as follows:

$$\begin{cases} s_i < s_j \text{ if } i < j, \\ t_i < t_j \text{ if } i < j, \\ t_i < s_j \text{ if } i \leq j, \\ s_i < t_j \text{ if } i < j. \end{cases}$$

The ord-lex ranking is an orderly ranking (see the appendix B.2). We use this ranking on the differential ring $\mathbb{K}(x)\{s,t\}$ from now on.

Definition 2.2.9. Let $F, G \in \mathbb{K}(x)\{s,t\}$. F is said to be *of higher rank than G in s* iff one of the following conditions holds:

1. $\operatorname{ord}_s(F) > \operatorname{ord}_s(G)$;

2. $\operatorname{ord}_s(F) = \operatorname{ord}_s(G) = n$ and $\deg_{s_n}(F) > \deg_{s_n}(G)$.

If F is of higher rank than G in s, then we also say G is *of lower rank than F in s*. Analogously these notions are defined for t.

Definition 2.2.10. Let F be a differential polynomial in $\mathbb{K}(x)\{s,t\}$. The *leader* of F is the highest derivative occurring in F with respect to the ord-lex ranking on the set of derivatives $\mathcal{V}$. The *initial* of F is the leading coefficient of F with respect to its leader. The *separant* of F is the partial derivative of F with respect to its leader.

Observe that the separant of F is also the initial of any proper derivative $\delta^i(F)$ of F.

Definition 2.2.11. Let F and G be differential polynomials in $\mathbb{K}(x)\{s,t\}$. G is said to be *reduced* with respect to F iff G is of lower rank than F in the indeterminate defining the leader of F.

Let $\mathcal{A}$ be an autoreduced set in the differential ring $\mathcal{R} = \mathbb{K}(x)\{s,t\}$. Let $G \in \mathbb{K}(x)\{s,t\}$. By Ritt's reduction, Ritt (1950), Kolchin (1973)[§], there exist $R \in \mathcal{R}$, $s_A, i_A \in \mathbb{N}$ such that R is reduced w.r.t. $\mathcal{A}$, the rank of R is lower than or equal to that of G and

$$\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{s_A} G - R$$

can be written as a linear combination over $\mathcal{R}$ of derivatives $\{\delta^i(A) \mid A \in \mathcal{A}, \delta^i(u_A) \leq u_G\}$, where u_A and u_G are the leader of A and G, respectively. The differential polynomial R is called the *differential pseudo remainder* of G with respect to $\mathcal{A}$, denoted by

$$R = \operatorname{prem}(G, \mathcal{A}).$$

From now on, we consider $M_i, N_i \in \mathbb{K}[s,t]$, $N_i \neq 0$, $\gcd(M_i, N_i) = 1$ for $i = 1, 2$ and two special differential polynomials F_1 and F_2 in $\mathcal{R}$ defined as follows

$$F_1 := N_1 s' - M_1, \quad F_2 := N_2 t' - M_2.$$

In fact, F_1 and F_2 are in the subring $\mathbb{K}\{s,t\}$ of autonomous differential polynomials of $\mathcal{R}$. The leaders of F_1 and F_2 are s' and t', respectively. Moreover, $\deg_{s'}(F_1) = \deg_{t'}(F_2) = 1$. It follows that the initial and separant of F_1 (respectively, of F_2) are the same. The differential ideal generated by F_1 and F_2 is denoted by $[F_1, F_2]$. In applications, we will take M_1, M_2, N_1, N_2 to be the polynomials in the numerators and the denominators of the right hand side of the associated system (2.13).

[§]Kolchin (1973), I, Proposition 1.

The set $\mathcal{A} = \{F_1, F_2\}$ is an *autoreduced set* relative to the ord-lex ranking because F_1 is reduced with respect to F_2 and F_2 is reduced with respect to F_1. Now by Ritt's reduction, for any $G \in \mathbb{K}(x)\{s,t\}$, we can reduce G w.r.t. the autoreduced set $\mathcal{A}$ to obtain the differential pseudo remainder $R = \operatorname{prem}(G, \mathcal{A})$, which is also denoted by $R = \operatorname{prem}(G, F_1, F_2)$ in an explicit form.

Observation 2.2.1. *Observe that the differential pseudo remainder $R = prem(G, F_1, F_2)$ is always a polynomial in $\mathbb{K}(x)[s,t]$ because F_1 and F_2 are of order 1 and of degree 1 w.r.t. their leaders.*

Lemma 2.2.2. *Let*

$$\mathcal{I} = \{G \in \mathbb{K}(x)\{s,t\} \mid \operatorname{prem}(G, F_1, F_2) = 0\}.$$

Then $\mathcal{I}$ is a prime differential ideal in $\mathbb{K}(x)\{s,t\}$.

Proof. Let $H_{\mathcal{A}} = N_1 N_2$ and denote $H_{\mathcal{A}}^{\infty} = \{N_1^{m_1} N_2^{m_2} \mid m_1, m_2 \in \mathbb{N}\}$. Then

$$[F_1, F_2] : H_{\mathcal{A}}^{\infty} := \{G \in \mathbb{K}(x)\{s,t\} \mid \exists J \in H_{\mathcal{A}}^{\infty}, JG \in [F_1, F_2]\}$$

is a prime differential ideal (Ritt (1950), V, §3, page 107). We prove that

$$\mathcal{I} = [F_1, F_2] : H_{\mathcal{A}}^{\infty}.$$

In fact, it is clear that $\mathcal{I} \subseteq [F_1, F_2] : H_{\mathcal{A}}^{\infty}$. Let $G \in [F_1, F_2] : H_{\mathcal{A}}^{\infty}$. Then there exists $J \in H_{\mathcal{A}}^{\infty}$ such that $JG \in [F_1, F_2]$. On the other hand, let $R = \operatorname{prem}(G, F_1, F_2)$, we have

$$J_1 G - R \in [F_1, F_2]$$

for some $J_1 \in H_{\mathcal{A}}^{\infty}$. It follows that $JR \in [F_1, F_2]$. Since R and J are in $\mathbb{K}(x)[s,t]$, we have $JR \in [F_1, F_2]$ if and only if $JR = 0$. We must have $R = 0$ because $J \neq 0$. Therefore, $\mathcal{I} = [F_1, F_2] : H_{\mathcal{A}}^{\infty}$, i.e., $\mathcal{I}$ is a prime differential ideal. $\square$

Definition 2.2.12. Let $M_i, N_i \in \mathbb{K}[s,t],\ N_i \neq 0,\ \gcd(M_i, N_i) = 1$ for $i = 1, 2$. A rational solution $(s(x), t(x))$ of the autonomous system

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)},\ t' = \frac{M_2(s,t)}{N_2(s,t)} \right\} \tag{2.16}$$

is called a *rational general solution* iff it is a rational generic zero of the prime differential ideal $\mathcal{I}$, i.e., for any $G \in \mathbb{K}(x)\{s,t\}$, we have

$$G(s(x), t(x)) = 0 \Longleftrightarrow \operatorname{prem}(G, N_1 s' - M_1, N_2 t' - M_2) = 0. \tag{2.17}$$

Lemma 2.2.3. *Let* $(s(x), t(x))$ *be a rational general solution of the system* (2.16)*. Let* G *be a bivariate polynomial in* $\mathbb{K}(x)[s,t]$*. Then* $G(s(x), t(x)) = 0 \Longleftrightarrow G = 0$ *in* $\mathbb{K}(x)[s,t]$*.*

Proof. Since $G \in \mathbb{K}(x)[s,t]$, we have $\operatorname{prem}(G, N_1 s' - M_1, N_2 t' - M_2) = G$. By definition of general solutions, $G(s(x), t(x)) = 0 \Longleftrightarrow G = 0$ in $\mathbb{K}(x)[s,t]$. □

Lemma 2.2.4. *Let*

$$s(x) = \frac{a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0}{b_l x^l + b_{l-1} x^{l-1} + \cdots + b_0} \quad \text{and} \quad t(x) = \frac{c_m x^m + c_{m-1} x^{m-1} + \cdots + c_0}{d_n x^n + d_{n-1} x^{n-1} + \cdots + d_0}$$

be a non-trivial rational solution of the system (2.16)*, where* a_i, b_i, c_i, d_i *are in some field of constants* $\mathbb{L}$*, extended from* $\mathbb{K}$*; and* $b_l, d_n \neq 0$*. If* $(s(x), t(x))$ *is a rational general solution of the system* (2.16)*, then there exists a constant, which is transcendental over* $\mathbb{K}$*, among the coefficients of* $s(x)$ *and* $t(x)$*.*

Proof. Let $S = (b_l x^l + b_{l-1} x^{l-1} + \cdots + b_0)s - (a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0)$ and $T = (d_n x^n + d_{n-1} x^{n-1} + \cdots + d_0)t - (c_m x^m + c_{m-1} x^{m-1} + \cdots + c_0)$. Let $G = \operatorname{res}_x(S, T)$ be the resultant of S and T with respect to x. Then G is a polynomial in s and t with the coefficients depending on a_i, b_i, c_i, d_i. If all a_i, b_i, c_i, d_i were in $\mathbb{K}$, then $G \in \mathbb{K}[s,t]$ and $G(s(x), t(x)) = 0$. Since $(s(x), t(x))$ is a rational general solution, it follows by Lemma 2.2.3 that $G = 0$. But G is the implicit equation of the rational curve with parametrization $(s(x), t(x))$; compare Chapter 4 §4.5 in Sendra et al. (2008). So $G \neq 0$, in contradiction. Therefore, there is a coefficient of $s(x)$ or $t(x)$ that does not belong to $\mathbb{K}$. Since $\mathbb{K}$ is an algebraically closed field, a constant which is not in $\mathbb{K}$ must be a transcendental element over $\mathbb{K}$. □

This lemma gives us a necessary condition for $(s(x), t(x))$ to be a rational general solution of the system (2.16). It requires that any rational general solution of the system has to contain at least one coefficient transcendental over the constant field of the system itself. As an early discussion in the chapter, this transcendental coefficient is an arbitrary constant. Next we give a sufficient condition for a rational solution $(s(x), t(x))$ of the system (2.16) to be a rational general solution.

Lemma 2.2.5. *Let* $(s(x), t(x))$ *be a rational solution of the system* (2.16)*. Let* $H(s,t)$ *be the monic defining polynomial (w.r.t. a lexicographic order of terms in* s *and* t*) of the rational algebraic curve defined by* $(s(x), t(x))$*. If there is an arbitrary constant in the set of coefficients of* $H(s,t)$*, then* $(s(x), t(x))$ *is a rational general solution of the system* (2.16)*.*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the system (2.16). Let $G \in$

$\mathbb{K}(x)\{s,t\}$ be a differential polynomial such that $G(s(x),t(x))=0$. Let

$$R = \operatorname{prem}(G, N_1 s' - M_1, N_2 t' - M_2).$$

Then $R \in \mathbb{K}(x)[s,t]$ and $R(s(x),t(x)) = 0$. It is sufficient to assume that there is only one arbitrary constant c in the set of coefficients of $H(s,t)$. Then $H(s,t) \in \mathbb{K}(c)[s,t]$. Moreover, $H(s,t)$ is irreducible over $\overline{\mathbb{K}(c)}$ because it is a rational curve (Sendra et al. (2008), Theorem 4.4). Therefore, the polynomial $R(s,t)$ must be a multiple of $H(s,t)$. This happens if and only if $R = 0$. It follows that $(s(x),t(x))$ is a rational general solution of the system (2.16). $\square$

> **Q 3.** *What is the inverse image of a solution curve of a rational general solution of $F(x,y,y')=0$?*

Theorem 2.2.6. *Let $y = f(x)$ be a rational general solution of $F(x,y,y')=0$. Suppose that the solution curve of f is parametrizable by $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$. Let*

$$(s(x),t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$$

and $g(s,t) = \dfrac{\partial\chi_1(s,t)}{\partial s}\cdot\dfrac{\partial\chi_2(s,t)}{\partial t} - \dfrac{\partial\chi_1(s,t)}{\partial t}\cdot\dfrac{\partial\chi_2(s,t)}{\partial s}$. Then $g(s(x),t(x)) \neq 0$ and $(s(x),t(x))$ is a rational general solution of the system (2.13).

Proof. It is sufficient to prove the claim that if $R \in \mathbb{K}(x)[s,t]$ and $R(s(x),t(x)) = 0$, then $R = 0$. If this is done, then $g(s(x),t(x)) \neq 0$ because $g(s,t) \neq 0$ by (2.8). Suppose that $P \in \mathbb{K}(x)\{s,t\}$ is a differential polynomial such that $P(s(x),t(x)) = 0$. Let

$$R = \operatorname{prem}(P, N_1 s' - M_1, N_2 t' - M_2),$$

where M_1, M_2, N_1, N_2 are numerators and denominators of the right hand side of the system (2.13). Then $R \in \mathbb{K}(x)[s,t]$ and $P(s(x),t(x)) = 0$ implies that $R(s(x),t(x)) = 0$. By the claim, $R = 0$. Hence $(s(x),t(x))$ is a rational general solution of the system (2.13).

Now it remains to prove the claim. We have

$$R(s(x),t(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x))) = 0.$$

Let us consider the rational function $R(\mathcal{P}^{-1}(x,y,z)) = \dfrac{U(x,y,z)}{V(x,y,z)}$. Then $U(x,y,y')$ is a differential polynomial satisfying the condition

$$U(x, f(x), f'(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F(y)=0$ and $U(x,y,y')$ vanishes on $y = f(x)$, the differential pseudo remainder of U with respect to F must be zero. On the other hand, both F and U are differential polynomials of order 1, we only divide U by F and not by any of its derivatives. Hence, we have the reduction

$$I^m U = Q_0 F,$$

where I is the initial of F, $m \in \mathbb{N}$ and Q_0 is a differential polynomial of order at most 1 in $\mathbb{K}(x)\{y\}$. Therefore,

$$R(s,t) = R(\mathcal{P}^{-1}(\mathcal{P}(s,t))) = \frac{U(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = \frac{Q_0(\mathcal{P}(s,t))F(\mathcal{P}(s,t))}{I^m(\mathcal{P}(s,t))V(\mathcal{P}(s,t))} = 0$$

because $F(\mathcal{P}(s,t)) = 0$ and $I(\mathcal{P}(s,t)) \neq 0$. □

Q 4. *Suppose that we have a rational solution of $F(x,y,y') = 0$ but it is not parametrizable by $\mathcal{P}$? Could this solution be a rational general solution of $F(x,y,y') = 0$?*

Proposition 2.2.7. *If $y = f(x)$ is a rational solution of $F(x,y,y') = 0$ but it is not parametrizable by $\mathcal{P}$, then it can not be a rational general solution of $F(x,y,y') = 0$.*

Proof. Since $y = f(x)$ is not parametrizable by $\mathcal{P}$, the solution curve $(x, f(x), f'(x))$ must lie on the intersection of the solution surface $F(x,y,z) = 0$ and another surface $G(x,y,z) = 0$ defined by the denominators of the inverse map $\mathcal{P}^{-1}$. The resultant $R(x,y) = res_z(F,G)$ is vanished on $f(x)$ and reduced w.r.t. F. By definition, this can not be a general solution. □

Q 5. *How to construct a rational general solution of $F(x,y,y') = 0$ from a rational general solution of its associated system?*

Assume that $(s(x), t(x))$ is a rational general solution of the associated system (2.13). Substituting $s(x)$ and $t(x)$ into $\chi_1(s,t)$ and using the relation (2.11) we get $\chi_1(s(x),t(x)) = x + c$ for some constant c. Hence $\chi_1(s(x-c), t(x-c)) = x$. It follows that

$$y = \chi_2(s(x-c), t(x-c)) \tag{2.18}$$

is a solution of the differential equation $F(x,y,y') = 0$. Moreover, we will prove that $y = \chi_2(s(x-c), t(x-c))$ is a rational general solution of $F(x,y,y') = 0$.

Theorem 2.2.8. *Let $(s(x), t(x))$ be a rational general solution of the system* (2.13). *Let $c = \chi_1(s(x), t(x)) - x$. Then $y = \chi_2(s(x-c), t(x-c))$ is a rational general solution of $F(x,y,y') = 0$.*

Proof. By the above discussion, it is clear that $y = \chi_2(s(x-c), t(x-c))$ is a rational solution of $F(x,y,y') = 0$. Let G be an arbitrary differential polynomial in $\mathbb{K}(x)\{y\}$ such that $G(y) = 0$. Let $R = \mathrm{prem}(G, F)$ be the differential pseudo remainder of G with respect to F. It follows that $R(y) = 0$. We have to prove that $R = 0$. Assume that $R \neq 0$. Then

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \mathbb{K}(s,t).$$

On the other hand,

$$R(\mathcal{P}(s(x-c), t(x-c))) = R(x, y, y') = 0.$$

It follows that $W(s(x-c), t(x-c)) = 0$, hence, $W(s(x), t(x)) = 0$. By Lemma 2.2.3 we must have $W(s,t) = 0$. Thus $R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0$. Since F is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$ in $\mathbb{K}[x, y, z]$. Therefore, y is a rational general solution of $F(x, y, y') = 0$. □

2.2.3 Algorithm and example

Theorem 2.2.6 and Theorem 2.2.8 give a method to determine a rational general solution of the first-order parametrizable algebraic ODE $F(x, y, y') = 0$. We summarize the procedure by the following semi-algorithm. It depends on a method for solving the system (2.13).

Algorithm `GENERALSOLVER`
Input: $F(x, y, y') = 0$ and $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ such that $F(\mathcal{P}(s,t)) = 0$ and $\mathcal{P}(s,t)$ is proper, where $F \in \mathbb{K}[x, y, z]$ and $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s,t)$.
Output: a rational general solution of $F(x, y, y') = 0$ in the affirmative case.

1. Compute $f_1(s,t), f_2(s,t), g(s,t)$ as in (2.14).
2. Compute a rational general solution $(s(x), t(x))$ of the associated system

$$\left\{ s' = \frac{f_1(s,t)}{g(s,t)},\quad t' = \frac{f_2(s,t)}{g(s,t)} \right\}.$$

3. Compute the constant $c := \chi_1(s(x), t(x)) - x$.
4. Return $y = \chi_2(s(x-c), t(x-c))$.

Note that we still have to solve the associated system for its rational general solutions in general cases. The rest of this chapter and the next chapters will develop a method for determining a rational general solution of such systems.

Example 2.2.1. Consider the differential equation

$$F(x,y,y') \equiv y'^3 - 4xyy' + 8y^2 = 0. \tag{2.19}$$

Note that this differential equation appears in Piaggio (1933), Chapter XV, Arts 161, in Hubert (1996) and in Kamke (1948), equation I.525. Here we demonstrate our approach for this differential equation. The solution surface $z^3 - 4xyz + 8y^2 = 0$ has a proper parametrization¶

$$\mathcal{P}(s,t) = (t, -4s^2 \cdot (2s-t), -4s \cdot (2s-t)).$$

The inverse map is $\mathcal{P}^{-1}(x,y,z) = \left(\frac{y}{z}, x\right)$. We compute

$$g(s,t) = 8s \cdot (3s-t),$$

$$f_1(s,t) = 4s \cdot (3s-t), \quad f_2(s,t) = 8s \cdot (3s-t).$$

In this case, the associated system is very simple $\left\{s' = \frac{1}{2},\ \ t' = 1\right\}$. Solving this system we obtain a rational general solution $s(x) = \frac{x}{2} + c_2,\ t(x) = x + c_1$ for arbitrary constants c_1, c_2. The above algorithm follows that the rational general solution of the differential equation $F(x,y,y') = 0$ is

$$y = -4s(x-c_1)^2 \cdot (2s(x-c_1) - t(x-c_1)) = -c(x+c)^2, \tag{2.20}$$

where $c = 2c_2 - c_1$. The following is the graph of three solution curves

$$(x, -c(x+c)^2, -2c(x+c))$$

with $c = -1, 1$ and 2 on the solution surface $z^3 - 4xyz + 8y^2 = 0$.

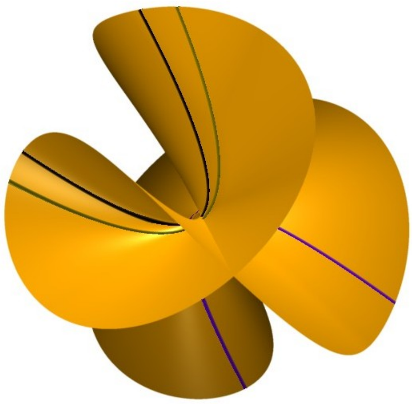


Figure 2.1: Some solution curves on the solution surface $z^3 - 4xyz + 8y^2 = 0$

¶It can be seen that the cubic curve $z^3 - 4xyz + 8y^2 = 0$ is rational over $\mathbb{K}(x)$.

Note that, in this example, $\gcd(g(s,t), f_1(s,t)) = 4s \cdot (3s - t)$. This defines a reducible algebraic curve and we still find some solutions of the differential equation (2.19), whose solution curves are parametrizable by $\mathcal{P}(s,t)$, by solving the system

$$\begin{cases} t(x) = x \\ -4s(x)^2 \cdot (2s(x) - t(x)) = f(x) \\ -4s(x) \cdot (2s(x) - t(x)) = f'(x) \\ 4s(x) \cdot (3s(x) - t(x)) = 0. \end{cases} \tag{2.21}$$

This system has two different solutions, namely

$$(s(x), t(x)) = (0, x) \quad \text{and} \quad (s(x), t(x)) = \left(\frac{x}{3}, x\right).$$

These solutions give us two other solutions of the equation (2.19), namely $y = 0$ and $y = \frac{4}{27}x^3$. The solution $y = 0$ can be obtained by specifying the constant $c = 0$ in the general solution (2.20). However, we can not get the solution $y = \frac{4}{27}x^3$ from the general solution (2.20). Note that the separant of F is $S = 3y'^2 - 4xy$. We can prove that the common solutions of F and S, which are called *singular solutions* of $F(x, y, y') = 0$, are only $y = 0$ and $y = \frac{4}{27}x^3$. Here is the graph of the cubic curve $\left(x, \frac{4}{27}x^3, \frac{4}{9}x^2\right)$ generated by the singular solution $y = \frac{4}{27}x^3$. ‖

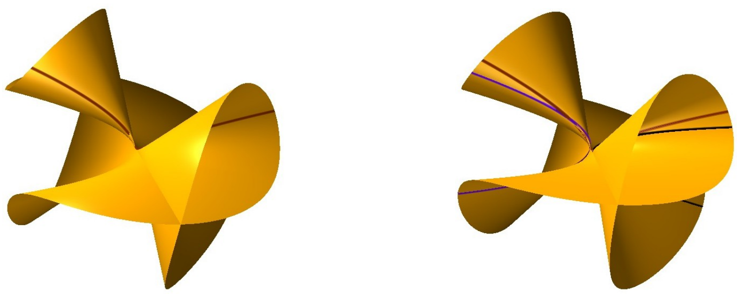


Figure 2.2: The cubic curve $\left(x, \frac{4}{27}x^3, \frac{4}{9}x^2\right)$ on the solution surface $z^3 - 4xyz + 8y^2 = 0$

Remark 2.2.1. Let $F(x, y, y') = 0$ be a first-order algebraic ODE and let $S = \frac{\partial F}{\partial y'}$ be its separant. The singular solutions of $F(x, y, y') = 0$ are defined by the system

$$\{F(x, y, y') = 0, \quad S(x, y, y') = 0\}.$$

‖The visualization is done by `surfex`, which is included as a SINGULAR library, see Labs (2001).

Assume that $y = f(x)$ is a rational singular solution of $F(x, y, y') = 0$. Then $y = f(x)$ must be a rational solution of the first-order autonomous ODE defined by $res_x(F, S) = 0$. This must define a rational curve and we can compute its rational solutions by parametrization. The rational solutions of $res_x(F, S) = 0$ are the candidates for rational singular solutions of $F(x, y, y') = 0$.

E.g. in the above example, $res_x(F, S) = 8y(-y'^3 + 4y^2)$. Solving this differential equation we obtain $y = 0$ and a rational general solution $y = \frac{4}{27}(x + c)^3$, where c is an arbitrary constant. Now, it is clear that $y = 0$ is a singular solution of $F(x, y, y') = 0$. In order that $y = \frac{4}{27}(x + c)^3$ is a singular solution of $F(x, y, y') = 0$, we must have $c = 0$. Hence, $y = \frac{4}{27}x^3$.

2.2.4 Specialize to first-order autonomous algebraic ODEs

We consider autonomous algebraic ODEs of order 1 $F(y, y') = 0$ as a special case of a possibly non-autonomous algebraic ODE. In this section, we show that if $F(y, z) = 0$ is a rational curve, then the associated system of $F(y, y') = 0$ is really simple.

By Feng and Gao (2004, 2006), in order that $F(y, y') = 0$ has a non-trivial rational solution, the algebraic curve $F(y, z) = 0$ must be rational. Suppose that $(f(t), g(t))$ is a proper rational parametrization of the curve $F(y, z) = 0$. Then we immediately have $\mathcal{P}(s, t) = (s, f(t), g(t))$ as a proper parametrization of the solution surface $F(x, y, z) = 0$. This is a special case of a pencil of rational curves, namely, a cylindrical surface. With respect to $\mathcal{P}(s, t)$ the associated system is

$$\left\{ s' = 1, \;\; t' = \frac{g(t)}{f'(t)} \right\}.$$

The second equation of the associated system is again autonomous but of degree 1 in the derivative. Therefore, its rational solution must be either a constant or a linear rational function of the form $\frac{ax + b}{cx + d}$, where a, b, c and d are constants such that $ad - bc \neq 0$.

Let $s = x + C,\ t = t(x)$ be a rational general solution of the above associated system, where C is an arbitrary constant. Then, by algorithm `GENERALSOLVER`, we obtain $y = f(t(x - C))$ as a rational general solution of $F(y, y') = 0$. This means that if we specialize the algorithm `GENERALSOLVER` to first-order autonomous algebraic ODEs, we obtain Algorithm 1 in Feng and Gao (2004). Moreover, in this specialization we can geometrically interpret the reason why we can get a rational general solution from a non-trivial rational solution $y = f(x)$ of $F(y, y') = 0$ by simply taking $y = f(x + C)$ for an arbitrary constant C, which is stated in Theorem 5 in Feng and Gao (2004).

2.2.5 Independence of the proper parametrization

We know that proper rational parametrizations of a rational surface are not unique. Indeed, two parametrizations are different by a birational map of the plane. Let

$$\phi(s_1, t_1) = (\phi_1(s_1, t_1), \phi_2(s_1, t_1))$$

be a birational map of the plane and $\psi(s_2, t_2) = \phi^{-1}(s_2, t_2)$. If $\mathcal{P}(s_1, t_1)$ is a proper parametrization of $F(x, y, z) = 0$, then $(\mathcal{P} \circ \psi)(s_2, t_2)$ is a new proper rational parametrization of $F(x, y, z) = 0$.

> **Q 6.** *How are the associated systems of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}(s_1, t_1)$ and $(\mathcal{P} \circ \psi)(s_2, t_2)$ related to each other?*

Suppose that

$$\begin{cases} s_1' = R_1(s_1, t_1), \\ t_1' = R_2(s_1, t_1), \end{cases}$$

is the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}(s_1, t_1)$. Then the associated system of $F(x, y, y') = 0$ w.r.t. the proper parametrization $(\mathcal{P} \circ \psi)(s_2, t_2)$ is

$$\begin{pmatrix} s_2' \\ t_2' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} s_1' \\ t_1' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} R_1(s_1, t_1) \\ R_2(s_1, t_1) \end{pmatrix} \Bigg|_{(s_1, t_1) = (\psi_1(s_2, t_2),\ \psi_2(s_2, t_2))} \tag{2.22}$$

where $J_\phi = \begin{pmatrix} \phi_{1s_1} & \phi_{1t_1} \\ \phi_{2s_1} & \phi_{2t_1} \end{pmatrix}$ is the Jacobian matrix of the map ϕ. The two associated systems have the same rational solvability although they are different and the complexity of these systems are not the same.

It can be proven that every birational map of the line is of the form $\phi(x) = \dfrac{ax + b}{cx + d}$, where $a, b, c, d \in \mathbb{K}$ and $ad - bc \neq 0$. Unfortunately, it is not known what are the forms of a birational map of the plane. Therefore, any description on the birational maps of the plane could help us to simplify the associated system and perhaps find the simplest one.

Example 2.2.2. Consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0.$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ can be parametrized by

$$\mathcal{P}(s, t) = \left(\frac{t}{s} + \frac{2s + t^2}{s^2}, -\frac{1}{s} - \frac{2s + t^2}{s^2}, \frac{t}{s} \right)$$

and also by

$$\mathcal{Q}(s,t) = \left(s, \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8}, t - \frac{3}{2}\right).$$

The transformation between them is $\phi(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, \frac{t}{s} + \frac{3}{2}\right)$, i.e., $\mathcal{Q} \circ \phi = \mathcal{P}$, with the inverse

$$\phi^{-1} = \psi(s,t) = \left(\frac{8}{-3+8t+4s-4t^2}, \frac{4(2t-3)}{-3+8t+4s-4t^2}\right).$$

Now, the associated system of $F(x,y,y')$ w.r.t. $\mathcal{P}(s,t)$ is $\{s' = st, \;\; t' = s + t^2\}$ while the associated system w.r.t. $\mathcal{Q}(s,t)$ is $\{s' = 1, \;\; t' = 1\}$.

Remark 2.2.2. The properness of a parametrization of the solution surface $F(x,y,z) = 0$ is important. We might consider non-proper parametrizations of $F(x,y,z) = 0$ as well. However, we then have no control on the associated system, i.e., a non-rational solution of the associated system might be mapped into a rational solution of $F(x,y,y') = 0$. For instance, let us consider the differential equation $F \equiv y'^2 - 4y = 0$. The solution surface $z^2 - 4y = 0$ can be parametrized by the improper map $\mathcal{P}(s,t) = \left(s, \frac{t^4}{4}, t^2\right)$. Its associated system is

$$\left\{s' = 1, \;\; t' = \frac{1}{t}\right\}. \tag{2.23}$$

Although this system has a non-rational general solution $(s(x), t(x)) = (x + c, \sqrt{2x})$, its image by $\mathcal{P}(s,t)$ gives us the rational general solution of $y'^2 - 4y = 0$, namely, $y = (x-c)^2$, where c is an arbitrary constant.

Nonetheless, this may well be a way to study more general classes of solutions of algebraic ODEs by parametrization.

2.2.6 A degree bound for rational solutions of the associated system

We have studied the algebraic ODE of order 1, $F(x,y,y') = 0$, provided a proper rational parametrization $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ of the solution surface $F(x,y,z) = 0$. We know that every rational solution $(s(x), t(x))$ of the associated system of $F(x,y,y') = 0$ w.r.t. $\mathcal{P}(s,t)$ satisfies the condition $\chi_1(s(x), t(x)) = x + c$ for some constant c.

Q 7. *What is the relation between* $\deg s(x)$ *and* $\deg t(x)$ *if* $\chi_1(s(x), t(x)) = x$*?*

From the condition $\chi_1(s(x), t(x)) = x$ we can deduce that the degree of $t(x)$ is bounded in terms of the degree of $s(x)$ and the degree of $\chi_1(s,t)$ with respect to s.

Theorem 2.2.9. *Let*

$$\chi_1(s,t) = \frac{a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_0(t)}{b_m(t)s^m + b_{m-1}(t)s^{m-1} + \cdots + b_0(t)} \in \mathbb{K}(s,t)$$

be such that $\chi_1(s,t) \notin \mathbb{K}(s)$ *and* $\chi_1(s,t) \notin \mathbb{K}(t)$*, where* $m, n \in \mathbb{N}$ *and* $b_m(t) \neq 0$*. Suppose that* $s(x)$ *and* $t(x)$ *are rational functions in* $\mathbb{K}(x)$ *such that* $\chi_1(s(x), t(x)) = x$*. Let* $\delta = \deg s(x)$*. Then*

$$\deg t(x) \leq 1 + \delta \max\{m, n\}.$$

Proof. We have

$$\chi_1(s(x), t(x)) = x \iff \frac{a_n(t(x))s(x)^n + a_{n-1}(t(x))s(x)^{n-1} + \cdots + a_0(t(x))}{b_m(t(x))s(x)^m + b_{m-1}(t)s(x)^{m-1} + \cdots + b_0(t(x))} = x.$$

We know that for any rational function $t \in \mathbb{K}(x)$, x is algebraic over $\mathbb{K}(t)$ and

$$\deg t(x) = [\mathbb{K}(x) : \mathbb{K}(t)].$$

Therefore, in order to find a degree bound for t, it is enough to find an algebraic equation for x over $\mathbb{K}(t)$. Let $s(x) = \dfrac{P}{Q}$, where $P, Q \in \mathbb{K}[x]$, $Q \neq 0$. Let

$$\delta = \deg s(x) = \max\{\deg P, \deg Q\}, \quad l = \deg Q.$$

We have

$$\begin{aligned} x &= \frac{Q^m}{Q^n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)} \\ &= Q^{m-n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)}. \end{aligned}$$

This equation derives a non-zero algebraic equation of x over $\mathbb{K}(t)$ because $\chi_1(s,t) \notin \mathbb{K}(s)$ and $\chi_1(s,t) \notin \mathbb{K}(t)$. We can compute the degree of x in the above equation regarding $l \leq \delta$.

If $n \geq m$, then

$$\deg t(x) \leq \max\{1 + m\delta + l(n-m), n\delta\} \leq 1 + n\delta.$$

If $n < m$, then

$$\deg t(x) \leq \max\{1 + m\delta, n\delta + l(m-n)\} \leq 1 + m\delta.$$

Therefore, $\deg t(x) \leq 1 + \delta \max\{m, n\}$. □

Of course, the degree of $s(x)$ can also be bounded in the same way by the degree of $t(x)$ and the degree of the first component of $\mathcal{P}(s,t)$ with respect to t.

2.3 A criterion for the existence of a rational general solution

In this section, we derive a criterion for the existence of a rational general solution of the associated system of the equation $F(x, y, y') = 0$. The following lemma can be found in Feng and Gao (2006).

Lemma 2.3.1. *Let $n, m \in \mathbb{N}$. There exists a differential polynomial $D_{n,m}(y)$ such that every rational function*

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

is a solution of $D_{n,m}(y)$, where a_i, b_j are constants in $\mathbb{K}$. Moreover, the differential polynomial $D_{n,m}(y)$ has only rational solutions.

Definition 2.3.1. The differential polynomial in Lemma 2.3.1 is given by

$$D_{n,m}(y) = \begin{vmatrix} \binom{n+1}{0}y^{(n+1)} & \binom{n+1}{1}y^{(n)} & \cdots & \binom{n+1}{m}y^{(n+1-m)} \\ \binom{n+2}{0}y^{(n+2)} & \binom{n+2}{1}y^{(n+1)} & \cdots & \binom{n+2}{m}y^{(n+2-m)} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n+1+m}{0}y^{(n+1+m)} & \binom{n+1+m}{1}y^{(n+m)} & \cdots & \binom{n+1+m}{m}y^{(n+1)} \end{vmatrix}.$$

We call $D_{n,m}(y)$ a *Feng-Gao's differential polynomial.*

Using Feng-Gao's differential polynomials we have the following criterion.

Theorem 2.3.2. *Let $M_1, N_1, M_2, N_2 \in \mathbb{K}[s, t]$, $N_1, N_2 \neq 0$. The autonomous system (2.16), i.e.,*

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)}, \quad t' = \frac{M_2(s,t)}{N_2(s,t)} \right\}$$

has a rational general solution $(s(x), t(x))$ with $\deg s(x) \leq n$ and $\deg t(x) \leq m$ if and only if

$$\begin{cases} \operatorname{prem}(D_{n,n}(s), N_1 s' - M_1, N_2 t' - M_2) = 0, \\ \operatorname{prem}(D_{m,m}(t), N_1 s' - M_1, N_2 t' - M_2) = 0. \end{cases} \tag{2.24}$$

Proof. Suppose that the system (2.16) has a rational general solution $(s(x), t(x))$ with $\deg s(x) \leq n$ and $\deg t(x) \leq m$. Then $(s(x), t(x))$ is a solution of both $D_{n,n}(s)$ and $D_{m,m}(t)$. By definition of rational general solutions of the system (2.16) we have

$$\begin{cases} \operatorname{prem}(D_{n,n}(s), N_1 s' - M_1, N_2 t' - M_2) = 0, \\ \operatorname{prem}(D_{m,m}(t), N_1 s' - M_1, N_2 t' - M_2) = 0. \end{cases}$$

Conversely, if these two conditions hold, then $D_{n,n}(s)$ and $D_{m,m}(t)$ belong to

$$\mathcal{I} := \{G \in \mathbb{K}(x)\{s,t\} \mid \operatorname{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0\}.$$

Since $\mathcal{I}$ is a prime differential ideal, $\mathcal{I}$ has a generic zero. This generic zero is a zero of $D_{n,n}(s)$ and $D_{m,m}(t)$. By Lemma 2.3.1, these two differential polynomials have only rational solutions. Therefore, the generic zero of I must be rational. □

Remark 2.3.1. If we know a degree bound of the rational solutions of the system (2.16), then Theorem 2.3.2 gives us a criterion for the existence of a rational general solution of the system (2.16). In fact, there is a generic degree bound presented in the next chapter when we study the "invariant algebraic curves" of this associated system.

2.3.1 Application—Linear systems with rational general solutions

We have seen that the associated system of the algebraic ODE $F(x, y, y') = 0$ is an autonomous system. In this section, we consider the linear system of autonomous ODEs of the form

$$\begin{cases} s' = as + bt + e \\ t' = cs + dt + h \end{cases} \tag{2.25}$$

where a, b, c, d, e, h are constants in $\mathbb{K}$.

> **Q 8.** *When does the system has a rational general solution? What are the possible degrees of the rational general solutions?*

In fact, we prove that the rational solutions of that system are polynomials; moreover, their degrees are at most 2. Before studying rational solutions of the system (2.25) we need to introduce the notation of order of an irreducible polynomial in a rational function.

Definition 2.3.2. Let $\mathbb{K}$ be a field. Let $s \in \mathbb{K}(x)$ be a rational function in x. Suppose that s has a complete decomposition as follows

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}$$

where $A \in \mathbb{K}[x]$ and p_i are distinct irreducible polynomials over $\overline{\mathbb{K}}$ and $\gcd(A, p_i) = 1$ for all $i = 1, \dots, n$. The power α_i in this representation of s is called the *order* of s with respect to p_i, denoted by $\operatorname{ord}_{p_i}(s)$. By convention, if an irreducible polynomial p does not effectively appear in the denominator of s, then we define $\operatorname{ord}_p(s) = 0$.

Lemma 2.3.3. *Every rational solution of the linear system* (2.25) *is a polynomial solution.*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the linear system (2.25). If $s(x)$ or $t(x)$ is not a polynomial, then we can assume without loss of generality that

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}},$$

where $A \in \mathbb{K}[x]$, p_i are irreducible polynomials over $\overline{\mathbb{K}}$, $\gcd(A, p_i) = 1$ and $\alpha_i > 0$ for all $i = 1, \ldots, n$. Let

$$\beta_i = \operatorname{ord}_{p_i}(t) \geq 0 \quad \forall i = 1, \ldots, n.$$

Since $\alpha_i > 0$, computing the derivative of $s(x)$ we have

$$\operatorname{ord}_{p_i}(s') = \alpha_i + 1 \quad \forall i = 1, \ldots, n.$$

On the other hand,

$$\operatorname{ord}_{p_i}(as + bt + e) \leq \max\{\alpha_i, \beta_i\}, \quad \operatorname{ord}_{p_i}(cs + dt + h) \leq \max\{\alpha_i, \beta_i\}.$$

Let us compare the orders with respect to p_i of the left and the right hand sides of the linear system (2.25). There are two cases as follows.

- Either $\alpha_i \geq \beta_i$, then $\operatorname{ord}_{p_i}(as + bt + e) \leq \alpha_i < \operatorname{ord}_{p_i}(s')$, which is impossible;
- or $0 < \alpha_i < \beta_i$, then $\operatorname{ord}_{p_i}(cs + dt + h) \leq \beta_i < \operatorname{ord}_{p_i}(t')$, which is also impossible.

Therefore, $\alpha_i = 0$ for all $i = 1, \ldots, n$. Thus s is a polynomial. Replacing the role of s and t we also prove that t is a polynomial. Therefore, $(s(x), t(x))$ is a polynomial solution. $\square$

Theorem 2.3.4. *Every rational general solution of the linear system* (2.25) *is a couple of polynomials of degree at most 2.*

Proof. By Lemma 2.3.3, every rational solution of the linear system (2.25) is a polynomial solution. In this case the Gao's differential polynomials for checking polynomial general solutions of the system are of simple forms $s^{(n+1)}$ and $t^{(n+1)}$ for some n. We can write the linear system in the matrix form

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} e \\ h \end{pmatrix}.$$

Hence

$$\begin{pmatrix} s'' \\ t'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} s^{(n+1)} \\ t^{(n+1)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix},$$

for $n \in \mathbb{N}$. By Theorem 2.3.2, the system (2.25) has a polynomial general solution of degree at most n if and only if

$$\begin{cases} \operatorname{prem}(s^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0 \\ \operatorname{prem}(t^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0. \end{cases}$$

or equivalently when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix} = 0.$$

We will prove that these relations hold for $n \geq 2$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0.$$

Then the conclusion of the theorem follows immediately.

Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix} = 0$$

for all $n \geq 2$.

Conversely, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} = 0$$

for some $n \geq 1$. Then $ad - bc = 0$ and the Jordan canonical form of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is either $\begin{pmatrix} 0 & 0 \\ 0 & a+d \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In the first case, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} = 0$, we have $a + d = 0$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the second case, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0.$ $\square$

The above proof also tell us the necessary and sufficient conditions of the linear system for having rational general solutions. Namely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0.$$

We can easily find all possibilities of the coefficients a, b, c, d. In fact,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0 \Longleftrightarrow \begin{cases} a^2 + bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 0. \end{cases}$$

Solving this algebraic system we obtain the following cases

- if $b = 0$, then $a = d = 0$;
- if $b \neq 0$, then $a = -d$ and $c = -\dfrac{d^2}{b}$.

Thus the explicit polynomial solutions of the linear system are given by the following table, where C_1, C_2 are arbitrary constants. Note that the last line of the table also covers the

System	Rational general solution
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = hx + C_2 \end{cases}$
$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = ce\frac{x^2}{2} + (cC_1 + h)x + C_2 \end{cases}$
$\begin{pmatrix} -d & b \\ -\frac{d^2}{b} & d \end{pmatrix}$	$\begin{cases} s(x) = \frac{hb - ed}{2}x^2 + (bC_1 + e)x + C_2 \\ t(x) = \frac{(hb - ed)d}{2b}x^2 + (dC_1 + h)x + \frac{d}{b}C_2 + C_1 \end{cases}$

Table 2.1: Linear systems with rational general solutions

other symmetric cases, for instance

$$d = 0 \longmapsto \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}; \quad d = -a,\ b = -\frac{a^2}{c} \longmapsto \begin{pmatrix} a & -\frac{a^2}{c} \\ c & -a \end{pmatrix}.$$

We can prove that the solutions in the table are rational general solutions of the corresponding system. For instance, consider a simple system $\{s' = e, \ \ t' = h\}$, where e and h

are constants but not all zero. It turns out that the system has a solution given by

$$s(x) = ex + C_1,\ t(x) = hx + C_2,$$

where C_1, C_2 are arbitrary constants. The implicit defining polynomial of $(s(x), t(x))$ is

$$H(s,t) = hs - et - hC_1 + eC_2.$$

Since the coefficients of $H(s,t)$ contain an arbitrary constant, namely $-hC_1 + eC_2$, it follows from Lemma 2.2.5 that $(s(x), t(x))$ is a rational general solution.

Using a similar argument for the other systems in the table we prove that those solutions are rational general solutions of the corresponding systems.

Chapter 3

Planar rational systems of autonomous ODEs

3.1 Introduction to planar rational systems

In the previous chapter, we are motivated to studying the rational general solutions of the systems of the form (2.16). From the point of view of differential algebra, the solution set of the system (2.16) can be seen as an algebraic differential manifold (Ritt (1950), II, §1). We have described them by means of prime differential ideals in a differential ring. By studying the structure of such prime differential ideals, we are able to see, in the linear cases, when the system (2.16) has a rational general solution and what they are.

On the other hand, we can also study the rational (general) solutions of the system (2.16) from the point of view of algebraic geometry, i.e., by looking at the usual ideal of all polynomials vanishing on a rational solution of the system. In this direction, a treatment on the algebraic solutions of a polynomial system

$$\begin{cases} s' = P(s,t), \\ t' = Q(s,t), \end{cases} \tag{3.1}$$

has already been studying by Darboux (1878). In his work, G. Darboux has introduced the notion of an invariant algebraic curve, i.e., an algebraic relation between $s(x)$ and $t(x)$ of a solution $(s(x), t(x))$ of the polynomial system (3.1). This notion is essential for the Darboux's theory of integrability of a polynomial system. By Darboux, the system is integrable iff it has a first integral, i.e., a non-constant function such that its values on every solution of the system is constant. Invariant algebraic curves are the main ingredient to build up a first integral of the system.

Recall that the associated system in Chapter 2 is a rational system of autonomous

ODEs of the form (2.16), i.e.,

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)}, \ \ t' = \frac{M_2(s,t)}{N_2(s,t)} \right\}.$$

The rational system (2.16) and the polynomial system defined by $P = M_1N_2$ and $Q = M_2N_1$ have the same invariant algebraic curves and the same first integrals. Therefore, the Darboux's theory is applied to study rational (general) solutions of the system (2.16). In fact, we apply the theory of rational parametrization of algebraic curves for the invariant algebraic curves of the system in order to obtain an explicit rational (general) solution of the system.

> **Goal 1.** *The goal of this chapter is to study the existence of a rational general solution of the system* (2.16) *and in the affirmative case we determine an explicit one.*

The fact is that any rational solution of the system (2.16) is corresponding to an invariant algebraic curve of the system. Furthermore, this curve is rational. Therefore, we first need to determine the invariant algebraic curves of the system. Then we apply the theory of rational parametrization of algebraic curves for those invariant algebraic curves. In the end, we give an explicit procedure to computing a rational general solution of the system (2.16). The result of this chapter is based on Ngô and Winkler (2011)

Before going to details, we notice that the polynomial system (3.1) has been discussed in the context of holomorphic singular foliations of the complex projective plane $\mathbb{CP}^2$ (Darboux (1878); Jouanolou (1979); Lins Neto (1988); Carnicer (1994)). Most of the time we stay in the affine plane but in some discussions we need a result—the degree bound of an invariant algebraic curve—that holds in the complex projective plane. Therefore, in this chapter, we also mention the description of the system (3.1) in the complex projective plane.

Note that, one can derive from the system (2.16) and the polynomial system (3.1) to the single differential equation

$$\frac{dt}{ds} = \frac{Q(s,t)}{P(s,t)}, \tag{3.2}$$

or the 1-form

$$Q(s,t)ds - P(s,t)dt = 0, \tag{3.3}$$

or the polynomial vector field

$$\mathcal{D} := P\frac{\partial}{\partial s} + Q\frac{\partial}{\partial t}. \tag{3.4}$$

However, the correspondence is not one-to-one. Hence, from the equation (3.2), we can not construct the system (2.16) or the polynomial system (3.1). But it is enough to have the form (3.2) or (3.3) or (3.4) for studying the invariant algebraic curves of the system (2.16) or of the polynomial system (3.1).

3.2 Invariant algebraic curves of a planar rational system

In this section, we present the notion of invariant algebraic curves of the system (2.16) and a method to determining these curves.

Let $P = M_1N_2$ and $Q = M_2N_1$, be two polynomials in $\mathbb{K}[s,t]$; denote that $\mathcal{D} = P\dfrac{\partial}{\partial s} + Q\dfrac{\partial}{\partial t}$. Suppose that $(s(x), t(x))$ is a solution of the system (2.16) such that there exists an irreducible polynomial $G(s,t)$ with $G(s(x), t(x)) = 0$. Then we have

$$\mathcal{D}G = GK,$$

where K is some polynomial of degree at most $m-1$ and $m = \max\{\deg(P), \deg(Q)\}$.

Definition 3.2.1. An algebraic curve $G(s,t) = 0$ is called an *invariant algebraic curve* of the system (2.16) iff

$$\mathcal{D}G = GK,$$

for some polynomial K. The polynomial K is called the *cofactor* of G.

By definition, the invariant algebraic curves of the rational system (2.16) and that of the polynomial (3.1), where $P = M_1N_2$ and $Q = M_2N_1$, are the same. Therefore, when we are interested in the invariant algebraic curves of those systems, we will consider the analysis on polynomial systems.

Definition 3.2.2. An invariant algebraic curve $G(s,t) = 0$ of the system (2.16) is called a *general invariant algebraic curve* iff $G(s,t)$ is a monic polynomial w.r.t. a lexicographic order of terms in s, t and there exists a coefficient of G such that it is transcendental over $\mathbb{K}$.

In this case, $G(s,t) = 0$ can be seen as either one curve over $\overline{\mathbb{K}(c)}$, where c is a transcendental constant over $\mathbb{K}$, or a family of curves over $\mathbb{K}$. By Lemma 2.2.5, general invariant algebraic curves would be the algebraic curves of a potential general solution of the system (2.16).

Lemma 3.2.1. *Let $G(s,t) = \prod_{i=1}^{l} G_i^{n_i}$ be the decomposition of $G(s,t)$ into relatively prime irreducible factors over $\mathbb{K}$. Then $G(s,t) = 0$ is an invariant algebraic curve of the system* (2.16) *with cofactor $K(s,t)$ if and only if the curve $G_i(s,t) = 0$ is an invariant algebraic curve of that system with cofactor K_i for all $i = 1, \cdots, l$ and $K = \sum_{i=1}^{l} n_iK_i$.*

Proof. Suppose that $G(s,t) = 0$ is an invariant algebraic curve, i.e., we have

$$\mathcal{D}G = \sum_{i=1}^{l} n_i G_i^{n_i-1} \mathcal{D}G_i \prod_{j\neq i} G_j^{n_j} = G \cdot K,$$

where K is the cofactor of G. It implies that G_i devides $\mathcal{D}G_i \prod_{j\neq i} G_j^{n_j}$. Since G_i and G_j are relatively prime for $i \neq j$, we must have that G_i devides $\mathcal{D}G_i$. In other words, $G_i = 0$ is an invariant algebraic curve for all $i = 1, \ldots, l$.

Conversely, let $G_i = 0$ be an invariant algebraic curve with cofactor K_i. Then

$$\begin{aligned}\mathcal{D}G &= \sum_{i=1}^{l} n_i G_i^{n_i-1} \mathcal{D}G_i \prod_{j\neq i} G_j^{n_j} \\ &= \prod_{i=1}^{l} G_i^{n_i} \left(\sum_{i=1}^{l} n_i K_i \right).\end{aligned}$$

Hence, $G = 0$ is an invariant algebraic curve with cofactor $K = \sum_{i=1}^{l} n_i K_i$. □

Therefore, from now on, we only consider the irreducible invariant algebraic curves of the system (2.16). Computing an irreducible invariant algebraic curve $G(s,t) = 0$ of the system (2.16) can be performed via undetermined coefficients method as long as an upper bound for the degree of the polynomial $G(s,t)$ is setting up.

Let $H = \gcd(P,Q)$, $P = P_1 H$ and $Q = Q_1 H$. Then every invariant algebraic curve of the system

$$\begin{cases} s' = P_1(s,t), \\ t' = Q_1(s,t), \end{cases} \tag{3.5}$$

is an invariant algebraic curve of the system (2.16). Conversely, suppose that $G(s,t) = 0$ is an irreducible invariant algebraic curve of the system (2.16). Then

$$(G_s P_1 + G_t Q_1) H = GK$$

for some polynomial K. Since $G(s,t)$ is irreducible, either $G|H$ or $G|(G_s P_1 + G_t Q_1)$. In the latter case, $G(s,t) = 0$ is an invariant algebraic curve of the system (3.5). In the first case, $G(s,t)$ is an irreducible factor of $H(s,t)$ and for any parametrization $(s(x), t(x))$ of $G(s,t) = 0$ we have

$$P(s(x), t(x)) = 0 = Q(s(x), t(x)).$$

In this case, a parametrization $(s(x), t(x))$ of $G(s,t) = 0$ is a solution of the system (2.16) only if $s(x)$ and $t(x)$ are constants, i.e., any point of the curve $G(s,t) = 0$ gives a trivial rational solution of the system (2.16).

Example 3.2.1. Consider the polynomial differential system

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases} \tag{3.6}$$

We first ask for the invariant algebraic curves of degree 1. Consider the graded lexicographic order with $s > t$. Then there are two case distinctions, namely,

$$G(s,t) = t + c, \quad G(s,t) = s + bt + c.$$

The first polynomial can not define an invariant algebraic curve because

$$G_s P + G_t Q = s + t^2$$

is not divisible by G. Now we consider the second polynomial, and the remainder of the division of $G_s P + G_t Q$ by G is

$$(-c - b^2)t - bc.$$

It follows that $G(s,t) = s + bt + c$ defines an invariant algebraic curve if and only if $b = c = 0$. Therefore, $G(s,t) = s$ is an invariant algebraic curve of degree 1.

Similarly, we ask for the invariant algebraic curves of degree 2. Again take the graded lexicographic order with $s > t$. There are three case distinctions, namely,

$$G(s,t) = t^2 + ds + et + f, \quad G(s,t) = st + ct^2 + ds + et + f,$$

$$G(s,t) = s^2 + bst + ct^2 + ds + et + f.$$

If $G(s,t) = t^2 + ds + et + f$, then the remainder of the division of $G_s P + G_t Q$ by G is

$$(2 - d)st + (de + e)s + (e^2 - 2f)t + ef.$$

So we need to have $d = 2$ and $e = f = 0$. Hence, $G(s,t) = t^2 + 2s$ is an invariant algebraic curve of the system. With the same procedure we can see that $G(s,t) = st + ct^2 + ds + et + f$ is not an invariant algebraic curve for any choice of its coefficients; and $G(s,t) = s^2 + bst + ct^2 + ds + et + f$ is an invariant algebraic curve if and only if $b = e = f = 0$ and $d = 2c$, i.e., $G(s,t) = s^2 + ct^2 + 2cs$, where c is an arbitrary constant.

The family of irreducible curves $s^2 + ct^2 + 2cs = 0$ corresponds to the level curves of the surface $z = \frac{s^2}{2s+t^2}$. Later, the function $\frac{s^2}{2s+t^2}$ is a rational first integral of the given polynomial system. Hence, we have two ways of visualizing this family of curves as in Figure 3.1.

Of course, we can keep on increasing the degree of the curve for computing the irre-

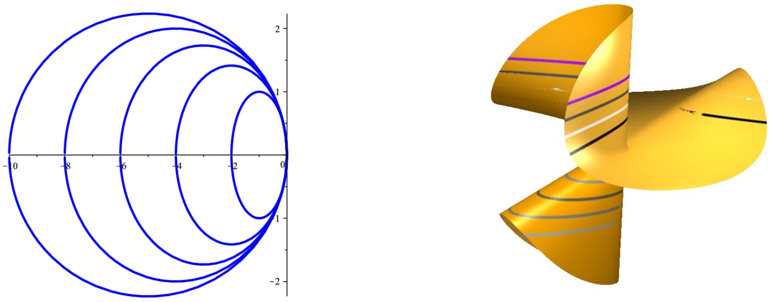


Figure 3.1: Family of irreducible invariant algebraic curves: $s^2 + ct^2 + 2cs = 0$

ducible invariant algebraic curves of this system. In this example, however, we will see that a theorem by Darboux guarantees that the system has no irreducible invariant algebraic curve of degree higher than 2.

The method consists of two steps: computing a normal form of $\mathcal{D}G$ in the ideal generated by G w.r.t. an ordering and then solving the result algebraic system on the coefficients of G. In Man (1993), one can find a discussion on the efficiency of different implementations of computing invariant algebraic curves in some computer algebra systems (MACSYMA and REDUCE).

3.3 Rational solutions of planar rational systems of first-order autonomous ODEs

In this section, we give an algorithm to determine a rational solution of the system (2.16). A *rational solution* of the system (2.16) is a pair of rational functions $\mathcal{C}(x) = (s(x), t(x))$ satisfying the system (2.16).

Definition 3.3.1. A rational solution $(s(x), t(x))$ of (2.16) is said to be *trivial* if both $s(x)$ and $t(x)$ are constants.

A trivial solution of (2.16) can be easily found by intersecting the two algebraic curves $M_1(s,t) = 0$ and $M_2(s,t) = 0$. Otherwise, a non-trivial rational solution $(s(x), t(x))$ of (2.16) defines a rational curve. Let us see what properties of that algebraic curve are.

Lemma 3.3.1. *Let* $\mathcal{C}(x) = (s(x), t(x))$ *be a non-trivial rational solution of the system* (2.16)*. Let* $G(s,t)$ *be the defining polynomial of the curve parametrized by* $\mathcal{C}(x)$*. Then*

$$G_s M_1 N_2 + G_t M_2 N_1 = GK,$$

where G_s *and* G_t *are the partial derivatives of* G *w.r.t.* s *and* t*;* K *is some polynomial.*

Proof. Since $G(s,t)$ is the defining polynomial of the curve parametrized by $(s(x),t(x))$, we have

$$G(s(x),t(x)) = 0.$$

Differentiating this equation with respect to x we obtain

$$G_s(s(x),t(x)) \cdot s'(x) + G_t(s(x),t(x)) \cdot t'(x) = 0.$$

$\mathcal{R}(x) = (s(x),t(x))$ is a solution of the system, so we have

$$G_s(s(x),t(x)) \cdot \frac{M_1(s(x),t(x))}{N_1(s(x),t(x))} + G_t(s(x),t(x)) \cdot \frac{M_2(s(x),t(x))}{N_2(s(x),t(x))} = 0.$$

Hence, the polynomial $G_sM_1N_2+G_tM_2N_1$ is in the ideal of the curve generated by $G(s,t)$. In other words, we have

$$G_sM_1N_2 + G_tM_2N_1 = GK$$

for some polynomial K. □

Every non-trivial rational solution of the system (2.16) determines a rational curve, which is also an invariant algebraic curve of the system. Therefore, we first look for the invariant algebraic curves of the system and then parametrize them to obtain rational solutions. Of course, if none of the invariant algebraic curves is rational, then we immediately conclude that there is no rational solution.

Definition 3.3.2. An invariant algebraic curve $G(s,t) = 0$ of the rational system (2.16) is called a *rational invariant algebraic curve* iff $G(s,t) = 0$ is a rational curve.

Lemma 3.3.2. *Let $G(s,t) = 0$ be an irreducible rational invariant algebraic curve of the system* (2.16)*. Let $(s(x),t(x))$ be a rational parametrization of the curve $G(s,t) = 0$. Then we have*

$$s'(x) \cdot M_2(s(x),t(x))N_1(s(x),t(x)) = t'(x) \cdot M_1(s(x),t(x))N_2(s(x),t(x)).$$

Moreover, if $G \nmid N_1$ and $G \nmid N_2$, then

$$s'(x) \cdot \frac{M_2(s(x),t(x))}{N_2(s(x),t(x))} = t'(x) \cdot \frac{M_1(s(x),t(x))}{N_1(s(x),t(x))}.$$

Proof. Since $G(s(x),t(x)) = 0$, we have

$$G_s(s(x),t(x))s'(x) + G_t(s(x),t(x))t'(x) = 0.$$

Moreover, $G(s,t)=0$ is an invariant algebraic curve, so we have

$$G_s(s(x),t(x))M_1(s(x),t(x))N_2(s(x),t(x))+G_t(s(x),t(x))M_2(s(x),t(x))N_1(s(x),t(x))=0.$$

Note that the irreducibility of G implies $(G_s(s(x),t(x)),G_t(s(x),t(x)))\neq(0,0)$. Therefore,

$$\begin{vmatrix} s'(x) & t'(x) \\ M_1(s(x),t(x))\cdot N_2(s(x),t(x)) & M_2(s(x),t(x))\cdot N_1(s(x),t(x)) \end{vmatrix}=0.$$

Moreover, if $G\nmid N_1$ and $G\nmid N_2$, then $N_1(s(x),t(x))\neq 0$ and $N_2(s(x),t(x))\neq 0$. Hence,

$$s'(x)\cdot\frac{M_2(s(x),t(x))}{N_2(s(x),t(x))}=t'(x)\cdot\frac{M_1(s(x),t(x))}{N_1(s(x),t(x))}.$$

□

The lemma tells us that not every rational parametrization of a rational invariant algebraic curve will provide a rational solution of the system. But they are good candidates for rational solutions of the system. Therefore, we still have to determine whether any of the infinitely many rational parametrizations leads to a solution of the system. We know that a rational parametrization of a curve is completely determined by reparametrization of a proper parametrization of the curve (see Appendix A).

Q 9. *Which reparametrizations of a proper rational parametrization of an invariant algebraic curve lead to solutions of the system* (2.16)*?*

Definition 3.3.3. A rational invariant algebraic curve of the system (2.16) is called a *rational solution curve* iff it possesses a rational parametrization which is a solution of the system.

From now on, we are only interested in non-trivial rational solutions of the system (2.16). Let us recall that if $(s(x),t(x))$ is a rational parametrization of an algebraic curve $G(s,t)=0$, then at least one of the components of $(s(x),t(x))$ must be non-constant.

The following theorem provides a necessary and sufficient condition for a rational invariant algebraic curve to be a rational solution curve.

Theorem 3.3.3. *Let $G(s,t)=0$ be a rational invariant algebraic curve of the system* (2.16) *such that $G\nmid N_1$ and $G\nmid N_2$. Let $(s(x),t(x))$ be a proper rational parametrization of $G(s,t)=0$. Then $G(s,t)=0$ is a rational solution curve of the system* (2.16) *if and only if one of the following differential equations has a rational solution $T(x)$:*

1. $T'=\dfrac{1}{s'(T)}\cdot\dfrac{M_1(s(T),t(T))}{N_1(s(T),t(T))}$ *when* $s'(x)\neq 0$,
2. $T'=\dfrac{1}{t'(T)}\cdot\dfrac{M_2(s(T),t(T))}{N_2(s(T),t(T))}$ *when* $t'(x)\neq 0$.

If there is such a rational solution $T(x)$, then the rational solution of the system (2.16) *corresponding to $G(s,t)=0$ is given by $(s(T(x)),\ t(T(x)))$.*

Proof. Assume that $(\bar{s}(x),\bar{t}(x))$ is a rational solution of the system (2.16) corresponding to $G(s,t)=0$, i.e.,

$$\begin{cases} \bar{s}'(x) = \dfrac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))}, \\ \bar{t}'(x) = \dfrac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))}. \end{cases}$$

Since $(s(x),t(x))$ is a proper parametrization of $G(s,t)=0$, there exists a rational function $T(x)$ such that

$$\bar{s}(x) = s(T(x)),\ \ \bar{t}(x) = t(T(x)).$$

It implies that

$$\begin{cases} \bar{s}'(x) = s'(T(x)) \cdot T'(x), \\ \bar{t}'(x) = t'(T(x)) \cdot T'(x). \end{cases}$$

Therefore,

$$T'(x) \cdot s'(T(x)) = \frac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))}$$

and

$$T'(x) \cdot t'(T(x)) = \frac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))}.$$

When $s(x)$ or $t(x)$ are non-constants, we have

$$T'(x) = \frac{1}{s'(T(x))} \cdot \frac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))} \ \text{ or } \ T'(x) = \frac{1}{t'(T(x))} \cdot \frac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))},$$

respectively. Conversely, assume w.l.o.g. that $s(x)$ is non-constant and $T(x)$ is a rational solution of the first differential equation. By Lemma 3.3.2 we have

$$s'(T) \cdot \frac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = t'(T) \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}.$$

If $t'(T)=0$, then $\dfrac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = 0$ and $t(x)=c$, for some constant c. It is obvious that $(s(T(x)),c)$ is a rational solution of the system (2.16). Hence $G(s,t)=0$ is a rational solution curve. If $t'(T) \neq 0$, then

$$\frac{1}{t'(T)} \cdot \frac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = \frac{1}{s'(T)} \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}.$$

Therefore, $T(x)$ is also a rational solution of the second differential equation. It follows that $(s(T(x)),t(T(x)))$ is a rational solution of the system (2.16). Hence $G(s,t)=0$ is a rational solution curve. □

We note that if $s(x)$ and $t(x)$ are both non-constant, the two differential equations in Theorem 3.3.3 are the same because the expressions on the right hand side are equal by Lemma 3.3.2. According to Theorem 3.3.3, assuming that we are in case (1), we need to compute a rational solution of the autonomous differential equation of order 1 and of degree 1 in T'

$$T'(x) = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}.$$

Those features of the differential equation are inherited by the associated system. In what follows, we see that it is simple to deal with the rational solvability of this differential equation.

In the next theorem, we prove that the rational solvability of this differential equation does not depend on the choice of a proper parametrization of the rational invariant algebraic curve $G(s,t) = 0$.

Theorem 3.3.4. *Let $G(s,t)$ be a rational invariant algebraic curve of the system* (2.16) *such that $G \nmid N_1$ and $G \nmid N_2$. Let $\mathcal{P}_1(x) = (s_1(x), t_1(x))$ and $\mathcal{P}_2(x) = (s_2(x), t_2(x))$ be two proper rational parametrizations of the curve $G(s,t) = 0$ such that $s_1'(x) \neq 0$ and $s_2'(x) \neq 0$. Then the two autonomous differential equations*

$$T_1' = \frac{1}{s_1'(T_1)} \cdot \frac{M_1(s_1(T_1), t_1(T_1))}{N_1(s_1(T_1), t_1(T_1))} \tag{3.7}$$

and

$$T_2' = \frac{1}{s_2'(T_2)} \cdot \frac{M_1(s_2(T_2), t_2(T_2))}{N_2(s_2(T_2), t_2(T_2))} \tag{3.8}$$

have the same rational solvability, i.e., one of them has a rational solution if and only if the other one has. Moreover, we can choose T_1 and T_2 such that

$$\mathcal{P}_1(T_1) = \mathcal{P}_2(T_2).$$

Proof. Suppose that (3.7) has a rational solution $T_1(x)$. Then the rational solution of (2.16) corresponding to $G(s,t) = 0$ is $(s_1(T_1(x)), t_1(T_1(x)))$. Since $(s_2(x), t_2(x))$ is a proper rational parametrization of the same curve $G(s,t) = 0$, there exists a rational function $T_2(x)$ such that

$$s_2(T_2(x)) = s_1(T_1(x)), \;\; t_2(T_2(x)) = t_1(T_1(x)).$$

Hence,

$$s_2'(T_2(x))T_2'(x) = s_1'(T_1(x))T_1'(x) = \frac{M_1(s_1(T_1), t_1(T_1))}{N_1(s_1(T_1), t_1(T_1))} = \frac{M_1(s_2(T_2), t_2(T_2))}{N_1(s_2(T_2), t_2(T_2))}.$$

This means that $T_2(x)$ is a rational solution of (3.8). □

Theorem 3.3.5. *Suppose that $s(x)$ is a non-constant rational function and $N_1(s(x), t(x)) \neq 0$. Then every rational solution of*

$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}$$

is of the form $T(x) = \dfrac{ax+b}{cx+d}$, where a, b, c and d are constants. In particular, every non-trivial rational solution of the system (2.16) *is proper in the sense of proper parametrization.*

Proof. Assume that $T(x)$ is a non-constant rational solution of the above differential equation. Then, by Theorem 3.7 in Feng and Gao (2006), $(T(x), T'(x))$ forms a proper parametrization of the algebraic curve $H(T, U) = 0$ defined by the numerator of

$$s'(T) \cdot N_1(s(T), t(T)) \cdot U - M_1(s(T), t(T)).$$

Using the degree bound of proper parametrizations in Sendra and Winkler (2001) and observing that the degree of $H(T, U)$ with respect to U is 1, we see that the degree of $T(x)$ must be 1. Therefore, by Theorem 3.3.3, a non-trivial rational solution of the system (2.16) is a composition of a proper parametrization with a non-constant linear rational function, hence a proper parametrization. □

Remark 3.3.1. There is another way of seeing this result. By Theorem 3.7 in Feng and Gao (2006), if $T(x)$ is a non-constant rational function, then we have $\mathbb{K}(x) = \mathbb{K}(T(x), T'(x))$. If $T'(x) = f(T(x))$, where $f(T)$ is a non-zero rational function in $\mathbb{K}(T)$, then

$$\mathbb{K}(x) = \mathbb{K}(T(x), T'(x)) = \mathbb{K}(T(x), f(T(x))) = \mathbb{K}(T(x)).$$

Therefore, $T(x)$ is a linear rational function in x.

By this theorem we can always decide whether the differential equation for $T(x)$ has a rational solution. Therefore, we can decide whether an invariant algebraic curve $G(s, t) = 0$ is a rational solution curve.

Note that if $(s(x), t(x))$ is a rational solution of the system (2.16), then, because of the autonomy of (2.16),

$$(s(x + \tilde{c}), t(x + \tilde{c}))$$

is also a rational solution of the system. In fact, this is the only way to generate rational solutions from the same rational solution curve.

Theorem 3.3.6. *Let* $(s_1(x), t_1(x))$ *and* $(s_2(x), t_2(x))$ *be non-trivial rational solutions of the differential system* (2.16) *corresponding to the same rational invariant algebraic curve. Then there exists a constant* $\tilde{c}$ *such that*

$$(s_1(x+\tilde{c}), t_1(x+\tilde{c})) = (s_2(x), t_2(x)).$$

Proof. As a corollary of Theorem 3.3.3 and Theorem 3.3.5, we have proven that these solutions are proper. Since $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ are rational parametrizations of the same invariant algebraic curve, there exists a linear rational function $T(x)$ such that

$$(s_2(x), t_2(x)) = (s_1(T(x)), t_1(T(x))).$$

Hence

$$\begin{cases} s_1'(T(x))T'(x) = s_2'(x) = \dfrac{M_1(s_2(x), t_2(x))}{N_1(s_2(x), t_2(x))} = \dfrac{M_1(s_1(T(x)), t_1(T(x)))}{N_1(s_1(T(x)), t_1(T(x)))} = s_1'(T(x)) \\ t_1'(T(x))T'(x) = t_2'(x) = \dfrac{M_2(s_2(x), t_2(x))}{N_2(s_2(x), t_2(x))} = \dfrac{M_2(s_1(T(x)), t_1(T(x)))}{N_2(s_1(T(x)), t_1(T(x)))} = t_1'(T(x)). \end{cases} \tag{3.9}$$

It follows that $T'(x) = 1$. Therefore, $T(x) = x + \tilde{c}$ for some constant $\tilde{c}$. In fact, we can compute the precise transformation $T(x) = (\mathcal{P}_1^{-1} \circ \mathcal{P}_2)(x)$, where $\mathcal{P}_1(x) = (s_1(x), t_1(x))$ and $\mathcal{P}_2(x) = (s_2(x), t_2(x))$. □

Remark 3.3.2. Let $(s(x), t(x))$ be a rational solution of the autonomous system (2.16). Then for any constant $\tilde{c}$, $(s(x+\tilde{c}), t(x+\tilde{c}))$ is also a rational solution of that system. In a sense, the latter solution is more general because we can evaluate any value for the constant $\tilde{c}$. From the point of view of parametrization, however, these two rational solutions parametrize the same algebraic curve. Later on, when we map this solution curve into a solution curve of the differential equation $F(x, y, y') = 0$, the constant $\tilde{c}$ will be eliminated and it plays no role in generating a rational general solution of $F(x, y, y') = 0$. Therefore, we can simply skip that arbitrary constant, appearing in the shifting way, of a the general solution of (2.16).

3.4 Algorithm and examples

We give our algorithm to compute a rational solution of the system (2.16) from an irreducible invariant algebraic curve of the system. If we apply the algorithm to a general invariant algebraic curve, then, in the affirmative case, its output is a rational general solution.

Algorithm `RATSOLVE`
Input: The system (2.16) and an irreducible invariant algebraic curve $G(s,t) = 0$ of the system such that $G \nmid N_1$ and $G \nmid N_2$.
Output: The corresponding rational solution of (2.16), if any.

1. `if` $G(s,t) = 0$ is not a rational curve, `then` return "there is no rational solution corresponding to $G(s,t) = 0$."
2. `else` compute a proper rational parametrization $(s(x), t(x))$ of $G(s,t) = 0$.
3. `if` $s'(x) \not\equiv 0$, then find the rational solution of the autonomous differential equation
$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}.$$
4. `else` find the rational solution of the autonomous differential equation
$$T' = \frac{1}{t'(T)} \cdot \frac{M_2(s(T), t(T))}{N_2(s(T), t(T))}.$$
5. `if` there exists $T(x)$ (a linear rational function), `then` return
$$(s(T(x)), t(T(x)))$$
6. `else` return "there is no rational solution corresponding to $G(s,t) = 0$."

Example 3.4.1. Consider the rational differential system
$$\begin{cases} s' = \dfrac{-2(-(t-1)^2 + s^2)(t-1)^2}{((t-1)^2 + s^2)^2}, \\ t' = \dfrac{-4(t-1)^3 s}{((t-1)^2 + s^2)^2}. \end{cases} \tag{3.10}$$
First we compute the set of invariant algebraic curves of degree less than or equal to 2,
$$\left\{ t - 1 = 0, s + \sqrt{-1}(t-1) = 0, s - \sqrt{-1}(t-1) = 0, s^2 + t^2 + (-1-c)t + c = 0 \right\},$$
where c is an arbitrary constant.

For computing rational solutions of the system we will not consider the two invariant algebraic curves $s + \sqrt{-1}(t-1) = 0$ and $s - \sqrt{-1}(t-1) = 0$ because they are divisors of the denominators of the system.

The invariant algebraic curve $t - 1 = 0$ can be parametrized by $(x, 1)$. The corresponding differential equation for reparametrization is $T' = 0$. Hence, $T(x) = c$, where

c is an arbitrary constant. This implies that $s(x) = c, t(x) = 1$ is a rational solution corresponding to the rational solution line $t - 1 = 0$.

It remains to consider the general invariant algebraic curve $s^2 + t^2 + (-1-c)t + c = 0$. This determines a rational curve in $\mathbb{A}^2(\overline{\mathbb{K}(c)})$, having the proper rational parametrization

$$\mathcal{P}(x) = \left(\frac{(c-1)x}{1+x^2}, \frac{cx^2+1}{1+x^2}\right).$$

By the algorithm RATSOLVE, the corresponding autonomous differential equation for reparametrization is $T' = \dfrac{-2T^2}{c-1}$. Hence $T(x) = \dfrac{c-1}{2x}$. Now we subsitute $T(x)$ into $\mathcal{P}(x)$ to obtain a rational (general) solution of the system (3.10), namely,

$$s(x) = \frac{2(c-1)^2 x}{4x^2 + (c-1)^2}, \quad t(x) = \frac{c(c-1)^2 + 4x^2}{4x^2 + (c-1)^2}.$$

Later, we can prove that the system has no irreducible invariant algebraic curve of degree higher than 2 (by Darboux's theorem).

Remark 3.4.1. Proper rational parametrizations of a rational curve are not unique. We can also parametrize the curve $s^2 + t^2 + (-1-c)t + c = 0$ by

$$\mathcal{P}_1(x) = \left(\frac{-ci + (i+ic)x - ix^2}{1+c-2x}, \frac{c-x^2}{1+c-2x}\right),$$

where i is the imaginary unit. In this case, we obtain another rational general solution, namely,

$$s(x) = \frac{(2ix - c + 1)(c-1)^2}{4x(ix - c + 1)}, \quad t(x) = \frac{i(4x^2 + 4i(c-1)x + (c-1)^3)}{4x(ix - c + 1)}.$$

This solution is transformable into the first one by the change of variable

$$\varphi(x) = x + \frac{i(c-1)}{2}.$$

Here we give another example for a complete algorithm combining the two algorithms GENERALSOLVER and RATSOLVE.

Example 3.4.2. Considering the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0. \tag{3.11}$$

The corresponding algebraic surface $z^2+3z-2y-3x=0$ can be parametrized by

$$\mathcal{P}_0(s,t)=\left(\frac{t}{s}+\frac{2s+t^2}{s^2},-\frac{1}{s}-\frac{2s+t^2}{s^2},\frac{t}{s}\right).$$

This is a proper parametrization and the corresponding associated system is

$$\begin{cases} s'=st,\\ t'=s+t^2.\end{cases}$$

We compute the set of irreducible invariant algebraic curves of the system and obtain

$$\{s=0,t^2+2s=0,s^2+ct^2+2cs=0 \mid c \text{ is an arbitrary constant}\}.$$

The general invariant algebraic curve $s^2+ct^2+2cs=0$ can be parametrized by

$$\mathcal{Q}(x)=\left(-\frac{2cx^2}{x^2+c},-\frac{2cx}{x^2+c}\right).$$

By the algorithm `RATSOLVE`, we have to solve an auxiliary differential equation for the reparametrization, namely:

$$T'=\frac{1}{\mathcal{Q}_1(T)'}\mathcal{Q}_1(T)\mathcal{Q}_2(T)=-T^2.$$

Hence, $T(x)=\dfrac{1}{x}$. So the rational general solution of the associated system is

$$s(x)=\mathcal{Q}_1(T(x))=-\frac{2c}{1+cx^2},\quad t(x)=\mathcal{Q}_2(T(x))=-\frac{2cx}{1+cx^2}.$$

We observe that

$$\chi_1(s(x),t(x))=x-\frac{1}{c}.$$

Therefore, according to algorithm `GENERALSOLVER`, the rational general solution of (3.11) is

$$y=\chi_2\left(s\left(x+\frac{1}{c}\right),t\left(x+\frac{1}{c}\right)\right)=\frac{1}{2}x^2+\frac{1}{c}x+\frac{1}{2c^2}+\frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y=\frac{1}{2}((x+c)^2+3c).$$

3.5 Singularities of a planar polynomial system and degree bounds

In order to determine all possible invariant algebraic curves of the system (3.1), it is sufficient to have an upper bound for the degree of these curves. The analysis of singularities of the system (3.1) will be a tool for determining this bound. As we mentioned, it is enough to consider polynomial systems when we look for the invariant algebraic curves of a planar rational system.

Geometrically, a solution $(s(x), t(x))$ of the system (3.1) represents a branch in a certain domain of the plane. Since the right hand sides of the system (3.1) are given by polynomials, the system has a unique solution at every point on the plane by the Picard's Existence and Uniqueness Theorem. If a solution $(s(x), t(x))$ is algebraic, i.e., there is a polynomial $G(s, t)$ such that $G(s(x), t(x)) = 0$, then it represents a plane algebraic curve. Therefore, $G(s, t) = 0$ is an invariant algebraic curve of the system (3.1) and it is the algebraic closure of the branch defined by $(s(x), t(x))$. Suppose that $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ are two distinct algebraic solutions of the system (3.1). Although they define distinct branches on the plane, their algebraic closures might have a common point. This means that there might be one or several invariant algebraic curves of the system passing through a point. The number of invariant algebraic curves of the system (3.1) at a point is an essential information to study a degree bound for those curves. This section shortly summarize some known properties of the invariant algebraic curves of the system (3.1) at a point of the plane.

First of all, the singular points of the system (3.1), in the affine plane, are the common points of two curves $P(s, t) = 0$ and $Q(s, t) = 0$. There is at most one invariant algebraic curve at a non-singular point (cf. Singer (1992), Lemma A1 or Seidenberg (1968), Theorem 2). It is more complicated to see the number of invariant algebraic curves at a singular point of the system. In his paper Seidenberg (1968), Seidenberg gave several structure theorems to describe the invariant curves defined by formal power series. In particular, the information of the invariant algebraic curves can be deduced from here since they are special cases defined by polynomials.

Let us have a look at the system (3.1) at a singular point. Without loss of generality one can assume that $(0, 0)$ is a singular point of the system, i.e., $P(0, 0) = Q(0, 0) = 0$. Then the system (3.1) can be written as

$$\begin{cases} s' = P_r(s,t) + P_{r+1}(s,t) + \cdots + P_m(s,t), \\ t' = Q_r(s,t) + Q_{r+1}(s,t) + \cdots + Q_m(s,t), \end{cases} \tag{3.12}$$

where P_i and Q_i are the homogeneous components of degree i of P and Q, respectively; r is the lowest degree such that either $P_r(s,t) \neq 0$ or $Q_r(s,t) \neq 0$; and m is the highest degree such that either $P_m(s,t) \neq 0$ or $Q_m(s,t) \neq 0$. Note that r is invariant under an invertible linear change of coordinates and according to Seidenberg it is an important measurement of the system. In fact, the study of those systems with $r > 1$ can be reduced to the one with $r = 1$. Let

$$J_0(P,Q) = \begin{pmatrix} \frac{\partial P}{\partial s}(0,0) & \frac{\partial P}{\partial t}(0,0) \\ \\ \frac{\partial Q}{\partial s}(0,0) & \frac{\partial Q}{\partial t}(0,0) \end{pmatrix} \tag{3.13}$$

be the Jacobian matrix of the system (3.12) at $(0,0)$. It turns out that if $r > 1$, then $J_0(P,Q) = 0$. If $r = 1$, then the eigenvalues of $J_0(P,Q)$ can possibly be one of the following cases:

- $\lambda_1 = \lambda_2 = 0$;
- $\lambda_1 = 0, \lambda_2 \neq 0$; or $\lambda_1 \neq 0, \lambda_2 = 0$;
- $\lambda_1 \neq 0, \lambda_2 \neq 0, \dfrac{\lambda_1}{\lambda_2} \in \mathbb{Q}_+$;
- $\lambda_1 \neq 0, \lambda_2 \neq 0, \dfrac{\lambda_1}{\lambda_2} \notin \mathbb{Q}_+$;

Assume that $P_1 = as + bt$ and $Q_1 = cs + dt$. It can happen in the first two cases if $ad - bc = 0$. The others happen if $ad - bc \neq 0$.

Definition 3.5.1. The singular point $(0,0)$ of the system (3.12) is called a *simple singularity* iff either $\lambda_1 = 0, \lambda_2 \neq 0$; or $\lambda_1 \neq 0, \lambda_2 = 0$; or $\lambda_1 \neq 0, \lambda_2 \neq 0, \dfrac{\lambda_1}{\lambda_2} \notin \mathbb{Q}_+$.

Let $\mathcal{D} := P\frac{\partial}{\partial s} + Q\frac{\partial}{\partial t}$. A solution of $\mathcal{D}$ is a function $f(s,t)$ such that $\mathcal{D}(f) \equiv 0 \bmod (f)$. In Seidenberg (1968), f is considered in the set of formal power series. In this section, the word "solution" is meant a solution of $\mathcal{D}$ in that sense.

If $(0,0)$ is a simple singularity of the system (3.12), then there are only two formal power series solutions at $(0,0)$ (Seidenberg (1968), Theorem 8(a); Theorem 10A). In particular, there are at most two invariant algebraic curves of the system at a simple singularity.

Seidenberg also showed that if $r > 1$, then it can be reduced to the case $r = 1$ by a certain number of birational transformations of the form

$$s = u, \quad t = uv.$$

Moreover, after finitely many steps of such transformations, one obtains a collection of systems with all simple singularities (Seidenberg (1968), Theorem 12; Theorem 10A; Theorem 10B and its proof). The transformation is known as the blowup technique in algebraic geometry and Carnicer (1994) has applied this reduction process to study an upper bound for the degree of an irreducible invariant algebraic curve of the polynomial system (3.1).

The reduction process distinguishes a singular point into two different classes: *dicritical singularities* and *non-dicritical singularities*. In fact, the blowup transforms the system (3.12) into a new system

$$\begin{cases} u' = u^r(P_r(1,v) + \cdots), \\ v' = u^{r-1}(Q_r(1,v) - vP_r(1,v) + \cdots). \end{cases} \tag{3.14}$$

At this step we have two possibilities either $Q_r(1,v) - vP_r(1,v) = 0$ identically or not. Of course, the second possibility is the generic case: $Q_r(1,v) - vP_r(1,v) \neq 0$.

- If $Q_r(1,v) - vP_r(1,v) \neq 0$, then dividing by u^{r-1}, the system (3.14) is simplified as

$$\begin{cases} u' = u(P_r(1,v) + \cdots), \\ v' = Q_r(1,v) - vP_r(1,v) + \cdots . \end{cases} \tag{3.15}$$

  In this case, the line $u = 0$ is an invariant algebraic curve. The singularities of the system (3.15) are of the form $(0, v_i)$, where v_i is a root of $Q_r(1,v) - vP_r(1,v)$. Since $Q_r(1,v) - vP_r(1,v) \neq 0$, it has only finitely many roots. Therefore, there are only finite such singular points $(0, v_i)$. The local solutions of (3.15) at $(0, v_i)$ go over into the local solutions of (3.12) at $(0,0)$.

- If $Q_r(1,v) - vP_r(1,v) = 0$, then dividing by u^r, the system (3.14) is simplified as

$$\begin{cases} u' = P_r(1,v) + uP_{r+1}(1,v) + \cdots , \\ v' = uQ_{r+1}(1,v) - uvP_{r+1}(1,v) + \cdots . \end{cases} \tag{3.16}$$

  In this case, the line $u = 0$ is not an invariant algebraic curve. Observe that most of the points on this line are non-singular points of the system (3.16) except for those points of the form $(0, v_i)$ where v_i is a root of the polynomial $P_r(1,v)$. (In this case, $P_r(1,v) \neq 0$ otherwise $Q_r(1,v) = 0$, which contradicts to the assumption that P_r and Q_r are not both zero). Since there is a unique formal power series solution at a non-singular point, there is an infinite family of local solutions of the system (3.16) intersecting the line $u = 0$ transversally at those non-singular points. The blowing down of these local solutions yields an infinite family of local solutions passing through the singular point $(0,0)$ of the system (3.12). Therefore, the system (3.12) might have infinite family of invariant algebraic curves at $(0,0)$.

Definition 3.5.2. The blowup of the system (3.12) at $(0,0)$ is called *dicritical* iff

$$Q_r(1,v) - vP_r(1,v) = 0. \tag{3.17}$$

The singular point $(0,0)$ of the system (3.12) is called a *dicritical singularity* iff there exists a dicritical blowup at some step of the blowup process. Otherwise, it is called a *non-dicritical singularity.*

If $(0,0)$ is a non-dicritical singularity, then there are only finitely many local solution curves of the system passing through the point. In particular, the number of invariant algebraic curves at $(0,0)$ is finite in this case.

3.5.1 The homogenization of a polynomial system

In order to obtain a global information of an irreducible invariant algebraic curve, one has to study it in the complex projective plane $\mathbb{CP}^2$, where one takes into account the points at infinity. In this case, the configuration obtained after one step of the blowup process is called a *holomorphic singular foliation.*

After finitely many steps of blowups, one obtains a foliation whose singularities are all simple and further blowups at a simple singularity will not give more information about the foliation at that point. Therefore, one can stop the blowup process when all singularities are simple.

It remains to describe the system (3.12) in its completion, i.e., its homogenization in the projective space $\mathbb{CP}^2$. Note that the system (3.12) can be associated to an affine polynomial vector field

$$\mathcal{D} = P(s,t)\frac{\partial}{\partial s} + Q(s,t)\frac{\partial}{\partial t}$$

or the 1-form

$$P(s,t)dt - Q(s,t)ds = 0.$$

These can be homogenized in the projective space $\mathbb{CP}^2$ as follows. Let

$$\begin{cases} s = \dfrac{u}{w}, \\ t = \dfrac{v}{w}. \end{cases}$$

Then $ds = \dfrac{wdu - udw}{w^2}$ and $dt = \dfrac{wdv - vdw}{w^2}$. It follows from the 1-form

$$P(s,t)dt - Q(s,t)ds = 0$$

that
$$P\left(\frac{u}{w},\frac{v}{w}\right)\cdot\frac{wdv-vdw}{w^2}-Q\left(\frac{u}{w},\frac{v}{w}\right)\cdot\frac{wdu-udw}{w^2}=0.$$
Let $m=\max\{\deg P,\deg Q\}$. Then we have the homogeneous 1-form in the projective space $\mathbb{CP}^2$ as follows
$$-w^{m+1}Q\left(\frac{u}{w},\frac{v}{w}\right)du+w^{m+1}P\left(\frac{u}{w},\frac{v}{w}\right)dv+\left(uw^mQ\left(\frac{u}{w},\frac{v}{w}\right)-vw^mP\left(\frac{u}{w},\frac{v}{w}\right)\right)dw=0. \tag{3.18}$$

- If we let $w=1$ and $dw=0$, we return the given 1-form
$$P(u,v)dv-Q(u,v)du=0.$$
- If we let $u=1$ and $du=0$, we return the 1-form
$$w^{m+1}P\left(\frac{1}{w},\frac{v}{w}\right)dv+\left(w^mQ\left(\frac{1}{w},\frac{v}{w}\right)-vw^mP\left(\frac{1}{w},\frac{v}{w}\right)\right)dw=0.$$
- If we let $v=1$ and $dv=0$, we return the 1-form
$$-w^{m+1}Q\left(\frac{u}{w},\frac{1}{w}\right)du+\left(uw^mQ\left(\frac{u}{w},\frac{1}{w}\right)-w^mP\left(\frac{u}{w},\frac{1}{w}\right)\right)dw=0.$$

The last two equations are the two 1-forms in the other affine charts of the projective space $\mathbb{CP}^2$. So we also have the representation of the given polynomial vector field at infinity.

Let $A=-w^{m+1}Q\left(\frac{u}{w},\frac{v}{w}\right)$, $B=w^{m+1}P\left(\frac{u}{w},\frac{v}{w}\right)$ and
$$C=uw^mQ\left(\frac{u}{w},\frac{v}{w}\right)-vw^mP\left(\frac{u}{w},\frac{v}{w}\right).$$
Then A,B and C are homogeneous polynomials with
$$uA+vB+wC=0. \tag{3.19}$$
The above condition is called the *Euler's condition.* We refer to Lins Neto (1988); Carnicer (1994) for the definition of a foliation in the projective space $\mathbb{CP}^2$. Denote $\mathcal{F}$ to be such a foliation.

Definition 3.5.3. A *singularity* of the polynomial system (3.1) in the projective space $\mathbb{CP}^2$ is a common point of the three projective curves $A=0,B=0$ and $C=0$. The set of all singularities of the system is denoted by $\mathrm{Sing}(\mathcal{F})$.

Note that a common point of $A=0,B=0$ and $C=0$ is either (at finite) of the form $(u:v:1)$, where $P(u,v)=Q(u,v)=0$, or (at infinity) of the form $(u:v:0)$, where u

and v satisfy the equation

$$uQ_m(u,v) - vP_m(u,v) = 0,$$

with $m = \max\{\deg(P), \deg(Q)\}$. The singularities of the foliation at infinity contain the common points at infinity of the two curves $P(u,v) = Q(u,v) = 0$.

It may happen that $uQ_m(u,v) - vP_m(u,v) = 0$ identically, then the system (3.1) is called *degenerated*. In this case, the line at infinity is not an invariant algebraic curve. In fact, the line at infinity is an invariant algebraic curve if and only if $uQ_m(u,v) - vP_m(u,v) \neq 0$ (Lemma 2, Lins Neto (1988)).

3.5.2 Degree of a polynomial system and a degree bound

In algebraic geometry, the degree of a projective plane curve can be defined as the number of intersection points—counting with the multiplicity—of the curve with a generic line in the projective plane $\mathbb{KP}^2$. In particular, if $\mathcal{C}$ is an algebraic curve of degree d and a generic line is tangent to $\mathcal{C}$ at $P \in \mathcal{C}$ of multiplicity r, then it can have at most $d - r$ other distinct intersection points. In this way, we see that the sum of multiplicities of a generic line tangent to $\mathcal{C}$ is a close lower bound to the degree of $\mathcal{C}$. Later, this number is defined as the degree of a polynomial system (Lins Neto (1988), Section 1.4).

Here we recall that notion of degree of the polynomial system (3.1). First of all, a generic line in this context is a line which is not an invariant algebraic curve of the given system.

Consider a generic line L in the affine space, for instance, passing through (s_0, t_0) with the direction (a, b). Let $p \in L$ and p be not a singular point of the polynomial system (3.1). Then p is called a *tangency point* of L and the solution curve* of the system if the tangent vector of the solution curve at p is coincided with the direction of the line L, i.e. there exists a parameter λ such that

$$bP(s_0 + a\lambda, t_0 + b\lambda) - aQ(s_0 + a\lambda, t_0 + b\lambda) = 0. \tag{3.20}$$

Definition 3.5.4. The multiplicity of the root λ of this polynomial equation is called the *multiplicity of tangency* of the solution curve and the line L at p. Let d be the sum of all multiplicities over all the tangency points of L with the solution curves of the system (3.1) in the projective space. Then d is called the *degree of the system* (3.1), denoted by $\deg \mathcal{F}$.

The definition of the degree of a polynomial system does not depend on the choice of a generic line L. There are two possible values for the degree of the system (3.1) depending whether the system is degenerated or not. Namely, if $uQ_m(u,v) - vP_m(u,v) \neq 0$,

*There is a unique solution at a non-singular point.

then $\deg \mathcal{F} = m$; otherwise, $uQ_m(u,v) - vP_m(u,v) = 0$ and $\deg \mathcal{F} = m-1$, where $m = \max\{\deg(P), \deg(Q)\}$ (Lins Neto (1988), Lemma 2).

Example 3.5.1. Consider the system

$$\{s' = st,\ t' = s + t^2\}. \tag{3.21}$$

This system is degenerated (i.e. the line at infinity is not an invariant algebraic curve). Hence, the degree of the system is 1. If we look at the system at infinity, it is represented by

$$\{v' = 1,\ w' = -v\} \tag{3.22}$$

Observe that the system at infinity has no singularity, hence the given system has only one singularity at $(0,0)$.

Let us consider the line $L := \{s = 1 + ax,\ t = 1 + bx\}$ passing through $(1,1)$ with direction (a,b). This line is tangent to a solution curve of the system if the following equation has a solution

$$b(1+ax)(1+bx) - a(1 + ax + (1+bx)^2) = 0. \tag{3.23}$$

This equation has a unique solution $x = \dfrac{2a-b}{b^2-a^2-ab}$ of multiplicity 1. This means that the line has only one tangency point, namely $\left(\dfrac{-(a-b)^2}{a^2-b^2+ab}, \dfrac{a(a-b)}{a^2-b^2+ab}\right)$.

If we look at the line at infinity, then the equation of this line is

$$L_\infty := \{v = 1 + (a-b)x,\ w = 1 + ax\}.$$

The tangency points are defined, according to the system (3.22), by the equation

$$a + (a-b)(1 + (a-b)x) = 0. \tag{3.24}$$

Hence, the tangency point is $\left(\dfrac{-a}{a-b}, \dfrac{-a^2-ab+b^2}{(a-b)^2}\right)$. In fact, this is the same point in the other coordinate.

Let us see what is the type of the singularity at $(0,0)$. The first blow up is not dicritical and it gives us two systems

$$\{s' = s^2u,\ u' = 1\},\ \ \{t' = -v^2,\ v' = vt + t^2\}. \tag{3.25}$$

We only have to blow up the sencond system at $(0,0)$ again. It is not dicritical and we obtain two systems

$$\{t' = -u^2 t,\ u' = u^3 + u + 1\},\ \ \{s' = -v^3 - v^2 - 1,\ v' = vs(v+1)\}. \tag{3.26}$$

The first system has only one simple singularity at $(0, u_0)$, where $u_0^3 + u_0 + 1 = 0$. The second system also has only one simple singularity at $(0, v_0)$, where $v_0^3 + v_0^2 + 1 = 0$. Therefore, $(0,0)$ is a non-dicritical singularity of the given system.

We have seen that the degree of the system (3.1) is a close lower bound for the degree of an invariant algebraic curve of that system. If the system (3.1) has no dicritical singularities, then this degree is not far away from the degree of its invariant algebraic curves. The following is the main result in Carnicer (1994) on the Poincaré's problem in the non-dicritical case.

Theorem 3.5.1. *Let $\mathcal{F}$ be a foliation of $\mathbb{CP}^2$ and $\mathcal{C} \subset \mathbb{CP}^2$ be an invariant algebraic curve of $\mathcal{F}$ such that there are no dicritical singularities of $\mathcal{F}$ in $\mathcal{C}$. Then*

$$\deg(\mathcal{C}) \leq \deg(\mathcal{F}) + 2.$$

Taking into account the genus of the invariant algebraic curves we have a property of the system without dicritical singularities.

Theorem 3.5.2. *Let $\mathcal{F}$ be a foliation of $\mathbb{CP}^2$ without dicritical singularities. If $\mathcal{F}$ has a rational general invariant algebraic curve $\mathcal{C}$, then either there exists a singular point at which infinitely many invariant algebraic curve passing through or* $\deg(\mathcal{C}) \leq 2$.

Proof. Assume that there are only finitely many invariant algebraic curves at each singular point. Since the singularities of an invariant algebraic curve are among the singularities of the system, most of the invariant algebraic curves in $\mathcal{C}$ have no singularities. Now the curves of genus zero without singularities are of degree at most 2. □

3.6 Rational first integrals and Darboux's theorem

In this section, we recall the notion of rational first integrals of a rational system and state the Darboux's theorem on the existence of a rational first integral. The irreducible invariant algebraic curves of the system (3.1) are the main ingredients to build up a rational first integral of the system.

Let $\mathcal{W}$ be the set of all differentiable functions in s and t. Let

$$\begin{aligned} \mathcal{D} : \mathcal{W} &\longrightarrow \mathcal{W} \\ W &\longmapsto \mathcal{D}W := PW_s + QW_t, \end{aligned} \tag{3.27}$$

where W_s and W_t are the partial derivatives of W w.r.t. s and t. Then $\mathcal{D}$ is a derivation on $\mathcal{W}$. Indeed, one can easily check that $\mathcal{D}$ satisfies the following properties:

- $\mathcal{D}(W_1 + W_2) = \mathcal{D}W_1 + \mathcal{D}W_2$,
- $\mathcal{D}(W_1 W_2) = (\mathcal{D}W_1)W_2 + W_1(\mathcal{D}W_2)$.

Definition 3.6.1. A bivariate function $W \in \mathcal{W}$ is called a *first integral* of the system (2.16) iff $\mathcal{D}W = 0$. In addition, if W is a rational function, then W is called a *rational first integral* of the system (2.16).

Since $\mathcal{D}$ is a derivation on $\mathcal{W}$, it follows that the set of all first integrals of the system (2.16) forms a field and its intersection with $\mathbb{K}(s,t)$ is the subfield of all rational first integrals of the system (2.16).

Observation 3.6.1. *Every first integral of the system* (2.16) *is constant on any solution of that system.*

In fact, let $(s(x), t(x))$ be a solution of the system (2.16) and $W(s,t)$ be a first integral of the same system. Consider the function $\phi(x) = W(s(x), t(x))$ we have

$$\begin{aligned}\phi(x)' =& W_s(s(x),t(x))s'(x) + W_t(s(x),t(x))t'(x)\\ =& W_s(s(x),t(x))\frac{M_1(s(x),t(x))}{N_1(s(x),t(x))} + W_t(s(x),t(x))\frac{M_2(s(x),t(x))}{N_2(s(x),t(x))}\\ =& \frac{(\mathcal{D}W)(s(x),t(x))}{N_1(s(x),t(x))N_2(s(x),t(x))} = 0.\end{aligned}$$

Hence, the first integral $W(s,t)$ is constant on any solution of the system (2.16).

One can construct a first integral of the system (2.16) if it has sufficiently many irreducible invariant algebraic curves. Indeed, from the proof of Lemma 3.2.1 we can see that if $G_1, \ldots, G_l$ are relatively prime irreducible invariant algebraic curves with cofactors $K_1, \ldots, K_l$, then

$$\mathcal{D}\prod_{i=1}^{l} G_i^{n_i} = \prod_{i=1}^{l} G_i^{n_i}\left(\sum_{i=1}^{l} n_i K_i\right).$$

Since the degree of any cofactor is bounded by $m-1$, the dimension of the $\mathbb{K}$-vector space of all cofactors is at most $d = \dfrac{m(m+1)}{2}$. Therefore, if the number of relatively prime (irreducible) invariant algebraic curves is higher than d, then there exists a non-trivial linear combination of their cofactors, i.e., $\sum_{i=1}^{l} n_i K_i = 0$, where $n_i \in \mathbb{K}$. If $n_i \in \mathbb{C}$ (resp., in $\mathbb{Z}$) for all $i = 1, \ldots, l$, then $W = \prod_{i=1}^{l} G_i^{n_i}$ is a (resp., rational) first integral of the system (2.16). The Darboux's theorem guarantees that if $l \geq d+2$, then we can always find a rational first integral of the system. Moreover, every other irreducible invariant algebraic curves of the system can be determined from a rational first integral by factorization.

Theorem 3.6.1 (Darboux's theorem). *Let $P, Q \in \mathbb{C}[s,t]$ and $\mathcal{D} = P\dfrac{\partial}{\partial s} + Q\dfrac{\partial}{\partial t}$. If $G_1, \ldots, G_r$ are relatively prime irreducible polynomials in $\mathbb{C}[s,t]$ such that $G_i | \mathcal{D}G_i$ for all $i = 1, \ldots, l$, then either $l < \dfrac{m(m+1)}{2} + 2$ or there exist integers n_i not all zero such that $\mathcal{D}\prod_{i=1}^{l} G_i^{n_i} = 0$. In the later case, if G is any irreducible polynomial such that $G|\mathcal{D}G$, then either there exist $c_1, c_2 \in \mathbb{C}$ not both zero such that*

$$G | c_1 \prod_{n_i \geq 0} G_i^{n_i} - c_2 \prod_{n_j < 0} G_j^{-n_j}$$

or $G| \gcd(P, Q)$.

A proof of the Darboux's theorem can be found in Singer (1992). From the theorem we see that the degree of any irreducible invariant algebraic curve of the polynomial system (3.1) is bounded.

In order to decide the existence of a first integral of the system (2.16), it is sufficient to compute an upper bound for the degree of an irreducible invariant algebraic curve of the system. This is known as the Poincaré problem. There are several authors working on this problem (e.g. Carnicer (1994), Walcher (2000)), and one usually gets an upper bound when an additional assumption either on the invariant algebraic curves or on the system is provided. Otherwise, one can not expect such an upper bound depending only on the degree of the system because, for instance, the system $\{s' = ps,\ t' = t\}$ has $s - t^p = 0$ as an irreducible invariant algebraic curve of degree $p \in \mathbb{N}$ and the degree of the system is just 1.

3.7 Rational general solutions and rational first integrals

In this section, we present a closed relation between a rational general solution of the system (2.16) and its rational first integrals. Since Darboux's theorem (Singer (1992)) on the existence of a rational first integral is valid[†] for the field of complex numbers $\mathbb{C}$, we consider in this section that $\mathbb{K} = \mathbb{C}$.

As a corollary of Darboux's theorem, we know that either there are finitely many irreducible invariant algebraic curves or there is a rational first integral of the system (2.16). In any case, the degree of irreducible invariant algebraic curves is bounded. If the system (2.16) has no dicritical singularities, then there is an upper bound for the degree of the irreducible invariant algebraic curves in terms of the degree of the polynomial system (Carnicer (1994)). Using this upper bound, one can always check whether the system has a rational first integral and compute it in the affirmative case via the Prelle-Singer procedure (Prelle and Singer (1983), Man (1993)).

[†]In this field, the notion of powers of a polynomial is well-defined.

Q 10. *Suppose that the system* (2.16) *has a rational first integral. Can we determine a rational general solution of that system from this first integral?*

The goal of this section is to determine a rational solution curves $G(s,t) = 0$ of a rational general solution of the system (2.16) in terms of a given rational first integral of the system (2.16).

We recall some basic properties of rational first integrals.

Lemma 3.7.1. *Let* $\dfrac{U(s,t)}{V(s,t)}$ *be a rational first integral of the system* (3.1) *with* $\gcd(U,V) = 1$*. Then* U *and* V *are invariant algebraic curves of the polynomial system* (3.1)*. Moreover, for any constants* c_1, c_2*, every irreducible factor of* $c_1U - c_2V$ *determines an invariant algebraic curve of the system* (3.1)*.*

Proof. We have $D\left(\dfrac{U}{V}\right) = 0$. This implies that $DU.V = U.DV$. Since $\gcd(U,V) = 1$, we must have $U|DU$ and $V|DV$, meaning U and V are invariant algebraic curves of the polynomial system (3.1).

Let $c_1U - c_2V = \prod_i P_i^{\alpha_i}$. Then

$$D(c_1U - c_2V) = c_1DU - c_2DV = c_1DU - c_2\frac{DU.V}{U} = \frac{DU}{U}(c_1U - c_2V).$$

Hence,

$$\sum_i \alpha_i P_i^{\alpha_i - 1} DP_i \prod_{j\neq i} P_j^{\alpha_i} = K \prod_i P_i^{\alpha_i},$$

where $K = \dfrac{DU}{U}$. Therefore, $P_i|DP_i$ for all i, i.e., P_i defines an invariant algebraic curve. □

Note that the above property is an affine version of a more general result in the projective space that can be found in Jouanolou (1979), Lemma 3.2.3, page 100.

Remark 3.7.1. Let $U, V \in \mathbb{K}[s,t]$ with $\gcd(U,V) = 1$. Let c be a transcendental constant over $\mathbb{K}$. Consider the factorization of the polynomial $U - cV$ over the field $\overline{\mathbb{K}(c)}$. Since the degree with respect to c of $U - cV$ is 1, using a generalized version of Theorem 36 in Chapter 3 of Schinzel (2000), we have that the irreducible factors of $U - cV$ are conjugate over $\mathbb{K}(c)$ and they appear in the form $A + \alpha B$, where $A, B \in \mathbb{K}[s,t]$ and $\alpha \in \overline{\mathbb{K}(c)}$. Moreover, α is also a transcendental constant over $\mathbb{K}$ because c is so.

Lemma 3.7.2. *Let* $\sigma : \overline{\mathbb{K}(c)} \longrightarrow \overline{\mathbb{K}(c)}$ *be an automorphism of fields with* $\sigma(\alpha) = \alpha$ *for all* $\alpha \in \mathbb{K}(c)$*. Let* $\overline{\sigma}_1 : \overline{\mathbb{K}(c)}[s,t] \longrightarrow \overline{\mathbb{K}(c)}[s,t]$ *be the induced homomorphism of rings defined by*

$$\overline{\sigma}_1\left(\sum_{i,j} c_{ij}s^it^j\right) = \sum_{i,j}\sigma(c_{ij})s^it^j. \tag{3.28}$$

If $G(s,t)=0$ is a rational solution curve of the system (2.16)*, so is $\overline{\sigma}_1(G)$.*

Proof. Let $\overline{\sigma}_2 : \overline{\mathbb{K}(c)}(x) \longrightarrow \overline{\mathbb{K}(c)}(x)$ be the homomorphism of fields defined by

$$\overline{\sigma}_2\left(\frac{\sum_i a_i x^i}{\sum_j b_j x^j}\right) = \frac{\sum_i \sigma(a_i) x^i}{\sum_j \sigma(b_j) x^j},$$

where $a_i, b_j \in \overline{\mathbb{K}(c)}$. Suppose that $(s(x), t(x))$ is a rational parametrization of the curve $G(s,t)=0$, i.e., we have $G(s(x),t(x))=0$.

Let $G(s,t)=\sum_{ij} c_{ij} s^i t^j$. Let $\phi(x)=G(s(x),t(x))=\sum_{ij} c_{ij} s(x)^i t(x)^j$. Applying the homomorphism $\overline{\sigma}_2$ we obtain

$$0=\overline{\sigma}_2(\phi(x))=\sum_{ij}\sigma(c_{ij})\overline{\sigma}_2(s(x))^i\overline{\sigma}_2(t(x))^j=\overline{\sigma}_1(G)(\overline{\sigma}_2(s(x)),\overline{\sigma}_2(t(x))).$$

Therefore, $\overline{\sigma}_1(G)$ is a rational curve parametrized by $(\overline{\sigma}_2(s(x)),\overline{\sigma}_2(t(x)))$.

Since $G(s,t)=0$ is a rational solution curve of the system (2.16), the differential equation

$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}$$

has a rational solution $T(x)$. Applying the homomorphism $\overline{\sigma}_2$ on this equation and noting that the coefficients of $M_1(s,t)$ and $N_1(s,t)$ are invariant under the action of σ, so we have that $\overline{\sigma}_2(T(x))$ is a rational solution of the differential equation

$$T' = \frac{1}{\overline{\sigma}_2(s(T))'} \cdot \frac{M_1(\overline{\sigma}_2(s(T)),\overline{\sigma}_2(t(T)))}{N_1(\overline{\sigma}_2(s(T)),\overline{\sigma}_2(t(T)))}.$$

Therefore, $\overline{\sigma}_1(G)$ is a rational solution curve of the system (2.16). □

Theorem 3.7.3. *The system* (2.16) *has a rational general solution if and only if the system* (2.16) *has a rational first integral $\frac{U}{V} \in \mathbb{K}(s,t)$ with $\gcd(U,V)=1$ and any irreducible factor of $U-cV$ determines a rational solution curve for a transcendental constant c over $\mathbb{K}$.*

Proof. If the system (2.16) has a rational general solution $\mathcal{C}(x)=(s(x),t(x))$, then it has infinitely many irreducible rational solution curves. Therefore, by corollary of Darboux's theorem (Singer (1992)), there exists a rational first integral $\frac{U}{V} \in \mathbb{K}(s,t)$. By the observation right after Definition 3.6.1 and the generality of $\mathcal{C}(x)$, there is a constant c transcendental over $\mathbb{K}$ such that

$$U(s(x),t(x)) - cV(s(x),t(x)) = 0.$$

Let $G(s,t)=0$ be the rational curve parametrized by $\mathcal{C}(x)$. Then G is a factor of $U-cV$. By Remark 3.7.1, every other irreducible factors of $U-cV$ is conjugate to G over $\mathbb{K}(c)$. Since $G(s,t)=0$ is a rational solution curve, by Lemma 3.7.2, any irreducible factor of $U-cV$ determines a rational solution curve of the system (2.16).

Conversely, suppose that the system (2.16) has a rational first integral

$$W=\frac{U}{V}\in\mathbb{K}(s,t)$$

with $\gcd(U,V)=1$. Let c be a transcendental constant over $\mathbb{K}$. By Remark 3.7.1, we know that every irreducible factor of $U-cV$ is of the form $A+\alpha B$, where $A,B\in\mathbb{K}[s,t]$ and $\alpha\in\overline{\mathbb{K}(c)}$ is a transcendental constant over $\mathbb{K}$. By the assumption, the factor $A+\alpha B$ determines a rational solution of the system (2.16). Since α is transcendental over $\mathbb{K}$, this rational solution is a rational general solution of the system (2.16) by Lemma 2.2.5. $\square$

Theorem 3.7.3 gives us an algorithmic way of computing a rational general solution of the system (2.16). In fact, when we execute `RATSOLVE`$(A+\alpha B)$, we do not need to verify

Algorithm `FIRST-INT-RATSOLVE`
Input: The system (2.16) and $W(s,t)=\dfrac{U(s,t)}{V(s,t)}$ such that $\mathcal{D}(W)=0$.
Output: A rational general solution of (2.16), if any.

1. factorize $U-cV$ over $\overline{\mathbb{K}(c)}$, where c is a transcendental constant over $\mathbb{K}$;
2. take one of the irreducible factors of $U-cV$, say $A+\alpha B\in\overline{\mathbb{K}(c)}[s,t]$;
3. apply `RATSOLVE`$(A+\alpha B)$.

the condition in algorithm `RATSOLVE` because we always have $A+\alpha B\nmid N_1$ and $A+\alpha B\nmid N_2$ for transcendental constant α over $\mathbb{K}$.

Example 3.7.1 (Example 3.4.1, cont.)**.** In fact, the system (3.10) in Example 3.4.1 has a rational first integral, namely,

$$W(s,t)=\frac{s^2+t^2-t}{t-1}.$$

The irreducible invariant algebraic curve derived from this rational first integral is

$$G(s,t)=s^2+t^2-t-c(t-1).$$

Figure 3.2: Level curves of the rational first integral $W(s,t)$

This determines a rational curve in $\mathbb{A}^2(\overline{\mathbb{K}(c)})$, having the proper rational parametrization

$$\mathcal{P}(x) = \left(\frac{(c-1)x}{1+x^2}, \frac{cx^2+1}{1+x^2}\right).$$

Applying `RATSOLVE` to $G(s,t)$, we obtain a rational general solution of the system (3.10)

$$s(x) = \frac{2(c-1)^2 x}{4x^2+(c-1)^2}, \quad t(x) = \frac{c(c-1)^2+4x^2}{4x^2+(c-1)^2}.$$

Remark 3.7.2. Note that the system (3.10) in Example 3.4.1 has other rational first integrals, for instance,

$$W_1(s,t) = \frac{(s^2+t^2-t)^2}{(t-1)^2}.$$

The factorization

$$(s^2+t^2-t)^2 - c(t-1)^2 = (s^2+t^2-t+\sqrt{c}(t-1))(s^2+t^2-t-\sqrt{c}(t-1))$$

gives us two different irreducible factors. We can take any one of them and proceed in the same way as before to obtain a rational general solution of the system (3.10).

Example 3.7.2. There are rational invariant algebraic curves without corresponding rational solutions. For example, we consider the following rational differential system, which is similar to (3.10),

$$\begin{cases} s' = \dfrac{-2(t-1)^3(-(t-1)^2+s^2)}{((t-1)^2+s^2)^2}, \\ t' = \dfrac{-4(t-1)^4 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{3.29}$$

We can see that the set of invariant algebraic curves of this system is the same as the one of (3.10), namely,

$$\left\{t-1=0, s+\sqrt{-1}(t-1)=0, s-\sqrt{-1}(t-1)=0, s^2+t^2+(-1-c)t+c=0\right\}.$$

The first invariant algebraic curve $t-1=0$ is a rational solution curve as before. However, the invariant algebraic curve $s^2+t^2+(-1-c)t+c=0$ produces no rational solution. Because if we parametrize this curve by $\mathcal{P}(x)=\left(\frac{(c-1)x}{1+x^2}, \frac{cx^2+1}{1+x^2}\right)$, then the corresponding differential equation is $T'=\frac{-2T^4}{1+T^2}$. This equation has no rational solution. Therefore, the rational differential system (3.29) has no rational general solution.

Note that, a similar phenomenon happens with respect to liouvillian solutions: the polynomial system (3.1) may have a liouvillian first integral without having non-constant liouvillian solutions (conf. Singer (1992), page 674).

Chapter 4

Classification of algebraic ODEs with respect to rational solvability

We introduce a group of affine linear transformations and consider its action on the set of parametrizable algebraic ODEs. In this way, the set of parametrizable algebraic ODEs is partitioned into classes with an invariant associated system, and hence of equal complexity in terms of rational solvability. We study some special parametrizable algebraic ODEs: some well-known and obviously parametrizable classses of ODEs, and some classes of algebraic ODEs with special geometric shapes, whose associated systems are characterized by classical ODEs such as separable or homogeneous ones. The content of this chapter is essentially based on Ngô et al. (2011).

4.1 A group of affine linear transformations

Up to now, we have independently looked at a parametrizable algebraic ODE of order 1 without any relation to others. Obviously, some equations (or their associated systems) are easier to solve than others. So, the natural question is

> **Q 11.** *Whether a given equation can be transformed into an easier one, and thus is of the same low complexity?*

Such a classification is the main goal of this chapter. Since we are interested in rational solutions, the natural transformations are birational maps (i.e., invertible rational map with rational inverse). However, since we are working in a differential frame, it is expectable that not all birational transformations are suitable. Indeed, we investigate birational transformations preserving certain characteristics of the rational solutions of the corresponding equations.

In this section, we define a group of affine linear transformations on $\mathbb{A}^3(\mathbb{K}(x))$ that maps an integral curve (also called a solution curve) of the space to another one. By an integral curve of the space we mean a parametric curve of the form $\mathcal{C}(x) = (x, f(x), f'(x))$. Moreover, this group can act on the set of all algebraic ODEs of order 1 and it is compatible with the solution curves of the corresponding differential equations. Therefore, it gives a partition on the set of all algebraic ODEs of order 1.

Let $L : \mathbb{K}(x)^3 \longrightarrow \mathbb{K}(x)^3$ be an affine linear transformation defined by

$$L(v) = Av + B,$$

where A is an invertible 3×3 matrix over $\mathbb{K}$, B is a column vector over $\mathbb{K}$ and v is a column vector over $\mathbb{K}(x)$. We want to determine A and B such that for any $f \in \mathbb{K}(x)$, there exists $g \in \mathbb{K}(x)$ with

$$L\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} = A\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + B = \begin{pmatrix} x \\ g(x) \\ g'(x) \end{pmatrix},$$

i.e., L maps an integral curve to an integral curve. By choosing some special rational functions for $f(x)$, we see that A and B must be of the forms

$$A := \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ c \\ b \end{pmatrix},$$

where a, b and c are in $\mathbb{K}$ and $a \neq 0$. Let $\mathcal{G}$ be the set of all such affine linear transformations. We represent the elements of $\mathcal{G}$ by a pair of matrices $[A, B]$. Let

$$L_i := \left[\begin{pmatrix} 1 & 0 & 0 \\ b_i & a_i & 0 \\ 0 & 0 & a_i \end{pmatrix}, \begin{pmatrix} 0 \\ c_i \\ b_i \end{pmatrix}\right], \quad i = 1, 2,$$

be two elements in $\mathcal{G}$. The usual composition of maps defines a multiplication on $\mathcal{G}$ as

$$L_1 \circ L_2 = \left[\begin{pmatrix} 1 & 0 & 0 \\ b_1 + a_1 b_2 & a_1 a_2 & 0 \\ 0 & 0 & a_1 a_2 \end{pmatrix}, \begin{pmatrix} 0 \\ c_1 + a_1 c_2 \\ b_1 + a_1 b_2 \end{pmatrix}\right]$$

and an inverse operation as

$$L_1^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 \\ -\frac{b_1}{a_1} & \frac{1}{a_1} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{c_1}{a_1} \\ -\frac{b_1}{a_1} \end{pmatrix}\right].$$

Hence $\mathcal{G}$ is a group with the unit element (the identity map)

$$I = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right].$$

This group can be naturally generalized to higher dimensional spaces; i.e., to the case of higher order AODEs.

Lemma 4.1.1. *The group $\mathcal{G}$ defines a group action on $\mathcal{AODE}$ by*

$$\begin{array}{ccl} \mathcal{G} \times \mathcal{AODE} & \to & \mathcal{AODE} \\ (L, F) & \mapsto & L \cdot F = (F \circ L^{-1})(x, y, y') = F\left(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}, -\frac{b}{a} + \frac{1}{a}y'\right), \end{array}$$

where

$$L := \left[\begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ b \end{pmatrix} \right].$$

Proof. We have

$$\begin{aligned} (L_1 \circ L_2) \cdot F = F \circ (L_1 \circ L_2)^{-1} &= F \circ (L_2^{-1} \circ L_1^{-1}) \\ &= (F \circ L_2^{-1}) \circ L_1^{-1} \\ &= L_1 \cdot (L_2 \cdot F), \end{aligned}$$

and $I \cdot F = F$. Therefore, this is an action of the group $\mathcal{G}$ on the set $\mathcal{AODE}$. □

Remark 4.1.1. Let $F \in \mathcal{PODE}$ and $\mathcal{P}(s, t)$ be a proper parametrization of the solution surface of F, then $(L \circ \mathcal{P})(s, t)$ is a proper parametrization of the solution surface of $(L \cdot F)$, because

$$(L \cdot F)((L \circ \mathcal{P})(s, t)) = F(L^{-1}((L \circ \mathcal{P})(s, t))) = F(\mathcal{P}(s, t)) = 0.$$

Therefore, $(L \cdot F) \in \mathcal{PODE}$. Moreover, the group $\mathcal{G}$ also defines a group action on $\mathcal{PODE}$.

The action of $\mathcal{G}$ on $\mathcal{PODE}$ induces a partition of $\mathcal{PODE}$ into classes for which the solvability, and in particular the rational solvability, is an invariant property. In the next theorem we state that the associated system is also invariant.

Theorem 4.1.2. *Let $F \in \mathcal{PODE}$, and $L \in \mathcal{G}$. For every proper rational parametrization $\mathcal{P}$ of the surface $F(x, y, z) = 0$, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal.*

Proof. Let $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ be a proper rational parametrization of $F(x,y,z)=0$. Then $L \cdot F$ can be parametrized by $(L \circ \mathcal{P})(s,t)$. The associated system of $F(x,y,y')=0$ w.r.t $\mathcal{P}(s,t)$ is $\left\{ s' = \dfrac{f_1}{g}, t' = \dfrac{f_2}{g} \right\}$ where

$$f_1 = \begin{vmatrix} 1 & \chi_{1t} \\ \chi_3 & \chi_{2t} \end{vmatrix}, \quad f_2 = \begin{vmatrix} \chi_{1s} & 1 \\ \chi_{2s} & \chi_3 \end{vmatrix}, \quad \text{and} \quad g = \begin{vmatrix} \chi_{1s} & \chi_{1t} \\ \chi_{2s} & \chi_{2t} \end{vmatrix}.$$

We have

$$(L \circ \mathcal{P})(s,t) = (\chi_1, b\chi_1 + a\chi_2 + c, b + a\chi_3),$$

where a, b and c are constants and $a \neq 0$. So the associated system of $(L \cdot F)(x,y,y') = 0$ w.r.t $(L \circ \mathcal{P})$ is $\left\{ s' = \dfrac{\tilde{f}_1}{\tilde{g}}, t' = \dfrac{\tilde{f}_2}{\tilde{g}} \right\}$ where

$$\tilde{f}_1 = \begin{vmatrix} 1 & \chi_{1t} \\ b + a\chi_3 & b\chi_{1t} + a\chi_{2t} \end{vmatrix} = af_1, \quad \tilde{f}_2 = \begin{vmatrix} \chi_{1s} & 1 \\ b\chi_{1s} + a\chi_{2s} & b + a\chi_3 \end{vmatrix} = af_2$$

and

$$\tilde{g} = \begin{vmatrix} \chi_{1s} & \chi_{1t} \\ b\chi_{1s} + a\chi_{2s} & b\chi_{1t} + a\chi_{2t} \end{vmatrix} = ag.$$

Therefore, the associated system of $F(x,y,y')=0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x,y,y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal. $\square$

Clearly, specially interesting classes of $\mathcal{PODE}$ are those containing first-order autonomous parametrizable ODEs. Algorithmically, if we are given an equation in $\mathcal{PODE}$ and we want to check whether it is in the autonomous class, we may apply to the equation a generic element in $\mathcal{G}$ (i.e., introducing undetermined elements in the description of $L \in \mathcal{G}$) and afterwards require the coefficients of the resulting equation not to depend on x. In the next corollary we describe the type of associated system we get for these equations.

Corollary 4.1.3. *Let $F \in \mathcal{PODE}$ and $L \in \mathcal{G}$ such that $(L \cdot F)(x,y,y') = 0$ is an autonomous algebraic ODE. There exists a proper rational parametrization $\mathcal{P}(s,t)$ of $F(x,y,z)=0$ such that its associated system is of the form*

$$\left\{ s' = 1,\ t' = \frac{M(t)}{N(t)} \right\}. \tag{4.1}$$

Proof. Since $(L \cdot F)(x,y,y') = 0$ is an autonomous paramatrizable ODE, the plane algebraic curve $(L \cdot F)(y,z) = 0$ is rational, and for every proper rational parametrization $(f(t), g(t))$ of $(L \cdot F)(y,z) = 0$ the associated system of $(L \cdot F)(x,y,y') = 0$ w.r.t

$\mathcal{P}(s,t) = (s, f(t), g(t))$ is of the form $\{s' = 1, t' = g(t)/f'(t)\}$. $\square$

Remark 4.1.2. The converse of Corollary 4.1.3 is not true. Indeed, we consider the equation

$$F(x,y,y') = y - y'^2 - y' - y'x = 0.$$

It belongs to $\mathcal{PODE}$ and it can be properly parametrized by

$$\mathcal{P}_1(s,t) = (s, t^2 + t + ts, t).$$

The associated system w.r.t. $\mathcal{P}_1(s,t)$ is $\{s' = 1, t' = 0\}$ that is of the form (4.1). Let us see that the class of $F(x,y,y') = 0$ does not contain any autonomous equation. A generic transformation yields

$$(L \cdot F)(x,y,y') = -\frac{1}{a^2}y'^2 - \frac{1}{a}y' + 2\frac{b}{a^2}y' - \frac{1}{a}xy' + \frac{1}{a}y + \frac{b}{a} - \frac{b^2}{a^2} - \frac{c}{a},$$

and from here the conclusion is clear.

Example 4.1.1. We consider the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0.$$

We first check whether in the class of F there exists an autonomous AODE. For this, we apply a generic L to F to get

$$(L \cdot F)(x,y,y') = \frac{1}{a^2}y'^2 + \frac{3}{a}y' - \frac{2b}{a^2}y' - \frac{2}{a}y + \frac{2b}{a}x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a}.$$

Therefore, for every $a \neq 0$ and b such that $2b - 3a = 0$, we get an autonomous AODE. In particular, for $a = 1, b = 3/2$ and $c = 0$ we get

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} \right],$$

i.e., we obtain $F(L^{-1}(x,y,y')) \equiv y'^2 - 2y - \frac{9}{4} = 0$. The last equation can be parametrized by $\mathcal{P}_2(s,t) = \left(s, \frac{t^2}{2} - \frac{9}{8}, t\right)$. Its associated system is $\{s' = 1, t' = 1\}$. Therefore, this is also the associated system of the given differential equation w.r.t. the parametrization

$$(L \circ \mathcal{P}_2)(s,t) = \left(s, \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8}, t - \frac{3}{2}\right).$$

The general invariant algebraic curve of this associated system is $s - t + \tilde{c} = 0$, where $\tilde{c}$

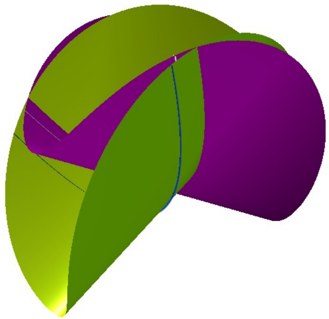


Figure 4.1: The solution surface $z^2 + 3z - 2y - 3x = 0$ transformed into $z^2 - 2y - \frac{9}{4} = 0$

is an arbitrary constant. Using the algorithm RATSOLVE we obtain a rational general solution of this associated system, namely: $s(x) = x$, $t(x) = x + \tilde{c}$. Therefore, we see that the rational general solution of the given differential equation is

$$y = \frac{t(x)^2}{2} - \frac{3}{2}s(x) - \frac{9}{8} = \frac{1}{2}\left(\left(x + \tilde{c} - \frac{3}{2}\right)^2 + 3\left(\tilde{c} - \frac{3}{2}\right)\right).$$

Now, it is clear that this rational general solution is equivalent to the rational general solution computed in Example 3.4.2 up to a change of the arbitrary constant. In fact, we have (see $\mathcal{P}_0(s,t)$ in Example 3.4.2)

$$((L \circ \mathcal{P}_2)^{-1} \circ \mathcal{P}_0)(s,t) = \left(\frac{t}{s} + \frac{2s + t^2}{s^2}, \frac{t}{s} + \frac{3}{2}\right).$$

This birational map transforms the plane curve $\left(-\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2}\right)$ in Example 3.4.2 into the plane curve $\left(x - \frac{1}{c}, x + \frac{3}{2}\right)$, whose defining equation is $s - t + \frac{1}{c} + \frac{3}{2} = 0$.

4.2 Solvable AODEs and their associated systems

Based on these observations, the study of parametrizable AODEs can be reduced to the study of their normal forms with respect to, for instance, an affine linear transformation in $\mathcal{G}$. In this section, we describe some of the special parametrizable AODEs that can be seen as normal forms. They are classified in Piaggio (1933), Chapter V; and in Murphy (1960), Chapter A2, Part I as *those solvable for* y', *those solvable for* y and *those solvable for* x. One can derive from these special AODEs new differential equations of order 1 and of degree 1, which are of the same complexity in terms of rational solvability. In fact, the three special types are, under minimal requirements, in $\mathcal{PODE}$ and they have an obvious proper parametrization. So we can interpret the results in the light of our algebraic geometric approach.

4.2.1 Equations solvable for y'

We consider a differential equation solvable for y', i.e., $y' = G(x, y)$, where $G(x, y)$ is a rational function. Then we need not change the variable because it is already in the desired form for applying Darboux's theory (see equation (3.2)).

Since $G(x, y)$ is rational, $(s, t, G(s, t))$ is a parametrization of the solution surface, and hence the equation belongs to $\mathcal{PODE}$; moreover, it is proper because $\mathbb{K}(s, t, G(s, t)) = \mathbb{K}(s, t)$.

If we apply an affine linear transformation $L \in \mathcal{G}$ to $F = y' - G(x, y)$, then

$$(L \cdot F)(x, y, y') = -\frac{b}{a} + \frac{1}{a}y' - G\left(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}\right).$$

Therefore, the new differential equation is of the same form. In other words, the property of being solvable for y' is invariant in the set, and we do not enlarge this set by applying the transformations in $\mathcal{G}$.

The associated system, via the parametrization $(s, t, G(s, t))$, is

$$\{s' = 1, t' = G(s, t)\}$$

and the single rational ODE derived from the system (see equation 3.2) is the original equation

$$\frac{dt}{ds} = G(s, t).$$

4.2.2 Equations solvable for y

Let the differential equation be of the form $y = G(x, y')$, where $G(x, y)$ is a rational function. A typical example is Clairaut's equation in Example 4.2.1. Let us assume that G is a rational function. Clearly this type of equations belongs to $\mathcal{PODE}$ since $(s, G(s, t), t)$ is a proper parametrization of the solution surface $y = G(x, z)$.

In this set, if we apply an affine linear transformation $L \in \mathcal{G}$ to $F = y - G(x, y')$, then

$$(L \cdot F)(x, y, y') = -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a} - G\left(x, -\frac{b}{a} + \frac{1}{a}y'\right).$$

Therefore, this set is also closed under the group action of $\mathcal{G}$, i.e., we do not enlarge this set by applying the transformations in $\mathcal{G}$.

The associated system, via the parametrization $(s, G(s,t), s)$, is

$$\left\{s' = 1, t' = \frac{t - G_s(s,t)}{G_t(s,t)}\right\}$$

where G_s and G_t are the partial derivatives of G w.r.t. s and t, respectively. Moreover, the single rational ODE derived from the system (see equation (3.2)) is

$$\frac{dt}{ds} = \frac{t - G_s(s,t)}{G_t(s,t)},$$

which is of the desired form.

Let us see that one gets the same equation using the classical reasoning. One can differentiate the equation w.r.t. x to obtain

$$y' = G_x(x, y') + G_{y'}(x, y') \cdot y'',$$

where G_x and $G_{y'}$ are the partial derivatives of $G(x, y')$ w.r.t x and y', respectively. Denoting y' by $\tilde{y}$, one can rewrite the above differential equation in the form

$$\tilde{y} = G_x(x, \tilde{y}) + G_{\tilde{y}}(x, \tilde{y}) \cdot \frac{d\tilde{y}}{dx},$$

or equivalently,

$$\frac{d\tilde{y}}{dx} = \frac{\tilde{y} - G_x(x, \tilde{y})}{G_{\tilde{y}}(x, \tilde{y})}.$$

Example 4.2.1. [*Clairaut's equation*] Let f be a smooth function of one variable. We consider Clairaut's differential equation

$$y = y'x + f(y').$$

This is a differential equation solvable for y and it can be parametrized by

$$\mathcal{P}_3(s,t) = (s, st + f(t), t).$$

If f is rational, then $\mathcal{P}_3(s,t)$ is a proper rational parametrization of the differential equation. The associated system w.r.t. $\mathcal{P}_3(s,t)$ is $\{s' = 1, t' = 0\}$. The set of irreducible invariant algebraic curves is

$$\{t - c = 0 \mid c \text{ is an arbitrary constant}\}.$$

Again using the algorithm RATSOLVE, we obtain $(s(x), t(x)) = (x, c)$ as a rational general solution of the associated system. So we get the rational general solution of Clairaut's differential equation, namely:

$$y = cx + f(c).$$

4.2.3 Equations solvable for x

We consider a differential equation of the form $x = G(y, y')$. Assuming that G is rational, this AODE belongs to $\mathcal{PODE}$, because $(G(s,t), s, t)$ is a proper parametrization of the solution surface.

If we apply an affine linear transformation $L \in \mathcal{G}$ to $F = x - G(y, y')$, then

$$(L \cdot F)(x, y, y') = x - G\left(-\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}, -\frac{b}{a} + \frac{1}{a}y'\right).$$

The degree of x in this equation is no longer linear. So this set is not closed under the action of the group $\mathcal{G}$.

The associated system, via the parametrization $(G(s,t), s, t)$, is

$$\left\{s' = t,\ t' = \frac{1 - tG_s(s,t)}{G_t(s,t)}\right\}$$

where G_s and G_t are the partial derivatives of G w.r.t. s and t, respectively. Moreover, the single rational ODE derived from the system (see equation (3.2)) is

$$\frac{dt}{ds} = \frac{1 - tG_s(s,t)}{tG_t(s,t)}.$$

Let us see that one gets the same equation using the classical reasoning. One can differentiate the equation w.r.t. y to obtain

$$\frac{dx}{dy} = G_y(y, y') + G_{y'}(y, y') \cdot \frac{dy'}{dy}.$$

Let $\tilde{y} = y'$, then we have

$$\frac{1}{\tilde{y}} = G_y(y, \tilde{y}) + G_{\tilde{y}}(y, \tilde{y}) \cdot \frac{d\tilde{y}}{dy}.$$

So we have transformed the differential equation $x = G(y, y')$ to a new differential equation of order 1 and of degree 1 in the desired form, namely

$$\frac{d\tilde{y}}{dy} = \frac{1 - \tilde{y}G_y(y, \tilde{y})}{\tilde{y}G_{\tilde{y}}(y, \tilde{y})}.$$

We summarize the three classes, and their geometric interpretation, in the following table.

Example 4.2.2. As we have already mentioned, if $F \in \mathcal{PODE}$ is solvable for y', then all

	Solvable for y'	Solvable for y	Solvable for x
AODE	$y' = G(x, y)$	$y = G(x, y')$	$x = G(y, y')$
Proper Parametrization	$(s, t, G(s, t))$	$(s, G(s, t), t)$	$(G(s, t), s, t)$
Associated System	$\begin{cases} s' = 1 \\ t' = G(s, t) \end{cases}$	$\begin{cases} s' = 1 \\ t' = \dfrac{t - G_s(s, t)}{G_t(s, t)} \end{cases}$	$\begin{cases} s' = t \\ t' = \dfrac{1 - tG_s(s, t)}{G_t(s, t)} \end{cases}$
Equation (3.2)	$\dfrac{dt}{ds} = G(s, t)$	$\dfrac{dt}{ds} = \dfrac{t - G_s(s, t)}{G_t(s, t)}$	$\dfrac{dt}{ds} = \dfrac{1 - tG_s(s, t)}{tG_t(s, t)}$

Table 4.1: Solvable algebraic ODEs and their associated systems

elements in the class are solvable for y'; similarly if $F \in \mathcal{PODE}$ is solvable for y. However, this is not the case for equations solvable in x. So, if we are given $F \in \mathcal{PODE}$ we may try to check whether there exists $L \in \mathcal{G}$ such that $(L \cdot F)$ is solvable for x. For this purpose, we apply a generic transformation in $\mathcal{G}$, and afterwards require that $(L \cdot F)$ be linear in x. For instance, let us consider the equation

$$F(x, y, y') \equiv -3x - 4x^2 + 4xy - y^2 + 2xy' + 2y - yy' + 8 - 8y' + 2y'^2 = 0,$$

which belongs to $\mathcal{PODE}$. Note that

$$(s^2 + st - 2t^2,\ 2s^2 + 2st - 4t^2 + s,\ 2 + t)$$

is a proper parametrization of $F(x, y, z) = 0$. Applying a generic transformation in $\mathcal{G}$ one gets a quadratic polynomial in x, and the coefficient of x^2 is

$$\frac{-(2a + b)^2}{a^2}.$$

So if we take, for instance, $a = 1, b = -2$ and $c = 0$, we get an equation in the class solvable for x; indeed, we get $x - y^2 - y\,y' + 2y'^2 = 0$.

Remark 4.2.1. A linear algebraic ODE of order 1, $a(x)y' + b(x)y + c(x) = 0$ where $a(x), b(x)$ and $c(x)$ are polynomials in x, is in the intersection of the two sets: equations solvable for y' and equations solvable for y. The set of linear algebraic ODEs of order 1 is closed under the group action of affine linear transformations. Moreover, if $a(x)$ or $b(x)$ is a non-constant polynomial in x, then there is no autonomous ODE of order 1 in the equivalence class of $a(x)y' + b(x)y + c(x) = 0$ w.r.t. this group action.

4.2.4 Rational solutions of the equation $y' = R(x, y)$

Let us have a closer look at the parametrizable algebraic ODE $y' = R(x, y)$, where $R(x, y)$ is a rational function. This equation is an interesting class, of which Riccati equation is a special case. First of all, we want to relate a linear homogeneous ODE of order 2 to its associated Riccati equation, which is a non-linear algebraic ODE of the form $y' = A(x)y^2 + B(x)y + C(x)$. Then we present a description of an invariant algebraic curve corresponding to a rational solution of the equation $y' = R(x, y)$. Precisely, the equation can be transformed into infinity; in the new coordinate, we give a simplification on the form of an invariant algebraic curve of the system at infinity: if the invariant algebraic curve is of degree n, then it has the form $G_n(s, t) + G_{n-1}(s, t) = 0$, where G_i is a homogeneous polynomial of degree i. This is an essential improvement because we have discarded a big number of unknown coefficients of $G(s, t)$, which is important when we use undetermined coefficients method.

The associated Riccati equation

Let us consider a linear homogeneous ODE of order 2,

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0, \tag{4.2}$$

where $A_i(x)$ is a polynomial over an algebraically closed field $\mathbb{K}$, $i = 0, 1, 2$. This differential equation can be transformed into a non-linear ODE of order 1 as follows.

Let $u = \dfrac{y'}{y}$. Then we have the relations

$$y' = uy, \quad y'' = (u' + u^2)y. \tag{4.3}$$

Now, subsituting these identities to the equation (4.2), we obtain

$$(A_2(x)(u' + u^2) + A_1(x)u + A_0(x))y = 0.$$

Therefore, every non-zero solution $y(x)$ of (4.2) corresponds to a solution $u(x) = \dfrac{y'(x)}{y(x)}$ of the non-linear algebraic ODE

$$A_2(x)(u' + u^2) + A_1(x)u + A_0(x) = 0 \tag{4.4}$$

and vice versa.

Definition 4.2.1. The differential equation (4.4) is called the *associated Riccati equation* of (4.2).

It is clear that this definition can be generalized to a linear homogeneous ODE of any order (see Bronstein (1992)).

Now, a rational solution $u(x)$ of the associated Riccati equation (4.4) corresponds to a solution $y(x)$ of (4.2) with $\dfrac{y'(x)}{y(x)} = u(x) \in \mathbb{K}(x)$, hence $y = e^{\int u(x)dx}$. This type of solution is called an *exponential solution* of (4.2).

Rational solutions of the equation $y' = R(x, y)$

In Bronstein (1992), there is a method for determining all rational solutions of an associated Riccati equation. However, the associated Riccati equation (4.4) is a first-order parametrizable algebraic ODE (see Section 4.2.1) and we can put the equation in a general form to analyze it as well.

Let us consider the differential equation

$$y' = R(x, y), \tag{4.5}$$

where $R(x, y)$ is a bivariate rational function. We are interested in the rational solutions $y(x) = \dfrac{A(x)}{B(x)}$ of this equation. Of course, this equation is the same as the system

$$\left\{s' = 1, \;\; t' = R(s, t)\right\}, \tag{4.6}$$

which is the associated system of (4.5) w.r.t the parametrization $(s, t, R(s, t))$.

Theorem 4.2.1. *The system* (4.6) *has a non-zero rational solution* $(s(x), t(x)) = \left(x, \dfrac{A(x)}{B(x)}\right)$ *if and only if the rational system*

$$\begin{cases} u' = -uvR\left(\dfrac{u}{v}, \dfrac{1}{v}\right) + v, \\ v' = -R\left(\dfrac{u}{v}, \dfrac{1}{v}\right) v^2, \end{cases} \tag{4.7}$$

has a non-zero rational solution $(u(x), v(x)) = \left(x\dfrac{B(x)}{A(x)}, \dfrac{B(x)}{A(x)}\right)$. *In this case, let* $d = \max\{\deg A(x), \deg B(x)\}$. *Then the rational invariant algebraic curve parametrized by* $\mathcal{C}(x) = (u(x), v(x))$ *is of the form*

$$G(u, v) = \begin{cases} G_{d-1}(u, v) + G_d(u, v) = 0 & \text{if } \deg A(x) > \deg B(x), \\ G_d(u, v) + G_{d+1}(u, v) = 0 & \text{if } \deg A(x) \leq \deg B(x), \end{cases} \tag{4.8}$$

where G_i *is the homogeneous polynomial of degree* i.

Proof. Let $\left(x, \dfrac{A(x)}{B(x)}\right)$ be a non-zero rational solution of the system (4.6). This solution parametrizes a rational curve defined by $(xB(x) : A(x) : B(x))$ in the projective plane. Hence, in another affine coordinate it is defined by $\mathcal{C}(x) = \left(x\dfrac{B(x)}{A(x)}, \dfrac{B(x)}{A(x)}\right)$.

Let $t(x) = \dfrac{A(x)}{B(x)}$, $u(x) = x\dfrac{B(x)}{A(x)}$ and $v(x) = \dfrac{B(x)}{A(x)}$. From the relations $u(x) = xv(x)$, $v(x) = \dfrac{1}{t(x)}$, and note that $t'(x) = R(x, t(x))$, it is straight forward to prove that $(u(x), v(x))$ is a rational solution of the system (4.7).

Conversely, let $(u(x), v(x))$ be a rational solution of the system (4.7). We have

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = 1.$$

Hence $u(x) = (x + c)v(x)$ for some constant c. Let $s(x) = x$, $t(x) = \dfrac{1}{v(x-c)}$. Then $\left(x, \dfrac{1}{v(x-c)}\right)$ is a rational solution of the system (4.6).

Consider the curve parametrized by $\left(x, \dfrac{A(x)}{B(x)}\right)$ with the defining equation

$$B(s)t - A(s) = 0.$$

The homogenization equation is

$$\begin{cases} v^d\left(B\left(\frac{s}{v}\right)\frac{t}{v} - A\left(\frac{s}{v}\right)\right) = 0 & \text{if } \deg A(x) > \deg B(x), \\ v^{d+1}\left(B\left(\frac{s}{v}\right)\frac{t}{v} - A\left(\frac{s}{v}\right)\right) = 0 & \text{if } \deg A(x) \le \deg B(x), \end{cases} \tag{4.9}$$

where $d = \max\{\deg A(x), \deg B(x)\}$. Now, the dehomogenization w.r.t t gives us the equation of the form (4.8). □

This theorem gives us a simplification of an invariant algebraic curve of the system (4.7). Indeed, if n is the degree of an invariant algebraic curve of the system, then we can discard all the homogeneous parts of degree lower than $n - 1$ in determining the curve.

Definition 4.2.2. The system (4.7) is called the *system at infinity* of the differential equation (4.5).

Example 4.2.3. Consider the Riccati equation (in Bronstein (1992))

$$y' = \frac{(x^4 - x^2)y^2 + (8x^3 - 5x + 1)y + 12x^2 - 3}{x^2 - x^4} \tag{4.10}$$

In this example, the system at infinity is

$$\begin{cases} u' = \dfrac{u^4 - u^2v^2 + 9\,v^2u^3 - 6\,v^4\,(u) + v^5 + 12\,u^2v^4 - 3\,v^6}{uv\,(u^2 - v^2)}, \\ v' = \dfrac{u^4 - u^2v^2 + 8\,v^2u^3 - 5\,v^4\,(u) + v^5 + 12\,u^2v^4 - 3\,v^6}{u^2\,(u^2 - v^2)}. \end{cases} \tag{4.11}$$

Using the observation on the degree, we can find all invariant algebraic curves of degree up to 4:

$$\{u,\ v,\ u+v,\ u-v,\ u^3 - vu^2 + v^4 - 4\,uv^3 + 4\,u^2v^2\}.$$

However, only the curve $u^3 - vu^2 + v^4 - 4\,uv^3 + 4\,u^2v^2 = 0$ leads to a rational solution of the equation (4.11):

$$u(x) = -\frac{x^3(x-1)}{(2x-1)^2},\quad v(x) = -\frac{x^2(x-1)}{(2x-1)^2}.$$

Therefore, a rational solution of the given Riccati equation is $y(x) = \dfrac{1}{v(x)} = -\dfrac{(2x-1)^2}{x^2(x-1)}$.

4.3 Parametrizable ODEs with special geometric shapes

In Feng and Gao (2004, 2006), an autonomous AODE of order 1 is associated to a plane algebraic curve. Accordingly, an autonomous AODE of order 1 possessing a rational general solution is associated to a rational plane curve. In fact, these are special AODEs in $\mathcal{PODE}$, whose solution surfaces are cylindrical surfaces over a rational plane curve. Observe that the action of an element in $\mathcal{G}$ on an autonomous AODE typically results in a non-autonomous one. Hence, the resulting AODE has the same associated system and the same rational solvability. Therefore, autonomy is not an intrinsic property of an AODE with respect to rational solvability. In this section, we consider some classes in $\mathcal{PODE}$ having special geometric shapes and one of the classes is a generalization of autonomous AODEs of order 1.

4.3.1 Differential equations of pencil type

We first consider parametrizable ODEs whose solution surface is a pencil of rational curves. More precisely, we assume that $F(x, y, z) = 0$ is the defining equation of a rational algebraic curve over the algebraic closure $\overline{\mathbb{K}(x)}$ of $\mathbb{K}(x)$ and that it is $\mathbb{K}(x)$-parametrizable; i.e., $\mathbb{K}(x)$ is the optimal field of the parametrization of the curve. The latter assumption is always fulfilled if the degree of the curve is odd (cf. Sendra et al. (2008), Chapter 5). With these assumptions, the surface $F(x, y, z) = 0$ has a proper parametrization of the form

$$\mathcal{P}_4(s,t) = (s, f(s,t), g(s,t)), \tag{4.12}$$

where f and g are rational functions in s and t. Indeed, letting $(f(s,t), g(s,t)) \in \mathbb{K}(s)(t)^2$ be a proper parametrization of the curve (recall that Lüroth's theorem is valid over every field), then $\mathbb{K}(s)(f(s,t), g(s,t)) = \mathbb{K}(s)(t)$. $\mathcal{P}_4$ parametrizes the surface $F(x, y, z) = 0$ and $\mathbb{K}(s, f(s,t), g(s,t)) = \mathbb{K}(s,t)$; hence it is proper.

The surface parametrized by (4.12) is called a *pencil of rational curves*. In this case, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}_4(s,t)$ is

$$\left\{ s' = 1,\ t' = \frac{-f_s(s,t) + g(s,t)}{f_t(s,t)} \right\}, \tag{4.13}$$

where f_s and f_t are the partial derivatives of f w.r.t. s and t, respectively. The derived differential equation from the associated system (see equation (3.2)) is

$$\frac{dt}{ds} = \frac{-f_s(s,t) + g(s,t)}{f_t(s,t)}. \tag{4.14}$$

In fact, there are several cases, in which the associated system (4.13) and the derived ODE (4.14) are simple: it can be separable or homogeneous. For instance, if $f(s,t)$ and $g(s,t)$ are homogeneous polynomials of degree $m+1$ and m, respectively, then the derived differential equation (4.14) is homogeneous. In this case, we can write

$$f(s,t) = s^{m+1} f\left(1, \frac{t}{s}\right),\ g(s,t) = s^m g\left(1, \frac{t}{s}\right).$$

So the birational change of parameters $s^* = s,\ t^* = \dfrac{t}{s}$ transforms $(s, f(s,t), g(s,t))$ into the parametrization

$$(s, s^{m+1} f_1(t), s^m f_2(t)).$$

We consider, in the next subsections, the following two cases:

- [Cylindrical type] $f(s,t) = \lambda s + f_1(t)$ and $g(s,t) = f_2(t)$;

- [Quasi-cylindrical type] $f(s,t) = s^{m+1} f_1(t)$ and $g(s,t) = s^m f_2(t)$;

where f_1, f_2 are non-constant rational functions such that $(f_1(t), f_2(t))$ is proper; i.e., $\mathbb{K}(f_1(t), f_2(t)) = \mathbb{K}(t)$.

Differential equations of cylindrical type

Definition 4.3.1. Let $F \in \mathcal{PODE}$. F is of *cylindrical type* iff $F(x, y, z) = 0$ has a proper rational parametrization of the form

$$\mathcal{P}_5(s,t) = (0, f_1(t), f_2(t)) + s(1, \lambda, 0) = (s, \lambda s + f_1(t), f_2(t)), \tag{4.15}$$

where λ is a constant and $f_1(t)$ is non-constant; i.e., F can be written as

$$G(y - \lambda x, y') = 0, \tag{4.16}$$

where $G(u, v) = 0$ is a rational curve over $\mathbb{K}$.

It is clear that an autonomous AODE of order 1 with rational solutions is a special case of cylindrical type, corresponding to $\lambda = 0$.

Note that the properness of $\mathcal{P}_5(s,t)$ is equivalent to the properness of $(f_1(t), f_2(t))$ because

$$\mathbb{K}(s, \lambda s + f_1(t), f_2(t)) = \mathbb{K}(s)(f_1(t), f_2(t)) = \mathbb{K}(s)(t).$$

If an AODE can be parametrized by a proper rational parametrization of the form

$$\mathcal{P}_6(s,t) = (f_1(t), f_2(t), f_3(t)) + s(1, \lambda, 0), \tag{4.17}$$

where λ is a constant and $f_2'(t) - \lambda f_1'(t) \neq 0$, then by a change of parameters we can bring it to the standard cylindrical type. Indeed, one can apply the birational transformation $\{s^* = f_1(t) + s,\ t^* = t\}$ in order to transform $\mathcal{P}_6(s,t)$ into $\mathcal{P}_5(s^*, t^*)$.

Theorem 4.3.1. *Every parametrizable ODE of cylindrical type is transformable into an autonomous AODE by the transformation*

$$L := \left[\begin{pmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\lambda \end{pmatrix} \right].$$

As a consequence (see Theorem 4.1.2 and Corollary 4.1.3), every parametrizable ODE of cylindrical type has a parametrization w.r.t. which its associated system is of the form $\left\{ s' = 1,\ t' = \dfrac{M(t)}{N(t)} \right\}$*, where M, N are polynomials in one variable over $\mathbb{K}$.*

Proof. We have

$$L \cdot F = F(L^{-1}(x, y, y')) = G(y, y' + \lambda),$$

which is an autonomous AODE. $\square$

The associated system w.r.t. the parametrization in (4.15) is

$$\left\{ s' = 1,\ t' = \frac{f_2(t) - \lambda}{f_1'(t)} \right\}. \tag{4.18}$$

A rational general solution of this system, if it exists, is of the form

$$(s(x), t(x)) = \left(x + c, \frac{\alpha x + \beta}{\gamma x + \delta} \right),$$

where $\alpha, \beta, \gamma, \delta$ are constants and c is an arbitrary constant. Here we use the fact that the second differential equation in the associated system is autonomous. So from Feng and Gao (2004, 2006) we know the exact degree of a possible rational solution, which in this case is 1. This exact degree bound is derived from an exact degree bound for curve parametrizations in Sendra and Winkler (2001). In this case, a rational general solution of $G(y - \lambda x, y') = 0$ is

$$y(x) = f_1(t(x - c)) + \lambda x = f_1\left(\frac{\alpha(x - c) + \beta}{\gamma(x - c) + \delta} \right) + \lambda x,$$

where c is an arbitrary constant.

Remark 4.3.1. If the integral $\varphi(t) = \displaystyle\int \frac{f_1'(t)}{f_2(t) - \lambda} dt = \frac{P_1(t)}{P_2(t)}$ is a rational function (which is algorithmically detectable as in Bronstein (2004), Chapter 2), then the general irreducible invariant algebraic curve of the system (4.18) is $P_1(t) - sP_2(t) - cP_2(t) = 0$, where c is an arbitrary constant. Hence, the system (4.18) has a general solution of the form $(x, t(x))$, where $t(x)$ is an algebraic function satisfying the equation

$$P_1(t(x)) - xP_2(t(x)) - cP_2(t(x)) = 0.$$

So a general solution of $G(y - \lambda x, y') = 0$ is an algebraic solution given by $y = f_1(t(x)) + \lambda x$.

By Theorem 4.3.1, the autonomous AODEs of order 1 are the representatives of parametrizable ODEs of cyclindrical type. In order to check whether an $F(x, y, y') = 0$ in $\mathcal{PODE}$ is equivalent to a parametric ODE of cylindrical type, we proceed as follows. First we apply a generic transformation $L \in \mathcal{G}$, say $G(a, b, c, x, y, y') = (L \cdot F)(x, y, y')$. Then we consider the differential equation

$$G(a, b, c, x, y, y') = 0 \tag{4.19}$$

and determine a, b, c such that the new differential equation is an autonomous AODE; i.e., the coefficients of $G(a, b, c, x, y, y')$ w.r.t. x must be all zero except for the coefficient of degree 0.

Differential equations of quasi-cylindrical type

Definition 4.3.2. Let $G(u, v) = 0$ be a rational plane curve over $\mathbb{K}$. A differential equation of the form

$$G\left(\frac{y}{x^{m+1}}, \frac{y'}{x^m}\right) = 0, \tag{4.20}$$

is called of *quasi-cylindrical type*.

Of course, there are other differential equations which are transformable into this type via linear affine transformations. Those are of the form

$$G\left(\frac{ay + bx + c}{x^{m+1}}, \frac{ay' + b}{x^m}\right) = 0, \tag{4.21}$$

where a, b and c are constants and $a \neq 0$.

Suppose that $(f_2(t), f_3(t))$ is a proper rational parametrization of $G(u, v) = 0$. Then the solution surface of (4.20) can be properly parametrized by

$$\mathcal{P}_7(s, t) = (s, s^{m+1} f_2(t), s^m f_3(t)). \tag{4.22}$$

Note that $\mathcal{P}_7(s, t)$ is proper, because it is a special case of the parametrization considered in (4.12). With respect to $\mathcal{P}_7(s, t)$ the associated system is separable

$$\left\{ s' = 1,\ t' = \frac{-(m+1)f_2(t) + f_3(t)}{s f_2'(t)} \right\}. \tag{4.23}$$

Therefore, we can always decide whether the differential equation (4.20) has a rational general solution or not.

4.3.2 Differential equations of cone type

A rational conical surface (say with vertex at the origin) can be parametrized as $s\,\mathcal{E}(t)$ where $\mathcal{E}(t)$ is a space curve parametrization; if the curve $\mathcal{E}(t)$ is contained in a plane passing through the origin, the surface is that plane. This motivates the following definition.

Definition 4.3.3. A parametrizable ODE of order 1 is of *cone type* if its solution surface can be parametrized by a parametrization of the form

$$\mathcal{P}_8(s, t) = (s^{m_1} f_1(t), s^{m_2} f_2(t), s^{m_3} f_3(t)), \tag{4.24}$$

where f_1, f_2, f_3 are rational functions and m_1, m_2, m_3 are integers.

We know that the associated system w.r.t. the parametrization (4.24) is

$$\begin{cases} s' = \dfrac{s^{1-m_1} f_2'(t) - s^{1-m_2+m_3} f_3(t) f_1'(t)}{m_1 f_1(t) f_2'(t) - m_2 f_1'(t) f_2(t)}, \\ \\ t' = \dfrac{-m_2 s^{-m_1} f_2(t) + m_1 s^{-m_2+m_3} f_3(t) f_1(t)}{m_1 f_1(t) f_2'(t) - m_2 f_1'(t) f_2(t)}. \end{cases} \tag{4.25}$$

In fact, we consider the case $m_2 = m_1 + m_3$, i.e., the parametrization is

$$\mathcal{P}_9(s,t) = (s^{m_1} f_1(t), s^{m_1+m_3} f_2(t), s^{m_3} f_3(t)), \tag{4.26}$$

in which $\mathcal{P}_7(s,t)$ is a special case (let $m_1 = 1, f_1(t) = 1$ and $m_2 = m_3 + 1$), we obtain a quasi-cylindrical surface. In this case, the derived differential equation is separable, namely:

$$\frac{dt}{ds} = \frac{-m_2 f_2(t) + m_1 f_1(t) f_3(t)}{(f_2'(t) - f_3(t) f_1'(t))s}. \tag{4.27}$$

By integration of rational functions (Bronstein (2004), Chapter 2) we can decide if the associated system has a general invariant algebraic curve and proceed as in the algorithm RATSOLVE to check the existence of a rational general solution of the system (4.25) with $m_2 = m_1 + m_3$.

The differential equation corresponding to the parametrization (4.26) is of the form

$$G\left(\frac{y^{m_1}}{x^{m_1+m_3}}, \frac{y'^{m_1}}{x^{m_3}}\right) = 0, \tag{4.28}$$

where $G(u,v) = 0$ is a rational planar curve over $\mathbb{K}$. In general, from the form (4.28) we do not know whether the surface is rational or not. However, in some special cases, we can decide this property. For instance, if the rational curve $G(u,v) = 0$ has a rational parametrization of the form $(f_2(t)^{m_1}, f_3(t)^{m_1})$, then the surface defined by (4.28) can be parametrized by

$$\mathcal{P}_{10}(s,t) = (s^{m_1}, s^{m_1+m_3} f_2(t), s^{m_3} f_3(t)). \tag{4.29}$$

This parametrization is proper if $(f_2(t)^{m_1}, f_3(t)^{m_1})$ is proper and $\gcd(m_1, m_3) = 1$. In this case, the associated system w.r.t. $\mathcal{P}_{10}(s,t)$ is

$$\begin{cases} s' = \dfrac{s^{1-m_1}}{m_1}, \\ t' = \dfrac{(m_1 f_3(t) - (m_1 + m_3) f_2(t))}{m_1 s^{m_1} f_2'(t)}. \end{cases} \tag{4.30}$$

We can immediately conclude that if $m_1 \neq 1$, then the system (4.30) has no rational solution. Therefore, the differential equation 4.28 has no rational general solution for all rational curves $G(s,t)=0$ specified as above.

Class/Equation	Parametrization	Associated system
Autonomous $G(y,y')=0$ $G(y,z)=0$ is rational $G(f_1(t),f_2(t))=0$	$(s, f_1(t), f_2(t))$	$\begin{cases} s' = 1 \\ t' = \dfrac{f_2(t)}{f_1'(t)} \end{cases}$
Cylinder $G(y-\lambda x, y')=0$ $G(y,z)=0$ is rational $G(f_1(t),f_2(t))=0$	$(s, \lambda s + f_1(t), f_2(t))$	$\begin{cases} s' = 1 \\ t' = \dfrac{f_2(t)-\lambda}{f_1'(t)} \end{cases}$
Quasi cylinder $G\left(\dfrac{y}{x^{m+1}}, \dfrac{y'}{x^m}\right)=0$ $G(y,z)=0$ is rational $G(f_2(t),f_3(t))=0$	$(s, s^{m+1}f_2(t), s^m f_3(t))$	$\begin{cases} s' = 1 \\ t' = \dfrac{-(m+1)f_2(t)+f_3(t)}{sf_2'(t)} \end{cases}$
Cone $G\left(\dfrac{y^{m_1}}{x^{m_1+m_3}}, \dfrac{y'^{m_1}}{x^{m_3}}\right)=0$ $G(y,z)=0$ is rational $G(f_2^{m_1}(t),f_3^{m_1}(t))=0$	$(s^{m_1}, s^{m_1+m_3}f_2(t), s^{m_3}f_3(t))$	$\begin{cases} s' = \dfrac{s^{1-m_1}}{m_1} \\ t' = \dfrac{(m_1 f_3(t)-(m_1+m_3)f_2(t))}{m_1 s^{m_1} f_2'(t)} \end{cases}$

Table 4.2: Some classes of first-order parametrizable algebraic ODEs

Note that, the differential equation of the form $G(\frac{y}{x}, y')=0$, which is considered in Kamke (1948) (Part A, Section 4.6), is a special case of the quasi-cylindrical class when $m=0$.

Supplement: Examples from Kamke

We demonstrate our method via several examples that come from the book "Differentialgleichungen: Lösungsmethoden und Lösungen" by E. Kamke (Kamke (1948)). The numbers and the solutions in the following table are given in the book, respectively.

Number	Differential equation	General solution
I.415	$xy'^2 + yy' - y^4 = 0$	$y = \frac{c}{x-c^2}$
I.417	$xy'^2 - yy' + a = 0$	$y = cx + \frac{a}{c}$
I.423	$xy'^2 - 2yy' + 2y + x = 0$	$y = \frac{1}{2}cx^2 + x + \frac{1}{c}$
I.424	$xy'^2 - 2yy' + bx = 0$	$y = cx^2 + \frac{b}{4c}$
I.425	$(x+1)y'^2 - (y+x)y' + y = 0$	$y = cx + \frac{c^2}{c-1}$
I.441	$x^2y'^2 - 4x(y+2)y' + 4y(y+2) = 0$	$y = \frac{1}{2}c^2x^2 - 2cx$
I.444	$x^2y'^2 - y(y-2x)y' + y^2 = 0$	$y = \frac{c^2}{c-x}$
I.525	$y'^3 - axyy' + 2ay^2 = 0$	$y = \frac{ac}{4}(x-c)^2$
I.527	$y'^3 - xy^4y' - y^5 = 0$	$y = \frac{c^3}{c^2x-1}$

Table 4.3: Some differential equations in Kamke's book and their rational general solutions

Remark 4.3.2. Note that I.417 and I.425 are Clairaut's equations. The equations I.417, I.423, I424, I.425 are equations solvable for y. All of these equations in Table 4.4 are differential equations of pencil type. Moreover, from the parametrizations of the equations I.423 and I.424, it is clear that they belong to the quasi-cylindrical class with $m = 0$ and $(f_2(t), f_3(t)) = \left(\frac{t^2+1}{2(t-1)}, t\right)$ and $(f_2(t), f_3(t)) = \left(\frac{t^2+b}{2t}, t\right)$, respectively. In fact, when $m = 0$, the quasi-cylindrical class has the form $G(\frac{y}{x}, y') = 0$. This special form is already considered in Kamke's book, Part A, Section 4.6.

Number	Proper parametrization	Associated system	Gen.Inv.Alg.Curve	Rational general solution
I.415	$\mathcal{P}(s,t) = \left(s, \frac{t}{t^2-s}, -\frac{t^3}{s(s-t^2)^2}\right)$	$\{s' = 1,\ t' = \frac{t}{s}\}$	$s - ct = 0$	$s = x,\ t = \frac{x}{c},\ y = \frac{c}{x-c^2}$
I.417	$\mathcal{P}(s,t) = \left(s, \frac{st^2+a}{t}, t\right)$	$\{s' = 1,\ t' = 0\}$	$t - c = 0$	$s = x,\ t = c,\ y = cx + \frac{a}{c}$
I.423	$\mathcal{P}(s,t) = \left(s, \frac{s(t^2+1)}{2(t-1)}, t\right)$	$\{s' = 1,\ t' = \frac{t-1}{s}\}$	$cs - (t-1) = 0$	$s = x,\ t = cx+1,\ y = \frac{1}{2}cx^2 + x + \frac{1}{c}$
I.424	$\mathcal{P}(s,t) = \left(s, \frac{s(t^2+b)}{2t}, t\right)$	$\{s' = 1,\ t' = \frac{t}{s}\}$	$2cs - t = 0$	$s = x,\ t = 2cx,\ y = cx^2 + \frac{b}{4c}$
I.425	$\mathcal{P}(s,t) = \left(s, \frac{t((s+1)t-s)}{t-1}, t\right)$	$\{s' = 1,\ t' = 0\}$	$t - c = 0$	$s = x,\ t = c,\ y = cx + \frac{c^2}{c-1}$
I.441	$\mathcal{P}(s,t) = \left(s, -\frac{2st(st-4)}{(st-2)^2}, \frac{8t}{(st-2)^2}\right)$	$\{s' = 1,\ t' = -\frac{1}{2}t^2\}$	$cst - 2t - 2c = 0$	$s = x,\ t = \frac{2c}{cx-2},\ y = \frac{1}{2}c^2x^2 - 2cx$
I.444	$\mathcal{P}(s,t) = \left(s, \frac{(s+t+1)^2}{t+1}, \frac{(s+t+1)^2}{(t+1)^2}\right)$	$\{s' = 1,\ t' = -1\}$	$s + t + 1 - c = 0$	$s = x,\ t = -x - 1 + c,\ y = \frac{c^2}{c-x}$
I.525	$\mathcal{P}(s,t) = (s, -t^2a(2t-s), -ta(2t-s))$	$\{s' = 1,\ t' = \frac{1}{2}\}$	$s - 2t - c = 0$	$s = x,\ t = \frac{1}{2}(x-c),\ y = \frac{ac}{4}(x-c)^2$
I.527	$\mathcal{P}(s,t) = \left(s, \frac{t^3}{-s^6+s^3t^2}, -\frac{t^5}{s^{10}-2s^7t^2+s^4t^4}\right)$	$\{s' = 1,\ t' = \frac{2t}{s}\}$	$cs^2 - t = 0$	$s = x,\ t = cx^2,\ y = \frac{c^3}{cx^2-1}$

Table 4.4: Solving differential equations by parametrization method

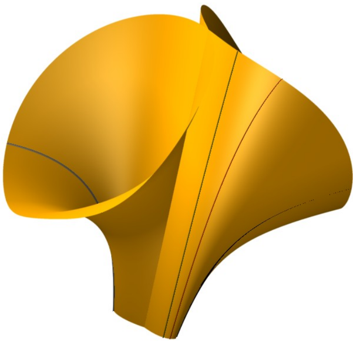
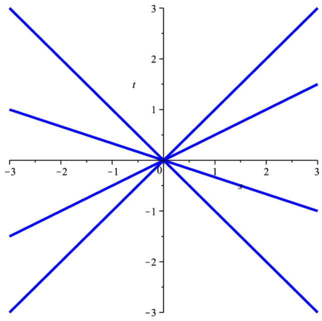
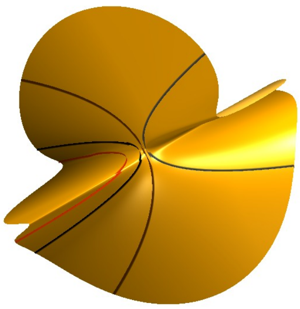
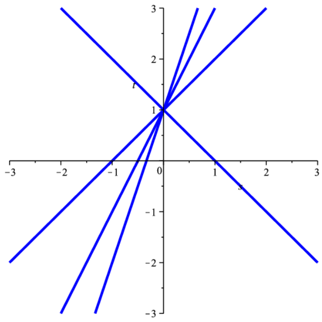
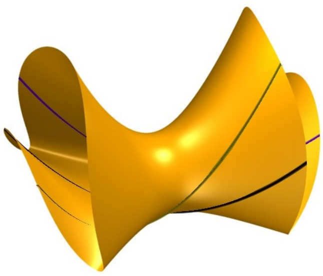
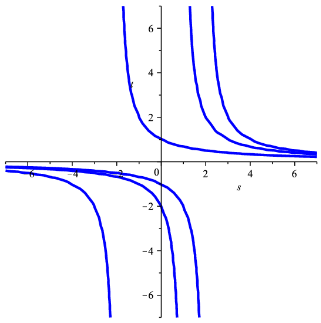
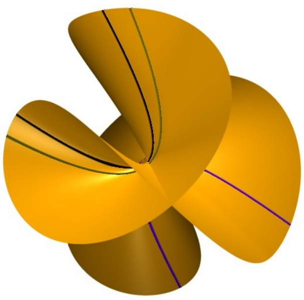
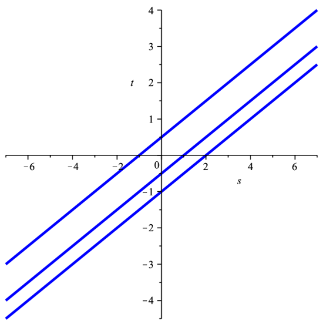
Solution surface and solution curve	Invariant algebraic curve
 I.415, $xz^2 + yz - y^4 = 0$	 $s - ct = 0$
 I.423, $xz^2 - 2yz + 2y + x = 0$	 $cs - (t-1) = 0$
 I.441, $x^2z^2 - 4x(y+2)z + 4y(y+2) = 0$	 $cst - 2t - 2c = 0$
 I.525, $z^3 - 4xyz + 8y^2 = 0$	 $s - 2t - c = 0$

Table 4.5: Examples of solution surfaces and solution curves

Summary and open problems

We have considered the following main problem: "*Given a trivariate polynomial $F(x, y, z)$, decide the existence of a rational general solution of the algebraic ODE $F(x, y, y') = 0$ of order 1; in the affirmative case, compute a rational general solution explicitly*". Here we have formulated the notion of a general solution of an algebraic ODE from the point of view of differential algebra.

In order to determine such a rational general solution of $F(x, y, y') = 0$ we first neglected the differential aspect of the problem, and we considered the algebraic solution surface defined by $F(x, y, z) = 0$. Then rational solutions of $F(x, y, y') = 0$ can be extracted from the rational solution curves on $F(x, y, z) = 0$. Having a proper rational parametrization $\mathcal{P}(s, t)$ of $F(x, y, z) = 0$, it amounts to looking for a rational general solution $(s(x), t(x))$ of the associated system of autonomous ODEs of order 1 and of degree 1 in s' and t'. We can solve for rational solutions of the associated system generically. Therefore, we can solve the main problem in the generic case.

The tools are developed in Chapter 2 and Chapter 3. The main contributions are the algorithm GENERALSOLVER in Section 2.2.3, the algorithm RATSOLVE in Section 3.4 and the algorithm FIRST-INT-RATSOLVE in Section 3.7. The correctness of these algorithms follows from Theorem 2.2.6, Theorem 2.2.8, Theorem 3.3.3, Theorem 3.3.4, Theorem 3.3.5, Lemma 3.7.2 and Theorem 3.7.3. Beside those results, the contributions in Section 2.3 are Theorem 2.3.2, Lemma 2.3.3, Theorem 2.3.4.

In order to simplify the main problem, we classified first-order parametrizable algebraic ODEs w.r.t rational solvability in Chapter 4. We gave a group of affine linear transformations so that the group action yields a partition of the set of first-order algebraic ODEs (Lemma 4.1.1). Moreover, the associated system of an ODE is invariant in its class w.r.t the group action (Theorem 4.1.2). Therefore, algebraic ODEs in the same equivalence class share important characteristics, such as the associated system, and the complexity of determining general rational solutions. Then we have analyzed some classes of algebraic ODEs having general rational solutions. It turns out that being autonomous is not a characteristic property of such a class. Some geometric properties of differential equations lead to representatives of their corresponding classes, which can obviously be solved rationally.

There are obviously several direction in which one could try to generalize our work. Here we list some open problem areas:

1. Generalize the method to higher order algebraic ODEs.

2. In the investigation of Chapter 4, consider similar questions w.r.t. a group of birational maps.

3. Extend the work to the case of algebraic general solutions of first-order algebraic ODEs: to decide the existence of an algebraic general solution and compute it in the affirmative case.

4. Given a rational curve parametrized by $\mathcal{C}(x) = (s(x), t(x))$, decide whether there is a planar system with a rational general solution and having $\mathcal{C}(x)$ as a special solution.

Appendix A

Basics of rational algebraic curves and surfaces

This appendix consists of some basic notions on algebraic curves, algebraic surface and rational parametrizations of rational curves and rational surfaces. A full introduction on these topics can be found in Sendra et al. (2008) and Schicho (1997).

Throughout this chapter we consider $\mathbb{K}$ to be an algebraically closed field of characteristic zero, i.e. $\mathbb{K}$ contains the field of rational numbers $\mathbb{Q}$. The affine plane over $\mathbb{K}$ is $\mathbb{A}^2(\mathbb{K})$ and the projective plane is $\mathbb{P}^2(\mathbb{K})$ or $\mathbb{KP}^2$.

A.1 Rational algebraic curves

Definition A.1.1. Let $f \in \mathbb{K}[x, y]$ be a non-constant polynomial. The set

$$\mathcal{C} = \{(a, b) \in \mathbb{A}^2(\mathbb{K}) \mid f(a, b) = 0\} \tag{A.1}$$

is called an *affine plane algebraic curve* over $\mathbb{K}$ defined by f.

The affine curve $\mathcal{C}$ can be embedded in the projective plane.

Definition A.1.2. Let $F(x, y, z) \in \mathbb{K}[x, y, z]$ be a non-constant homogeneous polynomial. The set

$$\mathcal{C}^* = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}) \mid F(a, b, c) = 0\} \tag{A.2}$$

is called a *projective plane algebraic curve* over $\mathbb{K}$ defined by F.

Sometimes, if the context is clear, we simply speak of an algebraic curve without mentioning the ambient space.

Definition A.1.3. The affine curve $\mathcal{C}$ in $\mathbb{A}^2(\mathbb{K})$ defined by the squarefree polynomial $f(x, y)$ is said to be *rational* iff there are rational functions $\chi_1(t), \chi_2(t) \in \mathbb{K}(t)$ such that

(a) for almost all $t_0 \in \mathbb{K}$ the point $(\chi_1(t_0), \chi_2(t_0))$ is on $\mathcal{C}$; and

(b) for almost all point $(x_0, y_0) \in \mathcal{C}$, there exists $t_0 \in \mathbb{K}$ such that $(x_0, y_0) = (\chi_1(t_0), \chi_2(t_0))$.

In this case, $(\chi_1(t), \chi_2(t))$ is called an *affine rational parametrization* of $\mathcal{C}$. A similar notion can be defined for projective plane algebraic curves.

A proof of the following results can be found in Sendra et al. (2008), Chapter 4.

Theorem A.1.1. *Any rational curve is irreducible.*

Theorem A.1.2. *An irreducible curve $\mathcal{C}$ defined by $f(x, y)$. The following are equivalent:*

1. *$\mathcal{C}$ is rational;*

2. *there exist rational functions $\chi_1(t), \chi_2(t) \in \mathbb{K}(t)$, not both constant, such that*

$$f(\chi_1(t), \chi_2(t)) = 0;$$

   in this case, $(\chi_1(t), \chi_2(t))$ is a rational parametrization of $\mathcal{C}$;

3. *the field of rational functions on $\mathcal{C}$ is isomorphic to $\mathbb{K}(t)$;*

4. *$\mathcal{C}$ is birationally equivalent to $\mathbb{K}$;*

5. $\text{genus}(\mathcal{C}) = 0$.

Definition A.1.4. An affine parametrization $\mathcal{P}(t)$ of a rational curve $\mathcal{C}$ is *proper* iff the map

$$\begin{aligned} \mathcal{P} : \mathbb{A}^1(\mathbb{K}) &\longrightarrow \mathcal{C} \\ t &\longrightarrow \mathcal{P}(t) \end{aligned}$$

is birational, i.e., there is the inverse rational map of $\mathcal{P}$.

Example A.1.1. The algebraic curve $y^2 - x^3 + 2x^2 - x = 0$ can be properly parametrized by $\mathcal{P}(t) = (t^2, t - t^3)$. The inverse of $\mathcal{P}$ is $(x, y) \mapsto \frac{y}{1-x}$.

As a direct corollary of Theorem A.1.2, every rational curve can be properly parametrized. We can reach every non-proper parametrization of a rational curve by a proper one.

Theorem A.1.3. *Let $\mathcal{P}(t)$ be a proper parametrization of an affine rational curve $\mathcal{C}$, and let $\mathcal{P}_1(t)$ be any other rational parametrization of $\mathcal{C}$.*

1. *There exists a non-constant rational function $R(t) \in \mathbb{K}(t)$ such that $\mathcal{P}_1(t) = \mathcal{P}(R(t))$.*

2. *$\mathcal{P}_1(t)$ is proper if and only if there exits a linear rational function $L(t) \in \mathbb{K}(t)$ such that $\mathcal{P}_1(t) = \mathcal{P}(L(t))$.*

Theorem A.1.4. *Let $\mathcal{C}$ be an affine rational curve defined over $\mathbb{K}$ with the defining polynomial $f(x,y) \in \mathbb{K}[x,y]$, and let $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ be a parametrization of $\mathcal{C}$. Then $\mathcal{P}(t)$ is proper if and only if* $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$. *Furthermore, if $\mathcal{P}(t)$ is proper and $\chi_1(t)$ is nonzero, then* $\deg(\chi_1(t)) = \deg_y(f)$*; similarly, if $\chi_2(t)$ is nonzero then* $\deg(\chi_2(t)) = \deg_x(f)$.

A.2 Rational algebraic surfaces

Definition A.2.1. An irreducible algebraic surface $\mathcal{S}$ in the affine space $\mathbb{A}^3(\mathbb{K})$ is defined as the zero set of an irreducible polynomial $F(x,y,z)$; i.e.,

$$\mathcal{S} = \{(a,b,c) \in \mathbb{A}^3(\mathbb{K}) \mid F(a,b,c) = 0\}.$$

A *rational parametrization* of $\mathcal{S}$ is a triple of rational functions satisfying

$$F(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0,$$

where the Jacobian matrix of $(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ has a generic rank 2. A surface having a rational parametrization is called a *unirational surface*.

Example A.2.1. The sphere $x^2 + y^2 + z^2 - 1 = 0$ can be parametrized by rational functions:

$$\mathcal{P}(s,t) = \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1} \right).$$

Definition A.2.2. A parametrization $\mathcal{P}(s,t)$ of the surface $\mathcal{S}$ is *proper* iff it is a birational isomorphism between the plane and the surface $\mathcal{S}$, i.e., it has a rational inverse.

A surface with a proper parametrization is called a *rational surface*. The above parametrization of the sphere is proper because it has the inverse $(x,y,z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$, which is known as the stereographic projection from the north pole onto the plane.

Theorem A.2.1 (Castelnuovo's theorem)**.** *Any unirational surface is rational.*

In higher dimension, unirationality of hypersurfaces is not equivalent to rationality. We refer to Schicho (1997) for further details on rational surfaces and algorithms to computing parametrizations of rational surfaces.

Appendix B

Basics of differential algebra

Throughout this chapter we consider $\mathbb{K}$ to be an algebraically closed field of characteristic zero, i.e. $\mathbb{K}$ contains the field of rational numbers $\mathbb{Q}$.

B.1 Notions

Definition B.1.1. Let $\mathcal{R}$ be a commutative ring and $\delta : \mathcal{R} \mapsto \mathcal{R}$ be a mapping such that

$$\delta(a+b) = \delta(a) + \delta(b), \text{ and } \delta(ab) = \delta(a)b + a\delta(b). \tag{B.1}$$

Then $(\mathcal{R}, \delta)$ is called a *differential ring* with the *derivation* δ. Moreover, if $\mathcal{R}$ is a field, then $(\mathcal{R}, \delta)$ is called a *differential field*.

Definition B.1.2. Let $(\mathcal{R}, \delta)$ be a differential ring. The set $C = \{c \in \mathcal{R} \mid \delta(c) = 0\}$ is called the set of *constants* of $\mathcal{R}$ w.r.t. the derivation δ. If $\mathcal{R}$ is a field, then C is a subfield of $\mathcal{R}$. In this case, we also call C the *field of constants* of $\mathcal{R}$ w.r.t. δ.

Example B.1.1. Let $\mathbb{K}(x)$ be the field of rational functions over $\mathbb{K}$. Let δ be a derivation on $\mathbb{K}(x)$. Then δ is uniquely defined by the two conditions: $\delta a = 0$ for all $a \in \mathbb{K}$ and $\delta x = 1$. This is the usual derivation w.r.t. the variable x, denoted by $\dfrac{d}{dx}$. Therefore, $\left(\mathbb{K}(x), \dfrac{d}{dx}\right)$ is a differential field and $\mathbb{K}$ is the field of constants of $\mathbb{K}(x)$.

Example B.1.2. Let $\left(\mathbb{K}(x), \delta = \dfrac{d}{dx}\right)$ be the differential field of rational functions. Consider the ring $\mathcal{R} = \mathbb{K}(x)[y, y', y'', \ldots]$. The derivation δ can be extended to the ring $\mathcal{R}$ as follows:

$$\delta\left(\sum_i a_i y^{(i)}\right) = \sum_i \left(\delta(a_i) y^{(i)} + a_i y^{(i+1)}\right), \tag{B.2}$$

where $a_i \in \mathbb{K}(x)$ for all i. Then $(\mathcal{R}, \delta)$ forms a differential ring, denoted by $\mathbb{K}(x)\{y\}$. The y is called a *differential indeterminate* over the differential field $\mathbb{K}(x)$. A polynomial in $\mathbb{K}(x)\{y\}$ is called a *differential polynomial*.

Similarly, this construction can be extended to several indeterminates (see Ritt (1950)). In that case, there may be several derivations. The differential ring is called *ordinary* if it is only equipped with one derivation. In fact, we only consider ordinary differential rings in this thesis.

Definition B.1.3. Let $(\mathcal{R}, \delta)$ be an ordinary differential ring. An ideal I of $\mathcal{R}$ is called a *differential ideal* iff I is closed under the derivation δ, i.e., for all $a \in I$ we have $\delta(a) \in I$.

Definition B.1.4. Let B be a set of differential polynomials in $\mathcal{R}$. The differential ideal generated by B, denoted by $[B]$, is the ideal generated by all elements in B and their derivatives. The radical differential ideal generated by B, denoted by $\{B\}$, is the radical of $[B]$.

Let I be a differential ideal in the differential ring $\mathcal{R} = (\mathcal{K}\{y_1, y_2, \ldots, y_n\}, \delta)$, where $\mathcal{K}$ is a differential field. Let $\mathcal{L}$ be a differential field extension of $\mathcal{K}$. An element $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{L}^n$ is called a *zero* of I if for all $A \in I$ we have $A(\xi) = 0$. The *defining differential ideal* of ξ in $\mathcal{R}$ is $\wp = \{A \in \mathcal{R} \mid A(\xi) = 0\}$.

Let $\wp$ be a prime differential ideal in $\mathcal{R}$. A point $\xi \in \mathcal{L}^n$ is called a *generic zero* of $\wp$ if $\wp$ is the defining differential ideal of ξ in $\mathcal{R}$. One can construct a differential field extension $\mathcal{L}$ so that $\mathcal{L}^n$ contains a generic zero of $\wp$, namely, the quotient field of the integral domain $\mathcal{R}/\wp$. The situation becomes complex when one needs to construct such a common extension field for several prime differential ideals in various differential rings. The notion of *semiuniversal* extension over $\mathcal{K}$ will solve this problem (Kolchin (1973), Chapter II, section 2).

Definition B.1.5. Let $\mathcal{L}$ be a differential extension field of $\mathcal{K}$. Let $\xi, \eta \in \mathcal{L}^n$. ξ is said to be a *differential specialization* of η over $\mathcal{K}$ iff the defining differential ideal of η is contained in the defining differential ideal of ξ.

By definition, if η is a generic zero of $\wp$, then every zero of $\wp$ is a differential specialization of η over $\mathcal{K}$.

Example B.1.3. Let $\mathcal{R} = \mathbb{K}(x)\{y\}$ and $\wp = \{y' - 1\}$, the radical differential ideal generated by $y' - 1$. It turns out that, for any $c \in \mathbb{K}$, $y = x + c$ is a zero of $\wp$ and $\wp \subset [y - x - c]$ for all $c \in \mathbb{K}$, i.e., $y = x + c$ is a differential specialization of a generic zero of $y' - 1 = 0$.

B.2 Rankings

We focus on the notion of a ranking for an ordinary differential ring. The more general setting can be found in Kolchin (1973), Chapter I, §8.

Definition B.2.1. Let $(\mathcal{K}, \delta)$ be a differential field and $\mathcal{R} = (\mathcal{K}\{y_1, y_2, \ldots, y_n\}, \delta)$ be the differential ring with indeterminates $y_1, y_2, \ldots, y_n$. A *ranking* of $(y_1, y_2, \ldots, y_n)$ is a total ordering of the set of all derivatives

$$\Delta y := \{\delta^j(y_i) \mid i = 1, \ldots, n;\ j \in \mathbb{N}\} \tag{B.3}$$

such that

$$u \leq \delta^j(u) \text{ and } u \leq v \Longrightarrow \delta^j(u) \leq \delta^j(v) \tag{B.4}$$

for all $u, v \in \Delta y$ and $j \in \mathbb{N}$.

Definition B.2.2. A ranking is said to be *orderly* iff $j \leq k \Longrightarrow \delta^j y_{i_1} \leq \delta^k y_{i_2}$ for any $i_1, i_2 \in \{1, \ldots, n\}$.

In the orderly ranking, it turns out that there are only finitely many derivatives of lower rank than $\delta^j y_i$ (this property is said to be *sequential*); and for any $\delta^j y_{i_1}, \delta^k y_{i_2}$, there exists $l \in \mathbb{N}$ such that $\delta^j y_{i_1} \leq \delta^{k+l} y_{i_2}$ (this property is said to be *integrated*).

B.3 Reduction

In the differential ring $\mathcal{R}$, it is important to study the structure of the derivative of a differential polynomial. Suppose that $A \in \mathcal{R}$ and $A \notin \mathcal{K}$. Let us fix a ranking of $(y_1, y_2, \ldots, y_n)$, since this ranking is a total order on the set of derivatives Δy, there exists the highest ranking derivative in A. This is called the *leader* of A, denoted by u_A. Then we can write A as a univariate polynomial in u_A, i.e.

$$A = \sum_{0 \leq i \leq d} I_i u_A^i, \tag{B.5}$$

where $I_0, \ldots, I_d \in \mathcal{R}$ are free of u_A and d is the degree of A w.r.t. u_A. The non-zero differential polynomial I_d is called the *initial* of A. Of course, all the derivatives appear in I_d and others I_i are lower than u_A. We have

$$\delta(A) = \sum_{0 \leq i \leq d} \delta(I_i) u_A^i + \left(\sum_{0 \leq i \leq d} i I_i u_A^{i-1} \right) \delta(u_A), \tag{B.6}$$

and it is clear that $\delta(u_A)$ becomes the leader of $\delta(A)$ and the initial of $\delta(A)$ is

$$S_A = \sum_{0\le i\le d} iI_i u_A^{i-1}. \tag{B.7}$$

The S_A is called the *separant* of A.

Now, for the reducing purpose, it is natural and possible to extend the ranking of the derivatives of $(y_1, y_2, \dots, y_n)$ to two arbitrary differential polynomials and therefore, to the whole differential ring $\mathcal{R}$. Precisely, let A and B are in $\mathcal{R}$. If $A, B \in \mathcal{K}$, then A and B have the same rank. If $A \in \mathcal{K}$ and $B \notin \mathcal{K}$, then A is of lower rank than B. If $A, B \notin \mathcal{K}$, then we compare the two leaders of A and B: A is of lower rank than B if $u_A < u_B$ or $u_A = u_B$ and $\deg_{u_A}(A) < \deg_{u_A}(B)$.

Definition B.3.1. Let us fix a ranking on the differential ring $\mathcal{R}$. Let $A \in \mathcal{R}$ be a differential polynomial such that $A \notin \mathcal{K}$. A differential polynomial F is said to be *lower*[*] than A iff $\deg_{u_A}(F) < \deg_{u_A}(A)$, where u_A is the leader of A. A differential polynomial F is said to be *reduced*[†] w.r.t. A iff F is lower than A and all the derivatives of A.

It turns out that if F is of lower rank than A, then F is lower than A (because if $u_F < u_A$, then $\deg_{u_A}(F) = 0 < \deg_{u_A}(A)$). Fix an *orderly ranking*, if F is of lower rank than A, then F is also reduced w.r.t. A (because, in this ranking, if F is lower than A, then F is also lower than all the derivatives of A).

Definition B.3.2. $\mathcal{A} \subset \mathcal{R}$ is called an *autoreduced set* iff $\mathcal{A} \cap \mathcal{K} = \emptyset$ and each element in $\mathcal{A}$ is reduced w.r.t. all the others.

Given an autoreduced set $\mathcal{A}$ w.r.t. a ranking and a differential polynomial F. In his work, Ritt (1950) has introduced a reduction of F w.r.t. $\mathcal{A}$, i.e., there exist $i_A, s_A \in \mathbb{N}$ and $F_0 \in \mathcal{R}$ such that

1. F_0 is reduced w.r.t. $\mathcal{A}$; (i.e., reduced w.r.t all elements in $\mathcal{A}$;)
2. the rank of F_0 is lower than or equal the rank of F;
3. $\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{s_A} F - F_0$ can be written as a linear combination over $\mathcal{R}$ of derivatives $\delta^j A$, where $A \in \mathcal{A}$ and $\delta^j u_A \le u_F$.

A proof of the termination of Ritt's reduction can be found in Ritt (1950); Kolchin (1973). In this representation, F_0 is called the *differential pseudo remainder* of F w.r.t. $\mathcal{A}$, denoted by $\mathrm{prem}(\mathrm{F}, \mathcal{A})$.

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[*]It could be that the leader of F is of higher rank than A.

[†]This is equivalent to the definition of being reduced in Kolchin (1973).

Appendix C

Maple code and demo examples

We have implemented some procedures described in this thesis using Maple. Here is a short description of those procedures.

1. **AssociatedSystem** – Given a rational mapping $\mathcal{P}(s,t)$, which parametrizes a rational surface $F(x,y,z)=0$. Compute the associated system of the differential equation $F(x,y,y')=0$ w.r.t. $\mathcal{P}(s,t)$.

   > `AssociatedSystem(P,v);`

   where `P` is a list of three rational functions in two variables, and `v` is the list of the variables of `P`.

```
AssociatedSystem:=proc(P::list,v::list)::list;
  local f1,f2,g;
  f1:=diff(P[2],v[2])-P[3]*diff(P[1],v[2]);
  f2:=P[3]*diff(P[1],v[1])-diff(P[2],v[1]);
  g:=diff(P[1],v[1])*diff(P[2],v[2])
  -diff(P[1],v[2])*diff(P[2],v[1]);
  if g=0 then
    error ``This is not a parametrization of an algebraic ODE";
  else
    return [factor(simplify(f1/g)),factor(simplify(f2/g))];
  fi;
end proc:
```

2. **InvariantAlgebraicCurves** – Given a planar rational system and a positive integer number d. Compute a set of invariant algebraic curves of the system of degree $\leq d$.

   > `InvariantAlgebraicCurves(V,v,d);`

   where `V` is a list of two rational functions in two variables, `v` is the list of the variables of `V` and `d` is an upper bound for the degree of the invariant algebraic curves.

We use the undetermined coefficients method for finding the invariant algebraic curves of a planar rational system. We need few auxiliary procedures before actually computing the invariant algebraic curves. The procedure `GeneratePolynomial` generates a bivariate polynomial of a certain degree d with unknown coefficients c_{ij}.

```
GeneratePolynomial:=proc(d::nonnegative, L::list, c::symbol)
  ::polynom;
  local i,j,P;
  i:=0;
  P:=0;
  while d-i>=0 do
    for j from 0 to d-i do
          P:=P+c[d-i-j,j]*L[1]^(d-i-j)*L[2]^j;
        od;
        i:=i+1;
  od;
  P;
end proc:
```

Note that if $G(s,t)=0$ is the defining equation of an algebraic curve, then for any constant $c \neq 0$ the equation $cG(s,t)=0$ defines the same algebraic curve. In order that the representation of the curve is unique, we ask for $G(s,t)$ to be a monic polynomial in a certain term order, i.e., the leading coefficient of $G(s,t)$ w.r.t. that term order is 1. The procedure `MonicPolynomial` generates the set of all monic polynomials of degree $\leq d$ w.r.t. the lexicographic order with $s>t$.

```
MonicPolynomial:=proc(k::posint, var::list, c::symbol)::set;
  # var[1]>var[2]
  local d,S,FHead,FRest,i,j;
  d:=1;S:={};
  while d<=k do
    FRest:=GeneratePolynomial(d-1,var,c);
    for i from 0 to d do
      j:=0;
      FHead:=0;
      while j<=i do
        if j=i then
          FHead:= FHead+var[2]^(d-i)*var[1]^i;
        else
          FHead:= FHead+c[j,d-j]*var[2]^(d-j)*var[1]^j;
        fi;
        j:=j+1;
      od;
```

```
      S:=S union {FRest+FHead};
    od;
    d:=d+1;
  od;
  S;
end proc:
```

Now we have all auxiliary procedures to find the invariant algebraic curves of a planar polynomial system. Here we use the implemented procedure in Man (1993) for computing such invariant algebraic curves. In this implementation we did not discard the reducible invariant algebraic curves. Later, we will present an undirect way to detect the reducibility of bivariate polynomials by using the `algcurves` package.

```
ManPrelleSinger:=proc(P::polynom,Q::polynom,v::list,d::posint)
::set;
  # v=list of variables, d=degree bound
  local para,Sfg,k,A,f,F,a,DF,G,GG,Feqns,divisible,lcF,lmF,lcDF,
    lmDF,ltF,ltDF,q,FS,H,h,var,coeF;
  para:=op((indets(P) union indets(Q)) minus {v[1],v[2]});
  Sfg:={};
  k:=1;
  while k<=d do
    A:=MonicPolynomial(k,v,c);
    for a in A do
      F:=a;
      DF:=expand(P*diff(F,v[1])+Q*diff(F,v[2]));
      G:=0;
      divisible:=true;
      Feqns:={};
      if DF=0 then
        Sfg:=Sfg union {[F,0]};
      fi;
      while divisible=true and DF<>0 do
        lcF:=Groebner:-LeadingCoefficient(F,tdeg(v[1],v[2]));
        lmF:=Groebner:-LeadingMonomial(F,tdeg(v[1],v[2]));
        ltF:=lcF*lmF;
        lcDF:=Groebner:-LeadingCoefficient(DF,tdeg(v[1],v[2]));
        lmDF:=Groebner:-LeadingMonomial(DF,tdeg(v[1],v[2]));
        ltDF:=lcDF*lmDF;
        if Algebraic:-Divide(ltDF,ltF)=true then
          q:=ltDF/ltF;
          G:=G+q;
          DF:=simplify(DF-F*q);
```

```
        elif nops(indets(lcDF))=0 then
          divisible:=false;
        else
          Feqns:=Feqns union {lcDF};
          DF:=simplify(DF-ltDF);
        fi;
      od;
      if divisible=true and Feqns<>{} then
        H:=solve(Feqns,[op(indets(F) minus {v[1],v[2]})]);
        for h in H do
          F:=subs(h,a);
          GG:=subs(h,G);
          Sfg:=Sfg union {[F,GG]};
        od;
      fi;
    od;
    k:=k+1;
  od;
  return(Sfg);
end proc:
```

We know that a planar rational system and its associated polynomial system have the same set of invariant algebraic curves. Note that if the polynomial system is defined by two polynomials P and Q, then $\gcd(P, Q)$ defines an invariant algebraic curve of the system. An irreducible invariant algebraic curve of the polynomial system is either a factor of $\gcd(P, Q)$ or an irreducible invariant algebraic curve of the system defined by $\frac{P}{\gcd(P,Q)}$ and $\frac{Q}{\gcd(P,Q)}$. Therefore, when we transform a rational system into the associated polynomial system we return the system in the form $[\frac{P}{\gcd(P,Q)}, \frac{Q}{\gcd(P,Q)}, \gcd(P, Q)]$. Later, we only compute the invariant algebraic curves of the system defined by $\frac{P}{\gcd(P,Q)}$ and $\frac{Q}{\gcd(P,Q)}$.

```
AssociatedPolynomialSystem:=proc(V::list)::list;
  # V is a list of rational functions
  local N,M,P1,Q1,PP,QQ;
  N:=gcd(numer(V[1]),numer(V[2]));
  M:=gcd(denom(V[1]),denom(V[2]));
  P1:=factor(V[1]*M/N);
  Q1:=factor(V[2]*M/N);
  PP:=numer(P1)*denom(Q1);
  QQ:=numer(Q1)*denom(P1);
  [PP,QQ,N];
end proc:
```

Now, we can find the invariant algebraic curves of a planar rational system using the below procedure `InvariantAlgebraicCurves`. Note that the coefficients of the system may be algebraic numbers. For instance, the symbol $RootOf(12-12sZ+s^2Z^2)s$ represents two algebraic numbers $6+2\sqrt{6}$ and $6-2\sqrt{6}$. Then the type of $RootOf(12-12sZ+s^2Z^2)s$ in MAPLE is not polynomial. Therefore, in order to simplify the implementation, we elaborate a bit the input of the procedure `InvariantAlgebraicCurves` without lost of generality.

```
InvariantAlgebraicCurves:=proc(V::list,v::list,d::posint)::set;
  local POLY,AA,P1,P2,S1,S2,var1,var2,K;
  POLY:=AssociatedPolynomialSystem(V);
  S1:=indets(POLY[1]);
  S2:=indets(POLY[2]);
  P1:=POLY[1];
  for var1 in S1 do
    if type(var1,RootOf)=true then
      P1:=allvalues(POLY[1])[1];
    fi;
  od;
  P2:=POLY[2];
  for var2 in S2 do
    if type(var2,RootOf)=true then
      P2:=allvalues(POLY[2])[1];
    fi;
  od;
  AA:=ManPrelleSinger(P1,P2,v,d);
  if degree(POLY[3],v)<>0 then
    K:=diff(POLY[3],v[1])*POLY[1]+diff(POLY[3],v[2])*POLY[2];
    # K is the cofactor of POLY[3].
    return AA union {[POLY[3],K]};
  else return AA;
  fi;
end proc:
```

In general, the output of `InvariantAlgebraicCurves` does not contain all possible invariant algebraic curves of degree $\leq d$, but it contains all irreducible ones of degree $\leq d$. If the associated polynomial system of V is defined by P and Q, then the missing invariant algebraic curves are multiples of $\gcd(P,Q)$, which will not produce any new solution of the system. Therefore, they are not interesting to us.

3. **CorrespondingRationalSolution** – Given a planar rational system and an invariant algebraic curve of the system. Compute a rational solution of the given system whose solution curve is the given one.

> `CorrespondingRationalSolution(G,V,v,x);`

where `V` is a list of two rational functions in two variables, `G` is a bivariate polynomial defining an invariant algebraic curve of the rational system defined by `V`, `v` is the list of two variables of `V` and `x` is the variable of the derivation.

At a certain step of the procedure `CorrespondingRationalSolution` we have to find a rational solution of a linear autonomous ODE $T'(x) = E(T)$, where E is a rational polynomial of T. The rational solution of this differential equation is simple, namely, it is of the form $T(x) = \dfrac{ax+b}{cx+d}$, where a, b, c, d are constants. We need the procedure `AutonomousLinearSolver` for computing it.

```
AutonomousLinearSolver:=proc(E::ratpoly,T::symbol,x::symbol)::set;
  local TT,A1,BB,c,disc;
  if type(E,polynom(anything,T))=true then
    if degree(E,T)<=0 then
      if E=0 then
        return({TT=c});
      else
        return({TT=E*x});
      fi;
    elif degree(E,T)=2 then
        disc:=simplify(discrim(E,T));#the discriminal of E
        if disc=0 then
          A1:=coeff(E,T,2);
          BB:=coeff(E,T,1);
          return({TT=-BB/(2*A1)-1/(A1*x)});#double root -BB/(2*A1).
        else
          return({});
        fi;
    else return({});
    fi;
  else return({});
  fi;
end proc:
```

Now we can execute the procedure `CorrespondingRationalSolution`. We have to include the package `algcurves` in order to use the two commands `genus` and `parametrization`.

```
CorrespondingRationalSolution:=proc(F::polynom,V::list,v::list,
t::symbol)::set;
  local Solution,R,Test1,Test2,E,E1,EE,T,S1,S2;
  Solution:={};
```

```
  if irreduc(F)=true then
    if genus(F,v[1],v[2])=0 then
      try
        R:=parametrization(F,v[1],v[2],t);
      catch:
        R:=[];
      end try;
      if R<>[] then
        Test1:=subs({v[1]=R[1],v[2]=R[2]},denom(V[1]));
        Test2:=subs({v[1]=R[1],v[2]=R[2]},denom(V[2]));
        if Test1<>0 and Test2<>0 then
          if diff(R[1],t)<>0 then
            E1:=subs({v[1]=R[1],v[2]=R[2]},V[1])/diff(R[1],t);
            E:=subs(t=T,E1);
            EE:=AutonomousLinearSolver(E,T,t);
            if EE<>{} then
              S1:=simplify(subs(t=op(op(EE))[2],R[1]));
              S2:=simplify(subs(t=op(op(EE))[2],R[2]));
              Solution:=Solution union {[S1,S2]};
            fi;
          elif diff(R[2],t)<>0 then
            E1:=subs({v[1]=R[1],v[2]=R[2]},V[2])/diff(R[2],t);
            E:=subs(t=T,E1);
            EE:=AutonomousLinearSolver(E,T,t);
            if EE<>{} then
              S1:=simplify(subs(t=op(op(EE))[2],R[1]));
              S2:=simplify(subs(t=op(op(EE))[2],R[2]));
              Solution:=Solution union {[S1,S2]};
            fi;
          fi;
        fi;
      fi;
    fi;
  fi;
  Solution;
end proc:
```

4. **RationalSolutionsOfSystem** – Given a planar rational system and a positive integer number d. Compute a set of rational solutions of the system up to the degree bound d of its invariant algebraic curves.

   > `RationalSolutionsOfSystem(V,v,x,d);`

where `V` is a list of two rational functions in two variables, `v` is the list of two variables of `V`, `x` is the variable of the derivation and `d` is an upper bound for the degree of the invariant algebraic curves of the system.

It is now simple to execute the procedure `RationalSolutionsOfSystem`.

```
RationalSolutionsOfSystem:=proc(V::list,v::list,t::symbol,
d::posint)::set;
  local POLY,a,Sol,AA,Solution;
  AA:=InvariantAlgebraicCurves(V,v,d);
  Solution:={};
  for a in AA do
    Sol:=CorrespondingRationalSolution(a[1],V,v,t);
    Solution:=Solution union Sol;
  od;
  Solution;
end proc:
```

In order to obtain a rational general solution of the planar rational system we need to find a general invariant algebraic curve, i.e., the curve defined by a monic polynomial whose coefficients contains an arbitrary constant. Therefore, the following procedure allows us to check whether a curve is a general curve.

```
IsGeneralCurve:=proc(F::polynom,v::list,id::set)::boolean;
  # F is a polynomial in two variables v[1],v[2];
  # id is a set of ideterminates
  evalb({coeffs(F,v)} intersect id <>{});
end proc:
```

Now, we can find a rational general solution of the planar rational system using the same procedure as `RationalSolutionsOfSystem` but for the general invariant algebraic curve.

```
RationalGeneralSolutionsOfSystem:=proc(V::list,v::list,t::symbol,
d::posint)::set;
  local POLY,a,Sol,AA,Solution,A;
  AA:=InvariantAlgebraicCurves(V,v,d);
  A:={coeffs(GeneratePolynomial(d,v,c),v)};
  Solution:={};
  for a in AA do
    if IsGeneralCurve(a[1],v,A)=true then
      Sol:=CorrespondingRationalSolution(a[1],V,v,t);
      Solution:=Solution union Sol;
    fi;
  od;
  Solution;
```

```
end proc:
```

5. **RationalParametricGeneralSolver** – Given a proper rational parametrization $\mathcal{P}(s,t)$ of the solution surface $F(x,y,z)=0$ and a positive integer number d. Compute a rational general solution of the differential equation $F(x,y,y')=0$ up to the degree bound d.

   > `RationalParametricGeneralSolver(P,v,x,d);`

   where `P` is a proper rational parametrization of the solution surface $F(x,y,z)=0$ of the differential equation $F(x,y,y')=0$, `v` is the list of two variables of `P`, `x` is the variable of the derivation and `d` is an upper bound for the degree of the invariant algebraic curves of the system.

   Now, we simply combine all previous procedures for solving rational general solutions of a parametrizable algebraic ODE.

```
RationalParametricGeneralSolver:=proc(P::list,v::list,x::symbol,
d::posint)::set;
  local V,SS,Solution,a,c,Test1,Test2,Y;
  V:=AssociatedSystem(P,v);
  SS:=RationalGeneralSolutionsOfSystem(V,v,x,d);
  Solution:={};
  for a in SS do
    Test1:=simplify(subs({v[1]=a[1],v[2]=a[2]},denom(P[1])));
    Test2:=simplify(subs({v[1]=a[1],v[2]=a[2]},denom(P[2])));
    if Test1<>0 and Test2<>0 then
      c:=simplify(subs({v[1]=a[1],v[2]=a[2]},P[1])-x);
      Y:=subs({v[1]=subs(x=x-c,a[1]),v[2]=subs(x=x-c,a[2])},P[2]);
      Solution:=Solution union {y=simplify(Y)};
    fi;
  od;
  Solution;
end proc:
```

ParametrizationODE

AssociatedSystem

AssociatedSystem(**P, v**)

Example

```
> restart;
> read"ParametrizationODE.txt";
> ode1:=x*diff(y(x),x)^2+y(x)*diff(y(x),x)-y(x)^4;
```

$$ode1 := x\left(\frac{d}{dx}y(x)\right)^2 + y(x)\left(\frac{d}{dx}y(x)\right) - y(x)^4 \quad \textbf{(1.1.1)}$$

The solution surface is F1:=x*z^2+y*z-y^4.

```
> F1:=x*z^2+y*z-y^4;
```

$$F1 := xz^2 + yz - y^4 \quad \textbf{(1.1.2)}$$

This surface can be parametrized as follows.

```
> parametrization(F1,y,z,t);
```

$$\left[\frac{t}{-x+t^2}, -\frac{t^3}{x\,(x^2-2\,x\,t^2+t^4)}\right] \quad \textbf{(1.1.3)}$$

```
> PP:=[s,op(subs(x=s,parametrization(F1,y,z,t)))];
```

$$PP := \left[s, \frac{t}{-s+t^2}, -\frac{t^3}{s\,(s^2-2\,s\,t^2+t^4)}\right] \quad \textbf{(1.1.4)}$$

```
> V:=AssociatedSystem(PP,[s,t]);
```

$$V := \left[1, \frac{t}{s}\right] \quad \textbf{(1.1.5)}$$

InvariantAlgebraicCurves

InvariantAlgebraicCurves(**V, v, d**)

Example

```
> V;
```

$$\left[1, \frac{t}{s}\right] \quad \textbf{(2.1.1)}$$

```
> InvariantAlgebraicCurves(V,[s,t],1);
```

$$\{[t, 1], [c_{0,1}\,t + s, 1]\} \quad \textbf{(2.1.2)}$$

```
> InvariantAlgebraicCurves(V,[s,t],2);
```

$$\{[t, 1], [t^2, 2], [c_{0,1}\,t + s, 1], [c_{0,2}\,t^2 + t\,s, 2], [c_{0,2}\,t^2 + c_{1,1}\,t\,s + s^2, 2]\} \quad \textbf{(2.1.3)}$$

In this example, the invariant algebraic curves of degree 2 are reducible.

CorrespondingRationalSolution

CorrespondingRationalSolution(**G, V, v, x**)

Example

```
> G:=s+c[0,1]*t:
> CorrespondingRationalSolution(G, V, [s,t], x);
```

$$\left\{\left[x, -\frac{x}{c_{0,1}}\right]\right\} \tag{3.1.1}$$

RationalSolutionsOfSystem

RationalSolutionsOfSystem(**V, v, x, d**)

Example

```
> RationalSolutionsOfSystem(V,[s,t],x,1);
```

$$\left\{[x, 0], \left[x, -\frac{x}{c_{0,1}}\right]\right\} \tag{4.1.1}$$

```
> V2:=[1,2*t/s];
```

$$V2 := \left[1, \frac{2t}{s}\right] \tag{4.1.2}$$

```
> RationalSolutionsOfSystem(V2,[s,t],x,1);
```

$$\{[x, 0]\} \tag{4.1.3}$$

```
> RationalSolutionsOfSystem(V2,[s,t],x,2);
```

$$\left\{[x, 0], \left[x, -\frac{x^2}{c_{0,1}}\right]\right\} \tag{4.1.4}$$

RationalParametricGeneralSolver

RationalParametricGeneralSolver(**P, v, x, d**)

Example

```
> RationalParametricGeneralSolver(PP,[s,t],x,1);
```

$$\left\{y = -\frac{c_{0,1}}{-c_{0,1}^2 + x}\right\} \tag{5.1.1}$$

Further examples

In what follows, we demonstrate several examples where the solution surfaces are pencil of rational curves. Moreover, the solution surface F(x,y,z)=0 is parametrizable over K(x).

```
> F2:=x*z^2-y*z+a:
  ode2:=x*diff(y(x),x)^2-y(x)*diff(y(x),x)+a:
> PP2:=parametrization(F2,y,z,t);
```

$$PP2 := \left[\frac{x t^2 + a}{t}, t\right] \tag{6.1}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP2))],[s,t],x,1);
```

$$\left\{y = -\frac{x c_{0,0}^2 + a}{c_{0,0}}\right\} \tag{6.2}$$

```
> F3:=x*z^2-2*y*z+2*y+x:
  ode3:=x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+2*y(x)+x:
> PP3:=parametrization(F3,y,z,t);
```

(6.3)

$$PP3 := \left[\frac{1}{2}\, \frac{x\,(t^2+1)}{-1+t},\, t \right] \tag{6.3}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP3))],[s,t],x,1);
```

$$\left\{ y = -\frac{1}{2}\, \frac{x^2 - 2\,x\,c_{0,1} + 2\,c_{0,1}^2}{c_{0,1}} \right\} \tag{6.4}$$

```
> F4:=x*z^2-2*y*z+b*x:
  ode4:=x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+b*x:
> PP4:=parametrization(F4,y,z,t);
```

$$PP4 := \left[\frac{1}{2}\, \frac{x\,(t^2+b)}{t},\, t \right] \tag{6.5}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP4))],[s,t],x,1);
```

$$\left\{ y = -\frac{1}{2}\, \frac{x^2 + b\,c_{0,1}^2}{c_{0,1}} \right\} \tag{6.6}$$

```
> F5:=(x+1)*z^2-(y+x)*z+y:
  ode5:=(x+1)*diff(y(x),x)^2-(y(x)+x)*diff(y(x),x)+y(x):
> PP5:=parametrization(F5,y,z,t);
```

$$PP5 := \left[\frac{t\,(-x + t\,x + t)}{-1+t},\, t \right] \tag{6.7}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP5))],[s,t],x,1);
```

$$\left\{ y = -\frac{c_{0,0}\,(x + c_{0,0}\,x + c_{0,0})}{1 + c_{0,0}} \right\} \tag{6.8}$$

```
> F6:=x^2*z^2-4*x*(y+2)*z+4*y*(y+2):
  ode6:=x^2*diff(y(x),x)^2-4*x*(y(x)+2)*diff(y(x),x)+4*y(x)*(y(x)+2):
> PP6:=parametrization(F6,y,z,t);
```

$$\begin{aligned} PP6 := \Big[& \big(t\,x\,(48 - 6\,t\,x - 8\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2)\,x + t\,x^2\,\mathit{RootOf}(12 - 12\,x\,\_Z \\ & + x^2\,\_Z^2))\big) \big/ \big(-6\,x^2\,t^2 - 8\,t\,x^2\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2) + t^2\,x^3\,\mathit{RootOf}(12 - 12\,x\,\_Z \\ & + x^2\,\_Z^2) - 48 + 40\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2)\,x\big),\ \big(2\,t\,(-4\,\mathit{RootOf}(12 - 12\,x\,\_Z \\ & + x^2\,\_Z^2)\,x - 30\,t\,x + 24 + 3\,t\,x^2\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2))\big) \big/ \big(-6\,x^2\,t^2 \\ & - 8\,t\,x^2\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2) + t^2\,x^3\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2) - 48 \\ & + 40\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2)\,x\big) \Big] \end{aligned} \tag{6.9}$$

```
> AS:=AssociatedSystem([s,op(subs(x=s,PP6))],[s,t]);
```

$$AS := \left[1,\, -\frac{1}{8}\, t^2\, (\mathit{RootOf}(12 - 12\,s\,\_Z + s^2\,\_Z^2)\,s - 10) \right] \tag{6.10}$$

```
> allvalues(AS[2])[1];
```

$$-\frac{1}{8}\, t^2\, (-4 + 2\sqrt{6}) \tag{6.11}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP6))],[s,t],x,1);
```

$$\{\ \} \tag{6.12}$$

We need a higher degree bound, in this case, it is 2.

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP6))],[s,t],x,2);
```

$$\left\{ y = \frac{1}{2}\, \left(x\,\left(4\,c_{0,1}^3\,\sqrt{6}\,\mathit{RootOf}(12 - 12\,x\,\_Z + x^2\,\_Z^2)\,x - 3\,x^3\,\mathit{RootOf}(12 - 12\,x\,\_Z \right.\right.\right. \tag{6.13}$$

$$+x^2\_Z^2)\, c_{0,1}\sqrt{6} - 6x^2\, RootOf(12 - 12x\_Z + x^2\_Z^2)\, c_{0,1}^2 - 14\, c_{0,1}\, RootOf(12 - 12x\_Z$$
$$+x^2\_Z^2)\, x^3 + 8\, RootOf(12 - 12x\_Z + x^2\_Z^2)\, x c_{0,1}^3 + x^4\, RootOf(12 - 12x\_Z + x^2\_Z^2)\sqrt{6}$$
$$+ 12x^3 + 46x c_{0,1}^2 + 48x^2 c_{0,1} - 10x^2 c_{0,1}\sqrt{6} - 44c_{0,1}^3\sqrt{6} - 88c_{0,1}^3 - 6x^3\sqrt{6} - 8x$$
$$c_{0,1}^2\sqrt{6}))\Big/\left(c_{0,1}^2\left(60x^2 + 24x^2\sqrt{6} + 12x c_{0,1} + 4x c_{0,1}\sqrt{6} + c_{0,1}^2\right)\right)\Big\}$$

```
> F7:=x^2*z^2-y*(y-2*x)*z+y^2:
  ode7:=x^2*diff(y(x),x)^2-y(x)*(y(x)-2*x)*diff(y(x),x)+y(x)^2:
> PP7:=parametrization(F7,y,z,t);
```

$$PP7 := \left[\frac{1+2x+x^2+2t+2tx+t^2}{t+1}, \frac{1+2x+x^2+2t+2tx+t^2}{2t+t^2+1}\right] \tag{6.14}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP7))],[s,t],x,1);
```

$$\left\{y = -\frac{1-2c_{0,0}+c_{0,0}^2}{c_{0,0}+x-1}\right\} \tag{6.15}$$

```
> F8:=z^3-a*x*y*z+2*a*y^2:
  ode8:=diff(y(x),x)^3-a*x*y(x)*diff(y(x),x)+2*a*y(x)^2:
> PP8:=parametrization(F8,y,z,t);
```

$$PP8 := \left[-t^2 a(-x+2t), -t a(-x+2t)\right] \tag{6.16}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP8))],[s,t],x,1);
```

$$\left\{y = -\frac{1}{4}\left(c_{0,0}+x\right)^2 a\, c_{0,0}\right\} \tag{6.17}$$

```
> F9:=z^3-x*y^4*z-y^5:
  ode9:=diff(y(x),x)^3-x*y(x)^4*diff(y(x),x)-y(x)^5:
> PP9:=parametrization(F9,y,z,t);
```

$$PP9 := \left[\frac{t^3}{-x^6+t^2x^3}, -\frac{t^5}{x^{10}-2x^7t^2+t^4x^4}\right] \tag{6.18}$$

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP9))],[s,t],x,1);
```

$$\{\ \} \tag{6.19}$$

Here we again need a higher degree bound.

```
> RationalParametricGeneralSolver([s,op(subs(x=s,PP9))],[s,t],x,2);
```

$$\left\{y = -\frac{1}{c_{0,1}\left(-c_{0,1}^2+x\right)}\right\} \tag{6.20}$$

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