JOHANNES KEPLER
UNIVERSITÄT LINZ | JKU

TNF
Technisch-Naturwissenschaftliche
Fakultät

Symbolic Solutions of First-Order Algebraic Differential Equations

DISSERTATION

zur Erlangung des akademischen Grades

Doktor der Technischen Wissenschaften

im Doktoratsstudium der

Technischen Wissenschaften

Eingereicht von:
Dipl.-Ing. Georg Grasegger

Angefertigt am:
Institut für Symbolisches Rechnen,
Doktoratskolleg Computational Mathematics (W1214)

Beurteilung:
Univ.-Prof. Dipl.-Ing. Dr. Franz Winkler (Betreuung)
Prof. Dr. J. Rafael Sendra

Linz, Juni 2015

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch.

Linz, Juni 2015

Georg Grasegger

Kurzfassung

Differentialgleichungen werden seit langer Zeit intensiv studiert. Diverse Methoden für spezielle Fälle wurden entwickelt. Dennoch gibt es bisher keinen allgemeinen Algorithmus zur Berechnung expliziter exakter Lösungen. Das wichtigste Ziel dieser Dissertation ist die Entwicklung und Untersuchung neuer Methoden zur Berechnung exakter expliziter Lösungen von algebraischen Differentialgleichungen. Hierfür wird das differentielle Problem in ein algebraisch geometrisches umgewandelt, indem die Differentialgleichung als algebraische Gleichung betrachtet wird. Eine solche Gleichung beschreibt eine algebraische Varietät und somit können Werkzeuge der algebraischen Geometrie angewendet werden. Im Speziellen spielen Parametrisierungen von algebraischen Varietäten eine wesentliche Rolle bei der Lösung des Problems und beim Nachweis von Eigenschaften der erhaltenen Lösungen. Eine allgemeine Idee zum Lösen autonomer algebraischer Differentialgleichungen erster Ordnung wird präsentiert.

Das Hauptresultat der Dissertation ist die konkrete Anwendung der allgemeinen Idee auf gewöhnliche und partielle Differentialgleichungen. Die Idee wird für autonome algebraische gewöhnliche Differentialgleichungen erster Ordnung vorgestellt. Die präsentierte Methode ist eine Verallgemeinerung von bereits existierenden Algorithmen zur Berechnung rationaler Lösungen. Sie ermöglicht die Erweiterung zur Berechnung radikaler Lösungen. Außerdem erlaubt sie eine weitere Verallgemeinerung auf algebraische gewöhnliche Differentialgleichungen höherer Ordnung. Ein zweiter Fokus liegt in der Anwendung der allgemeinen Idee auf partielle Differentialgleichungen in beliebig vielen Variablen. Die präsentierte Methode reduziert das Problem auf ein anderes, für welches Lösungsmethoden existieren. Diverse bekannte Differentialgleichungen lassen sich mit dieser Methode lösen. Außerdem werden Klassen von Differentialgleichungen mit rationalen, radikalen oder algebraischen Lösungen präsentiert. Mit Hilfe linearer Transformationen wird eine Methode für gewisse nicht autonome Differentialgleichungen erreicht.

Die Methoden sind so konstruiert, dass die dadurch erhaltenen Lösungen bestimmte Kriterien erfüllen. Es wird gezeigt, dass algebraische Lösungen von gewöhnlichen Differentialgleichungen allgemeine Lösungen sind. Rationale Lösungen von partiellen Differentialgleichungen sind bewiesenermaßen echt und vollständig.

Abstract

Differential equations have been intensively studied for a long time. Various exact solution methods have been proposed for specific cases. Nevertheless, there is no general algorithm for finding explicit exact solutions. The main aim of this thesis is to develop and investigate new methods for computing explicit exact solutions of algebraic differential equations. For this purpose, the differential problem is transformed into an algebraic geometric one by considering the differential equation to be an algebraic equation. Such an equation defines an algebraic variety and hence, tools from algebraic geometry can be applied. In particular, parametrizations of algebraic varieties are intrinsically used to solve the problem and prove properties of the obtained solutions. A general idea for solving first-order autonomous algebraic differential equations is presented.

The main results of the thesis are applications of this general idea to ordinary and partial differential equations. The idea is introduced for first-order autonomous algebraic ordinary differential equations. The presented method is a generalization of an existing algorithm for computing rational solutions. It admits an extension to the computation of radical solutions. Moreover, it allows a further generalization to higher-order algebraic ordinary differential equations.

A second focus lies on the application of the general idea to partial differential equations in arbitrary many variables. The presented method reduces the problem to another one for which solution methods exist. Various well-known differential equations are solved by this method. Furthermore, classes of differential equations with rational, radical or algebraic solutions are presented. With the help of linear transformations a solution method for certain non-autonomous differential equations is achieved.

The procedures are constructed in such a way that the obtained solutions thereof satisfy certain requirements. It is shown that algebraic solutions of ordinary differential equations are general solutions. Rational solutions of partial differential equations are proven to be proper and complete.

Acknowledgments

First of all I want to thank my supervisor Franz Winkler for the chance to work in his research group. His valuable and motivating advices have been an important factor of my research. I appreciate a lot the chance to work on my own ideas and I also enjoyed his lectures and the joint discussions.

I also want to express my gratitude to J. Rafael Sendra for his hospitality during my research visit in Alcalá. I learned a lot from him in the many fruitful mathematical discussions. In this respect I also want to thank Alberto Lastra. Both of them made my visit a most valuable one.

I want to thank James H. Davenport for hosting me at the university of Bath and for discussions with him and Matthew England on the topic of my research. Similarly, I thank Moulay A. Barkatou for the chance to visit him at the university of Limoges.

Furthermore, I thank the colleagues and secretaries at the doctoral program "Computational Mathematics" and at RISC for their help and the friendly atmosphere at the institutes. The interesting research discussions at RISC have been a motivation for many new ideas.

My special thanks go to my family who helped me a lot to achieve this goal. It was due to their continuous support throughout the years, that I was able to focus on my studies and research.

This thesis is based on research supported by the Austrian Science Fund (FWF) in the frame of the doctoral program "Computational Mathematics", grant W1214-N15, project DK11.

Contents

1. Introduction

In the literature one can choose among several exact methods in order to solve ordinary and partial differential equations (see for instance [73, Section II.B]). The main aim of the present work is to provide an alternative novel exact method for solving first-order algebraic differential equations. Our method provides a tool for systematically solving various well-known equations.

In this chapter we introduce the topic and we describe some of the ideas which the method is based on. In Section 1.1 we give an overview of similar and more specific methods and their historical background. Later we briefly describe the important notions of differential algebra (Section 1.2) and algebraic geometry (Section 1.3). Basic key facts of both areas are summarized. In the last part of the introduction we investigate the relation between solutions of differential equations and algebraic hypersurfaces. This relation figures the general idea of our method (Section 1.4).

In Chapter 2 we show how this idea can be applied to find radical solutions of first-order autonomous algebraic ordinary differential equations. This generalizes some previously known methods which are therefore briefly presented as well. We also give an idea of further generalization to finding non-algebraic solutions. At the end of the chapter we elaborate some advantages of the method.

Next we present a generalization of the new method to partial differential equations in Chapter 3. We first bring up the idea for the case of two variables. All important aspects of the procedure can be found there. The method is illustrated by examples and classes of partial differential equations with rational solutions are given. As an intermediate step we show the case of three variable separately before we extend the method to an arbitrary number of variables.

Finally, in Chapter 4 we introduce linear transformations and their application for solving some non-autonomous differential equations. These transformations have been used for solving and classifying algebraic ordinary differential equations. We show how the ideas can be used for partial differential equations.

Additional information on differential algebra and algebraic geometry as well as an extensive list of well-known ordinary and partial differential equations with computed solutions can be found in the appendix.

The content of the thesis is based on the papers (co)authored by Grasegger [23, 21, 22]. Moreover, this thesis also contains further ideas and investigations. The main contributions of the author are novel procedures for explicitly solving various classes of algebraic differential equations. These include but are not restricted to

- autonomous algebraic ordinary differential equations of any order and rational respectively radical solutions thereof,
- first-order autonomous algebraic partial differential equations (for arbitrary many variables).

1.1. Historical Background

Recently algebraic-geometric solution methods for first-order algebraic ordinary differential equations (AODEs) have been investigated. A first result on computing solutions of AODEs was presented in [30]. In this paper Hubert introduces a method for finding implicit general and singular solutions of AODEs by computing Gröbner bases. Later in Eremenko [15] a degree bound for rational solutions of a given AODE is computed. Such a bound enables to find solutions by solving algebraic equations.

The starting point for algebraic-geometric methods, such as the one described in this thesis, was an algorithm by Feng and Gao [17, 19]. This algorithm decides whether or not an autonomous first-order AODE, $F(y, y') = 0$, has a rational solution and in the affirmative case computes it. This is done by transformation properties between different proper curve parametrizations and by a degree bound on such parametrizations [62] which leads to a degree bound on the solutions. Hence, existence of a rational solution can be decided. From a rational solution a rational general solution can be deduced. Efficiency of the algorithm is obtained by Laurent series and Padé approximants. The basic idea of the algorithm is presented in Section 2.1.1.

Using the ideas of Feng and Gao several generalizations have been investigated since then. Aroca, Cano, Feng and Gao [5] presented an algorithm for finding algebraic solutions of first-order autonomous AODEs. Ngô and Winkler [44, 45, 46] introduced a method for finding rational solutions of non-autonomous AODEs, $F(x, y, y') = 0$. Here, parametrizations of surfaces play an important role. On the basis of a proper parametrization, the algorithm builds a so called associated system of first-order linear ODEs for which solution methods exist. Based on invariant algebraic curves a solution of the associated system leads to a rational general solution of the differential equation. A short introduction to this method is given in Section 2.1.2.

First results on higher-order AODEs can be found in [27, 28, 29]. Given a proper rational hypersurface parametrization the solutions of the AODE correspond to solutions of an associated system.

As shown in [37] also one-dimensional systems of first-order AODEs can be treated in a similar way.

For partial differential equations much less is known. Of course several solution methods do exist and can be found in standard textbooks but the knowledge on explicit symbolic solutions is worth further investigation. For linear and quasilinear equations solution methods can be found in textbooks (see for instance [73]). The method of characteristics for instance can be applied to quasilinear equations. However, it might not always give an explicit solution. Using this method it is possible to transform a partial differential equation to a system of ordinary ones. The method of characteristic plays a role in our procedure for partial differential equations. A generalization of the method of characteristics to arbitrary first-order partial differential equations was investigated by Lagrange and Charpit (compare [66]). However, we show that our method is essentially different.

1.2. Differential Algebra

All necessary notions of differential algebra which are needed in this thesis can found in standard textbooks such as Ritt [55] or Kolchin [33]. We recall some important aspects here and refer to Appendix A for a little more information.

We consider the field of rational functions $\mathbb{K}(x_1, \ldots, x_n)$ for some algebraically closed field $\mathbb{K}$ of characteristic 0; in practice, one may think of $\mathbb{K}$ as the field $\mathbb{C}$ of complex numbers. By $\frac{\partial}{\partial x_i}$ we denote the usual derivative by x_i. Sometimes we might use the abbreviations $u_{x_i} = \frac{\partial u}{\partial x_i}$. In case $n = 1$ we also write x for x_1 and $'$ or $\frac{\mathrm{d}}{\mathrm{d}x}$ for $\frac{\partial}{\partial x_i}$. For higher-order derivatives we use $u^{(1)} = u'$ and recursively $u^{(k)} = (u^{(k-1)})'$. In case $n = 2$ we also write x for x_1 and y for x_2. Then $\mathbb{K}(x_1, \ldots, x_n)$ together with the derivations is a *(partial) differential field*, i. e. the derivations are linear with respect to addition and fulfill the Leibniz rule and moreover $\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\right)$. Hence, we might use shorthand notation

$$\begin{aligned} \frac{\partial^2}{\partial x_j^2} &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}\right), \\ \frac{\partial^k}{\partial x_j^k} &= \frac{\partial}{\partial x_j}\left(\frac{\partial^{k-1}}{\partial x_j^{k-1}}\right), \\ \frac{\partial^{k_1+\ldots+k_n}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} &= \frac{\partial}{\partial x_1^{k_1}}\left(\ldots\left(\frac{\partial}{\partial x_{n-1}^{k_{n-1}}}\left(\frac{\partial}{\partial x_n^{k_n}}\right)\right)\right). \end{aligned}$$

As short hand notation we might also use $u^{(k_1,\ldots,k_n)} = \frac{\partial^{k_1+\ldots+k_n} u}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}$.

The *ring of differential polynomials* is denoted by $\mathbb{K}(x_1, \ldots, x_n)\{u\}$. It consists of all polynomials in u and its derivatives, i. e.

$$\mathbb{K}(x_1, \ldots, x_n)\{u\} = \mathbb{K}(x_1, \ldots, x_n)[u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_nx_n}, \ldots].$$

An algebraic differential equation (ADE) is defined by a differential polynomial $F \in \mathbb{K}(x_1, \ldots, x_n)\{u\}$ which is also a polynomial in $x_1, \ldots, x_n$. We write

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_1x_n}, \ldots, u_{x_nx_n}, \ldots) = 0$$

for the considered differential equation. In case $n = 1$ this is an algebraic ordinary differential equation (AODE). In case $n > 1$ it is an algebraic partial differential equation (APDE). An ADE is called *autonomous* iff $F \in \mathbb{K}\{u\}$, i. e. if the coefficients of F do not depend on the variables of differentiation $x_1, \ldots, x_n$. We call an ADE *non-autonomous* if it is not necessarily autonomous.

1.2.1. Solutions of AODEs

Usually we want to describe solutions of ADEs by an expression which is as general as possible, i. e. an expression which declares almost all solutions. Such an expression shall be called a general solution. In the following we give a precise definition for the case of AODEs. Let Σ be a prime differential ideal in $\mathbb{K}(x)\{u\}$. Then we call η a *generic zero* of Σ iff for any differential polynomial $P \in \mathbb{K}(x)\{u\}$ we have $P(\eta) = 0 \iff P \in \Sigma$. Such an η exists in a suitable extension field since Σ is prime.

Let F be an irreducible differential polynomial of order n. Then $\{F\}$, the radical differential ideal generated by F, can be decomposed essential prime differential ideals $\Sigma_1, \ldots, \Sigma_k$ (c. f. [55, Chapter II]).

$$\{F\} = \Sigma_1 \cap \ldots \cap \Sigma_k\,.$$

A prime divisor of an ideal is called essential if it does not contain any other prime divisor. There is one component where the *separant* $S := \frac{\partial F}{\partial u^{(n)}}$ does not vanish. It is a prime differential ideal

$$\Sigma_1 = \{F\} : \langle S\rangle = \{P \in \mathbb{K}(x)\{u\} \mid SA \in \{F\}\}\,.$$

This Σ_1 represents the general component.

The other part $\{F, S\}$ represents the *singular component*. It can be further decomposed in $\Sigma_2 \cap \ldots \cap \Sigma_k$.

A zero of $\{F\}$ is called a solution of F. A solution is called *non-singular* if it does not annul the separant. Otherwise it is called *singular*. The non-singular zeros are all contained in the general component.

A generic zero of Σ_1 is called a *general solution* of $F = 0$. A general solution depends on some transcendental constant. Every non-singular solutions can be expressed by a certain evaluation of the constants in the general solution. No choice of evaluating the constant yields a singular solution.

Example 1.1.
Let us consider the AODE, $F(u, u') = u'^2 + u' - 2u - x = 0$. *Then* $\frac{1}{2}(c + (c + x)^2)$ *is a general solution. The separant of* F *is* $S = 2u' + 1$ *which has solutions* $-\frac{x}{2} + k$ *for some constant* k. *We choose* $k = -\frac{1}{8}$ *such that the solution of the separant is also a solution of* F. *Then* $-\frac{x}{2} - \frac{1}{8}$ *is a singular solution. It is easy to see that no choice of* c *in the general solution would yield the singular one.*

We say a general solution is *rational* if it is of the form $u = \frac{a_k x^k + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$, where the a_i and b_i are constants in some field extension of $\mathbb{K}$. An *algebraic general solution* is a general solution $v(x)$ which satisfies an algebraic equation $g(x, u) = 0$ (c. f. [5]). In Section 2.2 we define the subclass of radical solutions.

1.2.2. Solutions of APDEs

Similarly to the case of AODEs we want to describe solutions of partial differential equations in a preferably general way. Again the ideal $\{F\}$, the radical differential ideal generated by F, can be decomposed into prime differential ideals $\Sigma_1, \ldots, \Sigma_k$ (c.f. [55, Chapter IX]),

$$\{F\} = \Sigma_1 \cap \ldots \cap \Sigma_k \,.$$

Furthermore, we can assume that all these prime differential ideals are essential. In partial differential algebra the separant is defined with respect to an ordering of the derivatives (for further details see Appendix A). Let v be the leader of F according to this ordering. Then the separant of F is defined as $\frac{\partial F}{\partial v}$. Let Σ_1 be the part where the separant does not vanish (such a component exists). In fact, Σ_1 does not contain any separant, whereas the other Σ_k contain all separants. Hence, Σ_1 is the general component, and the other Σ_k form the singular components. A zero of some Σ_k is called a solution. A solution is called *non-singular* if it does not annul any of the separants. Otherwise it is called *singular*. Then a general solution of F is the *manifold* of Σ_1, i.e. the set of all zeros of Σ_1. Since Σ_1 is a prime differential ideal, it has a generic zero. Sometimes this generic zero is also called a general solution.

A solution u of an APDE is called rational if it is of the form $u = \frac{a(x_1,\ldots,x_n)}{b(x_1,\ldots,x_n)}$ where a and $b \neq 0$ are polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. For APDEs, in difference to AODEs, there is also an intermediate level of generally describing solutions. Such solutions are called complete. This and other important properties of solutions of APDEs are described later in Section 1.4.

1.3. Algebraic Geometry

An *algebraic hypersurface* $\mathcal{S}$ is an algebraic variety of codimension 1, i.e. a zero set of a squarefree non-constant polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$,

$$\mathcal{S} = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0\} \,,$$

where $\mathbb{A}^n$ is the n-dimensional affine space over $\mathbb{K}$. In case $n = 2$ we call $\mathcal{S}$ an *algebraic curve*. In case $n = 3$ we call $\mathcal{S}$ an *algebraic surface*. The polynomial f is identified as the *defining polynomial* of $\mathcal{S}$. An important aspect of algebraic hypersurfaces is their rational parametrizability. We consider an algebraic hypersurface defined by an irreducible polynomial f. We write $\bar{s} = (s_1, \ldots, s_{n-1})$. A tuple of rational functions $\mathcal{P}(s_1, \ldots, s_{n-1}) = (p_1(\bar{s}), \ldots, p_n(\bar{s}))$ is called a *rational parametrization* of the hypersurface if $f(p_1(\bar{s}), \ldots, p_n(\bar{s})) = 0$ for all $\bar{s}$ and the Jacobian of $\mathcal{P}$ has generic rank $n-1$. We observe that the condition on the Jacobian is fundamental since, otherwise, we are parametrizing a lower dimensional subvariety on the hypersurface. A parametrization can be considered as a dominant map $\mathcal{P}(\bar{s}) : \mathbb{A}^{n-1} \to \mathcal{S}$. By abuse of notation we also call this map a parametrization. We call a parametrization $\mathcal{P}(\bar{s})$ *proper* iff it is a birational map or in other words if for almost every point $a = (a_1, \ldots, a_n)$ on the hypersurface we find exactly one tuple $(s_1, \ldots, s_{n-1})$ such that $\mathcal{P}(\bar{s}) = a$ or equivalently if $\mathbb{K}(\mathcal{P}(\bar{s})) = \mathbb{K}(\bar{s})$.

Remark 1.2.
The Jacobian of a proper parametrization $\mathcal{P}(s_1, \ldots, s_{n-1})$ of a hypersurface has generic rank $n-1$, where n is the dimension of the hypersurface. Since $\mathcal{P}$ is proper we know that $\mathbb{K}(s_1, \ldots, s_{n-1}) = \mathbb{K}(\mathcal{P}(\bar{s}))$. Hence, there is a rational function $R(a_1, \ldots, a_n) = (R_1(\bar{a}), \ldots, R_n(\bar{a})) \in \mathbb{K}(\bar{a})^n$ such that $R(\mathcal{P}(\bar{s})) = (s_1, \ldots, s_{n-1})$. Thus, $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_{R \circ \mathcal{P}} = \mathcal{J}_R(\mathcal{P}) \cdot \mathcal{J}_{\mathcal{P}}$. Taking into account, that the rank of a product of two matrices is less or equal the minimal rank of the two matrices, we get that $\mathrm{rank}(\mathcal{J}_{\mathcal{P}}) = n-1$.

A hypersurface which has a rational parametrization is called *unirational*. If it also has a proper rational parametrization it is called *rational*. Theorems of Lüroth and Castelnuovo, respectively, prove that all unirational curves and surfaces are rational. For higher dimensions there exist hypersurfaces which are unirational but not rational. See for instance [63] and [7] for further information.

There exist algorithms for curves and surfaces to decide rationality and to compute proper rational parametrizations (c. f. [63] and [56] respectively). However, for higher n so far no general algorithm for finding rational parametrizations is known. Nevertheless, there exist methods for special kinds of hypersurfaces. Consider for instance a hypersurface where one variable appears linearly in the defining polynomial. For further information on parametrizations and treatment of special cases we refer to Appendix B.1.

The following well-known property of proper parametrization is a main motivation of the idea for solving differential equations.

Lemma 1.3.
Let $\mathcal{P}(\bar{s})$, $\mathcal{Q}(\bar{s})$ be two proper parametrizations of some algebraic hypersurface $\mathcal{S} \subseteq \mathbb{A}^n$. Then there exists a rational function $R(\bar{s}) \in \mathbb{K}(\bar{s})$ such that $\mathcal{Q}(\bar{s}) = \mathcal{P}(R(\bar{s}))$.

- *In case $n = 2$, $R(\bar{s})$ is a* Möbius *transformation, i. e. a linear rational function $R(s_1) = \frac{a_0+a_1s_1}{b_0+b_1s_1}$ with $a_0b_1 - a_1b_0 \neq 0$.*
- *In case $n = 3$, $R(\bar{s})$ is a* Cremona transformation *(i. e. a birational map of the plane to itself), and hence by the theorem of Castelnuovo-Noether a finite composition of quadratic transformations $(s_1, s_2) \mapsto (\frac{a_0+a_1s_1+a_2s_2}{b_0+b_1s_1+b_2s_2}, \frac{c_0+c_1s_1+c_2s_2}{d_0+d_1s_1+d_2s_2})$ and projective linear transformations (c. f. [8, 64]).*
- *In general $R(\bar{s}) = \mathcal{P}(\mathcal{Q}^{-1}(\bar{s}))$.*

1.4. Differential Equations and Algebraic Hypersurfaces

Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE. We consider the *corresponding algebraic hypersurface* developed by replacing the derivatives by independent transcendental variables, $F(z, p_1, \ldots, p_n) = 0$. Whenever we talk about the differential equation and its solutions we use the variables $x_1, \ldots, x_n$. To distinguish the parametrization problem we use the variables $s_1, \ldots, s_n$ there.

Given any non-constant differentiable function $u(x_1, \ldots, x_n)$ which satisfies the APDE, $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$, the tuple $(u(s_1, \ldots, s_n), u_{x_1}(s_1, \ldots, s_n), \ldots, u_{x_n}(s_1, \ldots, s_n))$ is a parametrization. We call this parametrization the *corresponding parametrization of the solution* and denote it usually by $\mathcal{L}$. We observe that the corresponding parametrization of a solution is not necessarily a parametrization of the associated hypersurface, since the condition on the rank of the Jacobian may fail. For instance, let us consider the APDE, $u_x = 0$, with $n = 2$. A solution would be of the form $u(x, y) = g(y)$, with g differentiable. However, this solution generates $(g(s_2), 0, g'(s_2))$ that is a curve in the surface; namely the plane $p = 0$. Now, consider the APDE, $u_x = \lambda$, with λ a nonzero constant. Hence, the solutions are of the form $u(x, y) = \lambda x + g(y)$. Then, $u(x, y) = \lambda x + y$ generates the line $(\lambda s_1 + s_2, \lambda, 1)$ while $u(x, y) = \lambda x + y^2$ generates the parametrization $(\lambda s_1 + s_2^2, \lambda, 2s_2)$ of the associated plane $p = \lambda$. These examples motivate the following definition. Clearly, a solution of an APDE is a function $u(x_1, \ldots, x_n)$ such that $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$.

Definition 1.4.
A solution of an APDE is rational *iff* $u(x_1, \dots, x_n)$ *is a rational function over* $\mathbb{K}$.
A rational solution of an APDE is proper *iff the corresponding parametrization is proper.*

In the case of autonomous ordinary differential equations, every non-constant solution induces a proper parametrization of the associated curve (see [17]). However, this is not true in general for autonomous APDEs. For instance, the solution $x + y^3$ of $u_x = 1$, induces the parametrization $(s_1 + s_2^3, 1, 3s_2^2)$ which is, although its Jacobian has rank 2, not proper.

In addition, we observe that it can happen that none of the rational solutions of an APDE is proper. This is the case for instance, of $u_x = 0$, since all rational solutions are of the form $u = R(y)$, for some rational function R and $\mathbb{K}(R(s_1), 0, R'(s_1)) \subsetneq \mathbb{K}(s_1, s_2)$. Furthermore, we see that none of the solutions of this APDE generates a parametrization of the associated hypersurface, since the Jacobian has rank 1.

Every solution of the problem under consideration in this work can be attained by the knowledge of a set of complete solutions. We will see details later. For this reason, we focus on finding families of complete solutions. This notion of a complete solution is due to Lagrange (compare [14]). He calls a solution $u(x, y)$ of a first-order PDE complete, if it depends on two arbitrary constants, i. e. $u(x, y) = u(x, y, c_1, c_2)$, such that the elimination of the constants in the equations $z - u, p - u_x, q - u_y$ gives back the differential equation. Such a property for rational functions can be proven by Gröbner bases.

For the following we use a definition of completeness which is easier to check. This definition can be found for instance in [31, 50].

Definition 1.5.
Let $F(u, u_{x_1}, \dots, u_{x_n}) = 0$ *be an autonomous APDE. Let* u *be a rational solution depending on* n *arbitrary constants* $c_1, \dots, c_n$. *Let* $\mathcal{L} = (v_0, v_1, \dots, v_n)$ *be the parametrization induced by the solution, i. e.* $v_0 = u$ *and* $v_i = u_{x_i}$ *for* $i \geq 1$. *We call the solution* complete *if the Jacobian* $\mathcal{J}_{\mathcal{L}}^{c_1,\dots,c_n}$ *of* $\mathcal{L}$ *with respect to* $c_1, \dots, c_n$ *has generic rank* n.
We call the solution complete of suitable dimension *if it is complete and the Jacobian* $\mathcal{J}_{\mathcal{L}}^{s_1,\dots,s_n}$ *of* $\mathcal{L}$ *with respect to* $s_1, \dots, s_n$ *has generic rank* n.

Intuitively speaking, the notion of a complete solution is requiring that the corresponding parametrization of the solution parametrizes an algebraic set on the hypersurface, independently of the constants $c_1, \dots, c_n$. On the other hand, the notion of suitable dimension ensures that the corresponding parametrization really parametrizes the associated hypersurface and not a lower dimensional subvariety.

The following example illustrates proper, complete and non-complete solutions for some simple APDEs.

Example 1.6.
We consider the APDE, $F(u, u_x, u_y) = u_x = 0$, with the solution $u(x, y) = y + c_1 + c_2$. The corresponding parametrization is $\mathcal{L} = (s_2 + c_1 + c_2, 0, 1)$. Then

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence $u(x, y)$ is not complete. However, if we take $u(x, y) = c_1 y + c_2$, the Jacobian with respect to c_1, c_2 has generic rank 2, and u is complete but not of suitable dimension, since the Jacobian of $\mathcal{L}$ with respect to s_1, s_2 has rank 1.

Now, we take the APDE, $u_x = 1$. In Table 1.1 we see solutions and their properties. Note, that the solution $x+c_1+y^2+c_2$ is not complete and hence, not complete of suitable dimension. However, the other requirement of suitable dimension is fulfilled.

solution	complete	suitable dim	proper	rank($\mathcal{J}_{\mathcal{L}}^{s_1,s_2}$)
$x + c_1$	F	F	F	1
$x + y + c_1 + c_2$	F	F	F	1
$x + c_1 + c_2 y$	T	F	F	1
$x + c_1 + y^2 + c_2$	F	F*	T	2
$x + c_1 + c_2 y^2$	T	T	T	2
$x + c_1 + (y + c_2)^2$	T	T	T	2
$x + c_1 + (y + c_2)^3$	T	T	F	2
$x + c_1 + y^3 + c_2$	F	F*	F	2

Table 1.1.: Properties of some solutions of $u_x = 1$, where T means true, F false and F* false since not complete but condition on Jacobian is true

Note, that in this example all possible combinations of properties are found. Indeed, by Remark 1.2 the Jacobian of a proper solution has generic rank n and hence there is no rational solution which is proper and complete but not of suitable dimension.

In the case of AODEs, i. e. $n = 1$, the notion of complete and general solution is equivalent. This is not the case for APDEs. Nevertheless, as shown by Lagrange (c. f. [14]) from a complete solution one can compute singular and general solutions by envelopes. An envelope of a one-parameter family of surfaces, given implicitly by $g(x, y, z, a) = 0$ for parameters a, is the surface which touches each point on any of the surfaces in the family. It is defined by the solution of the system

$$g(x, y, z, a) = 0\,, \qquad\qquad g_a(x, y, z, a) = 0\,.$$

An envelope of a two-parameter family of surfaces, defined by $g(x, y, z, a_1, a_2) = 0$ for parameters a_1 and a_2, is the surface which touches each point on any of the surfaces in the family. It is defined by the solution of the system

$$g(x, y, z, a_1, a_2) = 0\,, \qquad g_{a_1}(x, y, z, a_1, a_2) = 0\,, \qquad g_{a_2}(x, y, z, a_1, a_2) = 0\,.$$

Let us consider a complete rational solution $u(x, y, c_1, c_2)$ of some APDE. Hence, the equation of the family of surfaces is $z - u(x, y, c_1, c_2) = 0$.

Assume now $c_2 = \varphi(c_1)$ for some function φ. Then, we consider the envelope of the family of surfaces $u(x, y, c_1, \varphi(c_1))$, i. e. we solve $0 = \frac{\partial u(x,y,c_1,\varphi(c_1))}{\partial c_1} = u_{c_1} + u_{c_2}\varphi'(c_1)$ for c_1. Let $c_1 = \psi(x, y)$ be the solution. Then $u(x, y, \psi(x, y), \varphi(\psi(x, y)))$ is a general solution of the APDE, where φ is an arbitrary function.

On the other hand we might compute the envelope with respect to c_1 and c_2, i. e. solve the equations $\frac{\partial u}{\partial c_1} = \frac{\partial u}{\partial c_2} = 0$ for c_1 and c_2. Let $c_1 = \varphi(x, y)$ and $c_2 = \psi(x, y)$. Then $u(x, y, \varphi(x, y), \psi(x, y))$ is a singular solution of the APDE.

These computations can be done as well for implicitly given solutions (as actually shown in [14]) and for APDEs in more variables.

General idea for solving AODEs and APDEs

We now want to present the general idea of the main procedure presented in this thesis. Here only the introductory part, which does not depend on a specific number of variables, is described. Details for AODEs can be found in Section 2.2. The procedure for APDEs is presented in Section 3.1 and 3.3 for 2 and n variables respectively.

Let $F(u, u_{x_1}, \dots, u_{x_n}) = 0$ be an algebraic (partial) differential equation, where F is an irreducible non-constant polynomial. We consider the hypersurface $F(z, p_1, \dots, p_n) = 0$ and assume it admits a proper (rational) hypersurface parametrization

$$\mathcal{Q}(s_1, \dots, s_n) = (q_0(s_1, \dots, s_n), q_1(s_1, \dots, s_n), \dots, q_n(s_1, \dots, s_n))\,.$$

A summary of results for the problem of hypersurface parametrization is given in Appendix B.1.

Assume that $\mathcal{L}(s_1, \dots, s_n) = (v_0, \dots, v_n)$ corresponds to a solution of the APDE. Furthermore we assume that the parametrization $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}(s_1, \dots, s_n) = \mathcal{L}(g(s_1, \dots, s_n)) \tag{1.1}$$

for some invertible function $g(s_1, \dots, s_n) = (g_1(s_1, \dots, s_n), \dots, g_n(s_1, \dots, s_n))$. This assumption is motivated by the fact that in case of rational algebraic curves every non-constant rational solution of an AODE yields a proper rational parametrization of the associated algebraic curve (c. f. [17]) and each proper rational parametrization can be obtained from any other proper one by a rational transformation (c. f. Lemma 1.3). In the case of APDEs, however, not all rational solutions provide a proper parametrization, as mentioned in the remark after Definition 1.4. What we still know is that any proper rational hypersurface parametrization can be obtained from any other proper one by a rational transformation. Assuming that g exists, it would be enough to find g^{-1} in order to get the solution $q_0(g^{-1}(x_1, \dots, x_n))$. Looking at the Jacobian of the parametrizations, equation (1.1) implies the following equation:

$$\mathcal{J}_{\mathcal{Q}}(s_1, \dots, s_n) = \mathcal{J}_{\mathcal{L}}(g(s_1, \dots, s_n)) \cdot \mathcal{J}_g(s_1, \dots, s_n)\,.$$

In particular we consider the first row,

$$\left.\begin{aligned} \frac{\partial q_0}{\partial s_1} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g)\frac{\partial g_i}{\partial s_1} = \sum_{i=1}^{n} q_i(s_1, \dots, s_n)\frac{\partial g_i}{\partial s_1}\,, \\ &\vdots \\ \frac{\partial q_0}{\partial s_n} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g)\frac{\partial g_i}{\partial s_n} = \sum_{i=1}^{n} q_i(s_1, \dots, s_n)\frac{\partial g_i}{\partial s_n}\,. \end{aligned}\right\} \tag{1.2}$$

This is a system of quasilinear equations in the unknown functions g_1 to g_n. In case of $n = 1$ the system reduces to a single ordinary differential equation. Details on that case are presented in Chapter 2. Further investigation of the APDE case can be found in Chapter 3.

2. Solution Method for AODEs

As mentioned in the introduction there exist several algorithms for solving first-order AODEs of some special kind. Especially linear AODEs are well-studied objects. In this chapter we briefly describe some existing algorithms for finding explicit rational solutions of (non-linear) first-order AODEs (Section 2.1).

We mainly focus on an algorithm by Feng and Gao for autonomous first-order AODEs (Section 2.1.1) which can be considered to be the basis of subsequent algebraic-geometric methods; among others the generalization of Ngô and Winkler for non-autonomous first-order AODEs (Section 2.1.2). Likewise based on these algorithms we present an extended method for finding radical solutions of first-order autonomous AODEs (Section 2.2). This new procedure generalizes existing algorithms and follows the framework presented in Section 1.4.

Using the general idea of finding a suitable transformation of a given parametrization, for $n = 1$ the system (1.2) reduces to a single ODE which is easily solvable by existing algorithms. Deduced from special properties of the components of a given proper rational or radical parametrization, several classes of AODEs with radical solutions are presented. Later we illustrate that this new method is not restricted to the computation of algebraic solutions (Section 2.2.2).

In Section 2.3 we present ideas for a generalization to higher-order AODEs. We start with an approach for special second-order AODEs and show difficulties arising in respect of further generalization. Later we use ideas from partial differential equations for solving general second-order and higher-order AODEs.

Examples are provided throughout the sections whenever suitable. For a more extensive list of (well-known) examples from literature which can be solved by the method we refer to Appendix C.1.

2.1. Rational Solutions

Recently two algorithms for solving first-order AODEs have been presented. Both of them intrinsically use rational parametrizations of curves respectively surfaces. In Section 2.1.1 we recall an algorithm for finding rational solutions of autonomous first-order AODEs. It is based on the computation of a proper rational parametrization and the fact that a solution of the AODE yields such a parametrization.

Several improvements and generalizations of this algorithm have been published. Some of these are presented in the following and references are given for others. First we recall an algorithm for finding rational solutions of non-autonomous AODEs (Section 2.1.2). Later, in Section 2.2 we present a new generalized method for finding non-rational solutions of autonomous AODEs.

2.1.1. Autonomous First-Order AODEs

In this section we briefly describe the algorithm of Feng and Gao [17] for finding rational solutions of autonomous first-order AODEs. As a key fact they show that any rational solution v of an autonomous AODE corresponds to a proper rational parametrization $\mathcal{L} = (v, v')$. Furthermore, it is known, that from any proper rational parametrization $\mathcal{Q}$ any other proper parametrization $\mathcal{P}$ can be obtained by a transformation with a linear rational function, i. e. $\mathcal{P}(s) = \mathcal{Q}(\frac{a_0+a_1 s}{b_0+b_1 s})$, (see Lemma 1.3).

Theorem 2.1. (Feng and Gao [17])
Let $F(u, u') = 0$ be an autonomous first-order AODE. It has a rational general solution if and only if there is a proper parametrization $\mathcal{Q}(s) = (q_0(s), q_1(s))$ over $\mathbb{Q}$ of the corresponding curve $F(z, p) = 0$ and for any such parametrization the indicator $A := \frac{q_1(s)}{q_0'(s)}$ is either in $\mathbb{Q}$ or equal to $a(s-b)^2$, where $a, b \in \mathbb{Q}$.

For both cases in Theorem 2.1 a constructive algorithm for computing an explicit rational general solution can be given. The general idea of the algorithm for deciding rational solvability and, in the affirmative case, for computing a solution of a given autonomous first-order AODE is the following.

Algorithm 1. (Feng and Gao [17])
<u>Input</u>: An autonomous AODE, $F(u, u') = 0$, where F is irreducible and non-constant.
<u>Output</u>: A rational general solution or a statement that it does not exist.

1. Compute a proper rational parametrization $\mathcal{Q}(s) = (q_0, q_1)$ of the corresponding curve $F(z,p) = 0$. Let $\mathbb{K}$ be the ground field of $\mathcal{Q}$.
   If such a parametrization does not exist there is no rational solution. Otherwise continue.

2. Compute $A = \frac{q_1}{q_0'}$.
   - If $A \in \mathbb{K}$ return $q_0(A(x-c))$.
   - If $A = a(s-b)^2$ with $a, b \in \mathbb{K}$ return $q_0(\frac{ab(x+c)-1}{a(x+c)})$.
   - Otherwise there is no rational solution.

The original algorithm in [17] is enhanced by the incorporation of an a priori test of degree bounds. Using Laurent series solutions and Padé approximations this algorithm was further improved to a polynomial time algorithm [19]. An algorithm for the special case of polynomial solution can be found in [18] and a generalization of the algorithm to algebraic solutions is described in [5]. We do not go into detail here. Instead we look at an example for Algorithm 1.

Example 2.2.
Let us consider the simple AODE, $F(u, u') = u^3 + u'^2 = 0$. The corresponding curve has a rational parametrization $\mathcal{Q}(s) = (s^2, s^3)$. Then, $A = \frac{1}{2}s^2$. Hence, $\left(\frac{-1}{\frac{1}{2}(x+c)}\right)^2$ is a rational general solution.

2.1.2. Non-autonomous First-Order AODEs

In this section we briefly describe the algorithm of Ngô and Winkler [44, 45, 46] for solving first-order AODEs which are not necessarily autonomous. An AODE, $F(x, u, u') = 0$, can be viewed as an algebraic surface, $F(x, z, p) = 0$, and the algorithm is based on a given proper rational parametrization of this corresponding surface. A solution of a non-autonomous AODE represents a curve on the surface. The aim is to find this specific curve. The key idea is to construct an associated system of ODEs which depends on the input parametrization. Given a parametrization $\mathcal{Q}(s,t) = (q_0, q_1, q_2)$, the associated system is defined as

$$s_1' = \frac{\frac{\partial q_1}{\partial s_2} - q_2 \frac{\partial q_0}{\partial s_2}}{\det(\mathcal{J}_{(q_0,q_1)})}, \qquad s_2' = \frac{\frac{\partial q_0}{\partial s_1} q_2 - \frac{\partial q_1}{\partial s_1}}{\det(\mathcal{J}_{(q_0,q_1)})}. \tag{2.1}$$

It is proven that there is a one to one correspondence between rational general solutions of the AODE and rational general solutions of the associated system. Furthermore, the associated system is of order 1 and degree 1 in the derivative. Hence, existing algorithms can be used to solve the system; for instance by computing invariant algebraic curves and their parametrizations $\mathcal{P}(t) = (p_0(t), p_1(t))$. The system has a rational solution if and only if one of the ODEs

$$T' = \frac{1}{p_0'(T)} s_1'(p_0(T), p_1(T)), \qquad T' = \frac{1}{p_1'(T)} s_1'(p_0(T), p_1(T)), \tag{2.2}$$

where s_i' is as in (2.1), has a rational solution (see [45, Theorem 2.2]).

The idea of the algorithm in the generic case is described in Algorithm 2. A proper rational parametrization of the corresponding surface has to be given. For details we refer to [45].

Algorithm 2. (Ngô and Winkler [45])
Input: A first-order AODE, $F(x, u, u') = 0$, where F is an irreducible non-constant polynomial, and a proper rational parametrization $\mathcal{Q}(s_1, s_2) = (q_0, q_1, q_2)$ of the corresponding surface.
Output: A rational general solution of the AODE, if it exists.

1. Create the associated system as in (2.1).

2. Compute the irreducible invariant algebraic curves of the system, if possible.

3. Chose a general rational invariant curve and compute a parametrization $\mathcal{P}(t) = (p_0, p_1)$, if possible.

4. Solve the ODEs in (2.2) if possible. Then $\mathcal{P}(T(x))$ is a solution of the associated system.

5. Compute $c = q_0(\mathcal{P}(T(x))) - x$.

6. Return $q_1(\mathcal{P}(x - c))$.

Example 2.3.
Consider the non-autonomous first-order AODE, from Example 1.1,

$$F(x, u, u') = u'^2 + 3u' - 2u - x = 0\,.$$

We briefly comment on the intermediate steps of Algorithm 2. The solution surface $p^2 + 3p - 2z - 3x = 0$ *has the proper rational parametrization*

$$\mathcal{Q}(s_1, s_2) = \left(s_1, \frac{s_2^2 - s_1 + s_2}{2}, s_2 \right) .$$

The associated system is

$$s' = 1 \,, \qquad\qquad t' = 1 \,.$$

There is a 1-1 correspondence between the rational solutions of the original AODE and the rational solutions of the associated system.

Now we consider the irreducible invariant algebraic curves of the associated system:

$$G(s_1, s_2) = s_1 - s_2 + c_0 \,, \qquad G(s_1, s_2) = s_1^2 - 2s_1 s_2 + s_1 c_1 + s_2^2 - s_2 c_1 + c_0 \,.$$

These invariant algebraic curves are candidates for generating rational solutions of the associated system. The first curve can be parametrized easily by $\mathcal{P}(t) = (t, t + c_0)$*. We compute a solution of the ODE,* $T' = 1$*, i. e.* $T(x) = x$*. Then we solve* $c = q_0(\mathcal{P}(T(x))) - x = x - x = 0$*. Finally,*

$$q_1(x - c, x - c + c_0) = \frac{1}{2}((x + c_0)^2 - x + (x + c_0)) = \frac{1}{2}(c_0 + (x + c_0)^2)$$

is exactly the general solution mentioned in Example 1.1.

2.2. Non-rational Solutions

In this section we present a method for finding radical solutions of first-order autonomous AODEs based on the content of the author's papers [21, 23]. However, additional information and improvements are incorporated.

Let $F(u, u') = 0$ be an autonomous AODE. For readability we ignore the index 1 in x_1, s_1, p_1, g_1 and h_1 and write x, s, p, g and h respectively instead. We consider the corresponding algebraic curve $F(z, p) = 0$. As we know, for a non-trivial, i. e. non-constant solution u of the AODE, $\mathcal{L}(s) = (u(s), u'(s))$ is a parametrization of F (not necessarily rational or radical). Assume we are given an arbitrary parametrization

$\mathcal{Q}(s) = (q_0(s), q_1(s))$. Following the general idea of Chapter 1 for $n = 1$ the system (1.2) reduces to a single ordinary differential equation

$$\frac{\mathrm{d}q_0}{\mathrm{d}s} = q_1(s)\frac{\mathrm{d}g}{\mathrm{d}s}\,. \tag{2.3}$$

We define $A = A_{\mathcal{Q}} = \frac{q_1(s)}{q_0'(s)}$ as in [17] (see Section 2.1.1). Then A acts as an indicator for information on the solvability in a certain class of functions. In case $A = 1$ a solution is already found. In case $q_0 \in \mathbb{K}(s)$ and $A \in \mathbb{K}$ or $A = a(b+t)^2$ there exists a rational solution (see in Section 2.1.1). Further investigation on properties of A are done later. For now we reason from (2.3) that

$$g'(s) = \frac{1}{A_{\mathcal{Q}}(s)}\,.$$

Using symbolic integration and algebraic computation of h, such that $g(h(x)) = x$ for all x, the procedure continues as follows.

$$\begin{aligned} g(s) &= \int g'(s)\,\mathrm{d}s = \int \frac{1}{A_{\mathcal{Q}}(s)}\,\mathrm{d}s\,, \\ u(x) &= q_0(h(x))\,. \end{aligned}$$

Kamke [32] already mentions such a procedure where he restricts to continuously differentiable functions q_0 and q_1 which satisfy $F(q_0(s), q_1(s)) = 0$. However, he does not mention where to get these functions from.

In general g is not a bijective function. Hence, when we talk about an inverse function we actually mean one branch of a multivalued inverse. Each branch inverse gives us a solution to the differential equation.

We might add any constant c to the solution of the indefinite integral. Assume $g(s)$ is a solution of the integral and h its inverse. Then also $\bar{g}(s) = g(s) + c$ is a solution and $\bar{h}(t) = h(s - c)$. We know that if $u(x)$ is a solution of the AODE, so is $u(x + c)$. Hence, we may postpone the introduction of c to the end of the procedure.

We summarize the procedure for the case of radical parametrizations and solutions (see Section 2.2.1).

Procedure 3.
<u>*Input*</u>: An autonomous AODE defined by $F(u, u') = 0$, where F is an irreducible non-constant polynomial, and a radical parametrization $\mathcal{Q}(s) = (q_0(s), q_1(s))$ of the corresponding curve.
<u>*Output*</u>: A general solution u of F or “fail”.

1. Compute $A_{\mathcal{Q}}(s) = \frac{q_1(s)}{q_0'(s)}$.
2. Compute $g(s) = \int \frac{1}{A_{\mathcal{Q}}(s)}\, \mathrm{d}s$.
3. Compute h such that $g(h(s)) = s$.
4. If h is not radical return “fail”, else return $q_0(h(x + c))$.

The procedure finds a solution if we can compute the integral and the inverse function. On the other hand it does not give us any clue on the existence of a solution in case either part does not work. Neither do we know whether we found all solutions.

Step 2 of Procedure 3 is an instance of the problem of integration in finite terms. Liouville's investigation of the problem ([39, 40, 41], c. f. also [9, 54]) led to his well-known theorem which proves that an elementary function integral of an algebraic function is (if it exists) of the form $\int u\, \mathrm{d}x = w_0(x) + \sum_{i=1}^{m} c_i \log(w_i(x))$, where w_i are algebraic functions and c_i are constants. Elementary functions are those obtained by finite compositions of algebraic functions, logarithms and exponential functions. Hence, an integral of an algebraic function is not necessarily algebraic again. Furthermore, an elementary integral of an algebraic function might not exist. In contrast, the integral of a rational function can always be expressed by elementary functions. Based on a procedure of Risch [51, 52] the theoretical theorem of Liouville led to algorithms deciding the existence of an elementary integral and computation thereof in the affirmative case. The case which is of interest in our procedure was solved by Trager [67, 68]. Improvements of this algorithm have been investigated and several generalizations of Liouville's theorem and algorithms resulting thereof have been found. Be refrain from listing them but refer to Bronstein [9] for details and a general treatment of symbolic integration.

For computing h in Step 3 there is no general algorithm for deciding whether h is rational, radical or algebraic. However, Ritt [53] investigated the special case when g is a polynomial. For details see Theorem 2.12 in Section 2.2.1. In any case it would be possible to either return an implicit solution or to solve the algebraic equation in Step 3 by approximation with truncated Puiseux series. Nevertheless, we restrict to finding exact solutions here.

So far Procedure 3 fails if it does not find a radical solution. In Section 2.2.2 we see how this case can be further investigated. Now, we focus on the case when the procedure computes a result and show that it is indeed a general solution.

Lemma 2.4. (Aroca, Cano, Feng, Gao [5, 17])
Let $F(u, u')$ be a first-order autonomous AODE and let $u(x)$ be an algebraic solution of $F = 0$. Then $u(x + c)$, with an arbitrary constant c, is an algebraic general solution. If $u(x)$ is rational, then $u(x + c)$ is a rational general solution.

Corollary 2.5.
Let $F(u, u')$ be a first-order autonomous AODE and let $u(x) = q_0(h(x + c))$ be an algebraic function computed by Procedure 3. Then u is an algebraic general solution of $F = 0$.

Proof. Let $u(x) = q_0(h(x + c))$. We need to show that $u'(x) = q_1(h(x + c))$:

$$\begin{aligned} u'(x) &= h'(x+c)q_0'(h(x+c)) = h'(x+c)\frac{q_1(h(x+c))}{A_{\mathcal{Q}}(h(x+c))} \\ &= h'(x+c)g'(h(x+c))q_1(h(x+c)) = \frac{\mathrm{d}g(h(x+c))}{\mathrm{d}x}q_1(h(x+c)) \\ &= q_1(h(x+c)) \end{aligned}$$

Hence, u is a solution of $F = 0$. Lemma 2.4 implies that it is a general algebraic solution. □

We show now that Algorithm 1 accords with Procedure 3. Assume we are given an AODE with a proper parametrization $\mathcal{Q}(s) = (q_0(s), q_1(s))$. Assume further that $A_{\mathcal{Q}}(s) = a \in \mathbb{K}$ or $A_{\mathcal{Q}}(s) = a(s - b)^2$. Then we get from Procedure 3

$$\begin{aligned} A_{\mathcal{Q}}(s) &= a\,, & A_{\mathcal{Q}}(s) &= a(s-b)^2\,, \\ g'(s) &= \frac{1}{a}\,, & g'(s) &= \frac{1}{a(s-b)^2}\,, \\ g(s) &= \frac{s}{a} + c\,, & g(t) &= -\frac{1}{a(s-b)} + c\,, \\ h(s) &= a(s-c)\,, & h(s) &= \frac{-1 + ab(s-c)}{a(s-c)}\,. \end{aligned}$$

We see that $q_0(h(s))$ is exactly what Feng and Gao found aside from the sign of c. Feng and Gao [17] already proved that there is a rational general solution if and only if A is of the special form mentioned above and all rational general solutions can be found by the algorithm.

However, as mentioned above, in case the indicator A is not of such a special type, Procedure 3 does not answer the question whether the AODE has a rational solution. It might, however, find non rational solutions for some AODEs.

2.2.1. Radical Solutions

The research area of radical parametrizations is rather new. Sendra and Sevilla [60] recently published a paper on parametrizations of curves using radical expressions. In this paper Sendra and Sevilla define the notion of radical parametrization and they provide algorithms to find such parametrizations in certain cases which include but are not restricted to curves of genus less or equal 4. Every rational parametrization is a radical one but obviously not the other way round. Further considerations of radical parametrizations can be found in Schicho and Sevilla [59], Harrison [24] and Schicho, Schreyer and Weimann [58]. First approaches for the computation of radical parametrization of surfaces can be found in [61]. Nevertheless, for the beginning we restrict to the case of first-order autonomous equations and hence to algebraic curves. We summarize important definitions here and refer to Appendix B.2 for further details on radical parametrizations.

Definition 2.6.
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A field extension $\mathbb{K} \subseteq \mathbb{L}$ is called a radical field extension *iff $\mathbb{L}$ is the splitting field of a polynomial of the form $x^k - a \in \mathbb{K}[x]$, where k is a positive integer and $a \neq 0$. A* tower of radical field extensions *of $\mathbb{K}$ is a finite sequence of fields*

$$\mathbb{K} = \mathbb{K}_0 \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \ldots \subseteq \mathbb{K}_m$$

such that for all $i \in \{1, \ldots, m\}$, the extension $\mathbb{K}_{i-1} \subseteq \mathbb{K}_i$ is radical.
A field $\mathbb{E}$ is a radical extension field *of $\mathbb{K}$ iff there is a tower of radical field extensions of $\mathbb{K}$ with $\mathbb{E}$ as its last element.*
A polynomial $h(x) \in \mathbb{K}[x]$ is solvable by radicals *over $\mathbb{K}$ iff there is a radical extension field of $\mathbb{K}$ containing the splitting field of h.*

Let now $\mathcal{C}$ be an affine plane curve over $\mathbb{K}$ defined by an irreducible polynomial $f(x, y)$. According to [60], $\mathcal{C}$ is parametrizable by radicals *iff there is a radical extension field $\mathbb{E}$ of $\mathbb{K}(s)$ and a pair $(p_1(s), p_2(s)) \in \mathbb{E}^2 \setminus \mathbb{K}^2$ such that $f(p_1(s), p_2(s)) = 0$. Then the pair $(p_1(s), p_2(s))$ is called a* radical parametrization *of the curve $\mathcal{C}$.*

We call a function $f(x)$ over $\mathbb{K}$ a radical function *iff there is a radical extension field of $\mathbb{K}(x)$ containing $f(x)$. Hence, a* radical solution *of an AODE is a solution that is a radical function. A* radical general solution *is a general solution which is radical.*

Computing radical parametrizations as in [60] goes back to solving algebraic equations of degree less or equal four. Depending on the degree we might therefore get more than one solution to such an equation. Each solution yields one branch of a parametrization. Therefore, we use the notation $a^{\frac{1}{n}}$ for any n-th root of a.

In fact we do not need to restrict to rational or radical parametrizations. More generally a parametrization of f is a generic zero of the prime ideal generated by f in the sense of van der Waerden [69].

Now we extend our set of possible parametrizations and also the set, in which we are looking for solutions, to functions including radical expressions. In the following we investigate special cases for which the procedure yields solvability information.

Theorem 2.7.
Let $\mathcal{Q}(t) = (q_0(s), q_1(s))$ be a radical parametrization of the curve $F(z,p) = 0$. Assume $A_{\mathcal{Q}}(s) = a(b+s)^n$ for some $n \in \mathbb{Q} \setminus \{1\}$, $a \neq 0$.
Then $q_0(h(x))$, with $h(x) = -b + (-(n-1)a(x+c))^{\frac{1}{1-n}}$, is a radical general solution of the AODE $F(u,u') = 0$.

Proof. From the procedure we get

$$
\begin{aligned}
g'(s) &= \frac{1}{A_Q(s)} = \frac{1}{a(b+s)^n}\,,\\
g(s) &= \int g'(s)\,\mathrm{d}s = \frac{(b+s)^{1-n}}{a(1-n)}\,,\\
h(s) &= -b + (-(n-1)as)^{\frac{1}{1-n}}\,.
\end{aligned}
$$

Then $u(x) = q_0(h(x))$ is a solution of F and $q_0(h(x+c))$, for some arbitrary constant c, is a general solution of F. □

The result of the algorithm of Feng and Gao is therefore a special case of Theorem 2.7 with $n = 0$ or $n = 2$ using rational parametrizations. In exactly these two cases h is a rational function. Feng and Gao [17] used rational parametrizations for finding rational solutions. The existence of a rational parametrization is of course a necessary condition for rational solvability. However, in Procedure 3 we might use a radical parametrization of the same curve which is not rational and we might still find a rational solution.

If $A_{\mathcal{Q}} = a(b+1)$, i.e. the exceptional case $n = 1$ of Theorem 2.7, then g contains a logarithmic part and hence its inverse contains an exponential term.

Example 2.8.
The differential equation $F(u, u') = u^5 - u'^2 = 0$ *gives rise to the radical parametrization* $\left(\frac{1}{s}, -\frac{1}{s^{5/2}}\right)$ *with corresponding* $A(s) = \frac{1}{\sqrt{s}}$. *We can compute* $g(s) = \frac{2s^{3/2}}{3}$ *and* $h(s) = \left(\frac{3}{2}\right)^{2/3} t^{2/3}$. *Hence,* $\frac{\left(\frac{2}{3}\right)^{2/3}}{(x+c)^{2/3}}$ *is a solution of the AODE.*

As a corollary of Theorem 2.7 we get the following statement for AODEs with another special type of the indicator A.

Corollary 2.9.
Let $F(u, u') = 0$ *be an autonomous AODE. Assume we have a radical parametrization* $\mathcal{Q}(s) = (q_0(s), q_1(s))$ *of the corresponding curve* $F(z, p) = 0$ *and assume* $A(s) = \frac{a(b+s^k)^n}{ks^{k-1}}$ *with* $k \in \mathbb{Q} \setminus \{0\}$. *Then* F *has a radical solution.*

Proof. Transforming the parametrization by $f(t) = s^{1/k}$ to the radical parametrization $\bar{\mathcal{Q}}(s) = (q_0(f(s)), q_1(f(s)))$ we get

$$A_{\bar{\mathcal{Q}}}(s) = \frac{q_1(f(s))}{\frac{\partial}{\partial s}(q_0(f(s)))} = \frac{A(f(s))}{f'(s)} = \frac{a(b + f(s)^k)^n}{kf(s)^{k-1}f'(s)}$$
$$= \frac{a(b + s^{\frac{k}{k}})^n}{ks^{\frac{k-1}{k}}\frac{1}{k}s^{\frac{1-k}{k}}} = a(b + s)^n\,,$$

which is of the form described in Theorem 2.7. ∎

In the rational situation there were exactly two possible cases for A in order to guarantee a rational solution. Here, in contrast, there are more possible forms for A. In the following we see another rather simple form of A which might occur. In this case we do not know immediately whether or not the procedure leads to a solution.

Theorem 2.10.
Let $\mathcal{Q}(s) = (q_0(s), q_1(s))$ *be a radical parametrization of the curve* $F(z, p) = 0$. *Assume* $A(s) = \frac{as^n}{b+s^m}$ *for some* $a, b \in \mathbb{Q} \setminus \{0\}$ *and* $m, n \in \mathbb{Q}$ *with* $m \neq n - 1$ *and* $n \neq 1$. *Then the AODE,* $F(u, u') = 0$, *has a radical solution if the equation*

$$b(m - n + 1)h^{1-n} - (n - 1)h^{m-n+1} + (n - 1)(m - n + 1)as = 0 \tag{2.4}$$

has a non-zero radical solution for $h = h(s)$. *A general solution of the AODE is then* $q_0(h(x + c))$.

Proof. The procedure yields

$$g'(s) = \frac{1}{A_{\mathcal{Q}}(s)} = \frac{b + s^m}{a s^n},$$
$$g(s) = \int g'(s)\,\mathrm{d}s = \frac{1}{a} s^{1-n} \left(\frac{b}{1-n} + \frac{s^m}{1+m-n} \right).$$

The inverse of g can be found by solving the equation

$$\frac{1}{a} h(s)^{1-n} \left(\frac{b}{1-n} + \frac{h(s)^m}{1+m-n} \right) = s$$

for $h(s)$. By a reformulation and the assumptions for m and n this is equivalent to (2.4). □

In the excluded cases $n = 1$ or $m = n - 1$ the integral in Step 2 of Procedure 3 is not radical.

Example 2.11.
For the AODE, $F(u, u') = -u^5 - u' + u^8 u' = 0$, we compute the radical parametrization $\mathcal{Q}(s) = \left(\frac{1}{s}, \frac{s^3}{1-s^8}\right)$ with corresponding $A(s) = \frac{s^5}{-1+s^8}$. Then equation (2.4) has a solution, e. g. $-\left(2s - \sqrt{-1+4s^2}\right)^{1/4}$. Hence, we get the solution of the AODE

$$u(x) = -\left(2(x+c) - \sqrt{4(x+c)^2 - 1}\right)^{-1/4}.$$

It remains to show when (2.4) is solvable by radicals (i. e. when $g(t)$ in the proof of Theorem 2.10 has an inverse which is expressible by radicals). The following theorem due to Ritt [53] helps us to do so.

Theorem 2.12.
A polynomial g has an inverse expressible by radicals if and only if it can be decomposed into

- *linear polynomials,*
- *power polynomials x^n for $n \in \mathbb{N}$,*
- *Chebyshev polynomials and*
- *degree 4 polynomials.*

Certainly also polynomials of degree 2 and 3 are invertible by radicals but it can be shown, that Theorem 2.12 applies to them. We prove now that a certain polynomial is not decomposable into non-linear factors.

Theorem 2.13.
Let $g(t) = C_1 t^\alpha + C_2 t^\beta \in \mathbb{K}[t]$ *where* $\mathbb{K}$ *is a field of characteristic zero,* $C_1, C_2 \in \mathbb{K} \setminus \{0\}$, $\alpha, \beta \in \mathbb{N}$, $\gcd(\alpha, \beta) = 1$, $\beta > \alpha > 0$ *and* $\beta > 4$. *Assume* $g = f \circ h$ *for some polynomials* f *and* h. *Then* $\deg f = 1$ *or* $\deg h = 1$.

Proof. Assume $g(t) = f(h(t))$ with $f = \sum_{i=0}^{n} a_i x^i$ and $h = \sum_{k=0}^{m} b_k x^k$, where $a_n \neq 0$, $b_m \neq 0$, $m, n > 1$. In case $b_0 \neq 0$ it follows that $g(t) = \bar{f}(\bar{h}(t))$ where $\bar{f}(t) = f(b_0 + t)$ and $\bar{h}(t) = h(t) - b_0$. Hence, without loss of generality we can assume that $b_0 = 0$ and therefore also $a_0 = 0$.

We denote the coefficient of order k in a polynomial g by $\mathrm{coef}_k(g)$. Let now $\tau \in \{1, \ldots, m\}$ such that $b_\tau \neq 0$ and $b_l = 0$ for all $l \in \{1, \ldots, \tau - 1\}$. Similarly let $\pi \in \{1, \ldots, n\}$ such that $a_\pi \neq 0$ and $a_l = 0$ for all $l \in \{1, \ldots, \pi - 1\}$. This implies that $\mathrm{coef}_l(g) = 0$ for all $l \in \{1, \ldots, \tau\pi - 1\}$ and $\mathrm{coef}_{\tau\pi}(g) = a_\pi b_\tau^\pi \neq 0$. Hence, $\alpha = \tau\pi$. Assume now that $\pi < n$. Then $\alpha = \tau\pi < m(n-1) + l$ and therefore

$$\begin{aligned} 0 &= \mathrm{coef}_{m(n-1)+l}(g) \\ &= \mathrm{coef}_{m(n-1)+l}\left(a_n \left(\sum_{j=\tau}^{m} b_j x^j \right)^n \right) \\ &= n a_n b_m^{n-1} b_l + \sum_{\bar{\varepsilon} \in E} \binom{n}{\varepsilon_{l+1}, \ldots, \varepsilon_m} a_n \prod_{j=l+1}^{m} b_j^{\varepsilon_j} \end{aligned}$$

for all $l \in \{\tau, \ldots, m-1\}$ where $\bar{\varepsilon} = (\varepsilon_{l+1}, \ldots, \varepsilon_m)$ and

$$E = \left\{ \bar{\varepsilon} \mid \sum_{k=l+1}^{m} \varepsilon_k = n, \sum_{k=l+1}^{m} k\varepsilon_k = m(n-1) + l \right\}.$$

This yields, that $0 = \mathrm{coef}_{mn-1}(g) = n a_n b_m^{n-1} b_{m-1}$, hence, $b_{m-1} = 0$. By induction it follows that $b_l = 0$ for all $l \in \{\tau, \ldots, m-1\}$ which contradicts $b_\tau \neq 0$.

Therefore, $\tau = m$ or $\pi = n$. But then we have $m \mid \alpha$ and $m \mid \beta$ or $n \mid \alpha$ and $n \mid \beta$ which contradicts $\gcd(\alpha, \beta) = 1$ since $m, n \neq 1$. ☐

Let us now consider the function

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right)$$

from the proof of Theorem 2.10. In general g is not a polynomial. Our aim is to find a radical function h with a radical inverse such that $\bar{g} = g(h)$ is a polynomial. Then g has a radical inverse if and only if $\bar{g}$ has a radical inverse.

Let $z_1, z_2 \in \mathbb{Z}$, $d_1, d_2 \in \mathbb{N}$ such that $1 - n = \frac{z_1}{d_1}$, $m - n + 1 = \frac{z_2}{d_2}$ and $\gcd(z_1, d_1) = \gcd(z_2, d_2) = 1$. Then $g(t) = \bar{g}(h(t))$ where $h(t) = t^{\frac{d}{d_1 d_2}}$ and

$$\bar{g}(t) = \frac{1}{a} t^{\bar{n}} \left(\frac{b}{1-n} + \frac{t^{\bar{m}-\bar{n}}}{1+m-n} \right), \tag{2.5}$$

with exponents $\bar{n} = \frac{(1-n)d_1 d_2}{d}$, $\bar{m} = \frac{(m-n+1)d_1 d_2}{d}$ and $d = \gcd(z_1 d_2, z_2 d_1)$. Hence, $\bar{m}$, $\bar{n}$ are integers with $\gcd(\bar{m}, \bar{n}) = 1$. The function h has an inverse expressible by radicals. If $m - n + 1$, $1 - n$ are positive integers, then also $\bar{m}, \bar{n}$ are positive. On the other hand if $n - m - 1$, $n - 1 \in \mathbb{N}$ we get a polynomial by a further composition with $f(t) = t^{-1}$.

If not both $\bar{n}$ and $\bar{m}$ are positive and not both are negative but $|\bar{m}| + |\bar{n}| \leq 4$, computing the inverse function of $\bar{g}$ is the same as solving an equation of degree less or equal 4, which can be done by radicals.

We summarize this discussion using Theorem 2.12 and 2.13 as follows.

Corollary 2.14.
The function g from the proof of Theorem 2.10,

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right),$$

has an inverse expressible by radicals in the following cases (where we use the notation from above):

- $b = 0$,
- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leq 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leq 4$,
- $-\bar{m}, \bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leq 4$,
- $\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leq 4$.

It has no inverse expressible by radicals in the cases

- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(\bar{m}, \bar{n}) > 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) > 4$.

Proof. In the cases where $\bar{m}$ and $\bar{n}$ have the same sign and $|\bar{m}| + |\bar{n}| > 4$ the function $\bar{g}$ as discussed above fulfills the requirements of Theorem 2.13. Hence, if $\bar{g} = f_1 \circ f_2$ either f_1 or f_2 is of degree one. It is not difficult to show, that the other one can neither be a power polynomial nor a Chebyshev polynomial. Hence, by the Theorem of Ritt $\bar{g}$ has no radical inverse.

The case $b = 0$ is obvious. In all the other cases mentioned in the theorem and not discussed so far we end up in solving an algebraic equation of degree less or equal four and hence, there is a radical inverse. □

Hence, in some cases we are able to decide the solvability of an AODE with properties as in Theorem 2.10. Nevertheless, the procedure is not complete, since even Corollary 2.14 does not cover all possible cases for m and n.

2.2.2. Other Solutions

So far we were looking for rational and radical solutions of AODEs. However, the procedure is not restricted to these cases but might also solve some AODEs with non-radical solutions as we can see in the following examples where trigonometric and exponential solutions are found. This is the case when Step 2 (integration) or 3 (solving algebraic equation) cannot be computed in the field of rational functions but in some field extension. For the integration problem the existing algorithms provide information on the necessary field extensions. Further investigation of non-algebraic results of the procedure are subject to further research.

Example 2.15. (c. f. Example 1.371 of [32])
Consider the equation $F(u, u') = \pm u^3 + u^2 + u'^2 = 0$. *The corresponding curve has the parametrization* $\mathcal{Q}(s) = \mp(1 + s^2, s(1 + s^2))$. *We get* $A(s) = \frac{1}{2}(1 + s^2)$ *and hence,* $g(s) = 2\arctan(s)$. *The inverse function is* $h(s) = \tan(\frac{s}{2})$ *and thus,* $u(x) = \mp(1 + \tan(\frac{x+c}{2})^2) = \mp\sec(\frac{x+c}{2})^2$ *is a solution.*

Example 2.16.
Consider the AODE, $F(u,u') = u^2 + u'^2 + 2uu' + u = 0$. We get the rational parametrization $\left(-\frac{1}{(1+s)^2}, -\frac{s}{(1+s)^2}\right)$. With $A(s) = -\frac{1}{2}s(1+s)$ we compute $g(s) = -2\log(s) + 2\log(1+s)$ and hence $h(s) = \frac{1}{-1+\mathrm{e}^{s/2}}$, which leads to the solution $-\mathrm{e}^{-(x+c)}(-1+\mathrm{e}^{(x+c)/2})^2$.

These examples show that we can find non-radical solutions even with rational parametrizations.

2.2.3. Investigation of the Procedure

In many books on differential equations we can find a method for transforming an autonomous ODE of any order $F(u, u', \ldots, u^{(n)}) = 0$ to an equation of lower order by substituting $v(u) = u'$ (see for instance [32, 73]). For the case of first-order ODEs this method yields a solution. It turns out that this method is somehow related to our procedure. The method does the following:

- Substitute $v(u) = u'$.
- Solve $F(u, v(u)) = 0$ for $v(u)$.
- Solve $\int \frac{1}{v(u)}\,\mathrm{d}u = x$ for u.

These computations are a special case of our general procedure where a specific form of parametrization is used, i. e. $\mathcal{Q}(s) = (s, q_1(s))$.

We now give some arguments concerning the possibilities and benefits of the general procedure. Since in the procedure any radical parametrization can be used we might take advantage of picking a good one as we see in the following example.

Example 2.17.
We consider the AODE, $F(u,u') = u'^6 + 49uu'^2 - 7 = 0$, and find a parametrization of the form $(s, q_1(s))$:

$$\left(s, \frac{\sqrt{\left(756+84\sqrt{28812s^3+81}\right)^{2/3} - 588s}}{\sqrt{6}\left(756+84\sqrt{28812s^3+81}\right)^{1/6}}\right).$$

Computing the corresponding integral is not very efficient and might not be done in some computer algebra systems. Nevertheless, we can input another parametrization to our procedure. An obvious one to try next is

$$(q_0(s), q_1(s)) = \left(-\frac{-7+s^6}{49s^2}, s\right).$$

It turns out that here we get $g(s) = \frac{2}{21s^3} - \frac{4s^3}{147}$. *Its inverse can be computed* $h(s) = \frac{1}{2}\left(-147s - \sqrt{7}\sqrt{32+3087s^2}\right)^{1/3}$. *Applying* $h(x+c)$ *to* $q_0(t)$ *we get the solution*

$$u(x) = \frac{7 - \frac{1}{64}\left(-147(x+c) - \sqrt{7}\sqrt{32+3087(x+c)^2}\right)^2}{\frac{49}{4}\left(-147(x+c) - \sqrt{7}\sqrt{32+3087(x+c)^2}\right)^{2/3}}.$$

The procedure might find a radical solution of an AODE by using a rational parametrization as we have seen in Example 2.11 and 2.17. As long as we are looking for rational solutions only, the corresponding curve has to have genus zero. Now we can also solve some examples where the genus of the corresponding curve is greater than zero and hence no rational parametrization exists. The AODE in Example 2.18 below corresponds to a curve with genus 1.

Example 2.18.
Consider the AODE, $F(u, u') = -u^3 - 4u^5 + 4u^7 - 2u' - 8u^2u' + 8u^4u' + 8uu'^2 = 0$. *We compute a parametrization and get*

$$\left(\frac{1}{s}, \frac{-4+4s^2+s^4}{s\left(4s^2-4s^4-s^6-\sqrt{-16s^4+16s^8+8s^{10}+s^{12}}\right)}\right)$$

as one of the branches. The procedure yields

$$A(s) = -\frac{s\left(-4+4s^2+s^4\right)}{4s^2-4s^4-s^6-\sqrt{-16s^4+16s^8+8s^{10}+s^{12}}},$$

$$g(s) = \frac{2s^4+s^6+\sqrt{s^4\left(2+s^2\right)^2\left(-4+4s^2+s^4\right)}}{4s^2+2s^4},$$

$$h(s) = -\frac{\sqrt{1+s^2}}{\sqrt{1+s}},$$

$$u(x) = -\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}.$$

2.3. Extension to Higher-Order AODEs

Higher-order AODEs have already been studied in [29]. We present how the key idea corresponds to these investigations. Before doing so, we show some simple methods for special kinds of higher-order AODEs.

In Section 2.2 we presented a procedure for solving AODEs of the form $F(u, u') = 0$. Obviously we can use this procedure just as well for solving differential equations $F(u^{(n-1)}, u^{(n)}) = 0$ for any $n \geq 1$. We do so by first solving $F(v, v') = 0$ and then computing u from $u^{(n-1)} = v$ by integration (compare [32, Example 7.18, p. 604]).

Second-Order AODEs of the form $F(u, u'') = 0$

In the following we show an attempt to extend the procedure to second-order AODEs of the form $F(u, u'') = 0$. Note, that the content of this section is mainly a collection of ideas. A more elaborate investigation is subject to further research. Already Kamke gave a special case of the method in [32, Section 23.1 and Example 7.19, p. 605]. He restricts to ODEs which are solvable for the highest derivative $u^{(n)}$. Later we show difficulties arising in the general setting for higher n.

Assume we are given a differential equation $F(u, u'') = 0$ and a rational or radical parametrization $\mathcal{Q} = (q_0, q_1)$ of the corresponding curve. The method will, as we see later, in general yield a radical solution. We assume (as usual) that $\mathcal{Q} = \mathcal{L}(g)$ for some function $g(s)$, where $\mathcal{L}$ is the parametrization corresponding to the solution. Then the following equations have to be fulfilled

$$q_0 = u(g)\,, \qquad\qquad q_1 = u''(g)\,.$$

We take derivatives of the first equation and get

$$\begin{aligned} q_0' &= g' u'(g)\,, \\ q_0'' &= g'^2 u''(g) + g'' u'(g) = g'^2 q_1 + g'' \frac{q_0'}{g'}\,. \end{aligned} \tag{2.6}$$

This yields an ODE of order 2 in g which can be reduced to an ODE in $\bar{g}$ of order 1 by setting $\bar{g} = g'$. This quasilinear ODE can be solved:

$$\begin{aligned} \bar{g} &= \frac{q_0'}{\sqrt{c_1 - 2\int -q_1 q_0' \,\mathrm{d}s}}\,, \\ g &= c_2 + \int \bar{g} \,\mathrm{d}s\,. \end{aligned}$$

Using this method we can solve differential equations of the form $F(u^{(n-2)}, u^{(n)}) = 0$ with $n \geq 2$ by first solving $F(v, v'') = 0$ and then $u^{(n-2)} = v$.

Example 2.19. (Example 2.6 of Kamke [32])
We consider the AODE, $F(u, u'') = u'' - u$. *It is easy to see that* $\mathcal{Q} = (s, s)$ *is a parametrization of the associated curve. Equation* (2.6) *simplifies to* $sg'^3 + g'' = 0$ *which can be reduced to* $s\bar{g}^3 + \bar{g}' = 0$. *Thus, we get* $\bar{g} = -\frac{1}{\sqrt{s^2-2c_1}}$ *and hence* $g = \log(s + \sqrt{s^2 - 2c_1}) + c_2$. *Computing the inverse* h *of* g *we get the solution* $h = \frac{1}{2}\mathrm{e}^{-s+c_2} + \mathrm{e}^{s-c_2}c_1$, *since* $q_0 = s$.

Now we would like to generalize this idea to AODEs of the form $F(u, u^{(n)}) = 0$. The Formula of Faà di Bruno describes the higher-order derivatives of a function composition:

$$q_0^{(\eta)} = \frac{\partial^\eta u(g)}{\partial s^\eta} = \sum_{(\kappa_1,\ldots,\kappa_\eta)\in R_\eta} \binom{\eta}{\kappa_1, \ldots, \kappa_\eta} \frac{\partial^{\kappa_1+\ldots+\kappa_\eta} u}{\partial s^{\kappa_1+\ldots+\kappa_\eta}}(g) \prod_{m=1}^{\eta} \left(\frac{\frac{\partial^m g}{\partial s^m}}{m!}\right)^{k_m}, \tag{2.7}$$

where $R_\eta = \{(\kappa_1, \ldots, \kappa_\eta) \in \mathbb{N}^\eta \mid \sum_{i=1}^{\eta} i\kappa_i = \eta\}$. Using Bell polynomials this can be denoted in a different way

$$\frac{\partial^\eta u(g)}{\partial s^\eta} = \sum_{k=1}^{\eta} \frac{\partial^k u}{\partial s^k}(g) B_{\eta,k}(g'(s), \ldots, g^{(\eta-k+1)}(s)),$$

where

$$B_{\eta,k}(x_1, \ldots x_{\eta-k+1}) = \sum_{\substack{j_1,\ldots,j_{\eta-k+1} \\ \sum j_i = k \\ \sum i j_i = \eta}} \binom{\eta}{j_1, \ldots, j_{\eta-k+1}} \prod_{i=1}^{\eta-k+1} \left(\frac{x_i}{i!}\right)^{j_i}.$$

Equation (2.7) yields a system of algebraic equations in the indeterminates $u^{(k)}(g)$. Indeed, it is possible to eliminate $u^{(k)}(g)$ for all $k \in \{1, \ldots, n-1\}$. In the resulting equation we replace the remaining $u^{(n)}(g)$ by q_1. Hence, we get a differential equation in g of order n. It is easy to see that g itself does not appear in the equation. Therefore, we can transform to a differential equation of order $n - 1$. However, this ODE is in general not linear and not quasilinear either. For $n = 3$ for instance we get

$$G(\bar{g}, \bar{g}', \bar{g}'') = 3\bar{g}\bar{g}' q_0'' + q_0' \left(\bar{g}\bar{g}'' - 3\bar{g}'^2\right) - \bar{g}^2 q_0^{(3)} + \bar{g}^5 q_1 = 0.$$

These considerations lead over to the investigation of general second-order autonomous AODEs, $F(u, u', u'') = 0$.

General second-order autonomous AODEs

Using our general idea, we need a parametrization of the corresponding surface. However, the solution only yields a parametrization of some curve on the surface. We can avoid these circumstances and consider the AODE to be in fact an APDE of the form, $F(u, u_x, u_{xx}) = 0$. Given a solution $v(x, y)$ of the APDE, we can compute a solution $v(x, c)$ of the AODE for some constant c. Of course this feels like cracking a nut with a sledgehammer but nevertheless it could work as we see on the next example. A method for solving the related APDE is presented in 3.4.

Example 2.20. (Example 6.107 of Kamke [32])
We consider the AODE, $F(u, u', u'') = uu'' + u'^2 = 0$. *For simplicity we chose the constant of the original Example in Kamke to be zero. In Example 3.28 we solve the APDE,* $F(u, u_x, u_{xx}) = uu_{xx} + u_x^2 = 0$, *and get* $u(x, y) = \frac{\sqrt{2}t^{3/2}}{\sqrt{s-t}} - \frac{\sqrt{2}s\sqrt{t}}{\sqrt{s-t}}$ *as a solution. Hence,* $v(x) = u(x + c_1, c_2)$ *is a solution of the AODE.*

This idea can be easily generalized to higher-order AODEs.

General higher-order AODEs

First we consider a general autonomous AODE, $F(u, u', \ldots, u^{(n)}) = 0$. We assume the equation to be in fact an APDE, $F(u, u^{(1,0,\ldots,0)}, \ldots, u^{(n,0,\ldots,0)}) = 0$, in n independent variables. Such APDEs can be solved with Procedure 6. Let $u(x_1, \ldots, x_n)$ be a solution of the APDE. Then $u(x_1 - c_1, c_2, \ldots, c_n)$, with arbitrary constants $c_1, \ldots, c_n$, is a solution of the AODE.

Note, that if u is rational, the arbitrary constants $c_1, \ldots, c_n$ are indeed independent since by Lemma 3.30 a rational solution computed by Procedure 6 is proper.

Example 2.21. (Example 3.3.13 of [49])
We consider the AODE, $F(u, u', u'') = uu''' - u'u'' = 0$. *This coincides with Example 3.3.13 of [49] for* $a = 1$. *In Example 3.29 we solve the APDE,* $F(u, u_x, u_{xx}) = uu_{xxx} - u_x u_{xx} = 0$, *and get* $u(x_1, x_2, x_3) = \frac{e^{x_1\sqrt{x_2}}\left(e^{-2x_1\sqrt{x_2}} - 2x_2x_3\right)}{2x_2}$ *as a solution. Hence,* $u(x + c_1, c_2, c_3)$ *is a solution of the AODE.*

The same can be done for non-autonomous AODEs, $F(x, u, u', \ldots, u^{(n)}) = 0$, by applying Procedure 7 to the APDE, $F(x_1, u, u^{(1,0,\ldots,0)}, \ldots, u^{(n,0,\ldots,0)}) = 0$, in $n+1$ independent variables. Let $u(x_1, \ldots, x_{n+1})$ be a solution of the APDE. Then $u(x_1, c_1, c_2, \ldots, c_n)$ is a solution of the AODE with arbitrary constants $c_1, \ldots, c_n$.

Remark 2.22.
In Procedure 7 the method of characteristics yields a system of ODEs. In case $n = 1$, this system coincides with the associated system from [44], which is described in (2.1) *in Section 2.1.2. For higher order, the system coincides with the associated system defined in [29]. Hence, the algorithms in [29, 44] are special cases of the Procedure for computing rational solutions. To this effect those algorithms (respectively the corresponding papers) also provide more information on solvability.*

3. Solution Method for APDEs

In this chapter we investigate the general idea of Section 1.4 for the case of two and more variables and present a method for solving first-order autonomous APDEs. The procedure is a generalization of Procedure 3.

For APDEs ($n \geq 2$), the system (1.2) definitely consists of more than one equation. We first (Section 3.1) present the case of two variables and show how the system can be transformed to a single quasilinear equation in one component which we know how to solve.

The separate presentation of the three variable case (Section 3.2) shall depict the essentially new steps needed for the generalization to an arbitrary number of variables. The difference is now that we transform system (1.2) to a set of independent quasilinear equations, i. e. PDEs which are linear in the derivatives but possibly non-linear in the dependent function and the variables (see Appendix D for further information). We show in detail how to find these equations but omit further elaboration in this section.

Finally, we show the general case for an arbitrary number of variables in Section 3.3 including all details and restrictions. Furthermore, we prove properties of the computed solutions. As in the case of AODEs we provide evidence that the method is not restricted to finding rational solutions.

Several examples illustrate the procedure and its capabilities. Further examples containing well-known equations from literature can be found in Appendix C.3.

At the end of this chapter, in Section 3.4 we present further approaches for solving APDEs. These techniques are related to the previously investigated procedures. They include a glimpse of an idea of a generalization to higher order and a systematic way of computing solutions that are not proper. Furthermore, a procedure for degenerate APDEs, where only derivatives by one variable appear, is presented. This procedure can be used for solving higher-order AODEs.

3.1. Two Variables

In this section we present the generalization of Procedure 3 to APDEs in two variables. The general approach is the same. However, new ideas are needed to solve system (1.2) which now really is a system of PDEs in the unknowns g_1 and g_2. It turns out that this system can be transformed to a quasilinear equation in one of the components of g. Such equations can be solved by a well-known method. We present all important details in this section and join them to a procedure similar to the AODE case.

The content of this section is in a large part based on results in [22]. Nevertheless, additional information is incorporated.

We recall the idea of Section 1.4 for the specific setting of $n = 2$. Let $F(u, u_x, u_y) = 0$ be an algebraic partial differential equation.

We consider the surface $F(z, p, q) = 0$ and assume it admits a proper (rational) surface parametrization

$$\mathcal{Q}(s,t) = (q_0(s,t), q_1(s,t), q_2(s,t)) .$$

An algorithm for computing a proper rational parametrization of a surface can be found for instance in [56]. For further information on rational parametrization of surfaces we refer to Appendix B.1.2. Here, we stick to rational parametrizations, but the procedure which we present works as well with other kinds of parametrizations, for instance radical ones. First results on radical parametrizations of surfaces can be found in [61]. Assume that $\mathcal{L}(s,t) = (v_0(s,t), v_1(s,t), v_2(s,t))$ is the corresponding parametrization of a solution of the APDE. Furthermore we assume that the parametrization $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}(s,t) = \mathcal{L}(g(s,t))$$

for some invertible function $g(s,t) = (g_1(s,t), g_2(s,t))$. This assumption is motivated by the fact that in case of rational algebraic curves every non-constant rational solution of an AODE yields a proper rational parametrization of the associated algebraic curve and each proper rational parametrization can be obtained from any other proper one by a linear rational transformation. However, in the case of APDEs, not all rational solutions provide a proper parametrization, as mentioned in the remark after Definition 1.4. Nevertheless, we will see later that it does still make sense to stick to the assumption. Now, using the assumption, if we can compute g^{-1} we have a solution $\mathcal{Q}(g^{-1}(s,t))$.

Let $\mathcal{J}$ be the Jacobian matrix. Then we have

$$\mathcal{J}_{\mathcal{Q}}(s,t) = \mathcal{J}_{\mathcal{L}}(g(s,t)) \cdot \mathcal{J}_g(s,t)\,.$$

Taking a look at the first row we get the specific case of system (1.2)

$$\begin{aligned} \frac{\partial q_0}{\partial s} &= \frac{\partial v_0}{\partial s}(g)\frac{\partial g_1}{\partial s} + \frac{\partial v_0}{\partial t}(g)\frac{\partial g_2}{\partial s} = q_1(s,t)\frac{\partial g_1}{\partial s} + q_2(s,t)\frac{\partial g_2}{\partial s}\,, \\ \frac{\partial q_0}{\partial t} &= \frac{\partial v_0}{\partial s}(g)\frac{\partial g_1}{\partial t} + \frac{\partial v_0}{\partial t}(g)\frac{\partial g_2}{\partial t} = q_1(s,t)\frac{\partial g_1}{\partial t} + q_2(s,t)\frac{\partial g_2}{\partial t}\,. \end{aligned} \tag{3.1}$$

This is a system of quasilinear equations in the unknown functions g_1 and g_2. In case q_1 or q_2 is zero the problem reduces to ordinary differential equations. Hence, from now on we assume that $q_1 \neq 0$ and $q_2 \neq 0$. First we divide by q_1 and simplify notation:

$$\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s} + b\frac{\partial g_2}{\partial s}\,, \\ a_2 &= \frac{\partial g_1}{\partial t} + b\frac{\partial g_2}{\partial t} \end{aligned} \tag{3.2}$$

with

$$a_1 = \frac{\frac{\partial q_0}{\partial s}}{q_1}\,, \qquad a_2 = \frac{\frac{\partial q_0}{\partial t}}{q_1}\,, \qquad b = \frac{q_2}{q_1}\,. \tag{3.3}$$

By taking derivatives we get

$$\begin{aligned} \frac{\partial a_1}{\partial t} &= \frac{\partial^2 g_1}{\partial s \partial t} + \frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} + b\frac{\partial^2 g_2}{\partial s \partial t}\,, \\ \frac{\partial a_2}{\partial s} &= \frac{\partial^2 g_1}{\partial t \partial s} + \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} + b\frac{\partial^2 g_2}{\partial t \partial s}\,. \end{aligned} \tag{3.4}$$

Subtraction of the two equations yields

$$\frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} - \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}\,. \tag{3.5}$$

This is a single quasilinear differential equation which can be solved by the method of characteristics (see for instance [73] and Appendix D). In case $\frac{\partial b}{\partial t} = 0$ or $\frac{\partial b}{\partial s} = 0$ equation (3.5) reduces to a simple ordinary differential equation.

Remark 3.1.
If both derivatives of b are zero then b is a constant. Hence, the left hand side of (3.5) is zero. In case the right hand side is non-zero we get a contradiction, and therefore, no

proper solution exists. In case the right hand side is zero as well we get from (3.5) *that*

$$\begin{aligned}
0 = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{\frac{\partial q_0}{\partial s}}{q_1}\right) - \frac{\partial}{\partial s}\left(\frac{\frac{\partial q_0}{\partial t}}{q_1}\right) \\
&= \frac{\frac{\partial^2 q_0}{\partial t \partial s} q_1 - \frac{\partial q_0}{\partial s}\frac{\partial q_1}{\partial t}}{q_1^2} - \frac{\frac{\partial^2 q_0}{\partial s \partial t} q_1 - \frac{\partial q_0}{\partial t}\frac{\partial q_1}{\partial s}}{q_1^2} \\
&= -\frac{\frac{\partial q_0}{\partial s}\frac{\partial q_1}{\partial t} - \frac{\partial q_0}{\partial t}\frac{\partial q_1}{\partial s}}{q_1^2},
\end{aligned}$$

hence,

$$0 = \frac{\partial q_0}{\partial s}\frac{\partial q_1}{\partial t} - \frac{\partial q_0}{\partial t}\frac{\partial q_1}{\partial s}.$$

Moreover, since b is constant, $q_1 = kq_2$ for some constant k. But this means that the rank of the Jacobian of $\mathcal{Q}$ is 1*, a contradiction to $\mathcal{Q}$ being proper.*

Therefore, we assume from now on, that the derivatives of b are non-zero. According to the method of characteristics, we need to solve the following system of first-order ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)}, \\
\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= \frac{\frac{\partial a_1}{\partial t}(s(t),t) - \frac{\partial a_2}{\partial s}(s(t),t)}{-\frac{\partial b}{\partial s}(s(t),t)}.
\end{aligned}$$

The second equation is linear and separable but depends on the solution of the first. The first ODE can be solved independently. Its solution $s(t) = \eta(t,k)$ depends on an arbitrary constant k. Hence, also the solution of the second ODE depends on k. Finally, the function g_2 we are looking for is $g_2(s,t) = v(t,\mu(s,t)) + \nu(\mu(s,t))$ where μ is computed such that $s = \eta(t,\mu(s,t))$ and ν is an arbitrary function. In case we are only looking for rational solutions we can use the algorithm of Ngô and Winkler [44, 45, 46] for solving these ODEs.

Knowing g_2 we can compute g_1 by using (3.1) which now reduces to a separable ODE in g_1. The remaining task is to compute h_1 and h_2 such that $g(h_1(s,t),h_2(s,t)) = (s,t)$. Then $q_0(h_1,h_2)$ is a solution of the original PDE.

Finally the method reads as

Procedure 4.

Input: An autonomous APDE, $F(u, u_x, u_y) = 0$, where F is irreducible and $F(z, p, q) = 0$ is a rational surface with a proper rational parametrization $\mathcal{Q} = (q_0, q_1, q_2)$.
Output: A solution of the APDE or "fail".

1. Compute the coefficients b and a_i as in (3.3).
2. If $\frac{\partial b}{\partial s} = 0$ and $\frac{\partial b}{\partial t} \neq 0$ compute $g_2 = \int \frac{\frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}}{\frac{\partial b}{\partial t}} \mathrm{d}s + \kappa(t)$ and go to step 6 otherwise continue.
   If $\frac{\partial b}{\partial s} = \frac{\partial b}{\partial t} = 0$ return "No proper solution".
3. Solve the ODE, $\frac{\mathrm{d}s(t)}{\mathrm{d}t} = -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)}$, for $s(t) = \eta(t, k)$ with arbitrary constant k.
4. Solve the linear ODE, $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)}$, by computing the integral $v(t) = v(t, k) = \int \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)} \mathrm{d}t + \nu(k)$.
5. Compute μ such that $s = \eta(t, \mu(s, t))$ and then $g_2(s, t) = v(t, \mu(s, t))$.
6. Use the second equation of (3.2) to compute $g_1(s, t) = m(s) + \int a_2 - b\frac{\partial g_2}{\partial t} \mathrm{d}t$.
7. Determine $m(s)$ by using the first equation of (3.2).
8. Compute h_1, h_2 such that $g(h_1(s, t), h_2(s, t)) = (s, t)$.
9. Return the solution $q_0(h_1(x, y), h_2(x, y))$.

Observe that the proper rational parametrization $\mathcal{Q}$ can be computed applying Schicho's algorithm (see [56]). In addition, we also observe that the procedure can be extended to the non-rational algebraic case, if one has an injective parametrization, in that case non-rational, of the surface defined by $F(z, p, q) = 0$.

In general ν depends on a constant c_2 and m on a constant c_1. As a special case of the procedure we fix $\nu = c_2$. This choice is done for simplicity reasons but we may sometimes refer to cases with other choices which are a subject of further research.

Furthermore, the procedure can be considered symmetrically in Step 2 for the case that $\frac{\partial b}{\partial t} = 0$ and $\frac{\partial b}{\partial s} \neq 0$. In such a case the rest of the procedure has to be changed symmetrically as well. We do not go into further details.

Theorem 3.2.
Let $F(u, u_x, u_y) = 0$ be an autonomous APDE. If Procedure 4 returns a function $v(x, y)$ for input F, then v is a solution of F.

Proof. By the procedure we know that $v(x, y) = q_0(h_1(x, y), h_2(x, y))$ with h_i such that $g(h_1(s, t), h_2(s, t)) = (s, t)$. Since g is a solution of system (3.1) it fulfills the assumption that $u(g_1, g_2) = q_1$ for some solution u of the APDE. Hence, v is a solution. We have seen a more detailed description at the beginning of this section. □

Remark 3.3.
In Step 3 and 4 ODEs have to be solved. Depending on the class of functions to which the requested solution should belong, these ODEs do not necessarily have a solution. Furthermore, an explicit inverse (Step 8) does not necessarily exist.
It will be a subject of further research, to investigate conditions on cases for which the procedure does definitely not fail.

Now, we show that the result of Procedure 4 does not change if we postpone the introduction of c_1 and c_2 to the end of the procedure. It is easy to show that if $u(x, y)$ is a solution of an autonomous APDE then so is $u(x + c, y + d)$ for any constants c and d. From the procedure we see that in the computation of g_1 we use the derivative of g_2 only (and hence c_2 disappears). We can write

$$g_2 = \bar{g}_2 + c_2\,, \qquad\qquad g_1 = \bar{g}_1 + c_1$$

for some functions $\bar{g}_1$ and $\bar{g}_2$ which do not depend on c_1 and c_2. Let $g = (g_1, g_2)$ and $\bar{g} = (\bar{g}_1, \bar{g}_2)$. In Step 8 we are looking for a function h such that $g \circ h = \text{id}$. Now $g \circ h = \bar{g} \circ h + (c_1, c_2)$. Take $\bar{h}$ such that $\bar{g} \circ \bar{h} = \text{id}$. Then $g \circ \bar{h}(s - c_1, t - c_2) = \text{id}$. Hence, we can introduce the constants at the end.

In case the original APDE is in fact an AODE, the ODE in Step 4 turns out to be trivial and the integral in Step 7 is exactly the one which appears in the Procedure 3 for AODEs. Of course then g is univariate and so is its inverse. In this sense, this new procedure generalizes Procedure 3. We do not specify Procedure 4 to handle this case separately.

In the following we show some examples which can be solved by Procedure 4. Note, that the examples have more solutions than those computed below. In Example 3.4 for instance, other solutions can be found by choosing different ν, e. g. $\nu(x) = c_2 + x^2$.

However, the results might not be rational solutions then. In general the procedure, as stated in this thesis, yields only one solution containing two arbitrary independent constants. Hence, it cannot be a general solution in the sense of depending on an arbitrary function.

3.1.1. Rational Solutions

In this section we concentrate on APDEs with rational solutions. Further investigation on such solutions is done in Section 3.3.1 for the more general case of arbitrary many variables. Here, we give some well-known equations with rational solution and we prove that APDEs of a certain class have a rational solution which can be found by the procedure. We start with a simple well-known APDE which has a rational solution.

Example 3.4. (Inviscid Burgers Equation [3, p. 7])
We consider the autonomous APDE,

$$F(u, u_x, u_y) = uu_x + u_y = 0\,.$$

Since F is of degree one in each of the derivatives, it is easy to compute a parametrization $\mathcal{Q} = \left(-\frac{t}{s}, s, t\right)$. We compute the coefficients

$$a_1 = \frac{t}{s^3}\,, \qquad a_2 = -\frac{1}{s^2}\,, \qquad b = \frac{t}{s}\,.$$

In Step 3 we find $s(t) = kt$ and in Step 4 we compute $v(t) = \frac{1}{kt} + \nu(k)$. Then, $\mu(s,t) = \frac{s}{t}$ and hence (with $\nu = c_2$),

$$g_2 = \frac{1}{s} + c_2\,, \qquad g_1 = -\frac{t}{s^2} + m(s)\,.$$

Using Step 7 we find out that $m(s) = c_1$. Computing the inverse of g we get

$$h_1 = \frac{1}{t - c_2}\,, \qquad h_2 = \frac{-s + c_1}{(t - c_2)^2}\,.$$

Finally, $u(x,y) = \frac{x - c_1}{y - c_2}$ is a solution of the APDE.

The inviscid Burgers Equation is linear in u. This fact can be used for a more general treatment of such APDEs.

Lemma 3.5.
Let $F(u, u_x, u_y) = B(u_x, u_y)u + A(u_x, u_y) = 0$ be an APDE with coprime polynomials $A, B \in \mathbb{C}[p, q]$, $B \neq 0$. Let $\gamma(s,t) = \frac{A(s,t)}{B(s,t)}$ and assume that $\gamma(s,t) = \bar{\gamma}(\frac{t}{s})$ for some function $\bar{\gamma}$. Then $\gamma(t, -s)$ is a rational solution.

Proof. Due to the assumptions on A and B, we know that F is irreducible and $\mathcal{Q} = (-\gamma(s,t), s, t)$ is a proper parametrization of the surface $F(z,p,q) = 0$. We show that $\mathcal{P}(s,t) = \mathcal{Q}(-ta_2(-t,s), ta_1(-t,s))$ yields a solution, where a_i are defined as in (3.3). Indeed we have

$$\begin{aligned}\mathcal{P}(s,t) &= \left(-\bar{\gamma}(\frac{ta_1(-t,s)}{-ta_2(-t,s)}), -ta_2(-t,s), ta_1(-t,s)\right)\\ &= \left(-\bar{\gamma}(-\frac{s}{t}), \frac{1}{t}\bar{\gamma}'(-\frac{s}{t}), -\frac{s}{t^2}\bar{\gamma}(-\frac{s}{t})\right).\end{aligned}$$

Now it is easy to prove that this yields a solution

$$\begin{aligned}\frac{\partial\left(-\bar{\gamma}(-\frac{s}{t})\right)}{\partial s} &= \frac{1}{t}\bar{\gamma}'(-\frac{s}{t}),\\ \frac{\partial\left(-\bar{\gamma}(-\frac{s}{t})\right)}{\partial t} &= -\frac{s}{t^2}\bar{\gamma}'(-\frac{s}{t}).\end{aligned}$$

□

Procedure 4 can also handle more complicated APDEs.

Example 3.6.
We consider the APDE,

$$0 = F(u, u_x, u_y) = uu_x^4 + u_x^3u_y - uu_x^3u_y - u_x^2u_y^2 + uu_x^2u_y^2 + u_xu_y^3 - uu_xu_y^3 + uu_y^4\,.$$

A proper parametrization of the corresponding algebraic surface is for instance

$$\mathcal{Q} = \left(-\frac{t\,(1-t+t^2)}{1-t+t^2-t^3+t^4}, t\gamma(s,t), \gamma(s,t)\right),$$

with $\gamma(s,t) = \frac{t\left(-10+7t\right)\left(-9+t^2\right)\left(-1+2t-3t^2+3t^4-2t^5+t^6\right)}{2s(45-63t+5t^2)(1-t+t^2-t^3+t^4)^2}$. This parametrization is not easy to find. It is computed by first using parametrization by lines and then applying a linear transformation in s. Alternatively one could use the parametrization by lines directly.

In that case Procedure 4 would find the same solution, but the intermediate steps would need more writing space. Using the procedure with the parametrization $\mathcal{Q}$ we get

$$g_1 = s\left(\frac{7}{10-7t} - \frac{1}{t} + \frac{2t}{-9+t^2}\right), \qquad g_2 = \frac{2s\,(45-63t+5t^2)}{(-10+7t)\,(-9+t^2)},$$
$$h_1 = -\frac{t\,(-90s^3-63s^2t+10st^2+7t^3)}{2s\,(45s^2+63st+5t^2)}, \qquad h_2 = \frac{-t}{s},$$

and finally the solution $u(x,y) = \frac{xy\left(x^2+xy+y^2\right)}{x^4+x^3y+x^2y^2+xy^3+y^4}$. As mentioned before, $u(x+c_1, y+c_2)$ with constants c_1 and c_2 is also a solution.

Parametrization by lines (see Appendix B.1.2 and B.1.3) can be used if there is a $(d-1)$-fold point, where d is the degree of the APDE. In such cases the resulting parametrization is of the form $(sD(s,t), tD(s,t), D(s,t))$, assuming the singular point to be in the origin. The following Lemma describes solvability assuming that such a parametrization by lines fulfills further conditions.

Lemma 3.7.
Assume we have an APDE, $F(u, u_x, u_y) = 0$, with a parametrization of the form $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$ where $B(t) = \frac{N(t)}{D(t)} \notin \mathbb{K}$ with $N(t), D(t) \in \mathbb{K}[t]$, $\gcd(N,D) = 1$ and $n \in \mathbb{Z}$. Then F has an algebraic solution.

Proof. Assume we have $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$ with $B(t) \notin \mathbb{K}$ (the constant case is shifted to Lemma 3.8). Then we can use the procedure to get the following:

$$b = \frac{1}{t}, \qquad a_1 = \frac{n+1}{t}, \qquad a_2 = \frac{s}{t}\frac{B'}{B},$$

and hence, $\frac{\partial b}{\partial s} = 0$ but $\frac{\partial b}{\partial t} \neq 0$. Therefore

$$\begin{aligned}
g_2 &= \int \frac{\frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}}{\frac{\partial b}{\partial t}}\,\mathrm{d}s = \int \frac{-\frac{n+1}{t^2} - \frac{1}{t}\frac{B'}{B}}{-\frac{1}{t^2}}\,\mathrm{d}s = \int (n+1) + t\frac{B'}{B}\,\mathrm{d}s \\
&= s(n+1+t\frac{B'}{B}), \\
g_1 &= \int a_2 - b\frac{\partial g_2}{\partial t}\,\mathrm{d}t + m(s) = \int \frac{sB'}{tB} - \frac{s}{t}\frac{(tB''+B')B - B'B'}{B^2}\,\mathrm{d}t + m(s) \\
&= \int \frac{s}{t}\frac{B'B - tB''B - B'B + B'B'}{B^2}\,\mathrm{d}t + m(s) \\
&= \int \frac{s}{t}\frac{t(-B''B + B'B')}{B^2}\,\mathrm{d}t + m(s) = \int -s\left(\frac{B'}{B}\right)'\,\mathrm{d}t + m(s) \\
&= -s\frac{B'}{B} + m(s).
\end{aligned}$$

Now we need to find $m(s)$. We do so by using the equation as in the procedure

$$\begin{aligned}
\frac{\partial q_1}{\partial s} &= q_2 \frac{\partial g_1}{\partial s} + q_3 \frac{\partial g_2}{\partial s} \\
(n+1)s^n B &= s^n t B \left(-\frac{B'}{B} + m'(s) \right) + s^n B \left(n + 1 + t\frac{B'}{B} \right) \\
(n+1)s^n B &= -s^n t B' + s^n t B m' + s^n (n+1) B + s^n t B' \\
0 &= s^n t B m' \\
0 &= m' \,.
\end{aligned}$$

Hence, m is a constant and we choose $m = 0$. Finally we need to find h_1, h_2 fulfilling

$$\begin{aligned}
g_1(h_1, h_2) &= s \,, & g_2(h_1, h_2) &= t \,, \\
-h_1 \frac{B'(h_2)}{B(h_2)} &= s \,, & h_1 \left(n + 1 + h_2 \frac{B'(h_2)}{B(h_2)} \right) &= t \,.
\end{aligned}$$

Since B is non-constant we can eliminate h_1 and get

$$\begin{aligned}
-s \frac{B(h_2)}{B'(h_2)} &= t \left(n + 1 + h_2 \frac{B'(h_2)}{B(h_2)} \right)^{-1} \\
-sB(h_2) \left(n + 1 + h_2 \frac{B'(h_2)}{B(h_2)} \right) &= tB'(h_2) \\
B(h_2)s(n+1) + B'(h_2)(sh_2 + t) &= 0 \,.
\end{aligned}$$

Hence, we have an algebraic equation for h_2 and therefore also for h_1 (since the equations are linear in h_1). Thus, we get an algebraic solution. □

In Lemma 3.7 the case of $B(t) \in \mathbb{K}$ was excluded. It is shifted to the next Lemma.

Lemma 3.8.
Assume we have an APDE, $F(u, u_x, u_y) = 0$, with a parametrization of the form $\mathcal{Q} = (s^{n+1}\beta, ts^n\beta, s^n\beta)$ where $\beta \in \mathbb{K}$. Then F has a rational solution.

Proof. Assume we have $\mathcal{Q} = (s^{n+1}\beta, ts^n\beta, s^n\beta)$ with $B(t) = \beta \in \mathbb{K}$. According to the procedure we get $g_2 = (n+1)s + \kappa(t)$. We choose $\kappa(t) = t^2$. Then $g_1 = \int a_2 - b\frac{\partial g_2}{\partial t}\,\mathrm{d}t + m(s) = -\int \frac{\kappa'(t)}{t}\,\mathrm{d}t = -2t + m(s)$. As before we get that m is constant and we choose $m = 0$. Finally we need to find h_1, h_2 fulfilling

$$s = g_1(h_1, h_2) = -2h_2 \,, \qquad t = g_2(h_1, h_2) = h_1(n+1) + h_2^2 \,.$$

Then $h_2 = -\frac{s}{2}$ and hence $h_1 = \frac{t-h_2^2}{n+1}$. Thus, $q_0(h_1(x,y), h_2(x,y)) = h_1(x,y)^{n+1}\beta = \beta\left(\frac{y-\frac{x^2}{4}}{n+1}\right)^{n+1}$ is a rational solution. □

Note, that we might get algebraic solutions with other choices of κ.

Using the results of Lemma 3.7 we can prove that certain APDEs have rational solutions.

Lemma 3.9.
Assume we have an autonomous APDE, $F(u, u_x, u_y) = 0$, with a parametrization $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$. Let $B(t) = k\frac{N(t)}{D(t)}$, where $N(t), D(t) \in \mathbb{K}[t]$ are monic, $k \in \mathbb{K}$ and $\gcd(N, D) = 1$. The APDE has a rational solution if one of the following holds.

1. $n = -1$
2. $\deg(N) = \deg(D) = 0$
3. $\deg(N) = 0$, $\deg(D) > 0$ *and*
   (a) $D(t) = (t-\beta_1)^\delta (t-\beta_2)^{n+1-\delta}$ *with* $\beta_1 \neq \beta_2$, $\delta \geq 1$ *and* $n \geq 1$, *or*
   (b) $D(t) = (t-\beta_1)^\delta$ *with* $\delta \neq n+1$ *and* $\delta \geq 1$
4. $\deg(D) = 0$, $\deg(N) > 0$ *and*
   (a) $N(t) = (t-\alpha_1)^\varepsilon (t-\alpha_2)^{-n-1-\varepsilon}$, *with* $\alpha_1 \neq \alpha_2$, $\varepsilon \geq 1$ *and* $n \leq -3$, *or*
   (b) $N(t) = (t-\alpha_1)^\varepsilon$ *with* $-\varepsilon \neq n+1$ *and* $\varepsilon > 1$
5. $N = (t-\alpha_1)^\varepsilon$ *and* $D = (t-\beta_1)^{n+1+\varepsilon}$ *with* $\varepsilon \geq 1$ *and* $n+\varepsilon \geq 0$

Proof. In the proof of Lemma 3.7 we concluded that an APDE of the type under consideration has an implicitly given solution. The algebraic equation we found was

$$B(h_2)s(n+1) + B'(h_2)(sh_2+t) = 0\,. \tag{3.6}$$

Since the equations in the proof of Lemma 3.7 are linear in h_1 it suffices to show that in all the cases of this lemma, the algebraic equation (3.6) has a rational solution. In the following we will write $N = N(\zeta)$ and $D = D(\zeta)$. We need to show that the equation

$$E(\zeta) := s\left((n+1)DN + \zeta(N'D - ND')\right) + t(N'D - ND') = 0$$

has a rational solution for ζ.

1. $n = -1$: $E(\zeta)$ has the factor $s\zeta + t$.

2. This was shown in Lemma 3.8.

3. $\deg(N) = 0$, $\deg(D) > 0$: In this case $E(\zeta) = N\left(s\left((n+1)D - \zeta D'\right) - tD'\right)$.

   (a) $D(\zeta) = (\zeta - \beta_1)^\delta(\zeta - \beta_2)^{n+1-\delta}$ with $\beta_1 \neq \beta_2$, $\delta > 1$ and $n > 1$: Then $D' = \delta(\zeta - \beta_1)^{\delta-1}(\zeta - \beta_2)^{n+1-\delta} + (n+1-\delta)(\zeta - \beta_1)^\delta(\zeta - \beta_2)^{n-\delta}$ and

   $$\begin{aligned} E(\zeta) &= N(\zeta - \beta_1)^{\delta-1}(\zeta - \beta_2)^{n-\delta} \cdot \bar{E}(\zeta)\,, \\ \bar{E}(\zeta) &:= (n+1)(\beta_1 - \zeta)(\beta_2 s + t) - \delta(\beta_1 - \beta_2)(s\zeta + t)\,. \end{aligned}$$

   Thus, $\bar{E}$ is linear in ζ.

   (b) $D(\zeta) = (\zeta - \beta_1)^\delta$ with $\delta \neq n+1$ and $\delta > 1$: Then

   $$E(\zeta) = N(\zeta - \beta_1)^{\delta-1}((n+1)s(\zeta - \beta_1) - \delta(s\zeta + t))$$

   has a linear factor that depends on s and t.

4. $\deg(D) = 0$, $\deg(N) > 0$: In this case $E(\zeta) = D(s\left((n+1)N + \zeta N'\right) + tN')$.

   (a) $N(\zeta) = (\zeta - \alpha_1)^\varepsilon(\zeta - \alpha_2)^{-n-1-\varepsilon}$, with $\alpha_1 \neq \alpha_2$, $\varepsilon > 1$ and $n < -3$: Then

   $$\begin{aligned} E(\zeta) &= D(\zeta - \alpha_1)^{\varepsilon-1}(\zeta - \alpha_2)^{-n-2-\varepsilon}\bar{E}(\zeta)\,, \\ \bar{E}(\zeta) &:= \varepsilon(\alpha_1 - \alpha_2)(s\zeta + t) + (n+1)(\alpha_1 - \zeta)(s\alpha_2 + t)\,. \end{aligned}$$

   Thus, $\bar{E}(\zeta)$ is linear in ζ.

   (b) $N(\zeta) = (\zeta - \alpha_1)^\varepsilon$ with $-\varepsilon \neq n+1$ and $\varepsilon > 1$: Then

   $$E(\zeta) = D(\zeta - \alpha_1)^{\varepsilon-1}\left(s(n+1)(\zeta - \alpha_1) + \varepsilon(\zeta s + t)\right)$$

   has a linear factor that depends also on s and t.

5. $N = (\zeta - \alpha_1)^\varepsilon$ and $D = (\zeta - \beta_1)^{n+1+\varepsilon}$ with $\varepsilon \geq 1$ and $n + \varepsilon \geq 0$: Then

   $$\begin{aligned} E(\zeta) &= (\zeta - \alpha_1)^{\varepsilon-1}(\zeta - \beta_1)^{n+\varepsilon}\bar{E}(\zeta)\,, \\ \bar{E}(\zeta) &:= \varepsilon(\alpha_1 - \beta_1)(s\zeta + t) + (n+1)(\alpha_1 - \zeta)(s\beta_1 + t)\,. \end{aligned}$$

   Thus, $\bar{E}(\zeta)$ is linear in ζ.

□

Example 3.10.
We give examples for all cases of Lemma 3.9.

1. $F = -u_x^4 - 2uu_x^2u_y^2 + u_xu_y^3 - u_y^4$ *with* $\mathcal{Q} = \left(\frac{-1+t-t^4}{2t^2}, \frac{-1+t-t^4}{2st}, \frac{-1+t-t^4}{2st^2}\right)$

2. $F = 2u - u_y^2$ *with* $\mathcal{Q} = (2s^2, 2st, 2s)$

3. (a) $F = 4u - u_x^2 - 2u_xu_y - 2u_y^2$ *with* $\mathcal{Q} = \left(-\frac{4s^2}{-2-2t-t^2}, -\frac{4st}{-2-2t-t^2}, -\frac{4s}{-2-2t-t^2}\right)$
   $F = -u_x^5 + 8u_x^4u_y - 25u_x^3u_y^2 + 38u_x^2u_y^3 - 28u_xu_y^4 + 8u_y^5 + 3u^4$ *with* $\mathcal{Q} = \left(\frac{3s^5}{(-2+t)^3(-1+t)^2}, \frac{3s^4t}{(-2+t)^3(-1+t)^2}, \frac{3s^4}{(-2+t)^3(-1+t)^2}\right)$

   (b) $F = -10u^3u_x - 2u_y^3$ *with* $\mathcal{Q} = \left(-\frac{1}{5s^2t}, -\frac{1}{5s^3}, -\frac{1}{5s^3t}\right)$
   $F = -u_x^6 + 7u^2u_y^3$ *with* $\mathcal{Q} = \left(\frac{7s^3}{t^6}, \frac{7s^2}{t^5}, \frac{7s^2}{t^6}\right)$

4. (a) $F = -u^3 + u_x^2 - 10u_xu_y - u_y^2$ *with* $\mathcal{Q} = \left(\frac{-1-10t+t^2}{s^2}, \frac{t\left(-1-10t+t^2\right)}{s^3}, \frac{-1-10t+t^2}{s^3}\right)$

   $F = 6u^4 + 5u_x^3 + 5u_x^2u_y$ *with* $\mathcal{Q} = \left(-\frac{5t^2+5t^3}{6s^3}, -\frac{t\left(5t^2+5t^3\right)}{6s^4}, -\frac{5t^2+5t^3}{6s^4}\right)$

   (b) $F = 4u_x - 4u_y + u_y^2$ *with* $\mathcal{Q} = (-s(-4+4t), -t(-4+4t), 4-4t)$
   $F = -7u^2u_x^2 + 6u_y^5$ *with* $\mathcal{Q} = \left(\frac{7s^3t^2}{6}, \frac{7s^2t^3}{6}, \frac{7s^2t^2}{6}\right)$

5. $F = -u^4u_x + 4u_x^6 + u^4u_y$ *with* $\mathcal{Q} = \left(-\frac{s\left(s^4-s^4t\right)}{4t^6}, -\frac{s^4-s^4t}{4t^5}, -\frac{s^4-s^4t}{4t^6}\right)$

3.1.2. Algebraic Solutions

We have seen in Lemma 3.7 that solutions computed by the procedure might be considered implicitly. Hence, Procedure 4 is not restricted to rational solutions nor to rational parametrizations as we see in the following examples. In certain steps of the procedure we might do computations in a radical field extension. For instance Step 8 might be an algebraic equation without rational solution. In this case h is an algebraic function. The following example illustrates such a case.

Example 3.11. (Eikonal Equation [4, p. 2])
We consider the APDE,

$$F(u, u_x, u_y) = u_x^2 + u_y^2 - 1 = 0\,.$$

From the rational parametrization of a circle it is easy to see that

$$\mathcal{Q} = \left(s, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

is a parametrization of the corresponding surface. Using the procedure we get some rational g_1 *and* g_2 *which yield*

$$h_2 = \frac{-s + c_1 \pm \sqrt{s^2 + t^2 - 2sc_1 + c_1^2 - 2tc_2 + c_2^2}}{t - c_2},$$
$$h_1 = \pm\sqrt{s^2 + t^2 - 2sc_1 + c_1^2 - 2tc_2 + c_2^2}\,.$$

Finally, we get the radical solution

$$u(x,y) = \pm\sqrt{(x-c_1)^2 + (y-c_2)^2}\,.$$

We can use Lemma 3.7 to show that a certain class of APDEs has algebraic solutions, i. e. we show that a suitable parametrization exists to apply Lemma 3.7.

Corollary 3.12.
Let the APDE be of the form $F(u, u_x, u_y) = \lambda u^m + \gamma_{m-1}(u_x, u_y) = 0$, *where* $m \in \mathbb{N}, \lambda \in \mathbb{C} \setminus \{0\}$ *and* $\gamma_{m-1}(p,q)$ *be a form of degree* $m-1$. *Then* F *has an algebraic solution.*

Proof. Observe that $F(z,p,q)$ is irreducible, and can be parametrized as

$$\mathcal{Q}(s,t) = \left(-s\frac{\gamma_{m-1}(t,1)}{\lambda s^m}, -t\frac{\gamma_{m-1}(t,1)}{\lambda s^m}, -\frac{\gamma_{m-1}(t,1)}{\lambda s^m} \right),$$

which corresponds to the parametrization form in Lemma 3.7 with $n = -m$ and $B(t) = -\gamma_{m-1}(t,1)/\lambda$. Hence, there is an algebraic solution. □

3.1.3. Other Solutions

As in the AODE case we might do certain steps in the procedure by doing computations in some extension field. Investigation of such computation will be subject to future research. Nevertheless, we want to give a brief glimpse of what might be possible. In a further example we compute an exponential solution of an APDE.

Example 3.13. (Convection-Reaction Equation [3, p. 7])
We consider the APDE,

$$F(u, u_x, u_y) = u_x + cu_y - du = 0\,,$$

where $d \neq 0$ and $c \neq 0$. We compute a parametrization $\mathcal{Q} = \left(\frac{s+ct}{d}, s, t\right)$ and the coefficients

$$a_1 = \frac{1}{ds}\,, \qquad a_2 = \frac{c}{ds}\,, \qquad b = \frac{t}{s}\,.$$

Solving the ODEs of Step 3–6 we get

$$g_2 = \frac{c\log(t)}{d} + c_2\,, \qquad g_1 = c_1 + \frac{\log(s)}{d}\,.$$

Computing the inverse of g we find

$$h_1 = \mathrm{e}^{ds-dc_1}\,, \qquad h_2 = \mathrm{e}^{\frac{dt}{c}-\frac{dc_2}{c}}\,.$$

Finally, we get the solution $u(x, y) = \frac{\mathrm{e}^{d(x-c_1)}+c\mathrm{e}^{\frac{d(y-c_2)}{c}}}{d}$.

Table 3.1 presents a list of some well-known equations in two variables and the solutions found by the procedure. For the sake of readability we neglect the arbitrary constants and present only specific solutions.

Name	APDE	Parametrization	Solution
Burgers (inviscid) [73]	$uu_x + u_y$	$\left(-\frac{t}{s}, s, t\right)$	$\frac{x}{y}$
Traffic [10]	$u_y - u_x\left(\frac{2uv_m}{r_m} - v_m\right)$	$\left(\frac{r_m(t+sv_m)}{2sv_m}, s, t\right)$	$\frac{r_m(-x+yv_m)}{2v_m y}$
Eikonal [4]	$u_x^2 + u_y^2 - 1$	$\left(s, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$	$\pm\sqrt{x^2+y^2}$
Convection-Reaction [3]	$u_x + cu_y - du$	$\left(\frac{s+ct}{d}, s, t\right)$	$\frac{\mathrm{e}^{dx}+c\mathrm{e}^{\frac{dy}{c}}}{d}$
Generalized Burgers (special case) [73]	$u_y + uu_x + \alpha u + \beta u^2$	(sB, tB, B) $B = -\frac{(1+s\alpha)}{st+s^2\beta}$	$\frac{\mathrm{e}^{-x\beta}\left(1-\mathrm{e}^{x\beta}\right)\alpha}{(1+\mathrm{e}^{\alpha y})\beta}$

Table 3.1.: Well-known APDEs and their solutions found by Procedure 4.

3.2. Three Variables

In this section we consider the system (1.2) for $n = 3$. This case is discussed separately since it is still suitable to show step by step details but it is also general enough to intuitively lead to the overall picture presented in Section 3.3. Without loss of generality we assume that $q_1 \neq 0$. Hence, we might divide by q_1 and get

$$\left.\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s_1} + b_1 \frac{\partial g_1}{\partial s_1} + b_2 \frac{\partial g_2}{\partial s_1}\,, \\ a_2 &= \frac{\partial g_1}{\partial s_2} + b_1 \frac{\partial g_1}{\partial s_2} + b_2 \frac{\partial g_2}{\partial s_2}\,, \\ a_3 &= \frac{\partial g_1}{\partial s_3} + b_1 \frac{\partial g_1}{\partial s_3} + b_2 \frac{\partial g_2}{\partial s_3}\,, \end{aligned}\right\} \tag{3.7}$$

with $a_i = \frac{\frac{\partial q_0}{\partial s_i}}{q_1}$ and $b_i = \frac{q_i}{q_1}$. Now we take derivatives of these equations

$$\begin{aligned} \frac{\partial a_1}{\partial s_2} &= \frac{\partial^2 g_1}{\partial s_2 \partial s_1} + \frac{\partial b_1}{\partial s_2}\frac{\partial g_1}{\partial s_1} + b_1 \frac{\partial^2 g_1}{\partial s_2 \partial s_1} + \frac{\partial b_2}{\partial s_2}\frac{\partial g_2}{\partial s_1} + b_2 \frac{\partial^2 g_2}{\partial s_2 \partial s_1}\,, \\ \frac{\partial a_1}{\partial s_3} &= \frac{\partial^2 g_1}{\partial s_3 \partial s_1} + \frac{\partial b_1}{\partial s_3}\frac{\partial g_1}{\partial s_1} + b_1 \frac{\partial^2 g_1}{\partial s_3 \partial s_1} + \frac{\partial b_2}{\partial s_3}\frac{\partial g_2}{\partial s_1} + b_2 \frac{\partial^2 g_2}{\partial s_3 \partial s_1}\,, \\ \frac{\partial a_2}{\partial s_1} &= \frac{\partial^2 g_1}{\partial s_1 \partial s_2} + \frac{\partial b_1}{\partial s_1}\frac{\partial g_1}{\partial s_2} + b_1 \frac{\partial^2 g_1}{\partial s_1 \partial s_2} + \frac{\partial b_2}{\partial s_1}\frac{\partial g_2}{\partial s_2} + b_2 \frac{\partial^2 g_2}{\partial s_1 \partial s_2}\,, \\ \frac{\partial a_2}{\partial s_3} &= \frac{\partial^2 g_1}{\partial s_3 \partial s_2} + \frac{\partial b_1}{\partial s_3}\frac{\partial g_1}{\partial s_2} + b_1 \frac{\partial^2 g_1}{\partial s_3 \partial s_2} + \frac{\partial b_2}{\partial s_3}\frac{\partial g_2}{\partial s_2} + b_2 \frac{\partial^2 g_2}{\partial s_3 \partial s_2}\,, \\ \frac{\partial a_3}{\partial s_1} &= \frac{\partial^2 g_1}{\partial s_1 \partial s_3} + \frac{\partial b_1}{\partial s_1}\frac{\partial g_1}{\partial s_3} + b_1 \frac{\partial^2 g_1}{\partial s_1 \partial s_3} + \frac{\partial b_2}{\partial s_1}\frac{\partial g_2}{\partial s_3} + b_2 \frac{\partial^2 g_2}{\partial s_1 \partial s_3}\,, \\ \frac{\partial a_3}{\partial s_2} &= \frac{\partial^2 g_1}{\partial s_2 \partial s_3} + \frac{\partial b_1}{\partial s_2}\frac{\partial g_1}{\partial s_3} + b_1 \frac{\partial^2 g_1}{\partial s_2 \partial s_3} + \frac{\partial b_2}{\partial s_2}\frac{\partial g_2}{\partial s_3} + b_2 \frac{\partial^2 g_2}{\partial s_2 \partial s_3}\,. \end{aligned}$$

Choosing $a_{j,k} = \frac{\partial a_j}{\partial s_k} - \frac{\partial a_k}{\partial s_j}$ and $b_{i,k} = \frac{\partial b_i}{\partial s_k}$ and subtracting suitable equations we get

$$\left.\begin{aligned} a_{1,2} &= b_{2,2}\frac{\partial g_2}{\partial s_1} - b_{2,1}\frac{\partial g_2}{\partial s_2} + b_{3,2}\frac{\partial g_3}{\partial s_1} - b_{3,1}\frac{\partial g_3}{\partial s_2}\,, \\ a_{1,3} &= b_{2,3}\frac{\partial g_2}{\partial s_1} - b_{2,1}\frac{\partial g_2}{\partial s_3} + b_{3,3}\frac{\partial g_3}{\partial s_1} - b_{3,1}\frac{\partial g_3}{\partial s_3}\,, \\ a_{2,3} &= b_{2,3}\frac{\partial g_2}{\partial s_2} - b_{2,2}\frac{\partial g_2}{\partial s_3} + b_{3,3}\frac{\partial g_3}{\partial s_2} - b_{3,2}\frac{\partial g_3}{\partial s_3}\,. \end{aligned}\right\} \tag{3.8}$$

By a linear combination we get

$$\begin{aligned} &b_{2,3}a_{1,2} + b_{2,1}a_{2,3} - b_{2,2}a_{1,3} \\ &\quad = (b_{2,3}b_{3,2} - b_{2,2}b_{3,3})\frac{\partial g_3}{\partial s_1} + (b_{2,1}b_{3,3} - b_{2,3}b_{3,1})\frac{\partial g_3}{\partial s_2} + (b_{2,2}b_{3,1} - b_{2,1}b_{3,2})\frac{\partial g_3}{\partial s_3}\,. \end{aligned}$$

This is a quasilinear PDE in g_3. Hence, it can be solved by the well-known method of characteristics. Once we have g_3 we get a quasilinear PDE in g_2 adding the two first equations of (3.8):

$$a_{1,2} + a_{1,3} - \left((b_{3,2} + b_{3,3}) \frac{\partial g_3}{\partial s_1} - b_{3,1} \frac{\partial g_3}{\partial s_2} - b_{3,1} \frac{\partial g_3}{\partial s_3} \right) = (b_{2,2} + b_{2,3}) \frac{\partial g_2}{\partial s_1} - b_{2,1} \frac{\partial g_2}{\partial s_2} - b_{2,1} \frac{\partial g_2}{\partial s_3} \,.$$

Again, this can be solved by the method of characteristics. Finding g_1 is finally computing an integral from (3.7).

Note, here we have shown a recursive way. However, some computations can also be done in parallel, since there is no reason for the particular choice of the roles of the g_i. Indeed, we may as well consider another quasilinear PDE in g_2 which is similar to the one for g_3:

$$b_{3,3}a_{1,2} + b_{3,1}a_{2,3} - b_{3,2}a_{1,3} = (b_{2,2}b_{3,3} - b_{2,3}b_{3,2}) \frac{\partial g_2}{\partial s_1} + (b_{2,3}b_{3,1} - b_{2,1}b_{3,3}) \frac{\partial g_2}{\partial s_2} + (b_{2,1}b_{3,2} - b_{2,2}b_{3,1}) \frac{\partial g_2}{\partial s_3} \,.$$

In fact, the two quasilinear PDEs can be expressed as

$$\frac{1}{q_1^2} \det \begin{pmatrix} \frac{\partial q_0}{\partial s_1} & \frac{\partial q_0}{\partial s_2} & \frac{\partial q_0}{\partial s_3} \\ \frac{\partial q_1}{\partial s_1} & \frac{\partial q_1}{\partial s_2} & \frac{\partial q_1}{\partial s_3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial g_3}{\partial s_1} & \frac{\partial g_3}{\partial s_2} & \frac{\partial g_3}{\partial s_3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$

$$-\frac{1}{q_1^2} \det \begin{pmatrix} \frac{\partial q_0}{\partial s_1} & \frac{\partial q_0}{\partial s_2} & \frac{\partial q_0}{\partial s_3} \\ \frac{\partial q_1}{\partial s_1} & \frac{\partial q_1}{\partial s_2} & \frac{\partial q_1}{\partial s_3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial g_2}{\partial s_1} & \frac{\partial g_2}{\partial s_2} & \frac{\partial g_2}{\partial s_3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$

3.3. The General Case

Based on the ideas of Section 3.1 and 3.2 we can finally derive the generalization to arbitrary many variables. Furthermore, in this section we elaborate details which we omitted in the previous ones. These include but are not restricted to exceptional cases and properties of the computed solutions.

Note that, when we talk about the differential equation and its solution we use the variables $x_1, \ldots, x_n$, whereas in the parametrization problem we use the variables $s_1, \ldots, s_n$.

We recall system (1.2) from the introduction.

$$\left.\begin{aligned}\frac{\partial q_0}{\partial s_1} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g)\frac{\partial g_i}{\partial s_1} = \sum_{i=1}^{n} q_i(s_1,\ldots,s_n)\frac{\partial g_i}{\partial s_1}\,,\\ &\vdots\\ \frac{\partial q_0}{\partial s_n} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g)\frac{\partial g_i}{\partial s_n} = \sum_{i=1}^{n} q_i(s_1,\ldots,s_n)\frac{\partial g_i}{\partial s_n}\,.\end{aligned}\right\} \tag{3.9}$$

This is a system of quasilinear equations in the unknown functions g_1 to g_n. In case q_i is zero for some i the problem reduces to lower order. Since $\mathcal{Q}$ is a proper parametrization of a hypersurface, at most one of its components can be zero. So, we can ensure that there exists a non-zero q_i with $i > 0$. Let us assume that $q_1 \neq 0$. If this is not the case, we can always change the role of x_1 and x_i with $i > 1$. We proceed as in the case for three variables. First we divide by q_1:

$$\left.\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s_1} + \sum_{i=2}^{n} b_i \frac{\partial g_i}{\partial s_1}\,,\\ &\vdots\\ a_n &= \frac{\partial g_1}{\partial s_n} + \sum_{i=2}^{n} b_i \frac{\partial g_i}{\partial s_n}\,.\end{aligned}\right\} \tag{3.10}$$

with $a_i = \frac{\frac{\partial q_0}{\partial s_i}}{q_1}$ and $b_i = \frac{q_i}{q_1}$. For each $j \in \{1,\ldots,n\}$ we take derivatives of the j-th equation in (3.10) with respect to the variables s_k for $j \neq k$. This yields a new system of PDEs, which are henceforth of second order.

$$\frac{\partial a_j}{\partial s_k} = \frac{\partial^2 g_1}{\partial s_k \partial s_j} + \sum_{i=2}^{n} \frac{\partial b_i}{\partial s_k}\frac{\partial g_i}{\partial s_j} + b_i \frac{\partial^2 g_i}{\partial s_k \partial s_j} \qquad \text{for } j \neq k\,. \tag{3.11}$$

Obviously, we would like to get rid of the second-order terms. Thus, we take pairwise differences of equations in (3.11) and get the following equations where the second derivatives vanish.

$$a_{j,k} = \sum_{i=2}^{n} b_{i,k}\frac{\partial g_i}{\partial s_j} - b_{i,j}\frac{\partial g_i}{\partial s_k}\,, \qquad \text{for } j < k\,, \tag{3.12}$$

with $a_{j,k} = \frac{\partial a_j}{\partial s_k} - \frac{\partial a_k}{\partial s_j}$ and $b_{i,k} = \frac{\partial b_i}{\partial s_k}$.

The aim now is to take suitable linear combinations of the equations from (3.12) such that all derivatives of g_i vanish except for $i = n$, i. e. we are left with a quasilinear PDE in g_n. In Section 3.1 this was shown for $n = 2$ and in Section 3.2 we did so for $n = 3$. Now we prove the general case.

Theorem 3.14.
Let $n \geq 2$ be the number of independent variables. Let $M = (b_{k,\ell})_{2 \leq k \leq n, 1 \leq \ell \leq n}$, where $b_{i,j}$ are as in (3.12). Then system (3.10) yields a quasilinear PDE in g_n of the following form

$$\sum_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} a_{i,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) = \sum_{i=1}^{n} \frac{\partial g_n}{\partial s_i} (-1)^i \det(M_{\emptyset,\{i\}}) \,, \tag{3.13}$$

where $M_{R,S}$ denotes the matrix which is obtained from M by deleting all rows with index in R and all columns with index in S.

Proof. We proceed by rearranging the left hand side of equation (3.13) as long as equality to the right hand side is shown. Laplace expansion plays an important role in the proof. We use equation (3.12) for replacing the $a_{i,j}$ and we rearrange the sums. Then, the left hand side of (3.13) reads as

$$\begin{aligned}
&\sum_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} \left(\sum_{k=2}^{n} b_{k,j} \frac{\partial g_k}{\partial s_i} - b_{k,i} \frac{\partial g_k}{\partial s_j} \right) (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \\
&= \sum_{k=2}^{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(b_{k,j} \frac{\partial g_k}{\partial s_i} - b_{k,i} \frac{\partial g_k}{\partial s_j} \right) (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \\
&= \sum_{k=2}^{n} \left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{k,j} \frac{\partial g_k}{\partial s_i} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right. \\
&\qquad \left. - \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{k,i} \frac{\partial g_k}{\partial s_j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \\
&= \sum_{k=2}^{n} \left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{k,j} \frac{\partial g_k}{\partial s_i} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right. \\
&\qquad \left. - \sum_{i=2}^{n} \sum_{j=1}^{i-1} b_{k,j} \frac{\partial g_k}{\partial s_i} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=2}^{n} \left(\sum_{j=2}^{n} b_{k,j} \frac{\partial g_k}{\partial s_1} (-1)^{1+j+n} \det(M_{\{n\},\{i,j\}}) \right. \\
&\qquad + \sum_{i=2}^{n} \frac{\partial g_k}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right. \\
&\qquad\qquad\qquad \left.\left. - \sum_{j=1}^{i-1} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \right) \\
&= \sum_{k=2}^{n} \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right.\right. \\
&\qquad\qquad \left.\left. - \sum_{j=1}^{i-1} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \right) \\
&= \sum_{i=1}^{n} \frac{\partial g_n}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{n,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) - \sum_{j=1}^{i-1} b_{n,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \\
&= \sum_{i=1}^{n} \frac{\partial g_n}{\partial s_i} (-1)^i \det(M_{\emptyset,\{i\}}) \,.
\end{aligned}
$$

In the last two steps we used backward Laplace expansion and got a matrix with an additional line. This line does already appear in the matrix except for $k = n$. □

There is no reason for the special role of g_n. Hence, we can give a similar quasilinear equation for each g_ν for $\nu > 1$. Since there is no dependence between these equations we can solve them in parallel. The equations we have to solve are

$$
\left\{ \sum_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} a_{i,j} (-1)^{i+j+\nu} \det(M_{\{\nu\},\{i,j\}}) = \sum_{i=1}^{n} \frac{\partial g_\nu}{\partial s_i} (-1)^i \det(M_{\emptyset,\{i\}}) \right\}_{\nu \in \{2,\dots,n\}} . \tag{3.14}
$$

Finally, knowing $g_2, \dots, g_n$, we are left with computing g_1 by using the system (3.10).

The system of quasilinear PDEs in (3.14) can be expressed as (compare to the case of three variables)

$$
\left\{ \frac{(-1)^\nu}{q_1^2} \det \begin{pmatrix} \nabla q_0 \\ \nabla q_1 \\ M_{\{\nu\},\emptyset} \end{pmatrix} = \det \begin{pmatrix} \nabla g_\nu \\ M \end{pmatrix} \right\}_{\nu \in \{2,\dots,n\}} .
$$

This is a consequence of using backward Laplace expansion by the first row, of the right hand side determinant, and generalized Laplace expansion by the first two rows of the left hand side determinant.

Note, that the determinants on the right hand side of (3.14) do not depend on ν. In the following we see some cases where the determinants on the right hand side have special properties. Mainly, we are asking some or all of them to be zero.

Remark 3.15.
If $\det(M_{\emptyset,\{i\}}) = 0$ *for all but one index* $i \in \{1, \dots, n\} \setminus \{\ell\}$*, and* $\det(M_{\emptyset,\{\ell\}}) \neq 0$*, then the equations* (3.14) *reduce to* $n-1$ *ODEs with solution*

$$g_\nu = \int \frac{\sum_{\substack{i,j\in\{1\dots n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}})}{(-1)^\ell \det(M_{\emptyset,\{\ell\}})}\,\mathrm{d}s_\ell + K(s_1,...,s_{\ell-1},s_{\ell+1},...,s_n)\,.$$

In the following remark and theorem we see what happens if the right hand side of (3.14) is zero. Two possible cases might occur: Either the left hand side is zero as well, or it is not.

Remark 3.16.
If $\det(M_{\emptyset,\{i\}}) = 0$ *for every* $i \in \{1, \dots, n\}$ *and*

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}}) \neq 0$$

for some $\nu \in \{1, \dots, n\}$*, then we get a contradiction, and hence, the assumption* $\mathcal{Q} = \mathcal{L}(g)$ *was wrong. This, however, means that there is no proper rational solution (compare Lemma 1.3). Nevertheless, there might be a non-proper rational solution, which we cannot find with the procedure presented here.*

From this we conclude for instance that the linear transport equation cannot have a proper rational solution.

Example 3.17. (Linear Transport Equation [3])
We consider the transport equation $F(u, u_x, u_y) = u_x + du_y = 0$*. We choose the proper parametrization* $\mathcal{Q} = (s_1, -ds_2, s_2)$*. Then* $\det(M_{\emptyset,\{i\}}) = 0$ *but the left hand side of* (3.14) *is* $\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}}) = -\frac{1}{ds_2^2}$.

Indeed, the general solution of the linear transport equation is $u(x, y) = \varphi(y - dx)$ *for some arbitrary function* φ*, (see for instance [3]). The Jacobian of the corresponding parametrization has rank 1 and hence, this solution cannot be proper.*

We show now that the left hand side cannot be zero according to our assumptions. Note, that the proof can also be applied in the case when $\mathcal{Q}$ is not rational.

Theorem 3.18.
If $\det(M_{\emptyset,\{i\}}) = 0$ *for every* $i \in \{1, \dots, n\}$*, and*

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}}) = 0$$

for every $\nu \in \{1, \dots, n\}$*, then* $\mathcal{Q}$ *turns out to be a parametrization of a variety of dimension strictly less than* n*.*

Proof. In order to prove this statement, we take the matrix $M = (b_{k,\ell})_{\substack{2\le k\le n\\ 1\le \ell\le n}}$. From the fact that $\det(M_{\emptyset,\{i\}}) = 0$ for every $i \in \{1, \dots, n\}$, the rank of M is, at most, $n-2$. By definition of the $b_{k,\ell}$ we know

$$b_{k,\ell} = \frac{\partial b_k}{\partial s_\ell} = \frac{\partial}{\partial s_\ell}\left(\frac{q_k}{q_1}\right) = q_1^{-2}\left(\frac{\partial q_k}{\partial s_\ell}q_1 - \frac{\partial q_1}{\partial s_\ell}q_k\right)$$

for every $k \in \{2, \dots, n\}$ and $\ell \in \{1, \dots, n\}$. Let $M^\star = (\frac{\partial q_k}{\partial s_\ell})_{\substack{2\le k\le n\\ 1\le \ell\le n}}$. Then each row in M is obtained from a linear combination of the corresponding row in $M^\star$ and the vector $(\frac{\partial q_1}{\partial s_\ell})_{1\le \ell\le n}$. More precisely, the ν-th row in M is given by

$$\nabla(q_{\nu+1})\frac{1}{q_1} - \nabla(q_1)\frac{q_{\nu+1}}{q_1^2}$$

for every $\nu \in \{1, \dots, n-1\}$, and where $\nabla(q_j) = (\frac{\partial q_j}{\partial s_1}, ..., \frac{\partial q_j}{\partial s_n})$. So the rank of $\begin{pmatrix}\nabla q_1\\ M^\star\end{pmatrix}$ is upper bounded by $n-1$. It remains to prove that this rank is preserved when the vectors $(\frac{\partial q_0}{\partial s_j})_{1\le j\le n}$ and $(\frac{\partial q_1}{\partial s_j})_{1\le j\le n}$ are incorporated to $M^\star$ as new rows. If this occurs, then the matrix $(\frac{\partial q_j}{\partial s_k})_{\substack{0\le j\le n\\ 1\le \ell\le n}}$ would have rank strictly smaller than n, and the parametrization does not correspond to a variety of dimension n.

From their definition,

$$a_{i,j} = \frac{\partial a_i}{\partial s_j} - \frac{\partial a_j}{\partial s_i} = \frac{\partial}{\partial s_j}\left(\frac{\frac{\partial q_0}{\partial s_i}}{q_1}\right) - \frac{\partial}{\partial s_i}\left(\frac{\frac{\partial q_0}{\partial s_j}}{q_1}\right) = \frac{1}{q_1^2}\left(\frac{\partial q_0}{\partial s_j}\frac{\partial q_1}{\partial s_i} - \frac{\partial q_0}{\partial s_i}\frac{\partial q_1}{\partial s_j}\right).$$

By the assumption of the theorem the left hand side of equation (3.14) vanishes for every $\nu \in \{2, \ldots, n\}$, hence,

$$\sum_{\substack{i,j \in \{1,\ldots,n\} \\ i<j}} \left(\frac{\partial q_0}{\partial s_j} \frac{\partial q_1}{\partial s_i} - \frac{\partial q_0}{\partial s_i} \frac{\partial q_1}{\partial s_j} \right) (-1)^{i+j} \det(M_{\{\nu\},\{i,j\}}) = 0 \tag{3.15}$$

for $\nu \in \{2, \ldots, n\}$. Regarding the generalized Laplace expansion (see for instance [16]), the left hand side of (3.15) is the determinant of a single $n \times n$-matrix and we get

$$\det \begin{pmatrix} \nabla q_0 \\ \nabla q_1 \\ M_{\{\nu\},\emptyset} \end{pmatrix} = 0\,.$$

Hence, all such $n \times n$ matrices have rank $n-1$. We still need to show, that the rank of $\begin{pmatrix} \nabla q_0 \\ M^\star \end{pmatrix}$ is at most $n-1$. Assume to the contrary, that the rank is n. Then $(\nabla q_2, \ldots, \nabla q_n)$ are linearly independent. Since the rank of $\begin{pmatrix} \nabla q_1 \\ M^\star \end{pmatrix}$ is at most $n-1$, we know that $(\nabla q_1, \ldots, \nabla q_n)$ are linearly dependent. Hence, ∇q_1 can be written as a linear combination of $\nabla q_1 = \sum_{j=2}^n \lambda_j \nabla q_j$. We take k such that $\lambda_k \neq 0$. Then $\nabla q_k = \frac{1}{\lambda_k}\left(\nabla q_1 - \sum_{\substack{j=2 \\ j \neq k}}^n \lambda_j \nabla q_j \right)$. Hence, the rank of $\begin{pmatrix} \nabla q_0 \\ M^\star \end{pmatrix}$ equals the rank of $\begin{pmatrix} \nabla q_0 \\ \nabla q_1 \\ M_{\{k\},\emptyset} \end{pmatrix}$ which we have shown to be at most $n-1$ so we have a contradiction.

From this we conclude that the rank of $(\frac{\partial q_j}{\partial s_k})_{\substack{0 \le j \le n \\ 1 \le k \le n}}$ is, at most, $n-1$, and the parametrization does not correspond to a variety of dimension n. □

For the rest of the thesis we assume that the quasilinear equations (3.14) are non-trivial, i. e. we are not in one of the special cases described above.

Method of characteristics

The quasilinear equations (3.14) can be solved by using the method of characteristics (see for instance [73] and Appendix D). Doing so we need to solve the following system of ordinary differential equations.

$$\left.\begin{aligned}\frac{\partial s_i}{\partial t} &= (-1)^i \det(M_{\emptyset,\{i\}}) \qquad\qquad \text{for } 1 \leq i \leq n\,,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \sum_{\substack{i,j\in\{1,\ldots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu} \det(M_{\{\nu\},\{i,j\}})\,.\end{aligned}\right\} \tag{3.16}$$

In case $n = 2$ we have seen in Section 3.1 that this can be transformed to a decoupled system which can be solved by methods presented in [44, 45, 46]. For $n \geq 3$ system (3.16) is no longer uncoupled in general. The first n equations form a possibly coupled system, whereas (as in the case $n = 2$) the last one can then be solved by integration. Hence, an arbitrary constant is involved. We show later that the introduction of these constants can be postponed.

Constants also appear in the solutions of the first n equations of (3.16). We get $s_i(t) = \chi_i(t, k_2, \ldots, k_n)$ where k_i are arbitrary constants. Finally the solution of the last equation is $v(t) = v(t, k_2, \ldots, k_n) = \bar{v}(t, k_2, \ldots, k_n) + \omega(k_2, \ldots, k_n)$ for some $\bar{v}$ and an arbitrary function ω. To resolve these constants, we compute ξ_k such that $s_i = \chi_i(\xi_1, \ldots, \xi_n)$ for all i. Note, that it is not always possible to find an explicit solution. In the negative case the procedure fails to find a solution of the APDE and we do not know whether a solution exists. If we are able to find an explicit solution, then $g_\nu(s_1, \ldots, s_n) = \bar{v}(\xi_1, \ldots, \xi_n) + \omega$. In general ω depends on a constant c. For simplicity reasons we fix $\omega = c$. This might restrict the solution set which can be computed, but it still allows useful investigation. Other choices and a survey for arbitrary ω are a subject of further research.

Note, that the first n equations of (3.16) do not depend on ν since the right hand side of (3.14) did not either. This means we can solve this part of the system of ODEs once for each APDE. What remains is to solve the last equation of (3.16). This needs to be done for every $\nu > 1$, but it can be done in parallel.

Solution procedure

Finally, using the results from the previous sections we obtain a procedure for solving APDEs in n variables.

Procedure 5.
Input: An autonomous APDE, $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$, where F is an irreducible and non-constant polynomial, and a proper rational parametrization $\mathcal{Q}(s_1, \ldots, s_n) = (q_0, \ldots, q_n)$

of the corresponding hypersurface defined by F.
Output: A solution of the APDE or "fail".

1. Compute the coefficients $a_i = \frac{\frac{\partial q_0}{\partial s_i}}{q_1}$, and $b_i = \frac{q_i}{q_1}$. Compute further $a_{j,k} = \frac{\partial a_j}{\partial s_k} - \frac{\partial a_k}{\partial s_j}$ and $b_{i,\ell} = \frac{\partial b_i}{\partial s_\ell}$.

2. Compute $\det(M_{\emptyset,\{i\}})$ for all i. If only one of them is non-zero, solve the ODEs by integration as described in Remark 3.15 and continue with Step 4.
   If all determinants are zero, compute $\sum_{\substack{i,j\in\{1,\ldots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}})$. If this is non-zero, there is no proper rational solution. The procedure stops. If this is zero, then $\mathcal{Q}$ does not fulfill the requirements.

3. Solve (in parallel) the quasilinear PDEs (3.14) for g_ν, $n \geq \nu > 1$, respectively. Using the method of characteristics proceed as follows.

   a) Solve the system of ODEs, $\frac{\partial s_i}{\partial t} = (-1)^i \det(M_{\emptyset,\{i\}})$, for all $1 \leq i \leq n$ and get solutions $s_i(t) = \chi_i(t, k_2, \ldots, k_n)$.

   b) Solve the ODE, $\frac{\mathrm{d}v}{\mathrm{d}t} = \sum_{\substack{i,j\in\{1,\ldots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}})$, by integration.

   c) Compute ξ_k such that $s_i = \chi_i(\xi_1, \ldots, \xi_n)$ for all i.

   d) Compute $g_\nu(s_1, \ldots, s_n) = \bar{v}(\xi_1, \ldots, \xi_n) + c_\nu$.

4. Use (3.10) to compute g_1.

5. Compute $h_1, \ldots, h_n$ such that $g(h_1(s_1, \ldots, s_n), \ldots, h_n(s_1, \ldots, s_n)) = (s_1, \ldots, s_n)$.

6. Compute the solution $q_0(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n))$.

We want to prove now that Procedure 5 actually computes solutions of APDEs. Properties of the output solutions are presented later on.

Theorem 3.19.
Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE. If Procedure 5 returns a function $v(x_1, \ldots, x_n)$ for input F, then v is a solution of $F = 0$.

Proof. By the last step of the procedure we know that

$$v(x_1, \dots, x_n) = q_0(h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)) \,.$$

with h_i such that $g(h_1(s_1, \dots, s_n), \dots, h_n(s_1, \dots, s_n)) = (s_1, \dots, s_n)$. Since g is a solution of the system (3.12), it fulfills the assumption that $u(g_1, \dots, g_n) = q_0$ for some solution u of the APDE. Hence, v is a solution. We have seen a more detailed description at the beginning of this section. □

Now, we show that the result does not change if we postpone the introduction of the constants $c_1, \dots, c_n$ to the end of the procedure. It is easy to see that if $u(x_1, \dots, x_n)$ is a solution of an autonomous APDE then so is $u(x_1 + c_1, \dots, x_n + c_n)$ for any constants c_i, $1 \leq i \leq n$. From the procedure we get for $i \geq 2$ that $g_i = \bar{g}_i + c_i$ with $\bar{g}_i$ not depending on c_j for all j. Furthermore, we see that in the computation of g_1 we use the derivatives of g_i only (and hence the c_i disappear). Therefore, we have that $g_1 = \bar{g}_1 + c_1$. Let $g = (g_1, \dots, g_n)$ and $\bar{g} = (\bar{g}_1, \dots, \bar{g}_n)$. In Step 5 we are looking for a function h such that $g \circ h = \mathrm{id}$. Now $g \circ h = \bar{g} \circ h + (c_1, \dots, c_n)$. Take $\bar{h}$ such that $\bar{g} \circ \bar{h} = \mathrm{id}$. Then $g \circ \bar{h}(s_1 - c_1, \dots, s_n - c_n)) = \mathrm{id}$. Hence, we can introduce the constants at the end.

In case the original APDE is in fact an AODE, the ODE in (3.16) turns out to be trivial and the integral in Step 4 is exactly the one which appears in Procedure 3. Of course then g is univariate and so is its inverse. We do not specify Procedure 5 to handle this case. Furthermore, if $n = 2$ this procedure specifies to Procedure 4. In this sense, this new procedure generalizes Procedure 3 and 4.

Remark 3.20.
Procedure 5 might fail in several steps. First of all, we avoided to talk about parametrizability by assuming there is a parametrization of the corresponding hypersurface. In case such a parametrization does not exist in a certain class there cannot exist a solution in this class either. Further we use the method of characteristics which might not give an explicit solution (compare [73]).

Later we compute g_1 by integration where a solution might only be found in a field extension, i. e. we might get out of the class of functions we are looking for. Nevertheless, if we find an integral in a field extension and the subsequent steps are successful as well, we might still get a solution. See for instance the examples in Section 3.3.2.

Finally, in Step 5 it might happen that there is no explicit solution for h_i. In all of these cases, we say that the procedure fails and then we do not know anything about solvability of the input APDE. In the latter case, however, we might state the solution implicitly.

3.3.1. Rational Solutions

For first-order autonomous AODEs the algorithm of Feng and Gao [17] gives an answer on whether or not a rational solution exists. As Procedure 5 is a generalization of Procedure 3 for AODEs, it also generalizes this algorithm. As in Procedure 4, any final result of the procedure is a solution of the differential equation, but however, the procedure might fail and then it does not tell us whether a solution might exist. In the following we collect properties of rational solutions and compute some examples.

Properties of Rational Solutions

Now we discuss the properties of rational solutions computed by Procedure 5. In particular we show that these solutions are proper and complete of suitable dimension.

Lemma 3.21.
If Procedure 5 yields a rational solution, then the solution is proper.

Proof. Let $\mathcal{L}$ be the corresponding parametrization of the output solution. In the procedure we start with a proper parametrization $\mathcal{Q}$ of the associated surface. When the procedure is successful we know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is proper as well. □

Recall Remark 1.2 which proves that the Jacobian of the corresponding parametrization of a proper solution computed by the procedure has generic rank n.

Theorem 3.22.
Assume Procedure 5 yields a rational solution $u(x_1, \ldots, x_n)$. Then the solution u is complete of suitable dimension.

Proof. From the investigation below Theorem 3.19 we know that $u(x_1, \ldots, x_n) = u^*(x_1 + c_1, \ldots, x_n + c_n)$ for some u^*. As usual, let $\mathcal{L}$ be the corresponding parametrization of u. For the case of two variables we see that

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} u_x(x+c_1, y+c_2) & u_y(x+c_1, y+c_2) \\ u_{xx}(x+c_1, y+c_2) & u_{xy}(x+c_1, y+c_2) \\ u_{yx}(x+c_1, y+c_2) & u_{yy}(x+c_1, y+c_2) \end{pmatrix} = \mathcal{J}_{\mathcal{L}}^{x,y} = \mathcal{J}_{\mathcal{L}} \,.$$

The equation $\mathcal{J}_{\mathcal{L}}^{c_1,\ldots,c_n} = \mathcal{J}_{\mathcal{L}}^{x_1,\ldots,x_n}$ also holds in general. From Lemma 3.21 we know that $\mathcal{L}$ is proper and from Remark 1.2 we know that a proper solution has a Jacobian of rank n. □

APDEs with Rational Solutions

We start with a full computation of an example. Examples and classes of APDEs in two variables with rational solutions can be found in Section 3.1.1 and Appendix C.3. Therefore, we focus on an example with more than two variables.

Example 3.23. (Example 7.11 of Kamke [31])
We consider the APDE, $F(u, u_{x_1}, u_{x_2}, u_{x_3}) = d_1 u_{x_1}^2 + d_2 u_{x_2}^2 + d_3 u_{x_3}^2 - u = 0$*, where* d_1*,* d_2 *and* d_3 *are non-zero constants. A possible parametrization is*

$$\mathcal{Q} = \left(s_1, s_2, \frac{-\sqrt{-\frac{d_2}{d_3}}s_1 + d_1\sqrt{-\frac{d_2}{d_3}}s_2^2 + \sqrt{-\frac{d_2}{d_3}}d_3 s_3^2}{2d_2 s_3}, \frac{s_1 - d_1 s_2^2 + d_3 s_3^2}{2d_3 s_3} \right).$$

The coefficients as computed in the procedure are

$$a_1 = \frac{1}{s_2}, \quad a_2 = 0, \quad a_3 = 0\,,$$

$$b_2 = \frac{-\sqrt{-\frac{d_2}{d_3}}s_1 + d_1\sqrt{-\frac{d_2}{d_3}}s_2^2 + \sqrt{-\frac{d_2}{d_3}}d_3 s_3^2}{2d_2 s_2 s_3}, \quad b_3 = \frac{s_1 - d_1 s_2^2 + d_3 s_3^2}{2d_3 s_2 s_3}\,.$$

Then we have to solve the following quasilinear equations

$$\frac{-s_1 + d_1 s_2^2 + d_3 s_3^2}{2d_3 s_2^3 s_3^2} = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(s_3\frac{\partial g_2}{\partial s_3} + s_2\frac{\partial g_2}{\partial s_2} + 2s_1\frac{\partial g_2}{\partial s_1}\right)}{2d_2 s_2^3 s_3},$$

$$-\frac{\sqrt{-\frac{d_2}{d_3}}\,(s_1 - d_1 s_2^2 + d_3 s_3^2)}{2d_2 s_2^3 s_3^2} = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(s_3\frac{\partial g_3}{\partial s_3} + s_2\frac{\partial g_3}{\partial s_2} + 2s_1\frac{\partial g_3}{\partial s_1}\right)}{2d_2 s_2^3 s_3}.$$

Simplifying these equations and using the ideas of the method of characteristics, we have to solve the following system of ODEs.

$$s_1' = \frac{2s_1s_3}{\sqrt{-\frac{d_2}{d_3}d_3}}\,,$$

$$s_2' = -\frac{\sqrt{-\frac{d_2}{d_3}}s_2s_3}{d_2}\,,$$

$$s_3' = -\frac{\sqrt{-\frac{d_2}{d_3}}s_3^2}{d_2}\,,$$

$$v' = \frac{d_1s_2^2 + d_3s_3^2 - s_1}{d_3}\,, \qquad \textit{resp.} \qquad v = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(-d_1s_2^2 + d_3s_3^2 + s_1\right)}{d_2}\,.$$

The first three equations are independent from the last one. They yield solutions

$$s_1 = \frac{c_2}{\left(c_1\sqrt{-\frac{d_2}{d_3}}d_3 + t\right)^2}\,, \qquad s_2 = \frac{c_3}{c_1d_2 - \sqrt{-\frac{d_2}{d_3}}t}\,, \qquad s_3 = -\frac{d_2}{c_1d_2 - \sqrt{-\frac{d_2}{d_3}}t}\,,$$

for some arbitrary constants c_1, c_2, c_3*. Resolving* t *and the constants results in*

$$t = -\frac{\sqrt{-\frac{d_2}{d_3}}d_3}{s_3}\,, \qquad c_2 = -\frac{d_2d_3s_1}{s_3^2}\,, \qquad c_3 = -\frac{d_2s_2}{s_3}\,. \tag{3.17}$$

Solving the last equation of the system of ODEs by integration yields

$$v = \frac{\frac{c_3^2d_1}{d_2} + \frac{c_2}{d_3} + d_2d_3}{t}\,, \qquad \textit{resp.} \qquad v = \frac{\sqrt{-\frac{d_2}{d_3}}\left(c_3^2d_1d_3 + c_2d_2 - d_2^2d_3^2\right)}{d_2^2t}$$

Using (3.17) *we get the solutions*

$$g_2 = \frac{\sqrt{-\frac{d_2}{d_3}}\left(-s_1 + d_1s_2^2 + d_3s_3^2\right)}{s_3}\,, \qquad g_3 = \frac{s_1 - d_1s_2^2 + d_3s_3^2}{s_3}\,.$$

Now, we need to compute g_1*. We do so be taking the first equation of* (3.10)*. Thus,*

$$g_1 = m_1(s_2, s_3)\,,$$

where m_1 *is an arbitrary function. Using the second equation of* (3.10) *we compute* m_1 *and get*

$$m_1 = 2d_1s_2 + m_2(s_3)\,.$$

Finally, we compute $m_2 = c_1$ *using the last equation in* (3.10)*. We choose* c_1 *to be* 0*. Hence,*

$$g_1 = 2d_1 s_2 \,.$$

Solving the system $g_i(h) = s_i$*, we get*

$$h_1 = \frac{1}{4}\left(\frac{s_1^2}{d_1} + \frac{s_2^2}{d_2} + \frac{s_3^2}{d_3}\right), \qquad h_2 = \frac{s_1}{2d_1}, \qquad h_3 = \frac{\frac{s_2}{\sqrt{-\frac{d_2}{d_3}}} + s_3}{2d_3}\,.$$

Hence,

$$q_0(h(x_1, x_2, x_3)) = h_1(x_1, x_2, x_3) = \frac{1}{4}\left(\frac{x_1^2}{d_1} + \frac{x_2^2}{d_2} + \frac{x_3^2}{d_3}\right)$$

is a solution of the APDE and $q_0(h(x_1 + c_1, x_2 + c_2, x_3 + c_3))$ *is a complete one.*

3.3.2. Other Solutions

So far we concentrated on rational solutions. For the case of $n = 2$ we additionally saw on examples that non-rational solutions might be computed by the procedure. In the general case we can as well try to do computations in some field extension and as before this will be subject to further investigation. Though, before we show some examples for the general case, we also prove some properties of any result of the procedure regardless of the class of functions. Similarly to Lemma 3.21 we get the following.

Lemma 3.24.
If Procedure 5 yields a solution, then the corresponding parametrization is injective almost everywhere.

Proof. Let $\mathcal{L}$ be the corresponding parametrization of the output solution. In the procedure we start with a proper parametrization $\mathcal{Q}$ of the associated surface. When the procedure is successful we know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is injective almost everywhere. □

A parametrization which is injective almost everywhere is also called *almost injective*. Note, that the Jacobian of an almost injective parametrization $\mathcal{P}(s_1, \ldots, s_n)$ has generic rank n. Indeed, since $\mathcal{P}$ is almost injective, there exists a map R such that $\mathrm{id} = R \circ \mathcal{P}$

generically. Thus $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_{R\circ\mathcal{P}} = \mathcal{J}_R(\mathcal{P}) \cdot \mathcal{J}_{\mathcal{P}}$. Taking into account, that the rank of a product of two matrices is less than or equal to the minimal rank of the two matrices, we get that $\operatorname{rank}(\mathcal{J}_{\mathcal{P}}) = n$.

Theorem 3.25.
Assume Procedure 5 yields a solution $u(x_1, \ldots, x_n)$. Then the solution u is complete of suitable dimension.

Proof. As usual, let $\mathcal{L}$ be the corresponding parametrization of u. Then the equation $\mathcal{J}_{\mathcal{L}}^{c_1,\ldots,c_n} = \mathcal{J}_{\mathcal{L}}^{x_1,\ldots,x_n}$ holds in general. From Lemma 3.24 we know that $\mathcal{L}$ is almost injective and the notes above show that an almost injective solution has a Jacobian of expected rank. ☐

The following examples show that the method is not restricted to finding rational solutions. It might happen that the steps in Procedure 5 can be done by doing computations in some extension field. In this case we can of course continue in the procedure and might get a non-rational solution.

Example 3.26. (Eikonal Equation with 5 variables)
We consider the APDE, $F(u, u_{x_1}, \ldots, u_{x_5}) = \left(\sum_{i=1}^{5} u_{x_i}^2\right) - 1 = 0$. A possible rational parametrization of the corresponding hypersurface is

$$\mathcal{Q} = \left(s_1, \frac{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}{D}, \frac{2s_2}{D}, \frac{2s_3}{D}, \frac{2s_4}{D}, \frac{2s_5}{D}\right),$$

where $D = s_2^2 + s_3^2 + s_4^2 + s_5^2 + 1$. The parametrization is proper. Indeed, the inverse is given by

$$s_1 = z\,, \qquad s_2 = -\frac{p_2}{p_1 - 1}\,, \qquad s_3 = \frac{p_3(p_1+1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,,$$
$$s_4 = \frac{p_4(p_1+1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,, \qquad s_5 = \frac{p_5(p_1+1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,.$$

The coefficients appearing in the procedure are

$$a_1 = \frac{s_2^2 + s_3^2 + s_4^2 + s_5^2 + 1}{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}\,, \qquad a_i = 0\,, \qquad \text{for } i \geq 2\,,$$
$$b_i = \frac{2s_i}{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}\,.$$

Then we get the following quasilinear equations for $2 \leq i \leq 5$.

$$\frac{32 s_i}{\left(s_2^2+s_3^2+s_4^2+s_5^2-1\right)^5} = \frac{16\left(s_2^2+s_3^2+s_4^2+s_5^2+1\right)\frac{\partial g_i}{\partial s_1}}{\left(s_2^2+s_3^2+s_4^2+s_5^2-1\right)^5}.$$

Here we are in the case of Remark 3.15 and hence, by integration we get

$$g_i = \frac{2 s_1 s_i}{D}, \qquad \text{for } i \geq 2.$$

Note, that for simplicity we chose the arbitrary functions which occur in the solutions of the ODEs to be 0. *Now we need to compute* g_1. *We do so by taking the first equation of* (3.10). *As a solution we get* $g_1 = \frac{s_1\left(s_2^2+s_3^2+s_4^2+s_5^2-1\right)}{D} + m_1(s_2,s_3,s_4,s_5)$, *where* m_1 *is an arbitrary function. Step by step we compute* m_1 *by using the other equations of* (3.10). *Using the second equation we have an ODE in* m_1. *We get* $m_1 = m_2(s_3,s_4,s_5)$. *Continuing like this we finally get* $m_1 = c_1$ *for an arbitrary constant. Since, we can deal with constants at the end of the procedure, we choose* c_1 *to be zero for the moment. Now we have to solve the system* $g_i(h) = s_i$. *A solution of this system is*

$$h_1 = \frac{\sqrt{s_2^2\left(s_1^2+s_2^2+s_3^2+s_4^2+s_5^2\right)}}{s_2},$$

$$h_i = \frac{s_1 s_2 s_i - s_i\sqrt{s_2^2\left(s_1^2+s_2^2+s_3^2+s_4^2+s_5^2\right)}}{s_2\left(s_2^2+s_3^2+s_4^2+s_5^2\right)}, \qquad \text{for } i \geq 2.$$

Hence we conclude that

$$q_0(h(x)) = h_1(x) = \sqrt{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2}$$

is a solution of the APDE.

Example 3.27.

We consider the APDE, $F(u,u_{x_1},u_{x_2},u_{x_3}) = (u_{x_1}+d_1)u_{x_2} - (u+d_2)u_{x_3} = 0$. *A possible proper parametrization is* $\mathcal{Q} = (s_1, s_2, s_3, \frac{(s_2+d_1)s_3}{s_1+d_2})$. *The coefficients are*

$$a_1 = \frac{1}{s_2}, \qquad a_2 = 0, \qquad a_3 = 0,$$

$$b_2 = \frac{s_3}{s_2}, \qquad b_3 = \frac{(s_2+d_1)s_3}{(s_1+d_2)s_2}.$$

Then we have to solve the following quasilinear equations

$$\frac{d_1+s_2}{(d_2+s_1)s_2^3} = \frac{1}{(d_2+s_1)^2 s_2^3} s_3\left((d_1+s_2)s_3\frac{\partial g_2}{\partial s_3} + s_2\left((d_1+s_2)\frac{\partial g_2}{\partial s_2} + (d_2+s_1)\frac{\partial g_2}{\partial s_1}\right)\right),$$

$$-\frac{1}{s_2^3} = \frac{1}{(d_2+s_1)^2 s_2^3} s_3\left((d_1+s_2)s_3\frac{\partial g_3}{\partial s_3} + s_2\left((d_1+s_2)\frac{\partial g_3}{\partial s_2} + (d_2+s_1)\frac{\partial g_3}{\partial s_1}\right)\right).$$

Omitting the details and intermediate steps we get the solutions

$$g_2 = -\frac{(d_2+s_1)(d_1-\log(s_2)s_2)}{(d_1+s_2)s_3}\,, \qquad g_3 = \frac{(d_2+s_1)^2(d_1-\log(s_2)s_2)}{(d_1+s_2)^2 s_3}\,.$$

Now, we need to compute g_1*. We do so be taking the first equation of* (3.10)*. This results in*

$$g_1 = \frac{(1+\log(-s_2))s_1}{d_1+s_2} + m_1(s_2,s_3)\,,$$

where m_1 *is an arbitrary function. Using the second equation of* (3.10) *we compute*

$$m_1 = \frac{d_2(1+\log(-s_2))}{d_1+s_2} + m_2(s_3)$$

Finally, we compute $m_3 = c_1$ *using the last equation in* (3.10)*. We choose* c_1 *to be* 0*. Hence,*

$$g_1 = \frac{(1+\log(-s_2))(d_2+s_1)}{d_1+s_2}\,.$$

Solving the system $g_i(h) = s_i$*, we get*

$$\begin{aligned} h_1 &= -\frac{d_2 s_2 + d_1 s_3 - \mathrm{e}^{-1-\frac{s_1 s_2}{s_3}} s_3}{s_2}\,,\\ h_2 &= -\mathrm{e}^{-1-\frac{s_1 s_2}{s_3}}\,,\\ h_3 &= \frac{\mathrm{e}^{-1-\frac{s_1 s_2}{s_3}}\left(-s_1 s_2 + \left(-1 + d_1 \mathrm{e}^{1+\frac{s_1 s_2}{s_3}}\right) s_3\right)}{s_2^2}\,. \end{aligned}$$

Hence,

$$q_0(h(x)) = h_1(x_1,x_2,x_3) = -\frac{d_2 x_2 + d_1 x_3 - \mathrm{e}^{-1-\frac{x_1 x_2}{x_3}} x_3}{x_2}$$

is a solution of the APDE.

3.4. Further approaches

In this section we present some approaches which might be used for extending the class of APDEs for which we can compute solutions. The approaches under consideration connect methods described in Chapter 2 respectively 3 and other known methods. The first approach helps to find solutions of AODEs, whereas the second idea, which is based on a known method, computes non-proper solutions of APDEs. The ideas are briefly discussed but more elaborate treatment is subject to further research.

Solving AODEs via ideas from APDEs

In Section 2.3 we introduced some ideas for higher-order AODEs referring to a similar idea for APDEs which we present here. Let us consider an autonomous second-order APDE, $F(u, u_x, u_{xx}) = 0$. Assume we have a proper rational parametrization $\mathcal{Q} = (q_0, q_1, q_2)$ of the surface $F(z, p_1, p_2) = 0$. As usual, we assume that $\mathcal{Q} = \mathcal{L}(g)$ for some function g, where $\mathcal{L}$ is the parametrization corresponding to a solution. Then, similar to system (1.2) for the first-order case we now get the following equations.

$$\begin{aligned}
q_0(s,t) &= u(g_1(s,t), g_2(s,t))\,,\\
q_1(s,t) &= u_x(g_1(s,t), g_2(s,t))\,,\\
q_2(s,t) &= u_{xx}(g_1(s,t), g_2(s,t))\,,
\end{aligned}$$

which yield by taking first derivatives

$$\begin{aligned}
\frac{\partial q_0}{\partial s} &= \frac{\partial g_2}{\partial s} u_y(g_1,g_2) + \frac{\partial g_1}{\partial s} u_x(g_1,g_2) = \frac{\partial g_2}{\partial s} u_y(g_1,g_2) + \frac{\partial g_1}{\partial s} q_1\,,\\
\frac{\partial q_0}{\partial t} &= \frac{\partial g_2}{\partial t} u_y(g_1,g_2) + \frac{\partial g_1}{\partial t} u_x(g_1,g_2) = \frac{\partial g_2}{\partial t} u_y(g_1,g_2) + \frac{\partial g_1}{\partial t} q_1\,,\\
\frac{\partial q_1}{\partial s} &= \frac{\partial g_2}{\partial s} u_{xy}(g_1,g_2) + \frac{\partial g_1}{\partial s} u_{xx}(g_1,g_2) = \frac{\partial g_2}{\partial s} u_{xy}(g_1,g_2) + \frac{\partial g_1}{\partial s} q_2\,,\\
\frac{\partial q_1}{\partial t} &= \frac{\partial g_2}{\partial t} u_{xy}(g_1,g_2) + \frac{\partial g_1}{\partial t} u_{xx}(g_1,g_2) = \frac{\partial g_2}{\partial t} u_{xy}(g_1,g_2) + \frac{\partial g_1}{\partial t} q_2\,.
\end{aligned}$$

Elimination of $u_y(g_1, g_2)$ from the first two equations and $u_{xy}(g_1, g_2)$ from the last two equations yields

$$\frac{\partial g_2}{\partial s}\frac{\partial q_0}{\partial t} - \frac{\partial g_2}{\partial t}\frac{\partial q_0}{\partial s} = \left(\frac{\partial g_2}{\partial s}\frac{\partial g_1}{\partial t} - \frac{\partial g_2}{\partial t}\frac{\partial g_1}{\partial s}\right) q_1\,, \tag{3.18}$$

$$\frac{\partial g_2}{\partial s}\frac{\partial q_1}{\partial t} - \frac{\partial g_2}{\partial t}\frac{\partial q_1}{\partial s} = \left(\frac{\partial g_2}{\partial s}\frac{\partial g_1}{\partial t} - \frac{\partial g_2}{\partial t}\frac{\partial g_1}{\partial s}\right) q_2\,. \tag{3.19}$$

Combining these two equations we get a quasilinear PDE in g_2.

$$q_1\left(\frac{\partial q_1}{\partial t}\frac{\partial g_2}{\partial s} - \frac{\partial g_2}{\partial t}\frac{\partial q_1}{\partial s}\right) = q_2\left(\frac{\partial q_0}{\partial t}\frac{\partial g_2}{\partial s} - \frac{\partial g_2}{\partial t}\frac{\partial q_0}{\partial s}\right)\,. \tag{3.20}$$

We can solve the quasilinear equation by the method of characteristics to get g_2. Equation (3.18) is a quasilinear equation in g_1, when g_2 is given. Once knowing g_1 and g_2

we compute, as usual, h_1 and h_2 such that $(g_1(h_1,h_2), g_2(h_1,h_2)) = (s,t)$. The following example illustrates the procedure.

Example 3.28.
The usability of this example is shown in Example 2.20 where the solution of this APDE is needed to solve a second-order AODE. We consider the APDE, $F(u, u_x, u_{xx}) = uu_{xx} + u_x^2 = 0$. *It is easy to see that* $\mathcal{Q} = \left(s, t, -\frac{t^2}{s}\right)$ *is a proper parametrization of the corresponding surface. In this case equation* (3.20) *is*

$$t\frac{\partial g_2}{\partial s} - \frac{t^2}{s}\frac{\partial g_2}{\partial t} = 0\,.$$

Using the method of characteristics we need to solve the ODEs

$$\begin{aligned}\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= -\frac{s(t)}{t}\,,\\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= 0\,.\end{aligned}$$

The solution of the first equation is $s(t) = \frac{k}{t}$, *which yields* $k = st$. *The solution of the second equation is* $v(t) = \nu_2(k)$ *for some function* ν_2. *Then* $g_2(s,t) = \nu_2(st)$. *Now we take equation* (3.18) *to compute* g_1*:*

$$\begin{aligned}-q_1\frac{\partial g_2}{\partial t}\frac{\partial g_1}{\partial s} + q_1\frac{\partial g_2}{\partial s}\frac{\partial g_1}{\partial t} &= \frac{\partial g_2}{\partial s}\frac{\partial q_0}{\partial t} - \frac{\partial g_2}{\partial t}\frac{\partial q_0}{\partial s}\\ -st\nu_2'(st)\frac{\partial g_2}{\partial s} + t^2\nu_2'(st)\frac{\partial g_1}{\partial t} &= -s\nu_2'(st)\end{aligned}$$

Using again the method of characteristics we have to solve the ODEs

$$\begin{aligned}\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= -\frac{s(t)}{t}\,,\\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= -\frac{s(t)}{t^2}\,.\end{aligned}$$

The first equation yields $s(t) = \frac{k}{t}$, *i. e.* $k = st$. *Solving the second equation by integration we get* $v(t) = \int \frac{k}{t^3}\,\mathrm{d}t = -\frac{k}{2t^2} + \nu_1(k)$ *for some function* ν_1. *Then* $g_1(s,t) = -\frac{s}{2t} + \nu_1(st)$.

We choose $\nu_1 = \nu_2 = \mathrm{id}$. *Then we can compute* h_1 *and* h_2*:*

$$h_1 = \frac{\sqrt{2}t^{3/2}}{\sqrt{s-t}} - \frac{\sqrt{2}s\sqrt{t}}{\sqrt{s-t}}\,, \qquad h_2 = -\frac{\sqrt{t}}{\sqrt{2s-2t}}\,.$$

Since $q_0 = s$ *we know that* $h_1(x,y)$ *is a solution of the APDE and so is* $h_1(x+c_1, y+c_2)$.

This idea can be generalized to higher order. Let $F(u, u^{(1,0,\ldots,0)}, \ldots, u^{(n,0,\ldots,0)}) = 0$, where $u^{(k_1,\ldots,k_n)} = \frac{\partial^{k_1}}{\partial x_1^{k_1}}\left(\ldots\left(\frac{\partial^{k_n} u}{\partial x_n^{k_1}}\right)\right)$. Let $\mathcal{Q}(s_1, \ldots, s_n) = (q_0, \ldots, q_n)$ be a proper rational parametrization of the corresponding hypersurface. As usual, we assume that $\mathcal{Q} = \mathcal{L}(g)$, where g is an invertible function and $\mathcal{L}$ is the parametrization induced by a solution. Then the following equations have to be fulfilled.

$$q_k = u^{(k,0,\ldots,0)}(g)\,, \qquad \text{for } k \in \{0, \ldots, n\}\,, \tag{3.21a}$$

$$\left(\frac{\partial q_k}{\partial s_i}\right)_{i\in\{1,\ldots,n\}} = G \cdot \left(\frac{\partial u^{(k,0,\ldots,0)}}{\partial x_i}(g)\right)_{i\in\{1,\ldots,n\}}\,, \qquad \text{for } k \in \{0, \ldots, n-1\}\,, \tag{3.21b}$$

where $G = (\frac{\partial g_i}{\partial s_j})_{i,j\in\{1,\ldots,n\}}$. Since g is an invertible function, the matrix G has an inverse. We pick the first row in each equation of (3.21b) and use (3.21a) to get

$$(q_i)_{i\in\{1,\ldots,n\}} = M \cdot ((G^{-1})_1)^T\,,$$

where $M = \left(\frac{\partial q_i}{\partial s_j}\right)_{\substack{i\in\{0,\ldots,n-1\}\\ j\in\{1,\ldots,n\}}}$ and $(G^{-1})_1$ is the first row of the inverse of G. By Cramer's rule we know that

$$(G^{-1})_{1,k} = \frac{\det(\tilde{M}_k)}{\det(M)}\,,$$

where $\tilde{M}_k$ is constructed from M by replacing the i-th column by $(q_1, \ldots, q_n)^T$. Using the definition of an inverse we conclude that

$$\sum_{i=1}^{n} \frac{\partial g_1}{\partial s_i} \det(\tilde{M}_i) = \det(M)\,, \tag{3.22a}$$

$$\sum_{i=1}^{n} \frac{\partial g_k}{\partial s_i} \det(\tilde{M}_i) = 0\,, \qquad \text{for } k > 1\,. \tag{3.22b}$$

These are quasilinear PDEs and can be solved by the method of characteristics. Finally, we summarize the procedure.

Procedure 6.
Input: An autonomous APDE, $F(u, u^{(1,0,\ldots,0)}, \ldots, u^{(n,0,\ldots,0)}) = 0$, with an irreducible and non-constant polynomial F, and a proper rational parametrization $\mathcal{Q}(s_1, \ldots, s_n) = (q_0, \ldots, q_n)$ of the corresponding hypersurface defined by F.
Output: A solution $u(x_1, \ldots, x_n)$ of the APDE or "fail".

- Solve the quasilinear PDE from (3.22b) with the method of characteristics and take $n-1$ independent solutions to get $g_2, \ldots, g_n$.

- Solve the quasilinear PDE (3.22a) with the method of characteristics to get g_1.
- Compute h such that $g(h) = \mathrm{id}$.
- Return $q_0(h(x_1, \ldots, x_n))$.

Note, that for $n = 2$ this procedure indeed specializes to the idea from above.

Example 3.29.
The solution of this example is used in Example 2.21. We consider the APDE,

$$F(u, u_x, u_{xx}, u_{xxx}) = uu_{xxx} - u_x u_{xx} = 0\,.$$

It is easy to see that $\mathcal{Q} = \left(s_1, s_2, s_3, \frac{s_2 s_3}{s_1}\right)$ *is a proper parametrization of the corresponding hypersurface. In this case* $M = I_n$ *and hence,* (3.22) *yields*

$$\begin{aligned}
\frac{\partial g_1}{\partial s_1} s_2 + \frac{\partial g_1}{\partial s_2} s_3 + \frac{\partial g_1}{\partial s_3} \frac{s_2 s_3}{s_1} &= 1\,,\\
\frac{\partial g_2}{\partial s_1} s_2 + \frac{\partial g_2}{\partial s_2} s_3 + \frac{\partial g_2}{\partial s_3} \frac{s_2 s_3}{s_1} &= 0\,,\\
\frac{\partial g_3}{\partial s_1} s_2 + \frac{\partial g_3}{\partial s_2} s_3 + \frac{\partial g_3}{\partial s_3} \frac{s_2 s_3}{s_1} &= 0\,.
\end{aligned}$$

Using the method of characteristics we need to solve the system of ODEs

$$\begin{aligned}
\frac{\mathrm{d}s_2(t)}{\mathrm{d}t} &= \frac{s_3(t)}{s_2(t)}\,,\\
\frac{\mathrm{d}s_3(t)}{\mathrm{d}t} &= \frac{s_3(t)}{t}\,.
\end{aligned}$$

We get $s_2 = -\sqrt{t^2 k_1 + 2k_2}$ *and* $s_3 = tk_1$*. Since the right hand side of the quasilinear equations for* g_2 *and* g_3 *is zero we conclude that* $g_2 = \nu_2(k_1, k_2)$ *and* $g_3 = \nu_3(k_1, k_2)$ *for some arbitrary functions* ν_2, ν_3*. We choose* $g_2 = k_1$ *and* $g_3 = k_2$*. It remains to compute* g_1*. For this we integrate* $\frac{1}{s_2(t)}$ *and get* $-\frac{\log\left(\sqrt{k_1}\sqrt{k_1 t^2 + 2k_2} + k_1 t\right)}{\sqrt{k_1}}$*. We resolve* k_1 *and* k_2 *which yields* $k_1 = \frac{s_3}{t}$ *and* $k_2 = \frac{1}{2}(s_2^2 - ts_3)$*. Hence,*

$$g_1 = -\frac{\log\left(s_3 + \sqrt{\frac{s_3}{s_1}}\sqrt{s_2^2}\right)}{\sqrt{\frac{s_3}{s_1}}}\,, \qquad g_2 = \frac{s_3}{s_1}\,, \qquad g_3 = \frac{1}{2}\left(s_2^2 - s_1 s_3\right)\,.$$

Finally, we compute h such that $g(h) = \mathrm{id}$ and get

$$h_1 = \frac{\mathrm{e}^{s_1\sqrt{s_2}}\left(\mathrm{e}^{-2s_1\sqrt{s_2}} - 2s_2s_3\right)}{2s_2}\,,$$
$$h_2 = \frac{\mathrm{e}^{s_1\sqrt{s_2}}\left(2s_2s_3 + \mathrm{e}^{-2s_1\sqrt{s_2}}\right)}{2\sqrt{s_2}}\,,$$
$$h_3 = \frac{1}{2}\mathrm{e}^{s_1\sqrt{s_2}}\left(\mathrm{e}^{-2s_1\sqrt{s_2}} - 2s_2s_3\right)\,.$$

Therefore, $q_0(h_1(x_1,x_2,x_3), h_2(x_1,x_2,x_3), h_3(x_1,x_2,x_3)) = h_1(x_1,x_2,x_3)$ is a solution of the APDE.

The idea can be further generalized to non-autonomous APDEs. Let F be such an APDE, i. e. $F(x_1, u, u^{(1,0,\ldots,0)}, \ldots, u^{(n,0,\ldots,0)}) = 0$, in $n+1$ variables. Let $\mathcal{Q}(s_1, \ldots, s_{n+1}) = (q_{-1}, q_0, \ldots, q_n)$ be a proper rational parametrization of the corresponding hypersurface. As usual, we assume that $\mathcal{Q} = \mathcal{L}(g)$. Then the following equations have to be fulfilled.

$$q_k = u^{(k,0,\ldots,0)}(g)\,, \qquad \text{for } k \in \{0,\ldots,n\}\,, \tag{3.23a}$$
$$\left(\frac{\partial q_{-1}}{\partial s_i}\right)_{i\in\{1,\ldots,n+1\}} = G\cdot(1,0,\ldots,0)^T\,, \tag{3.23b}$$
$$\left(\frac{\partial q_k}{\partial s_i}\right)_{i\in\{1,\ldots,n+1\}} = G\cdot\left(\frac{\partial u^{(k,0,\ldots,0)}}{\partial x_i}(g)\right)_{i\in\{1,\ldots,n+1\}}\,, \qquad \text{for } k \in \{0,\ldots,n-1\}\,, \tag{3.23c}$$

where $G = (\frac{\partial g_j}{\partial s_i})_{i,j\in\{0,\ldots,n\}}$. Since g is an invertible function, the matrix G has an inverse. We pick the first row of each equation in (3.23c) and the first row of (3.23b). Using (3.23a) we get

$$(1, q_1, \ldots, q_n)^T = M\cdot((G^{-1})_1)^T\,,$$

where $M = \left(\frac{\partial q_i}{\partial s_j}\right)_{\substack{i\in\{-1,\ldots,n-1\}\\ j\in\{1,\ldots,n\}}}$ and $(G^{-1})_1$ is the first row of the inverse of G. By Cramer's rule we conclude that

$$\sum_{i=1}^{n}\frac{\partial g_k}{\partial s_i}\det(\tilde{M}_i) = 0\,, \qquad \text{for } k \geq 1\,, \tag{3.24}$$

where $\tilde{M}_k$ is constructed from M by replacing the i-th column by $(1, q_1, \ldots, q_n)^T$. These are quasilinear PDEs and can be solved by the method of characteristics. Finally, we summarize the procedure.

Procedure 7.
Input: A non-autonomous APDE, $F(x_1, u, u^{(1,0,\dots,0)}, \dots, u^{(n,0,\dots,0)}) = 0$, with an irreducible and non-constant polynomial F, and moreover a proper rational parametrization $\mathcal{Q}(s_1, \dots, s_{n+1}) = (q_{-1}, q_0, \dots, q_n)$ of the corresponding hypersurface defined by F.
Output: A solution $u(x_1, \dots, x_{n+1})$ of the APDE or "fail".

- Solve the quasilinear PDE from (3.24) with the method of characteristics and take n independent solutions to get $g_1, \dots, g_n$.
- Compute h such that $g(h) = \text{id}$.
- Return $q_0(h(x_1, \dots, x_{n+1}))$.

Lemma 3.30.
If Procedure 6 or Procedure 7 yields a rational solution, then the solution is proper.

Proof. Let $\mathcal{L}$ be the corresponding parametrization of the output solution. The input $\mathcal{Q}$ is a proper parametrization of the associated surface. We know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is proper as well. □

Complete solutions which are not of suitable dimension

The previous idea for solving APDEs might help to solve certain AODEs. Now we look at an idea for finding solutions of APDEs by transforming the problem to AODEs. In [31, p. 94] Kamke describes the basics of this method for finding a complete solution of a first-order autonomous partial differential equation.

We consider an APDE, $F(u, u_x, u_y) = 0$. We assume that $u(x, y)$ is a solution of the form $u(x, y) = f(\xi)$ where $\xi = x - cy$ for some constant c. Then the equation reduces to an AODE, $F(u, u_x, u_y) = F(f, f', -cf') = 0$. Note, that the ODE is not necessarily irreducible even if the PDE was. Kamke [31] continues only in case this equation can be solved for f'. Taking into account the methods presented in Chapter 2, this ODE might be solved, provided that a parametrization is given.

The general framework of the method works as follows:

Procedure 8.

<u>*Input*</u>: An autonomous APDE, $F(u, u_x, u_y) = 0$, where F is irreducible and non-constant.

<u>*Output*</u>: A set of solutions $(u_i(x, y))_{i\in\{1,\ldots,\eta\}}$ of the APDE, where each element is of the form $f(x - cy)$, or "fail".

- Compute $G(f, f') = F(f, f', -c_2 f')$.
- Take all irreducible factors of G, say $G_1, \ldots, G_\eta$.
- Compute parametrizations of $G_i(y, z) = 0$, if possible.
- Use methods from Chapter 2 for computing solutions f_i of $G_i(f, f') = 0$.
  Return "fail" if the computation fails.
- Compute $u = (u_i(x, y))_{i\in\{1,\ldots,\eta\}} = (f_i(x - c_2 y))_{i\in\{1,\ldots,\eta\}}$.

From the above discussion it is easy to show that the result is indeed a solution.

Theorem 3.31.
Let v be the result of Procedure 8 for a given input F. Then v_i is a solution of $F = 0$ for each i.

An autonomous APDE, $F(u, u_x, u_y) = 0$, has a rational solution of the form $f(x-cy)$ if and only if there is an i such that $G_i(f, f')$ has a rational solution. By Theorem 2.1 this is the case if and only if for any proper rational parametrization $\mathcal{P}(t) = (r_1(t), r_2(t))$ of G_i the quotient $\frac{r_2}{r_1'}$ is either in the ground field of $\mathcal{P}$ or a quadratic polynomial $a(t-b)^2$.

As usual, we are interested in the properties of solutions computed by Procedure 8.

Lemma 3.32.
If Procedure 8 computes a list of non-constant rational solutions u, then each component v is non-proper.

Proof. If v is a solution of an APDE computed by Procedure 8, then $v(x, y) = f(x-c_2 y)$ for some rational function f. Let $\mathcal{L} = (v, v_x, v_y)$ be the corresponding parametrization of v. Then the Jacobian of $\mathcal{L}$ is

$$\mathcal{J}_{\mathcal{L}}^{x,y} = \begin{pmatrix} f'(x - c_2 y) & -c_2 f'(x - c_2 y) \\ f''(x - c_2 y) & -c_2 f''(x - c_2 y) \\ -c_2 f''(x - c_2 y) & c_2^2 f''(x - c_2 y) \end{pmatrix}.$$

It is easy to prove that $\operatorname{rank}(\mathcal{J}_{\mathcal{L}}^{x,y}) \leq 1$. Due to Remark 1.2, $\mathcal{L}$ is not proper and so neither is v. □

Note, that rationality of the solution is only required in the definition of proper. The fact that the rank of the Jacobian equals 1 is also valid for non-rational solutions.

Lemma 3.33.
Let u be the output of Procedure 8. Let $v = u_i(x + c_1, y)$ for some i. Assume v is non-constant and rational. Then v is a complete solution.

Proof. If u is computed by Procedure 8, then $v(x, y) = f(x + c_1 - c_2 y)$ for some rational function f. Since u_i is a solution we know by Theorem 3.31, and the fact that the APDE is autonomous, that v is a solution as well. Let $\mathcal{L} = (v, v_x, v_y)$ be the corresponding parametrization of v. Then the Jacobian of $\mathcal{L}$ with respect to c_1 and c_2 is

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} f'(x + c_1 - c_2 y) & -y f'(x + c_1 - c_2 y) \\ f''(x + c_1 - c_2 y) & -y f''(x + c_1 - c_2 y) \\ -c_2 f''(x + c_1 - c_2 y) & -f'(x + c_1 - c_2 y) + c_2 y f''(x + c_1 - c_2 y) \end{pmatrix}.$$

Since, v is non-constant and rational, it is easy to verify that $\operatorname{rank}(\mathcal{J}_{\mathcal{L}}^{c_1,c_2}) = 2$. □

Note, if v is not rational, then the statement is not necessarily true. Assume for instance, that $f(\xi) = \mathrm{e}^{\xi}$. Then

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} \mathrm{e}^{c_1+x-c_2 y} & -c_2 \mathrm{e}^{c_1+x-c_2 y} \\ \mathrm{e}^{c_1+x-c_2 y} & -c_2 \mathrm{e}^{c_1+x-c_2 y} \\ -c_2 \mathrm{e}^{c_1+x-c_2 y} & c_2^2 \mathrm{e}^{c_1+x-c_2 y} \end{pmatrix},$$

which has only rank 1.

Corollary 3.34.
A solution computed by Procedure 8 is not complete of suitable dimension.

Proof. In the proof of Lemma 3.32 we have shown that the Jacobian has rank ≤ 1. □

We show on some examples how the procedure works and describe the connection and differences to the methods in Chapter 2 and 3.

Example 3.35.
We consider the APDE, $F(u, u_x, u_y) = 6u^4 + 5u_x^3 + 5u_x^2 u_y = 0$. *Using Procedure 4 we find the solution* $\frac{10}{3(x+c_1-y-c_2)^2(y+c_2)}$ *which is complete of suitable dimension.*

Now we use Procedure 8 to get $G(f, f') = 6f^4 + 5f'^3(1-c_2)$. *We find a proper rational parametrization* $\mathcal{P} = (\frac{5}{6}t^3(c_2-1), \frac{5}{6}t^4(c_2-1))$. *Hence,* $A = \frac{1}{3}t^2$. *This fulfills the requirements for a rational solution of the AODE (c.f. Theorem 2.1). A solution of the AODE is* $f(x) = \frac{5}{2}\frac{9(1-c_2)}{(x+c_1)^3}$. *Hence, we get a solution* $u(x, y) = \frac{5}{2}\frac{9(1-c_2)}{(x-c_2y+c_1)^3}$ *of* $F = 0$. *This solution is complete, but not of suitable dimension.*

In the following example G contains f' as a factor which always leads to a constant solution. Furthermore there is a factor not depending on f and f' at all, but only on c. Such factors give solutions with arbitrary functions f.

Example 3.36.
Consider the APDE, $F(u, u_x, u_y) = u_x - u_y = 0$. *By the transformation we get* $f'(1+c) = 0$. *Now either* $f' = 0$ *and hence* f *is constant, or* $c = -1$. *In the latter case we get that* $u(x, y) = f(x + y)$ *is a solution for any* f.

The following lemma shows some cases in which Procedure 8 finds a rational solution.

Lemma 3.37.
Let $F(u, u_x, u_y) = 0$ *be an APDE such that the origin is a* $(d-1)$*-fold point of* $F(z, p, q) = 0$, *where* d *is the total degree of* F. *Furthermore, we assume that there is a parametrization of* F *of the form* $\mathcal{Q} = (s^{n+1}B(t), ts^nB(t), s^nB(t))$ *where* $n \in \mathbb{Z} \setminus \{-1\}$ *and* $B(t) \in \mathbb{K}(t)$. *Then Procedure 8 yields a rational solution.*

Proof. Procedure 8 defined G to be $G(a, b) = F(a, b, -c_2b)$. Let $\mathcal{P}(s) = (p_1(s), p_2(s)) = (s^{n+1}B(-\frac{1}{c_2}), -s^n\frac{1}{c_2}B(-\frac{1}{c_2}))$. Then

$$G(p_1, p_2) = F(p_1, p_2, -c_2p_2) = F(\mathcal{Q}(s, -\frac{1}{c_2})) = 0\,.$$

Hence, $\mathcal{P}$ is a parametrization of G_i for some i. We compute $A = \frac{p_2}{p_1'} = \frac{-1}{c_2(n+1)}$. By Theorem 2.1 there is a rational solution $f(\xi) = (-\frac{\xi+c_1}{c_2(n+1)})^{n+1}B(-\frac{1}{c_2})$. Hence, $(-\frac{x-c_2y+c_1}{c_2(n+1)})^{n+1}B(-\frac{1}{c_2})$ is a rational solution of the APDE. □

Note, that by Lemma 3.7 Procedure 4 yields an algebraic (but not necessarily rational) solution for an APDE which fulfills the requirements of the above lemma. Recall, that Lemma 3.9 describes cases in which the solution is indeed rational. By Lemma 3.37 we can now find rational (but non-proper) solutions even if none of the cases of Lemma 3.9 is fulfilled.

Using Procedure 3 instead of Algorithm 1 for solving the ODEs in Procedure 8 might result in non-algebraic solutions.

Example 3.38.
We consider the APDE, $F(u, u_x, u_y) = u^2 + u_x u_y = 0$*. Procedure 8 computes* $G = f^2 + cf'^2 = (f + \sqrt{c_2} f')(f - \sqrt{c_2} f')$*. Hence, we get* $\mathcal{P} = (\pm\sqrt{c_2} s, s)$ *with* $A = \frac{s}{\pm\sqrt{c_2}}$*. Then* $g(s) = \int \frac{1}{A} \, \mathrm{d}s = \pm\sqrt{c_2} \log(s)$ *and its inverse* $h(s) = \mathrm{e}^{\pm\frac{s}{\sqrt{c_2}}}$*. Thus,* $f(\xi) = \pm\sqrt{c_2} \mathrm{e}^{\pm\frac{\xi}{\sqrt{c_2}}}$ *is a solution of* G *and hence,* $\pm\sqrt{c_2} \mathrm{e}^{\pm\frac{c_1 + x - c_2 y}{\sqrt{c_2}}}$ *is a solution of* F*.*

The ideas of Procedure 8 are easily generalized to an arbitrary number of independent variables. In this case we have two choices. Either we reduce to an AODE directly, or we just reduce to an APDE with less variables.

Reduction of the number of variables

We consider an APDE, $F(u, u_{x_1}, \dots, u_{x_n}) = 0$ and assume that $u(x_1, \dots, x_n)$ is a solution of the form $u(x_1, \dots, x_n) = f(\xi, x_3, \dots, x_n)$, where $\xi = x_1 - c_2 x_2$ for some constant c_2. Then the APDE reduces to another APDE of lower order, $F(u, u_{x_1}, \dots, u_{x_n}) = F(f, f_\xi, -c_2 f_\xi, f_{x_3}, \dots, f_{x_n}) = 0$. Now, in fact, this PDE (or to be precise, its factors) might be solved by Procedure 5. Note, that we could solve F directly by Procedure 5. However, we would get other solutions.

Procedure 9.
Input: An autonomous APDE, $F(u, u_{x_1}, \dots, u_{x_n}) = 0$, where F is irreducible and non-constant.
Output: A set of solutions $(u_i(x_1, \dots, x_n))_{i \in \{1, \dots, \eta\}}$ of the APDE, where each element is of the form $f(x_1 + c_1 - c_2 x_2, x_3, \dots, x_n)$, or "fail".

- Compute $G = F(f, f_\xi, -c_2 f_\xi, f_{x_3}, \dots, f_{x_n})$.
- Take all irreducible factors of G, say $G_1, \dots, G_\eta$.
- Compute parametrizations of G_i, if possible.

- Use Procedure 5 for computing solutions f_i of $G_i = 0$.
  Return "fail" if the computation fails.

- Compute $u = (u_i(x_1, \ldots, x_n))_{i\in\{1,\ldots,\eta\}} = (f_i(x_1 + c_1 - c_2x_2, x_3, \ldots, x_n))_{i\in\{1,\ldots,\eta\}}$.

Theorem 3.39.
Let v be the result of Procedure 9 for a given input F. Then v_i is a solution of $F = 0$ for each i.

Of course we can take any number of linear combinations $x_i - c_{i,j}x_j$ for reduction. Furthermore, we could take linear combinations involving more than two variables. A special case using relations for all variables reduces to an ODE as we see in the following.

Reduction to ODE

We consider an APDE, $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$. The following idea has been already mentioned in [31, 13.2]. However, we can now incorporate the methods from Chapter 2. We assume that $u(x_1, \ldots, x_n)$ is a solution of the form $u(x_1, \ldots, x_n) = f(\xi)$ where $\xi = x_1 - \sum_{i=2}^{n} c_ix_i$ for some constants c_i. Then the equation reduces to an ODE $F(u, u_{x_1}, \ldots, u_{x_n}) = F(f, f', -c_2f', \ldots, -c_nf') = 0$. Now, in fact, this ODE (or to be precise, its factors) might be solved by Algorithm 1 or by Procedure 3.

Procedure 10.
Input: An autonomous APDE, $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$, where F is irreducible and non-constant.
Output: A set of solutions $(u_i(x_1, \ldots, x_n))_{i\in\{1,\ldots,\eta\}}$ of the APDE, where each element is of the form $f(x_1 + c_1 - \sum_{i=2}^{n} c_ix_i)$), or "fail".

- Compute $G(f, f') = F(f, f', -c_2f', \ldots, -c_nf')$.

- Take all irreducible factors of G, say $G_1, \ldots, G_\eta$.

- Compute parametrizations of G_i, if possible.

- Use methods from Chapter 2 for computing solutions f_i of $G_i(f, f') = 0$.
  Return "fail" if the computation fails.

- Compute $u = (u_i(x_1, \ldots, x_n))_{i\in\{1,\ldots,\eta\}} = (f_i(x_1 + c_1 - \sum_{i=2}^{n} c_ix_i))_{i\in\{1,\ldots,\eta\}}$.

Theorem 3.40.
Let v be the result of Procedure 10 for a given input $F = 0$. Then v_i is a solution of F for each i.

The ideas from above can be easily extended to higher-order autonomous APDEs. We briefly discuss the case for two variables.

Higher-Order APDEs

We consider an APDE, $F(u, u_x, u_y, \ldots, \frac{\partial^\nu u}{\partial x^\nu}, \frac{\partial^\nu u}{\partial x^{\nu-1}\partial y}, \ldots, \frac{\partial^\nu u}{\partial y^\nu}) = 0$. We assume that $u(x, y)$ is a solution of the form $u(x, y) = f(\xi)$ where $\xi = x - cy$ for some constant c. Then our equation reduces to an ODE

$$F(u, u_x, u_y, \ldots, \frac{\partial^\nu u}{\partial y^\nu}) = F(f, f', -cf', \ldots, f^{(\nu)}, -cf^{(\nu)}, \ldots, (-c)^\nu f^{(\nu)}) = 0\,.$$

We can use the method described in Section 2.3 for solving the ODE. Note, that it is of course also possible to combine higher order and arbitrary many variables, but in the general case notation will be rather long and hence, we refrain from explicitly doing so.

Example 3.41. (Laplace Equation)
We consider the Laplace equation $F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = u_{xx} + u_{yy} = 0$. Then we get the ODE, $F(f, f', -cf', f'', -cf'', c^2 f'') = f'' + c^2 f'' = f''(1 + c^2) = 0$. This ODE has the solution $f(\xi) = c_1 + c_2\xi$. Hence, $u(x, y) = f(x - cy) = c_1 + c_2(x - cy)$ which is definitely a solution of $F = 0$.

Note, that we did not define the notions of complete and suitable dimension for higher order.

4. Linear Transformations

Transformations are widely used in different sense for solving differential equations (see for instance [73, Chapter I.B] for transformations of both ODEs and PDEs). The main aim is usually to transform a differential equation to another one which is easier to solve. After solving the new equation, the solution can be transformed back to obtain a solution of the original equation.

The kind of transformations we are investigating here are linear ones which transform implicit differential equations and likewise the parametrizations of their corresponding hypersurfaces. Linear transformations have been used in [42] for classifying AODEs. Using this classification it can be decided whether a given AODE can be transformed to an autonomous one, i. e. whether the class under consideration contains an autonomous AODE. It was shown in [42] that the associated system (see Section 2.1.2) remains invariant under linear transformations.

Here, we want to use the same idea for transforming non-autonomous APDEs to autonomous ones. This approach helps to solve some non-autonomous APDEs. Though, we do not have the associated system as an invariant, but there are still certain properties which are preserved by the transformation.

The idea of linear transformations can be generalized to transformations by birational maps (see [43] for AODEs). However, investigation of using this idea for APDEs is not topic of the current work.

In Section 4.1 we consider first-order APDEs in several variables. Specifying the number of variables to $n = 1$ yields the known transformations for AODEs. Many conclusions from the ODE case can be adopted rather easily. However, we will see that some nice results are not true for APDEs in general. A main question under consideration is whether some given APDE can be linearly transformed to an autonomous one. Later in Section 4.2 we show how the approach can be extended to higher order whereat for notational reasons we focus on the case of second order.

4.1. Linear Transformations of First-Order APDEs

In this section we introduce linear transformations for partial differential equations. For now we focus on first-order APDEs. The definitions work quite analogously to the case of AODEs in [42]. Let $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$ be a first-order APDE. Let L be a linear map defined by

$$\begin{aligned} L : \mathbb{K}(x_1, \ldots, x_n)^{2n+1} &\longrightarrow \mathbb{K}(x_1, \ldots, x_n)^{2n+1}\,, \\ w &\longmapsto Aw + B\,, \end{aligned}$$

where A is an invertible $((2n+1) \times (2n+1))$-matrix and B a column vector. We are interested in such L which map any tuple $(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) \in \mathbb{K}(x_1, \ldots, x_n)^{2n+1}$ to a similar one, say $(x_1, \ldots, x_n, v, v_{x_1}, \ldots, v_{x_n})$.

Obviously not every matrix is suitable. Using some simple examples for the function u, we find out that the matrix A and the vector B have to be of the form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \beta_2 & \cdots & \beta_n & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix},$$

with $\alpha, \delta, \beta_i \in \mathbb{K}$, $\alpha \neq 0$. In short notation we write

$$A = \left(\begin{array}{c|c} I_n & 0 \\ \hline \bar{\beta} & \\ 0 & \alpha I_{n+1} \end{array}\right), \qquad B = \begin{pmatrix} 0 \\ \delta \\ \bar{\beta}^T \end{pmatrix}, \tag{4.1}$$

where $\bar{\beta}$ is a row vector. Let $\bar{x} = (x_1, \ldots, x_n)^T$. Then we get that

$$L(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = A \cdot \begin{pmatrix} \bar{x} \\ u \\ \nabla u \end{pmatrix} + B = \begin{pmatrix} \bar{x} \\ \bar{\beta} \cdot \bar{x} + \alpha u + \delta \\ \alpha \nabla u + \bar{\beta}^T \end{pmatrix}.$$

We need to show that $\frac{\partial}{\partial x_j}(\bar{\beta} \cdot \bar{x} + \alpha u + \delta) = (\alpha \nabla u + \bar{\beta}^T)_j$. But this is easy to see.

Now we define the set $\mathcal{G}$ as the set of pairs $[A, B]$ with A and B of the form (4.1). Let $L_1 = [A_1, B_1]$ and $L_2 = [A_2, B_2]$ be elements of $\mathcal{G}$. We define the operation $\circ$ as follows.

$$L_1 \circ L_2 = [A_1 \cdot A_2, A_1 \cdot B_2 + B_1] = \left[\left(\begin{array}{c|c} I_n & 0 \\ \hline \bar{\beta}_1 + \alpha_1 \bar{\beta}_2 & \\ 0 & \alpha_1 \alpha_2 I_{n+1} \end{array} \right), \left(\begin{array}{c} 0 \\ \alpha_1 \delta_2 + \delta_1 \\ \alpha_1 \bar{\beta}_1^T + \bar{\beta}_2^T \end{array} \right) \right],$$

which is again in $\mathcal{G}$. There is a neutral element $N = [I_{2n+1}, 0]$ with respect to $\circ$. It is easy to see that

$$\left[\left(\begin{array}{c|c} I_n & 0 \\ \hline -\frac{1}{\alpha_1} \bar{\beta}_1 & \\ 0 & \frac{1}{\alpha_1} I_{n+1} \end{array} \right), \left(\begin{array}{c} 0 \\ -\frac{\delta_1}{\alpha_1} \\ -\frac{1}{\alpha_1} \bar{\beta}_1^T \end{array} \right) \right]$$

is the inverse of L_1. Hence, $(\mathcal{G}, \circ)$ is a group. We call it the group of *linear transformations*. Note, that this is a direct generalization of the ODE case which was investigated in [42]. For $n = 1$ the group $\mathcal{G}$ exactly accords with the group described in that paper.

Lemma 4.1.
The group $(\mathcal{G}, \circ)$ forms a group action on the set of algebraic partial differential equations, $\mathcal{APDE}$.

$$\begin{aligned} \diamond : \mathcal{G} \times \mathcal{APDE} &\longrightarrow \mathcal{APDE}\,, \\ (L, F) &\longmapsto L \diamond F = (F \circ L^{-1})\,. \end{aligned}$$

Proof. Obviously, $N \diamond F = F \circ N = F$. Furthermore

$$(L_1 \circ L_2) \diamond F = F \circ (L_1 \circ L_2)^{-1} = F \circ (L_2^{-1} \circ L_1^{-1}) = (F \circ L_2^{-1}) \circ L_1^{-1} = L_1 \diamond (L_2 \diamond F)\,.$$

□

From this we see that linear transformations keep the degrees of u and its derivatives u_{x_i} invariant. However, the degrees of the independent variables x_i might change.

Lemma 4.2.
The group action $(\mathcal{G}, \circ)$ as defined in Lemma 4.1 can be extended to the set of parametrizable APDEs. Indeed if $\mathcal{P}$ is a proper parametrization of the corresponding hypersurface, $F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$, then $L \circ \mathcal{P}$ is a proper parametrization of the hypersurface $L \diamond F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$.

Proof. The proof works the same way as in the ODE case [42]. Let $F \in \mathcal{APDE}$ and let $\mathcal{P}$ be a proper parametrization of the hypersurface $F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$. Then

$$(L \diamond F)(L \circ \mathcal{P}) = (F \circ L^{-1})(L \circ \mathcal{P}) = F(\mathcal{P}) = 0$$

and hence, $L \circ \mathcal{P}$ is a proper parametrization, since L is invertible. □

Remark 4.3.
Note, that the solution u of an APDE, $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$, induces a parametrization

$$\mathcal{P} = (x_1, \ldots, x_n, u(x_1, \ldots, x_n), u_{x_1}(x_1, \ldots, x_n), \ldots, u_{x_n}(x_1, \ldots, x_n))$$

of a lower dimensional hypersurface on the hypersurface, e. g. if $n = 2$ it parametrizes a surface on the hypersurface. In this sense the parametrization $\mathcal{P}$ cannot be proper. Nevertheless, we can consider properness according to the surface defined by the parametrization $(u(x, y), u_x(x, y), u_y(x, y))$.

We show by an example how linear transformations might be applied to solve non-autonomous APDEs.

Example 4.4.
Consider the APDE, $F(x, y, u, u_x, u_y) = -6 + u + 4u_x - u_x^2 + 2u_y + u_y^2 - 2x + y = 0$. We are interested in whether this can be linearly transformed to an autonomous APDE. Let $L \in \mathcal{G}$ be an arbitrary linear transformation. Then

$$\begin{aligned} L \diamond F &= F(x, y, -\frac{\beta_1}{\alpha}x - \frac{\beta_2}{\alpha}y + \frac{1}{\alpha}u - \frac{\delta}{\alpha}, \frac{u_x - \beta_1}{\alpha}, \frac{u_y - \beta_2}{\alpha}) \\ &= -3 - \frac{\delta}{\alpha} + \frac{u}{\alpha} - \left(-2 - \frac{\beta_1}{\alpha} + \frac{u_x}{\alpha}\right)^2 + \left(1 - \frac{\beta_2}{\alpha} + \frac{u_y}{\alpha}\right)^2 - 2x - \frac{\beta_1 x}{\alpha} + y - \frac{\beta_2 y}{\alpha}, \end{aligned}$$

which is autonomous for $-2 - \frac{\beta_1}{\alpha} = 0$ and $1 - \frac{\beta_2}{\alpha} = 0$. For instance take $\alpha = 1$, $\beta_1 = -2$, $\beta_2 = 1$. Then we get

$$L \diamond F = -3 - \delta + u - u_x^2 + u_y^2 .$$

This APDE can be solved by the procedure from Chapter 3. The solution is $\bar{u} = 3 + \delta + \frac{1}{4}(x - c_1)^2 - \frac{1}{4}(y - c_2)^2$. Hence, $(x, y, u, u_x, u_y) = L^{-1}(x, y, \bar{u}, \bar{u}_x, \bar{u}_y)$ yields a solution *of $F = 0$. We get the following complete solution*

$$\begin{aligned} u &= -\frac{\beta_1}{\alpha}x - \frac{\beta_2}{\alpha}y + \frac{1}{\alpha}\bar{u} - \frac{\delta}{\alpha} \\ &= 2x - y + \bar{u} - \delta \\ &= 2x - y + 3 + \frac{1}{4}(x - c_1)^2 - \frac{1}{4}(y - c_2)^2 \,. \end{aligned}$$

Note, that the solution is not complete of suitable dimension. Note further, that here we cannot, as in the autonomous case, add arbitrary constants to x and y respectively but of course c_1 and c_2 can still be chosen arbitrarily.

This is, however, not always possible. Some APDEs cannot be transformed to a non-autonomous one. Consider for instance the APDE, $F(u, u_x, u_y) = u + u_x + u_y + x^2 = 0$. Applying a general linear transformation yields $u - \beta_1 x - \beta_2 y - \delta + u_x - \beta_1 + u_y - \beta_2 + \alpha x^2 = 0$, which is not autonomous for any choice of α, β_1, β_2 and δ. The reason is the relation of degrees which is described in Section 4.1.2.

The question whether a given APDE can be linearly transformed to an autonomous one, is decidable. We can always apply a general linear transformation and solve a system of equations to check the existence of a suitable choice of the parameters α, β_i, δ. In some cases we can even decide linear transformability in advance as we see in Section 4.1.2.

4.1.1. Properties Preserved by Linear Transformations

In this section we investigate whether properties of solutions are preserved under linear transformations. Since, some of these properties were so far just defined for autonomous APDEs we first have to generalize them. For proving properties of a solution $u(x, y)$ we sometimes need the tuple (u, u_x, u_y).

Definition 4.5.
Let $u(x_1, \ldots, x_n)$ be a solution of some APDE. Then u yields a parametrization $\mathcal{L} = (x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})$ of the hypersurface defined by $F = 0$. We define $\mathcal{L}^* = (u, u_{x_1}, \ldots, u_{x_n})$ and call it the corresponding parametrization. *We call a solution* proper *iff the corresponding parametrization is proper. Similarly we define solutions to be* complete *or* complete of suitable dimension *as for APDEs using the corresponding parametrization $\mathcal{L}^*$.*

The fist result on preservation of properties is disillusioning.

Remark 4.6.
It is easy to see, that properness is not necessarily preserved. Consider for instance

$$F(x, y, u, u_x, u_y) = u_x^2 - 4u + 3 + 4y = 0\,.$$

Then obviously $v(x, y) = x^2 + x + 1 + y$ *is a solution. The corresponding parametrization is* $\mathcal{L}^*(s, t) = (s^2 + s + t + 1, 2s + 1, 1)$ *which is proper.*

Now we take the linear transformation with $\alpha = 1$*,* $\beta_2 = -1$ *and* $\beta_1 = \delta = 0$*. We get the new equation* $L \diamond F = u_x^2 - 4u + 3 = 0$ *and the corresponding solution is* $x^2 + x + 1$ *which is definitely not proper.*

The example shows that properness is not preserved in general. However, there are certain cases, in which properness can be proven to be preserved. The proofs are rather simple and are based on the following. Let u be a proper rational solution of some APDE, $F = 0$ and let v be the transformed solution. We assume that the rank of the Jacobian of the parametrization induced by v is less than or equal to 1, i. e. that determinants of all (2×2)-submatrices are 0.

$$0 = v_x v_{xy} - v_y v_{xx} = (\beta_1 + \alpha u_x)u_{xy} - (\beta_2 + \alpha u_y)u_{xx}\,, \tag{4.2a}$$

$$0 = v_x v_{yy} - v_y v_{xy} = (\beta_1 + \alpha u_x)u_{yy} - (\beta_2 + \alpha u_y)u_{xy}\,, \tag{4.2b}$$

$$0 = v_{xx} v_{yy} - v_{xy}^2 = u_{xx}u_{yy} - u_{xy}^2\,. \tag{4.2c}$$

From this we conclude the following lemmata.

Lemma 4.7.
Let u *be a proper rational solution of an APDE,* $F = 0$*, and let* v *be the transformed solution of the transformed APDE. Let* (p_0, p_1, p_2) *be the proper parametrization corresponding to* u*. Assume there is a rational function* φ *such that* $\varphi(p_1, p_2) = (\varphi_1(p_1, p_2), \varphi_2(p_1, p_2)) = (x, y)$*. Then* v *is proper.*

Proof. Since, u is proper there is a rational function R such that $R(p_0, p_1, p_2) = (x, y)$. Let (q_0, q_1, q_2) be the parametrization corresponding to v. We define a rational function T using R and φ:

$$\overline{T}(\omega_0, \upsilon_1, \upsilon_2) := R\left(\frac{\omega_0 - \beta_1\varphi_1(\upsilon_1, \upsilon_2) - \beta_2\varphi_2(\upsilon_1, \upsilon_2) - \delta}{\alpha}, \upsilon_1, \upsilon_2\right)\,,$$

$$T(\omega_0, \omega_1, \omega_2) := \overline{T}(\omega_0, \frac{\omega_1 - \beta_1}{\alpha}, \frac{\omega_2 - \beta_2}{\alpha})\,.$$

Then T proves the properness of v since

$$T(q_0, q_1, q_2) = R(p_0, p_1, p_2) = (x, y)\,.$$

$\square$

Lemma 4.8.
Let u be a proper rational solution of an APDE, $F = 0$, and let v be the transformed solution of the transformed APDE. Let $u_x \neq -\frac{\beta_1}{\alpha}$, $u_y \neq -\frac{\beta_2}{\alpha}$. Let further $u_{xy} = 0$ or $u_{xx} = 0$ or $u_{yy} = 0$. Then the Jacobian of the parametrization corresponding to v has rank 2.

Proof. If $u_{xx} = 0$ we conclude from (4.2c) that $u_{xy} = 0$. If $u_{yy} = 0$ we conclude from (4.2c) that $u_{xy} = 0$. Since, $u_{xy} = 0$ and $u_x \neq -\frac{\beta_1}{\alpha}$, $u_y \neq -\frac{\beta_2}{\alpha}$ we conclude from (4.2) that $u_{xx} = u_{yy} = 0$ which is a contradiction to u being proper. Hence, the Jacobian of the parametrization corresponding to v has rank 2. $\square$

Lemma 4.9.
Let u be a proper rational solution of an APDE, $F = 0$ and let v be the transformed solution of the transformed APDE. Assume that $u_x = -\frac{\beta_1}{\alpha} \in \mathbb{K}$ or $u_y = -\frac{\beta_2}{\alpha} \in \mathbb{K}$. Then v is not proper.

Proof. Let $u_x = -\frac{\beta_1}{\alpha} \in \mathbb{K}$. Then, obviously, $u_{xx} = u_{xy} = 0$. The equations in (4.2) are satisfied, hence, v is not proper. The same holds for the other case. $\square$

This can be easily generalized to the following Lemma.

Lemma 4.10.
Let $u(x, y) = (\lambda\beta_2 - \beta_1)x + G(y + \lambda x)$ or $u(x, y) = (\lambda\beta_1 - \beta_2)y + G(x + \lambda y)$, for some $\lambda \in \mathbb{K}$ and some rational function G, be a proper rational solution of an APDE, $F = 0$, and let v be the transformed solution of the transformed APDE. Then v is not proper.

Proof. Let $u(x, y) = (\lambda\beta_2 - \beta_1)x + G(y + \lambda x)$. From the definition of u we get that $\lambda u_y = u_x - (\lambda\beta_2 - \beta_1)$. Hence, the Jacobian of the corresponding parametrization of v is

$$\begin{pmatrix} \alpha(u_x + \beta_1) & \frac{\alpha}{\lambda}(u_x + \beta_1) \\ \lambda\alpha u_{xy} & \alpha u_{xy} \\ \alpha u_{xy} & \frac{\alpha}{\lambda} u_{xy} \end{pmatrix},$$

which has rank less than or equal to 1. Hence, v is not proper. The other case works analogously. □

After this short consideration of properness we continue with other possible properties of solutions. Though, properness is not preserved, we are more lucky with completeness of solutions.

Lemma 4.11.
If $\mathcal{L}^*$ is a parametrization corresponding to a complete solution of some APDE, $F = 0$, then $(L \circ \mathcal{L})^*$ is a parametrization corresponding to a complete solution of $L \diamond F$.

Proof. Let $u(x_1, \ldots, x_n, c_1, \ldots, c_n)$ be a complete solution of the APDE, $F = 0$, and let $\mathcal{L} = (x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})$. Then $L \circ \mathcal{L} = A \cdot \mathcal{L} + B = (x_1, \ldots, x_n, v, v_{x_1}, \ldots, v_{x_n})$, where $[A, B] \in \mathcal{G}$. Since u is complete, the Jacobian of $\mathcal{L}^* = (u_{x_1}, \ldots, u_{x_n})$ with respect to $c_1, \ldots, c_n$ has rank n. Hence, $\mathcal{J}^{c_1,\ldots,c_n}_{(L\circ\mathcal{L})^*} = \alpha \mathcal{J}^{c_1,\ldots,c_n}_{\mathcal{L}^*}$. Therefore, we have shown that v is a complete solution. □

Unfortunately, this proof does not extend to completeness of suitable dimension. The next remark shows why.

Remark 4.12.
It is easy to see, that completeness of suitable dimension is not necessarily preserved. Consider for instance

$$F(x, y, u, u_x, u_y) = u_x^2 - 4u + 3 + 4y = 0\,.$$

Then obviously $v(x, y) = (x - c_1 - c_2 y)^2 + \frac{3}{4} + y$ is a solution. The Jacobians of $\mathcal{L}^* = (v, v_x, v_y)$ with respect to c_1, c_2 and x, y respectively are

$$\mathcal{J}^{c_1,c_2}_{\mathcal{L}^*} = \begin{pmatrix} -2(x - c_1 - c_2 y) & -2y(x - c_1 - c_2 y) \\ -2 & -2y \\ 2c_2 & -2(x - c_1 - 2c_2 y) \end{pmatrix},$$

$$\mathcal{J}^{x,y}_{\mathcal{L}^*} = \begin{pmatrix} 2(x - c_1 - c_2 y) & -2c_2(x - c_1 - c_2 y) + 1 \\ 2 & -2c_2 \\ -2c_2 & 2c_2^2 \end{pmatrix},$$

which have rank 2 and hence, v is complete of suitable dimension. Now we take the linear transformation with $\alpha = 1$, $\beta_2 = -1$ and $\beta_1 = \delta = 0$. We get the new equation

$L \diamond F = u_x^2 - 4u + 3 = 0$ and the corresponding solution is $(x - c_1 - c_2 y)^2 + \frac{3}{4}$. This solution yields Jacobians of the corresponding parametrization with respect to c_1, c_2 and x, y respectively

$$\mathcal{J}^{c_1,c_2}_{(L\circ\mathcal{L})^*} = \begin{pmatrix} -2(x-c_1-c_2y) & -2y(x-c_1-c_2y) \\ -2 & -2y \\ 2c_2 & -2(x-c_1-2c_2y) \end{pmatrix},$$

$$\mathcal{J}^{x,y}_{(L\circ\mathcal{L})^*} = \begin{pmatrix} 2(x-c_1-c_2y) & -2c_2(x-c_1-c_2y) \\ 2 & -2c_2 \\ -2c_2 & 2c_2^2 \end{pmatrix},$$

which have rank 2 and 1 and hence, the solution is complete but not of suitable dimension.

4.1.2. Linear Transformations of Special Equations

In this section we investigate how some classes of APDEs behave under linear transformation. For instance, we consider APDEs which are solvable for some variables and examine whether the transformed APDE is again solvable for the same variable. Furthermore, we check whether degrees are preserved and how this is related to the question whether certain APDEs can be linearly transformed to an autonomous one.

Generalizing Example 4.4 we get the following result for APDEs which are linear in u and the independent variables.

Corollary 4.13.
Let $F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = G(u_{x_1}, \dots, u_{x_n}) + u + c + \sum_{i=0}^n a_i x_i = 0$ where G is a polynomial which does not depend on $x_1, \dots, x_n, u$. Then there is a linear transformation from F to an autonomous APDE, $\overline{F} = 0$, with $\overline{F}(0, \dots, 0) = 0$.

Proof. This is done by the choice $\alpha = 1$, $\beta_i = a_i$ for all $i \in \{1, \dots, n\}$, and $\delta = c + G(-\beta_1, \dots, -\beta_n)$. □

This corollary can be easily generalized.

Corollary 4.14.
Let $F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = \sum_{k=0}^m \left(u + c + \sum_{i=0}^n a_i x_i\right)^k G_k(u_{x_1}, \dots, u_{x_n}) = 0$ with polynomials G_k which do not depend on $x_1, \dots, x_n, u$. Then there is a linear transformation from F to an autonomous APDE.

Proof. This is done by the choice $\alpha = 1$, $\beta_i = a_i$ for all $i \in \{1, \ldots, n\}$, and arbitrary δ. $\square$

As mentioned before, the degrees in some variables are invariant whereas other degrees change under linear transformations. Starting from an autonomous APDE, we examine how the degrees might change under linear transformation.

Corollary 4.15.
Let $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = \overline{F}(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE and let $G = L \diamond F$. Then

$$\begin{aligned} \deg_{x_i}(G) &= \begin{cases} \deg_u(F) & \text{if } \beta_i \neq 0\,, \\ 0 & \text{if } \beta_i = 0\,, \end{cases} \\ \deg_u(G) &= \deg_u(F)\,, \\ \deg_{u_{x_i}}(G) &= \deg_{u_{x_i}}(F)\,. \end{aligned}$$

The converse statement of this corollary can be used in some cases for deciding whether an APDE can be linearly transformed to an autonomous one.

Corollary 4.16.
Let $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an arbitrary APDE. Then F can not be linearly transformed to an autonomous APDE if one of the following holds.

- $0 \neq \deg_{x_i}(F) \neq \deg_u(F)$ *for some* $i \in \{1, \ldots, n\}$.
- $0 \neq \deg_{x_i}(F) \neq \deg_{x_j}(F) \neq 0$ *for some pair* $(i, j) \in \{1, \ldots, n\}^2$.

Given some condition on the monomial with highest degree in u we can go further.

Corollary 4.17.
Let $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = \sum_{i=0}^{m} u^i G_i(x_1, \ldots, x_n, u_{x_1}, \ldots, u_{x_n})$ for some polynomials $G_i \in \mathbb{K}[x_1, \ldots, x_n, u_{x_1}, \ldots, u_{x_n}]$. If $G_m \notin \mathbb{K}[u_{x_1}, \ldots, u_{x_n}]$, then F cannot be linearly transformed to an autonomous APDE.

Proof. We see that

$$L \diamond F = u^m \frac{1}{\alpha^m}(L \diamond G_m) + \sum_{k=0}^{m-1} u^k \sum_{i=k}^{m} \binom{i}{k} \frac{1}{\alpha^i} \left(-\delta - \sum_{j=0}^{n} \beta_j x_j \right)^{i-k} (L \diamond G_i)\,.$$

Since $L \diamond G_m \in \mathbb{K}[x_1, \ldots, x_n, u_{x_1}, \ldots, u_{x_n}] \setminus \mathbb{K}[u_{x_1}, \ldots, u_{x_n}]$, Corollary 4.16 implies that $L \diamond F$ is not autonomous. □

The next paragraphs show results on APDEs which are solvable for some elements. Here, by solvable we mean rationally solvable, i. e. the APDE is linear in some variable.

Equations solvable for u_{x_i}

We consider an APDE whose defining polynomial is linear in some first derivative (w. l. o. g. say u_{x_1}), i. e.

$$0 = F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = u_{x_1} - G(x_1, \ldots, x_n, u, u_{x_2}, \ldots, u_{x_n})$$

for some rational function G. Then the transformed equation

$$L \diamond F = \frac{u_{x_1} - \beta_1}{\alpha} - G(x_1, \ldots, x_n, \frac{1}{\alpha}(u - \delta - \sum_{k=0}^{n} \beta_k x_k), \frac{u_{x_2} - \beta_2}{\alpha}, \ldots, \frac{u_{x_n} - \beta_n}{\alpha})$$

is again solvable for u_x.

Equations solvable for u

We consider an APDE whose defining polynomial is linear in u, i. e.

$$0 = F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = u - G(x_1, \ldots, x_n, u_{x_1}, \ldots, u_{x_n})$$

for some rational function G. Then the transformed equation

$$L \diamond F = \frac{1}{\alpha}\left(u - \delta - \sum_{k=0}^{n} \beta_k x_k\right) - G(x_1, \ldots, x_n, \frac{u_{x_1} - \beta_1}{\alpha}, \ldots, \frac{u_{x_n} - \beta_n}{\alpha})$$

is again solvable for u.

Equations solvable for x_i

We consider an APDE whose defining polynomial is linear in some independent variable (w. l. o. g. say x_1), i. e.

$$0 = F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = x_1 - G(x_2, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})$$

for some rational function G. In this case the transformed equation

$$L \diamond F = x_1 - G(x_2, \ldots, x_n, \frac{1}{\alpha}(u - \delta - \sum_{k=0}^{n} \beta_k x_k), \frac{u_{x_1} - \beta_1}{\alpha}, \ldots, \frac{u_{x_n} - \beta_n}{\alpha})$$

is not necessarily solvable for x_1, since the degree of x_1 in $L \diamond F$ depends on the degree of u in G.

4.2. Linear Transformations of Higher-Order APDEs

The group $\mathcal{G}$ from Section 4.1 can be easily generalized to higher-order PDEs. For notational reasons we restrict to second order. For the case of n variables and order 2 we have $n + \binom{n}{2} = \frac{n(n+1)}{2}$ different second derivatives $u_{x_i x_j}$ with $i \leq j$. By ∇^2 we denote the vector of these derivatives ordered lexicographically by the pair (i, j).

$$A = \left(\begin{array}{c|c} I_n & 0 \\ \hline \bar{\beta} & \\ 0 & \alpha I_{n(n+3)/2} \end{array}\right), \qquad B = \begin{pmatrix} 0 \\ \delta \\ \bar{\beta}^T \\ 0 \end{pmatrix},$$

where $\bar{\beta}$ is a row vector of dimension n. Similarly, let $\bar{x} = (x_1, \ldots, x_n)^T$. Then we get that

$$L(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots, u_{x_n x_n}) = A \cdot \begin{pmatrix} \bar{x} \\ u \\ \nabla u \\ \nabla^2 u \end{pmatrix} + B = \begin{pmatrix} \bar{x} \\ \bar{\beta} \cdot \bar{x} + \alpha u + \delta \\ \alpha \nabla u + \bar{\beta}^T \\ \alpha \nabla^2 u \end{pmatrix}.$$

We need to show that $\frac{\partial}{\partial x_j}(\bar{\beta} \cdot \bar{x} + \alpha u + \delta) = (\alpha \nabla u + \bar{\beta}^T)_j$ and $\frac{\partial}{\partial x_j}(\alpha \nabla u + \bar{\beta}^T)_i = (\alpha \nabla^2 u)_{(i-1)(2n-i)/2+j}$. But this is easy to see.

Again we let $\mathcal{G}$ be the group consisting of pairs $[A, B]$ with A and B of the form above. The proof, that $\mathcal{G}$ is indeed a group, works similarly to the first-order case. Also the proofs for the group action are the same.

Applying the group action in the case of two variables yields

$$L \diamond F = F \circ L^{-1} = F(x, y, \frac{u - \beta_1 x - \beta_2 y - \delta}{\alpha}, \frac{u_x - \beta_1}{\alpha}, \frac{u_y - \beta_2}{\alpha}, \frac{u_{xx}}{\alpha}, \frac{u_{xy}}{\alpha}, \frac{u_{yy}}{\alpha}).$$

This implies that the exponents of the second derivatives remain invariant under linear transformations, i. e.

$$L \diamond \left(\sum_{i,j,k} u_{xx}^i u_{xy}^j u_{yy}^k G_{i,j,k}(x, y, u, u_x, u_y) \right) = \sum_{i,j,k} \frac{u_{xx}^i u_{xy}^j u_{yy}^k}{\alpha^{i+j+k}} (L \diamond G_{i,j,k})(x, y, u, u_x, u_y).$$

Hence, the degree of second derivatives is preserved. Likewise the property of F being solvable for some second derivatives remains invariant. Similarly to the case of first order we can conclude, that the degrees of u and its derivatives are invariant under linear transformations. Furthermore, invariance of solvability for some variable is inherited:

- Solvability for x_i is not invariant as we have seen for the first-order case.
- Solvability for u is invariant, since second derivatives do not interfere in the degrees.
- Solvability for u_{x_i} is invariant, since second derivatives do not interfere in the degrees.
- Solvability for u_{x_i,x_j} is invariant, since second derivatives are transformed to constant factors of itself.

The following example illustrates the usage of linear transformations for higher-order APDEs.

Example 4.18.
We consider the non-autonomous APDE,

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = -2 - u + uu_x + u_y + x - u_x x + 2y - 2u_x y - u_{xx}\mu = 0\,.$$

By a general linear transformation we get

$$\begin{aligned} L \diamond F = {} & -2 - \frac{u}{\alpha} + \frac{uu_x}{\alpha^2} + \frac{u_y}{\alpha} + x - \frac{u_x x}{\alpha} + 2y - \frac{2u_x y}{\alpha} - \frac{u\beta_1}{\alpha^2} + \frac{2x\beta_1}{\alpha} - \frac{u_x x\beta_1}{\alpha^2} \\ & + \frac{2y\beta_1}{\alpha} + \frac{x\beta_1^2}{\alpha^2} - \frac{\beta_2}{\alpha} + \frac{y\beta_2}{\alpha} - \frac{u_x y\beta_2}{\alpha^2} + \frac{y\beta_1\beta_2}{\alpha^2} + \frac{\delta}{\alpha} - \frac{u_x\delta}{\alpha^2} + \frac{\beta_1\delta}{\alpha^2} - \frac{u_{xx}\mu}{\alpha}\,. \end{aligned}$$

With the choice $\alpha = 1$, $\beta_1 = -1$, $\beta_2 = -2$, $\delta = 0$ we get

$$L \diamond F = uu_x + u_y + \mu u_{xx}\,,$$

which is the viscid Burgers equation.

Elliptic, parabolic and hyperbolic APDEs

For second-order PDEs there is a classification by using the discriminant. There are three different types: elliptic, parabolic and hyperbolic. We recall this classification. Given an APDE, $F = 0$, by

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u - g(x,y) = 0\,.$$

Let $D = D(F) = a(x,y)c(x,y) - \frac{b(x,y)^2}{2}$. D is called the discriminant. Then the equation is called

Elliptic in (x, y) iff $D > 0$,

Parabolic in (x, y) iff $D = 0$,

Hyperbolic in (x, y) iff $D < 0$.

A second-order APDE is called elliptic, parabolic or hyperbolic if the respective property holds for all points (x, y). Now we apply a general linear transformation L to the equation $F = 0$ above and we get

$$\begin{aligned} F \circ L^{-1} = \frac{a(x,y)}{\alpha} u_{xx} + \frac{b(x,y)}{\alpha} u_{xy} + \frac{c(x,y)}{\alpha} u_{yy} \\ + \frac{d(x,y)}{\alpha}(-\beta_1 + u_x) + \frac{e(x,y)}{\alpha}(-\beta_2 + u_y) \\ + \frac{f(x,y)}{\alpha}(-\beta_1 x - \beta_2 y - \delta + u) - g(x,y)\,. \end{aligned}$$

Hence,

$$D(F \circ L^{-1}) = \frac{a(x,y)}{\alpha}\frac{c(x,y)}{\alpha} - \frac{b(x,y)^2}{2\alpha^2} = \frac{1}{\alpha^2} D(F)\,.$$

Thus, the type of a second-order APDE is invariant under linear transformations.

5. Conclusion

Several exact procedures are presented in the preceding chapters. They are all based on the same idea but work for different kinds of differential equations. As mentioned throughout the text the procedures generalize several existing methods. Summarizing the new achievements; there are methods for autonomous algebraic ordinary differential equations of any order (Procedure 3 and 6) and there is a procedure for autonomous first-order algebraic partial differential equations with an arbitrary number of independent variables (Procedure 5). Furthermore, previously known methods induce a procedure for higher-order autonomous APDEs (see Section 3.4) and linear transformations allow to solve some non-autonomous AODEs and APDEs (see Chapter 4).

All these methods have in common, that they assume the existence of a transformation from an arbitrary given parametrization of a hypersurface to a parametrization induced by a solution. This assumption is well reasoned but not always valid. Hence, the methods might fail to find a solution. This, however, does not mean that no solution exists. If the problem is computationally too expensive, a different input parametrization might lead to success. Furthermore, some intermediate steps of the procedures do not necessarily yield explicit symbolic solutions. But even if the methods fail to yield an explicit solution, they still very often lead to an implicit description.

It is desirable to extend the procedures to decision algorithms, i. e. algorithms which compute a solution in a certain class of functions or show that such a solution cannot exist. However, this remains to be an open problem.

Nevertheless, the procedures represent a useful tool to tackle differential equations. As shown in Appendix C the procedures work for a wide range of well-known differential equations from literature. The main procedures for AODEs and APDEs (Procedure 3 and 5 respectively) compute proper and complete solutions of suitable dimension. All other procedures for first-order ADEs yield at least complete solutions.

More detailed investigation of the idea for higher-order AODEs is currently under development. Further generalizations of the method might be worth considering:

- higher-order APDEs
- (1-dimensional) systems of AODEs
- non-autonomous APDEs

The latter is currently work in progress.

A. More Differential Algebra

In this chapter we recall further definitions of differential algebra which are not essential for the main part of the thesis but useful for a more comprehensive understanding of the basics. Most of these notions can be found in standard textbooks on differential algebra such as [33, 55].

We start with the main definition of a differential field.

Definition A.1. (c. f. Ritt [55])
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $\delta : \mathbb{K} \longrightarrow \mathbb{K}$ be an operation on that field with the following properties

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y)\,, && \textit{linearity,} \\ \delta(x \cdot y) &= \delta(a) \cdot b + a \cdot \delta(b)\,, && \textit{Leibniz rule.} \end{aligned} \tag{A.1}$$

Then we call $(\mathbb{K}, \delta)$ *a* differential field.

Let $\delta_1, \ldots, \delta_m$ be operations on the field $\mathbb{K}$ each of them fulfilling (A.1). *Assume also that*

$$\delta_i(\delta_j(z)) = \delta_j(\delta_i(z))\,,$$

for all $z \in \mathbb{K}$. Then we call $(\mathbb{K}, \delta)$ *a* partial differential field.

Similarly to commutative ring theory we can think about ideals which should now also be closed under derivation.

Definition A.2. (c. f. Ritt [55])
Let $\mathbb{K}(x)\{u\}$ be the ring of differential polynomials and let $I \subseteq \mathbb{K}(x)\{u\}$. Then, I is a differential ideal *iff the following conditions hold.*

- *For all $p_1, \ldots, p_k \in I$ and $c_1, \ldots, c_k \in \mathbb{K}(x)\{u\}$ the linear combination $c_1p_1 + \ldots + c_kp_k$ is also in I.*
- *For all $p \in I$ the derivative $\delta(p)$ is in I.*

Let $\mathbb{K}(x_1, \ldots, x_n)\{u\}$ *be the ring of partial differential polynomials and let* I *be a subset of* $\mathbb{K}(x_1, \ldots, x_n)\{u\}$*. Then,* I *is a* partial differential ideal *iff the following conditions hold.*

- *For all* $p_1, \ldots, p_k \in I$ *and* $c_1, \ldots, c_k \in \mathbb{K}(x_1, \ldots, x_n)\{u\}$ *the linear combination* $c_1 p_1 + \ldots + c_k p_k$ *is also in* I*.*
- *For all* $p \in I$ *the derivatives* $\delta_j(p)$ *are in* I *for all* $j \in \{1, \ldots, n\}$*.*

Standard notions from commutative algebra transfer to differential algebra.

Definition A.3. (c. f. Ritt [55])
Let I, J *be (partial) differential ideals.*

- I *is called* prime *iff* $\forall p, q \in \mathbb{K}(x_1, \ldots, x_n)\{u\} : pq \in I \Rightarrow (p \in I \vee q \in I)$*.*
- *Let* $\bar{J} \subseteq J$ *be an ideal. Then* J *is called a* divisor *of* $\bar{J}$*.*
  If J *is prime, then* J *is called a* prime divisor.
- *Let* J *be a prime divisor of* $\bar{J}$*. Then* J *is called* essential *iff* J *is not a divisor of any other prime divisor of* $\bar{J}$*.*
- *The (partial) differential ideal* $\sqrt{I} = \{p \in \mathbb{K}(x_1, \ldots, x_n)\{u\} \mid \exists n \in \mathbb{N} : p^n \in I\}$ *is called the* radical differential ideal *of* I*.*
- *A differential ideal* I *is called* perfect *iff* $I = \sqrt{I}$*.*

Let P *be a set of (partial) differential polynomials* $P \subseteq \mathbb{K}(x_1, \ldots, x_n)\{u\}$*. Let* $[P]$ *be the set of all linear combinations of elements of* P*. Then* $[P]$ *is a (partial) differential ideal. We say* $[P]$ *is* generated by P*. In fact,* $[P]$ *is the smallest (partial) differential polynomial containing* P*.*

We write $\{P\}$ *for the smallest perfect ideal containing* P*. We call* $\{P\}$ *the radical differential ideal generated by* P *or the perfect differential ideal generated by* P*.*

Let ν *be an element of some extension field of* $\mathbb{K}(x_1, \ldots, x_n)$*. We call* ν *a* zero *of* P *iff for all* $p \in P$ *we have* $p(\nu) = 0$*.*

The following definition of an ordering on the derivatives is needed for the notion of separants in partial differential algebra.

Definition A.4. (Ritt [55])
Let $\mathbb{K}(x_1, \ldots, x_n)\{u\}$ *be the ring of partial differential polynomials. We define an ordering on the derivatives as follows. For each* x_i *we fix positive integers* $\alpha_{i,1}, \ldots, \alpha_{i,\nu}$ *for some* ν*. Let* v_1 *and* v_2 *be compositions of derivations:*

$$v_1 = \frac{\partial^{i_1+\ldots+i_n}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}},$$
$$v_2 = \frac{\partial^{j_1+\ldots+j_n}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}.$$

We define the marks *of* v_1 *as* $\beta_j = \sum_{k=1}^{\nu} \alpha_{k,j} i_k$ *for each* j*. Let* γ_j *be the marks of* v_2*. We say that* v_1 *is of* higher rank *than* v_2 *iff there is an index* ι *such that* $\beta_\iota > \gamma_\iota$ *and* $\beta_j = \gamma_j$ *for all* $j < \iota$*.*

Let F *be a partial differential polynomial. We call a derivative* $v = \frac{\partial^{i_1+\ldots+i_n}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$ *the* leader *of* F *iff it is the derivative with highest rank occurring in* F*.*

B. Parametrizations

In this chapter we present some more details on the problem of parametrization of hypersurfaces. We mainly focus on special classes of curves, surfaces and general hypersurfaces which can be parametrized easily. We further give references of general algorithms if they exist and remarks on available implementations. Sometimes it might be interesting to use a known parametrization for finding another one which is in some sense simpler or better. This approach is called reparametrization. We also give references to such algorithms.

In the methods presented in Chapter 2 and Chapter 3 we assume a parametrization as input. Therefore, information on how to actually compute parametrizations is in large part omitted. We make up for this topic here. In Section B.1 we present rational parametrizations, whereas in Section B.2 we give further ideas of radical parametrizations. Finally, Section B.3 contains ideas on other kinds of parametrizations.

B.1. Rational Parametrization

In this section we summarize different algorithms from literature for computing rational parametrizations of algebraic curves, surfaces and hypersurfaces. The problem for curves and surfaces can be considered to be solved in general, whereas for hypersurfaces there is still no general algorithm.

All important definitions of rational parametrizations and their properties can be found in Chapter 1.

In the following sections we present algorithms, ideas and references for special kinds of curves, surfaces and hypersurfaces respectively. For the sake of the readers who might be interested in parametrization of varieties of a certain dimension, some of the methods are presented in each of the subsections even if they are just specialized versions of more general methods.

B.1.1. Rational Parametrization of Curves

The problem of rational parametrizability of curves is fully solved. An algebraic curve has a rational parametrization if and only if it has genus[1] zero. There is a general algorithm for computing proper rational parametrizations of curves (see for instance [63]). An implementation of such an algorithm can be found for instance in the computer algebra systems MAPLE[2] (package algcurves, available since Version V R5, based on an algorithm described in [70]) and MAGMA[3] (since version V2.8 [11]). For certain classes of curves, special algorithms exist.

Linear Occurrence of a Variable

Let $f(x, y)$ be the defining polynomial of an algebraic curve. We assume that $f(x, y) = g_0(x) + yg_1(x)$ for some polynomials g_0 and g_1. Then obviously $\mathcal{Q}(s) = \left(s, -\frac{g_0(s)}{g_1(s)}\right)$ is a rational parametrization of the curve. It is easy to see that the parametrization is also proper. This works analogously if x appears linearly.

Parametrization by Lines

Let $f(x, y)$ be the defining polynomial of an algebraic curve. Let d be the degree of f. We assume that $f(x, y) = f_d(x, y) + f_{d-1}(x, y)$, where f_k is a non-zero *form* of degree k, i. e. a *homogeneous polynomial* of degree k, i. e. a polynomial of the form $\sum_{i=0}^{k} c_i x^i y^{k-i}$. This means the curve has a $(d-1)$-fold point in the origin. Then $\mathcal{Q}(s) = \left(-\frac{f_{d-1}(1,s)}{f_d(1,s)}, -\frac{sf_{d-1}(1,s)}{f_d(1,s)}\right)$ is a proper parametrization of the curve. This parametrization is achieved by considering a pencil of lines through the origin. The idea can be easily generalized to curves with any $(d-1)$-fold point. See for instance [63] for further details.

Figure B.1 illustrates parametrization by lines for the ellipse defined by.

$$f(x, y) = (x + 4)^2 + 4y^2 - 16 = x^2 + 4y^2 + 8x = 0\,.$$

This curve is of degree $d = 2$ and hence, any point of the curve can be used for the parametrization. In the figure the origin is chosen.

[1]For a definition see for instance [63].
[2]Waterloo Maple Inc., MAPLE, Waterloo, Canada
[3]MAGMA is distributed by the Computational Algebra Group at the University of Sydney.

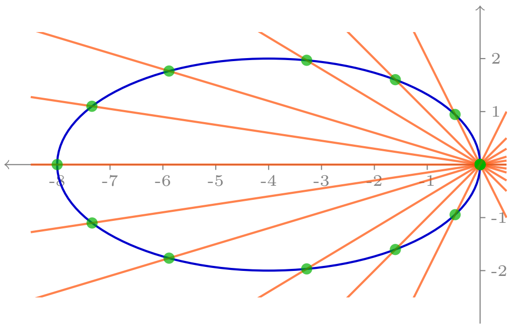

Figure B.1.: Parametrization by lines

B.1.2. Rational Parametrization of Surfaces

The problem of rational parametrizability of surfaces is fully solved. An algebraic surface has a rational parametrization if and only if the arithmetic genus and the second plurigenus[4] vanish. There is a general algorithm for computing proper rational parametrizations of surfaces [56]. An implementation based on the algorithm in [56] can be found for instance in the computer algebra system MAGMA[5] (since version V2.15 [12]). For certain classes of surfaces, special algorithms exist.

Linear Occurrence of a Variable

Let $f(x, y, z)$ be the defining polynomial of an algebraic surface. We assume that it is of the form $f(x, y, z) = g_0(x, y) + z g_1(x, y)$ for some polynomials g_0 and g_1. Then obviously $\mathcal{Q}(s_1, s_2) = (s_1, s_2, -\frac{g_0(s_1,s_2)}{g_1(s_1,s_2)})$ is a rational parametrization of the surface. It is easy to see that the parametrization is also proper. This works analogously if x or y appears linearly.

A Simple Approach

Let $f(x, y, z)$ be the defining polynomial of an algebraic surface. In this case we can try to compute a curve parametrization over the field $\mathbb{K}(z)$. Assume we have such a rational parametrization $\mathcal{P}(s_1) = (p_1(s_1, z), p_2(s_1, z))$. In general this parametrization might be over some field extension of $\mathbb{K}(z)$. Assume now, that this is not the case, i. e.

[4]For definitions of arithmetic genus and plurigenus see for instance [7, 64].
[5]MAGMA is distributed by the Computational Algebra Group at the University of Sydney.

$\mathcal{P}$ is also rational in z. Then, $\mathcal{Q}(s_1, s_2) = (p_1(s_1, s_2), p_2(s_1, s_2), s_2)$ is a rational surface parametrization. There is no reason for the special choice of z.

Parametrization by Lines

Let $f(x, y, z)$ be the defining polynomial of an algebraic surface. Let d be the degree of f. We assume that $f(x, y, z) = f_d(x, y, z) + f_{d-1}(x, y, z)$, where f_k is a non-zero *form* of degree k. This means the surface has a $(d-1)$-fold point in the origin. Then

$$\mathcal{Q}(s_1, s_2) = \left(-\frac{f_{d-1}(1, s_1, s_2)}{f_d(1, s_1, s_2)}, -s_1\frac{f_{d-1}(1, s_1, s_2)}{f_d(1, s_1, s_2)}, -s_2\frac{f_{d-1}(1, s_1, s_2)}{f_d(1, s_1, s_2)}\right)$$

is a proper parametrization of the surface. This can be easily generalized to surfaces with any $(d-1)$-fold point by using linear point transformations. See Section B.1.3 and for instance [47] for further details. Obviously, this method always works for quadrics, i. e. surfaces with defining polynomial of degree 2. Figure B.2 was created with MATHEMATICA[6].

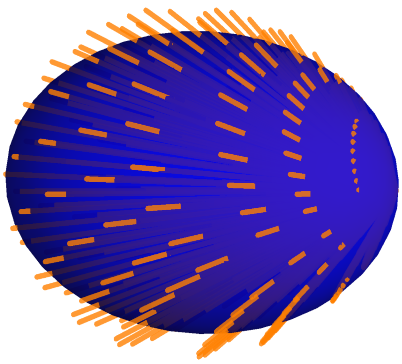

Figure B.2.: Parametrization by lines

Cubic Surfaces

Cubic surfaces are those with defining polynomial of degree 3. Non-singular cubic surfaces are rational (c. f. [64, p. 256]). In [6] non-singular cubic surfaces are divided in

[6]Wolfram Research, Inc., MATHEMATICA, Version 10.0, Champaign, IL (2014)

five families, for four of which a rational parametrization is computed, whereas the fifth involves square roots in the parametrization and hence is a radical one. For radical parametrizations see also Section B.2.

Revolution Surfaces

Revolution surfaces are constructed by a rotation of a so called *profile curve* around some axis. Obviously, if the profile curve is rational, the surface is rational as well, since the circle is rational. Let $\mathcal{P} = (p_1(s_1), p_2(s_1))$ be the rational parametrization of the profile curve in the yz-plane. We consider rotation around the z-axis. Then

$$\left(p_1(s_1)\frac{1-s_2^2}{1+s_2^2}, p_1(s_1)\frac{2s_2}{1+s_2}, p_2(s_1)\right)$$

is a rational parametrization of the surface. However, not all rational revolution surfaces can be constructed like this (compare for instance [1, Example 2.3]).

According to [1] the problem of deciding whether an implicitly given surface is a revolution surface, seems to be unsolved.

Figure B.3 shows a revolution surface constructed by rotating the semicubical parabola, i. e. $\mathcal{P} = (s_1^2, s_1^3)$. The implicit equation of the surface is $y^6 - z^4 + 3y^4x^2 + 3y^2x^4 + x^6 = 0$. Figure B.3 was created with SURFER[7].

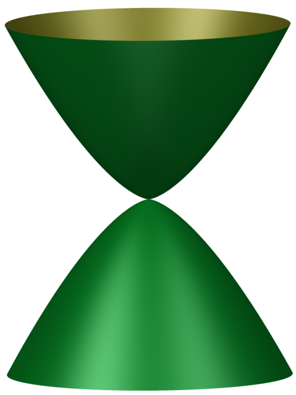

Figure B.3.: Revolution surface

[7] SURFER is a program by the Mathematisches Forschungsinstitut Oberwolfach (MFO) in collaboration with the Martin Luther University Halle-Wittenberg.

Swung Surfaces

By definition these are surfaces which can be parametrized by

$$\mathcal{Q}(s_1, s_2) = (p_1(s_1)q_1(s_2), p_1(s_1)q_2(s_2), p_2(s_1)) \ .$$

This kind of surfaces is a generalization of revolution surfaces. A swung surface can be constructed by two space curves [1]: a *profile curve* in the yz-plane and a *trajectory curve* in the xy-plane. We consider the swinging of the profile curve around the z-axis along the trajectory curve. The resulting surface is the one in question.

The parametrization of the profile curve is then $(0, p_1(s_1), p_2(s_1))$ and the trajectory curve is parametrized by $(q_1(s_2), q_2(s_2), 0)$.

Reparametrization of swung surfaces is considered in [1].

Figure B.4 shows a swung surface constructed with the semicubical parabola as profile curve and the astroid as trajectory curve i. e.

$$\begin{aligned} q_1(s,t) &= \frac{8t^3(1+6t+12t^2+8t^3)}{1+12t+63t^2+184t^3+315t^4+300t^5+125t^6} \, , \\ q_2(s,t) &= \frac{(1+2t+t^2)(1+10t+36t^2+54t^3+27t^4)}{1+12t+63t^2+184t^3+315t^4+300t^5+125t^6} \, . \end{aligned}$$

The implicit equation of the surface is

$$\begin{aligned} 0 =& x^{18} + 9x^{16}y^2 + 36x^{14}y^4 + 84x^{12}y^6 + 126x^{10}y^8 + 126x^8y^{10} + 84x^6y^{12} + 36x^4y^{14} \\ &+ 9x^2y^{16} + y^{18} - 3x^{12}z^4 + 468x^{10}y^2z^4 - 4662x^8y^4z^4 + 9417x^6y^6z^4 - 4662x^4y^8z^4 \\ &+ 468x^2y^{10}z^4 - 3y^{12}z^4 + 3x^6z^8 + 252x^4y^2z^8 + 252x^2y^4z^8 + 3y^6z^8 - z^{12} \, . \end{aligned}$$

Figure B.4 was created with SURFER[8].

[8] SURFER is a program by the Mathematisches Forschungsinstitut Oberwolfach (MFO) in collaboration with the Martin Luther University Halle-Wittenberg.

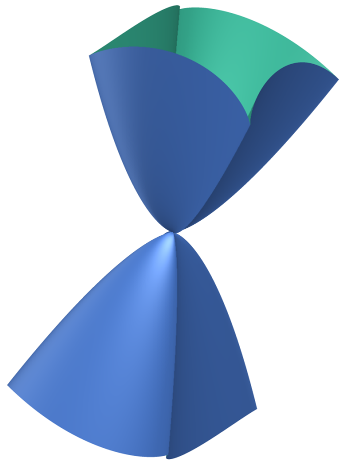

Figure B.4.: Swung surface

Canal Surfaces

Canal surfaces are defined as the envelope of spheres centered around a space curve (the *spine curve*). The spheres thereby might have a rationally changing radius. If the radius is constant the surface is also called a *pipe surface*. It is known that if the spine curve is rational then the canal surface is unirational (see [34, 48]).

Algorithms for computing rational parametrization of canal and pipe surfaces can be found in [35, 36, 48].

Ruled Surfaces

A main property of rational ruled surfaces is that they admit a parametrization of the form

$$\mathcal{Q} = (p_1(s_1) + s_2 q_1(s_1), p_2(s_1) + s_2 q_2(s_1), p_3(s_1) + s_2 q_3(s_1)) \ .$$

We omit a precise definition here and refer to [7, 64, 65] for more details. An algorithm for deciding whether an implicitly given algebraic surface is a rational ruled surface can be found in [65]. In the affirmative case, this algorithm also computes a parametrization.

Algorithms for the purpose of reparametrization in the case of ruled surfaces can be found for instance in [2, 38].

Figure B.5 shows the one-sheeted hyperboloid which is a ruled surface, in fact it is even doubly ruled. Its parametrization is $\mathcal{Q}(s_1, s_2) = \left(\frac{1-s_1^2}{s_1^2+1} - \frac{2s_1}{s_1^2+1}s_2, \frac{2s_1}{s_1^2+1} + \frac{1-s_1^2}{s_1^2+1}s_2, s_2\right)$ and the implicit equation is $x^2 + y^2 - z^2 = 1$. Note, the hyperboloid is also a revolution surface, rotating a hyperbola. Figure B.5 was created with MATHEMATICA[9].

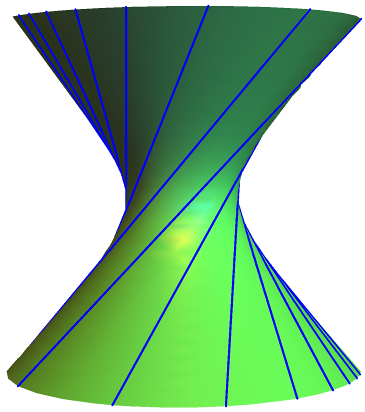

Figure B.5.: Ruled surface

Del Pezzo Surfaces

Del Pezzo surfaces play a role in the general parametrization algorithm in [56]. For instance smooth cubic surfaces are a subset of Del Pezzo surfaces (c. f. [7, Theorem IV.13]). We omit a definition here. A general approach is given in [56]. Further contributions to the parametrization of such surfaces can be found in [13, 20, 25]. The computer algebra system MAGMA[10] (c. f. [12]) contains particular functions for parametrizing Del Pezzo surfaces.

Tubular Surfaces

A tubular surface is defined by a polynomial of the form $f(x, y, z) = a(z)x^2 + b(z)y^2 + c(z)$. A parametrization algorithm can be found in [57].

[9] Wolfram Research, Inc., MATHEMATICA, Version 10.0, Champaign, IL (2014)
[10] MAGMA is distributed by the Computational Algebra Group at the University of Sydney.

B.1.3. Rational Parametrization of Hypersurfaces

So far there is no general algorithm for parametrizing algebraic hypersurfaces. Even parametrization of special kinds of hypersurfaces is not very much investigated. Nevertheless, for simple kinds of hypersurfaces some of the following approaches might be worth trying.

Linear Occurrence of a Variable

Let $f(x_1, \ldots, x_n)$ be the defining polynomial of an algebraic hypersurface. We assume that

$$f(x_1, \ldots, x_n) = g_0(x_1, \ldots, x_{n-1}) + x_n g_1(x_1, \ldots, x_{n-1})$$

for some polynomials g_0 and g_1. Then obviously

$$\mathcal{Q}(s_1, \ldots, s_{n-1}) = \left(s_1, \ldots, s_{n-1}, -\frac{g_0(s_1, \ldots, s_{n-1})}{g_1(s_1, \ldots, s_{n-1})}\right)$$

is a rational parametrization of the hypersurface. It is easy to see that the parametrization is also proper. This works analogously for any other variable appearing linearly.

A Simple Approach

Let $f(x_1, \ldots, x_n)$ be the defining polynomial of an algebraic surface. In this case we can try to compute a curve parametrization over the field $\mathbb{K}(x_\kappa, \ldots, x_n)$ for some $2 \leq \kappa \leq n$. Assume we have such a rational parametrization of a hypersurface of lower dimension,

$$\mathcal{P}(s_1, \ldots, s_{\kappa-2}) = (p_1(s_1, \ldots, s_{\kappa-2}, x_\kappa, \ldots, x_n), \ldots, p_{\kappa-1}(s_1, \ldots, s_{\kappa-2}, x_\kappa, \ldots, x_n)) \ .$$

In general such a parametrization might be found (if it exists) over some field extension of $\mathbb{K}(x_\kappa, \ldots, x_n)$. Assume now, that this is not the case, i. e. $\mathcal{P}$ is also rational in $x_\kappa, \ldots, x_n$. Then,

$$\mathcal{Q}(s_1, \ldots, s_{n-1}) = (p_1(s_1, \ldots, s_n), \ldots, p_{\kappa-1}(s_1, \ldots, s_n), s_{\kappa-1}, \ldots, s_{n-1})$$

is a rational hypersurface parametrization.

Parametrization by Lines

Let $f(x_1, \ldots, x_n)$ be the defining polynomial of an algebraic hypersurface. Let d be the degree of f. We assume that $f(x_1, \ldots, x_n) = f_d(x_1, \ldots, x_n) + f_{d-1}(x_1, \ldots, x_n)$, where f_k

is a non-zero *form* of degree k. This means the surface has a $(d-1)$-fold point in the origin. Let $R(s_1,\ldots,s_{n-1}) = -\frac{f_{d-1}(1,s_1,\ldots,s_{n-1})}{f_d(1,s_1,\ldots,s_{n-1})}$. Then

$$\mathcal{Q}(s_1,\ldots,s_{n-1}) = (R(s_1,\ldots,s_{n-1}), s_1R(s_1,\ldots,s_{n-1}),\ldots,s_{n-1}R(s_1,\ldots,s_{n-1}))$$

is a parametrization of the hypersurface. This can be easily generalized to hypersurfaces with any $(d-1)$-fold point by using linear point transformations. Let us briefly consider how the above parametrization is derived. We take a pencil of lines through the origin $x_k = s_{n-1}x_1$ for $k > 1$. Note, that there is no reason for the special role of x_1, any other x_i could be chosen as well. Each of these lines intersects the hypersurface in an additional point. We obtain

$$0 = f(x_1, s_1x_1\ldots,s_{n-1}x_1) = x_1^d f_d(1,s_1\ldots,s_{n-1}) + x_1^{d-1}f_{d-1}(1,s_1\ldots,s_{n-1})\,.$$

This has two factors. One yields the origin, the other the new intersection point.

B.2. Radical Parametrization

The definition of radical parametrizations can be found in Section 2.2.1. Since this definition accords to intuitive ideas on radicality, it is easy to come up with certain radical parametrizations of some specific hypersurfaces, e. g. those where all variables have degree less than or equal to four. Regardless of these ideas, detailed investigation of radical parametrization started rather recently. Here we give a brief overview on existing algorithms and other knowledge on computing such parametrizations for curves and surfaces. So far there seems to be no general treatment of radical parametrizations of hypersurfaces. Nevertheless, simple approaches might help.

B.2.1. Radical Parametrization of Curves

The curves of genus zero are exactly those that can be parametrized rationally. For curves of higher genus we want to find radical parametrizations if they exist. In [60] algorithms for computing radical parametrizations for curves of genus up to four are presented. For curves of genus five and six algorithms can be found in [24]. Zariski [71, 72] already proved that for the general curve of genus seven, there is no radical parametrization. Algorithms using gonality (i. e. the smallest possible degree d such that there is a d : 1-map from the curve to the projective line) can be found in [58, 59].

As for the rational case there are some classes of curves which allow a simple computation of radical parametrizations.

Variable with degree less than or equal to 4

Let $f(x,y)$ be the defining polynomial of an algebraic curve. We assume that $f(x,y) = \sum_{i=0}^{4} y^i g_i(x)$ for some polynomials g_i. Since the equation has degree less than or equal to 4, there is a solution for y in terms of radicals. This works analogously if x appears with degree less than or equal to 4.

Parametrization by Lines

Let $f(x,y)$ be the defining polynomial of an algebraic curve. Let d be the degree of f. We assume that $f(x,y) = \sum_{i=0}^{r} f_{d-i}(x,y)$ with $r \leq 4$, where f_k is a non-zero form of degree k, i. e. a polynomial of the form $\sum_{i=0}^{k} c_i x^i y^{k-i}$. This means the curve has a $(d-r)$-fold point in the origin. Intersecting a pencil of lines $y = sx$ with the curve yields

$$0 = f(x,sx) = \sum_{i=0}^{r} x^{d-i} f_{d-i}(1,s) = x^{d-r} \sum_{i=0}^{r} x^{r-i} f_{d-i}(1,s)\,,$$

which has a factor of degree $r \leq 4$ and hence is solvable by radicals for x. This can be easily generalized to curves with any $(d-r)$-fold point. See for instance [60] for further details.

B.2.2. Radical Parametrization of Surfaces

So far not very much investigation was done in this field. In [61] first algorithms for radical parametrization of surfaces are presented. For instance parametrization by lines is introduced and used for radically parametrizing irreducible curves of degree 5 and singular curves of degree 6. Furthermore, an algorithm for computing radical parametrizations of surfaces with a pencil of curves that has low genus is described. Again we can consider some special classes of surfaces which can be parametrized easily.

Variable with degree less than or equal to 4

Let $f(x,y,z)$ be the defining polynomial of an algebraic surface. We assume that $f(x,y,z) = \sum_{i=0}^{4} z^i g_i(x,y)$ for some polynomials g_i. Since the equation has degree less than or equal to 4, there is a solution for z in terms of radicals. This works analogously if x or y appears with degree less than or equal to 4. Obviously, cubic surfaces can be parametrized like this.

Parametrization by Lines

Let $f(x,y,z)$ be the defining polynomial of an algebraic surface. Let d be the degree of f. We assume that $f(x,y,z) = \sum_{i=0}^{r} f_{d-i}(x,y,z)$ with $r \leq 4$, where f_k is a non-zero form of degree k. This means the surface has a $(d-r)$-fold point in the origin. Intersecting a pencil of lines $y = s_1 x$, $z = s_2 x$ with the surface yields

$$0 = f(x, s_1 x, s_2 x) = \sum_{i=0}^{r} x^{d-i} f_{d-i}(1, s_1, s_2) = x^{d-r} \sum_{i=0}^{r} x^{r-i} f_{d-i}(1, s_1, s_2)\,,$$

which has a factor of degree $r \leq 4$ and hence is solvable by radicals for x. This can be easily generalized to curves with any $(d-r)$-fold point. See for instance [61] for further details.

B.2.3. Radical Parametrization of Hypersurfaces

So far no general methods for radical parametrization of hypersurfaces are known. Nevertheless, the two standard classes of hypersurfaces can be easily parametrized.

Variable with degree less than or equal to 4

Let $f(x_1,\ldots,x_n)$ be the defining polynomial of an algebraic hypersurface. We assume that $f(x_1,\ldots,x_n) = \sum_{i=0}^{4} x_n^i g_i(x_1,\ldots,x_{n-1})$ for some polynomials g_i. Since the equation has degree less than or equal to 4, there is a solution for x_n in terms of radicals. This works analogously if any other x_i appears with degree less than or equal to 4.

Parametrization by Lines

Let $f(x_1,\ldots,x_n)$ be the defining polynomial of an algebraic hypersurface. Let d be the degree of f. We assume that $f(x_1,\ldots,x_n) = \sum_{i=0}^{r} f_{d-i}(x_1,\ldots,x_n)$ with $r \leq 4$, where

f_k is a non-zero form of degree k. This means the hypersurface has a $(d-r)$-fold point in the origin. Intersecting a pencil of lines $x_k = s_{n-1}x_1$, $k > 1$, with the hypersurface yields

$$\begin{aligned} 0 = f(x_1, s_1x_1 \ldots, s_{n-1}x_1) &= \sum_{i=0}^{r} x_1^{d-i} f_{d-i}(1, s_1, \ldots, s_{n-1}) \\ &= x_1^{d-r} \sum_{i=0}^{r} x_1^{r-i} f_{d-i}(1, s_1, \ldots, s_{n-1})\,. \end{aligned}$$

which has a factor of degree $r \leq 4$ and hence is solvable by radicals for x_1. This can be easily generalized to curves with any $(d-r)$-fold point.

B.3. Other Parametrizations

In [26] *trigonometric parametrizations* of curves are defined to be of the form

$$\left(\sum_{k=0}^{m} a_{1,k} \sin(k\phi) + a_{2,k} \cos(k\phi), \sum_{k=0}^{n} b_{1,k} \sin(k\phi) + b_{2,k} \cos(k\phi) \right) .$$

A curve is called *trigonometric* if it admits such a parametrization. It is shown in [26] that the trigonometric curves are a subset of rational curves.

C. List of Differential Equations

This chapter contains a collection of examples together with the parametrizations in use and the solutions found by the procedures presented in Chapter 2 and 3. For reference we use the same numbering as in the literature. We prepend a letter to distinguish the referenced books.

K Kamke [32] and [31] for ODEs and PDEs respectively

PS Polyanin and Sajzew [49]

PZ Polyanin and Zaitsev [50]

In Section C.1 first-order AODEs are collected. Right afterward some examples of higher-order AODEs are solved. First-order APDEs can be found in Section C.3.

Intermediate steps of the procedures are skipped. Due to readability also the arbitrary constants are sometimes omitted. However, they can be easily introduced for autonomous first-order equations.

Note, that these lists are by no means complete. They are just a selection of examples which can be solved using the presented methods. Other equations also listed in the above mentioned collections might be solved as well. Furthermore, the methods do solve plenty of differential equations which are not listed in the collections.

C.1. First-Order AODEs

In this section we present a list of autonomous first-order AODEs which can be solved by Procedure 3. Table C.1 shows a selection of such examples from [32].

Note, that for K.1.209 the original ODE in [32] is not an algebraic one, but solutions of the AODE here imply solutions of the original ODE.

Source	AODE	Parametrization	Solution
K.1.12	$u^2+u'-1$	$(s,1-s^2)$	$\frac{e^{2x}-1}{e^{2x}+1}$
K.1.23	$u'+au^2-b$	$(s,b-as^2)$	$\frac{\sqrt{b}\tanh\left(\sqrt{a}\sqrt{b}x\right)}{\sqrt{a}}$
K.1.209	$u'^2u^2-au^2-b$	$\left(s,\frac{\sqrt{as^2+b}}{s}\right)$	$\frac{\sqrt{a^2x^2-b}}{\sqrt{a}}$
K.1.369	$u^2+u'^2-a^2$	$\left(a\frac{1-s^2}{s^2+1},\frac{2as}{s^2+1}\right)$	$a\cos(x)$
K.1.371	$u^2-u^3+u'^2$	$(1+s^2,s(1+s^2))$	$\sec^2\left(\frac{x}{2}\right)$
K.1.389	$(1+4u)(u-u')+u'^2$	$\left(-\frac{s(s-5)}{(2s-5)^2},\frac{5s}{(2s-5)^2}\right)$	$e^x\left(e^x+1\right)$
K.1.462	uu'^2-1	$\left(\frac{1}{s^2},s\right)$	$-\sqrt[3]{-1}\left(\frac{3}{2}\right)^{2/3}x^{2/3}$
K.1.486	$u^2+u^2u'^2-a^2$	$\left(s,-\frac{\sqrt{a^2-s^2}}{s}\right)$	$\pm\sqrt{a^2-x^2}$

Table C.1.: Well-known AODEs and their solutions found by Procedure 3.

C.2. Higher-Order AODEs

In this section we present examples of second-order autonomous AODEs which can be solved by the methods presented in Section 2.3. Table C.2 lists some AODEs where the first derivative does not appear and hence, equation (2.6) can be used for finding solutions. In Table C.3 a pool of examples can be found for which the approach from Section 3.4 using APDEs yields a solution of the AODE. Here erf is the non-elementary error function and sn the Jacobi elliptic function. The solutions of the examples in Table C.2 were computed by the method from Section 2.3 for AODEs of the form $F(u,u'')=0$. In Table C.3 the general method for higher-order AODEs (Procedure 6) is applied.

The solutions shown in Table C.3 are just an arbitrary choice. Other solutions might be found as well with the same procedure by choosing different intermediate solutions in the ODEs of the method of characteristics. Note, that K.6.71 can as well be solved using Procedure 3 for $8v'+v^4$ and integration. Example K.7.16 was solved by first solving $3uu''-5u'^2$.

Source	AODE	Parametrization	Solution
K.2.2	$u'' + u$	$(s, -s)$	$\sqrt{c_1}\sin(x - c_2)$
K.2.6	$u'' - u$	(s, s)	$\frac{1}{2}\mathrm{e}^{-x+c_2} + \mathrm{e}^{x-c_2}c_1$
K.2.9	$u'' + au$	$(s, -s)$	$-\frac{\sqrt{c_1}\sin\left(\sqrt{a}(c_2-x)\right)}{a}$
K.6.7	$u'' - au^3$	(s, as^3)	$-\frac{\sqrt[4]{-2}c_1\,\mathrm{sn}\left(\frac{1}{2}(-2)^{3/4}\sqrt[4]{a}c_1(x-c_2)\middle\vert -1\right)}{\sqrt[4]{a}}$
K.6.104	$uu'' - a$	$\left(s, \frac{a}{s}\right)$	$c_1\mathrm{e}^{-\operatorname{erf}^{-1}\left(-i\sqrt{\frac{2a}{\pi}}\frac{1}{c_1}(x-c_2)\right)^2}$
K.6.209	$u''u^3 - a$	$\left(s, \frac{a}{s^3}\right)$	$-\frac{\sqrt{a+c_1^2(c_2-x)^2}}{\sqrt{c_1}}$

Table C.2.: Well-known second-order AODEs with solutions found by the procedure from Section 2.3.

C.3. First-Order APDEs

In this section examples of first-order autonomous APDEs which are solvable by Procedure 5 are given. Table C.4 shows rational solutions, Table C.5 radical solutions and Table C.6 non-algebraic solutions of first-order APDEs in two variables. Table C.7 collects solutions of APDEs with more than two variables. In Table C.8 a list of non-autonomous APDEs is solved with the aid of linear transformations.

Note, that in Table C.6 W is the Lambert W function. Note, that in Table C.7 we used u_i for $u^{(i)}$. The polynomials $G, H \in \mathbb{K}[u_1, u_2, u_3]$ are used for abbreviation. They are defined as $G(u_1, u_2, u_3) = G_{a_1,a_2,a_3}(u_1, u_2, u_3) = a_1u_1^2 + a_2u_2^2 + a_3u3^2$ and $H(u_1, u_2, u_3) = H_{b_1,b_2,b_3}(u_1, u_2, u_3) = b_1u_1u_2 + b_2u_1u_3 + b_3u_2u_3$. The rational function S in the solution of Example PZ.4.1.4.4 is defined as

$$
\begin{aligned}
S = &-\frac{d}{c} + \frac{(x_3^2\,(4a_1a_2 - b_1^2) + x_2^2\,(4a_1a_3 - b_2^2) + x_1^2\,(4a_2a_3 - b_3^2))c}{4\,(4a_1a_2a_3 - a_1b_3^2 - a_2b_2^2 - a_3b_1^2 + b_1b_2b_3)} \\
&+ \frac{(x_3x_2(b_1b_2 - 2a_1b_3) + x_1x_3(b_1b_3 - 2a_2b_2) + x_1x_2(b_2b_3 - 2a_3b_1))c}{2\,(4a_1a_2a_3 - a_1b_3^2 - a_2b_2^2 - a_3b_1^2 + b_1b_2b_3)}\,.
\end{aligned}
$$

Using linear transformation we can also solve some non-autonomous APDEs. The transformations in Table C.8 are given in the form $\alpha, \bar{\beta}, \delta$. In Example K.6.33 Kamke [31] already describes the linear transformation.

Source	**AODE**	**Parametrization**	**Solution**
K.2.1	u''	$(s, t, 0)$	$(x - c_1)c_2$
K.6.71	$8u'' + 9u'^4$	$\left(s, t, -\frac{9}{8}t^4\right)$	$\frac{x-c_1}{\sqrt[3]{x-c_1}} + c_2$
K.6.107	$uu'' + u'^2 - a$	$\left(s, t, \frac{a-t^2}{s}\right)$	$\frac{\sqrt{x-c_1+a^2(x-c_1-c_2)^2}}{\sqrt{a}}$
K.6.110	$u''u - u'^2 + 1$	$\left(s, t, \frac{t^2-1}{s}\right)$	$\frac{1}{2}c_2 e^{-\frac{x-c_1}{c_2}}\left(e^{\frac{2(x-c_1)}{c_2}} - 1\right)$
K.6.111	$u''u - u'^2 - 1$	$\left(s, t, \frac{t^2+1}{s}\right)$	$c_2\sqrt{\sinh^2\left(\frac{x-c_1}{c_2}\right) + 1}$
K.6.125	$u''u - au'^2$	$\left(s, t, \frac{at^2}{s}\right)$	$-c_2((a-1)(x-c_1))^{\frac{1}{1-a}}$
K.6.138	$2u''u - u'^2 + a$	$\left(s, t, \frac{t^2-a}{2s}\right)$	$\frac{-4ac_2^2+(x-c_1)^2}{4c_2}$
K.6.150	$2u''u - 3u'^2$	$\left(s, t, \frac{3t^2}{2s}\right)$	$\frac{4c_2^3}{(c_2-x+c_1)^2}$
K.6.151	$-4u^2 - 3u'^2 + 2uu''$	$\left(s, t, \frac{4s^2+3t^2}{2s}\right)$	$\frac{4\left(\tan^2(x-c_1)+1\right)\cot^2(x-c_1)}{c_2}$
K.6.157	$3u''u - 5u'^2$	$\left(s, t, \frac{5t^2}{3s}\right)$	$\frac{3\sqrt{\frac{3}{2}}c_2^{5/2}}{2\sqrt{(c_2-x+c_1)^3}}$
K.6.158	$4u - 3u'^2 + 4uu''$	$\left(s, t, \frac{3t^2-4s}{4s}\right)$	$\frac{\left((x-c_1)^2c_2^2-64\right)^2}{256_2^2}$
K.6.164	$nu''u - (n-1)u'^2$	$\left(s, t, \frac{(n-1)t^2}{ns}\right)$	$\left(\frac{(x-c_1)c_2}{n}\right)^n$
K.6.168	$(au + b)u'' + cu'^2$	$\left(s, t, -\frac{ct^2}{as+b}\right)$	$-\frac{b}{a} - c_2\left(\frac{(a+c)(x-c_1)}{a}\right)^{\frac{a}{a+c}}$
K.6.191	$(1 + u^2)u'' + (1 - 2u)u'^2$	$\left(s, t, \frac{(2s-1)t^2}{s^2+1}\right)$	$\tan(\log((x - c_1)c_2))$
K.6.192	$(u^2 + 1)u'' - 3uu'^2$	$\left(s, t, \frac{3st^2}{s^2+1}\right)$	$-\frac{(x-c_1)c_2}{\sqrt{1-(x-c_1)^2c_2^2}}$
K.6.209	$u''u^3 - a$	$\left(s, t, \frac{a}{s^3}\right)$	$-\frac{\sqrt{-4ac_2^2(x-c_1-c_2)^2-1}}{\sqrt{2}\sqrt{c_2}}$
K.6.210	$(u^3 + u)u'' - (3u^2 - 1)u'^2$	$\left(s, t, \frac{\left(3s^2-1\right)t^2}{s(s^2+1)}\right)$	$-\frac{\sqrt{-2(x-c_1)c_2-1}}{\sqrt{2}\sqrt{x-c_1}\sqrt{c_2}}$
K.7.10	$2u'u''' - 3u''^2$	$\left(s_1, s_2, s_3, \frac{3s_3^2}{2s_2}\right)$	$-\frac{(x-c_1)c_2c_3+4}{(x-c_1)c_2^2}$
K.7.11	$(u'^2 + 1)u''' - 3u'u''^2$	$\left(s_1, s_2, s_3, \frac{3s_2s_3^2}{s_2^2+1}\right)$	$\frac{\sqrt{1-(x-c_1)^2c_2^2}}{c_2} - \frac{c_3}{c_2}$
K.7.16	$3u''u'''' - 5u'''^2$	$\left(s_1, s_2, \frac{5s_2^2}{3s_1}\right)$	$\frac{\sqrt{x-c_1}}{c_2} + c_3(x - c_1) + c_4$
PS.3.3.13	$uu''' - u'u''$	$\left(s_1, s_2, s_3, \frac{s_2, s_3}{s_1}\right)$	$\frac{e^{(x-c_1)c_2}\left(e^{-2(x-c_1)c_2}-2c_3\right)}{2c_2^2}$

Table C.3.: Well-known higher-order AODEs with solutions found by Procedure 6.

Source	**APDE**	**Parametrization**	**Solution**
PZ.1.1.1.18	$u_x + auu_y$	$\left(-\frac{s}{at}, s, t\right)$	$\frac{y}{ax}$
K.2.45	$uu_x + u_y - a$	$(s, t, a - st)$	$\frac{x}{y} + \frac{ay}{2}$
PZ.1.1.1.19	$u_x + auu_y - b$	$\left(\frac{b-s}{at}, s, t\right)$	$\frac{bx}{2} + \frac{y}{ax}$
K.2.47	$-uu_x + u_x + uu_y + u_y$	$\left(s, t, \frac{(s-1)t}{s+1}\right)$	$\frac{y-x}{x+y}$
PZ.1.1.1.28	$u(u_y - u_x) + a(u_x + u_y)$	$\left(\frac{a(s+t)}{s-t}, s, t\right)$	$\frac{a(y-x)}{x+y}$
PZ.1.1.1.29	$(a_1 - a_2u)u_x + (b_1 + b_2u)u_y - c$	$\left(\frac{a_1s+b_1t-c}{a_2s-b_2t}, s, t\right)$	$\frac{c(b_2x+a_2y)}{2(a_2b_1+a_1b_2)} + \frac{(a_1y-b_1y)}{(b_2x+a_2y)}$
K.2.68, PZ.1.1.1.47, $k = 1$	$u_x + (b + au)u_y - c$	$\left(\frac{c-s-bt}{at}, s, t\right)$	$\frac{cx}{2} + \frac{y-bx}{ax}$
K.6.23, PZ.2.1.1.8, $k = 3$	$u_xu_y - bu^3$	$\left(s, \frac{bs^3}{t}, t\right)$	$\frac{1}{bxy}$
K.6.66, PZ.2.1.6.14	$a^2u_x^2 + b^2u_y^2 - c^2u$	$\left(\frac{a^2s^2+b^2t^2}{c^2}, s, t\right)$	$\frac{1}{4}c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$
PZ.2.1.6.17, $k = 1$	$u_x^2 + u_y^2 - au$	$\left(\frac{s^2+t^2}{a}, s, t\right)$	$\frac{1}{4}a\left(x^2 + y^2\right)$
PZ.2.1.6.17, $k = 3$	$u_x^2 + u_y^2 - au^3$	$\left(s, \sqrt{as^3 - t^2}, t\right)$	$\frac{4}{a(x^2+y^2)}$
PZ.2.1.7.3	$bu + u_x(u_x + au_y)$	$\left(-\frac{s(s+at)}{b}, s, t\right)$	$\frac{by(y-ax)}{a^2}$
PZ.2.1.7.22	$u_x(uu_x + au_y) + b$	$\left(-\frac{b+ast}{s^2}, s, t\right)$	$\frac{ax}{y} - \frac{by^2}{3a^2}$
K.6.1	$u_x^2 - au_y - b$	$\left(s, t, \frac{t^2-b}{a}\right)$	$-\frac{ax^2}{4y} - \frac{by}{a}$
K.6.115	$u_x^3 - uu_y$	$\left(s, t, \frac{t^3}{s}\right)$	$-\frac{x^3}{27y}$
K.6.117	$u_x^3 - u_y^2$	(s, t^2, t^3)	$\frac{4x^3}{27y^2}$

Table C.4.: Well-known APDEs with rational solutions found by Procedure 5.

Source	APDE	Parametrization	Solution
K.6.21, PZ.2.1.1.1	$u_x u_y - a$	$\left(s, t, \frac{a}{t}\right)$	$2i\sqrt{a}\sqrt{-x}\sqrt{y}$
K.6.26, PZ.2.1.3.1	$(u_x - b)u_y - au_x$	$\left(\frac{s(b+at)}{t}, b+at, a+\frac{b}{t}\right)$	$bx + 2i\sqrt{b}\sqrt{-ax}\sqrt{y} + ay$
PZ.2.1.4.44	$u_x - au^4u_y^2$	(s, as^4t^2, t)	$\frac{\sqrt[5]{-5}y^{2/5}}{2^{2/5}\sqrt[5]{ax}}$
K.6.55	$-c^2 + u_x^2 + u_y^2$	$\left(s, \frac{c-ct^2}{t^2+1}, \frac{2ct}{t^2+1}\right)$	$-c\sqrt{x^2+y^2}$
K.6.56, PZ.2.1.6.1	$-c^2 + a^2u_x^2 + b^2u_y^2$	$\left(s, \frac{c-ct^2}{at^2+a}, \frac{2ct}{bt^2+b}\right)$	$-\frac{c\sqrt{b^2x^2+a^2y^2}}{ab}$
K.6.68, PZ.2.1.6.15	$-b^2 - au^2 + u_x^2 + u_y^2$	$\left(s, \frac{\sqrt{b^2+as^2}\left(1-t^2\right)}{t^2+1}, \frac{2t\sqrt{b^2+as^2}}{t^2+1}\right)$	$\frac{e^{-\sqrt{a}\sqrt{x^2+y^2}} - ab^2e^{\sqrt{a}\sqrt{x^2+y^2}}}{2a}$
PZ.2.1.6.16	$u^2\left(u_x^2 + u_y^2 + 1\right) - a^2$	$\left(s, \frac{\sqrt{\frac{a^2}{s^2}-1}\left(1-t^2\right)}{t^2+1}, \frac{2\sqrt{\frac{a^2}{s^2}-1t}}{t^2+1}\right)$	$-\sqrt{a^2-x^2-y^2}$
PZ.2.1.7.23	$b + uu_x(u_x + au_y)$	$\left(-\frac{b}{s^2+ats}, s, t\right)$	$\frac{\sqrt[3]{-1}3^{2/3}by\left((y-ax)^2\right)^{2/3}}{(ax-y)(-aby)^{2/3}}$
PZ.2.1.8.4	$(u_x + au_y + b)u_x + cu_y + d$	$\left(s, t, -\frac{d+t(b+t)}{c+at}\right)$	$\frac{2cy-a(cx+by)-2\sqrt{(da^2+c^2-abc)y(y-ax)}}{a^2}$
K.6.120	$\left(u_x^2 + a\right)u_y - (c+bu)u_x$	$\left(s, t, \frac{(c+bs)t}{t^2+a}\right)$	$\frac{1}{2}\left(-\frac{2c}{b} + bxy - x\sqrt{b^2y^2+4a}\right)$

Table C.5.: Well-known APDEs with radical solutions found by Procedure 5.

Source	APDE	Parametrization	Solution
PZ.1.1.1.22	$-bu + au_y u + u_x$	$\left(\frac{s}{b-at}, s, t\right)$	$\frac{be^{bx}y}{e^{bx}a+1}$
K.6.25, PZ.2.1.2.1	$u_x(a+u_y) - bu$	$\left(\frac{s(a+t)}{b}, s, t\right)$	$y(-\log(by)a + a + bx)$
PZ.2.1.8.10	$u_y u(bu + au_x) + u_x^2$	$\left(\frac{-t^2}{bs+at}, \frac{-t^3}{bs^2+ats}, \frac{-t^2}{bs^2+ats}\right)$	$\frac{-4by}{a^2 W\left(-\frac{2e^{\frac{bx}{a}-2}}{a}\right)\left(W\left(-\frac{2e^{\frac{bx}{a}-2}}{a}\right)+2\right)}$
K.2.20	$-au + u_x + u_y$	$(s, t, as - t)$	$\frac{e^{ay}(ax-\log(e^{ay}))}{a}$
K.2.35, n=1	$au_x + bu_y - u - c$	$\left(s, t, \frac{c+s-at}{b}\right)$	$-c + e^{y/b}x - ae^{y/b}\log\left(e^{y/b}\right)$
K.6.30, PZ.2.1.3.14	$(a+u_x)(bu+u_y) - c$	$\left(s, t, \frac{-abs-bst+c}{a+t}\right)$	$\frac{e^{-by}\left(a\left(abx+ce^{by}\right)-c\log\left(ae^{by}+1\right)\right)}{a^2 b}$

Table C.6.: Well-known APDEs with non-algebraic solutions found by Procedure 5.

Source	APDE	Parametrization	Solution
K.7.8	$u-(u_1+u_2)^2+2u_3$	$((s_1+s_2)^2-2s_3, s_1, s_2, s_3)$	$\frac{x_2^2}{4}-\mathrm{e}^{-\frac{x_3}{2}}x_2+\mathrm{e}^{-\frac{x_3}{2}}x_1$
K.7.9, PZ.4.1.2.1	$G(u_1,u_2,u_3)-1$	$\left(s_1, s_2, \frac{i\sqrt{\frac{a_2}{a_3}}R}{a_2}, \frac{-R}{a_3}\right)$, $R=\frac{a_1s_2^2+a_3s_3^2-1}{2s_3}$	$-\frac{\sqrt{a_1a_2x_3^2+a_1a_3x_2^2+a_2a_3x_1^2}}{\sqrt{a_1}\sqrt{a_2}\sqrt{a_3}}$
K.7.10, PZ.4.1.3.7	$H(u_1,u_2,u_3)-d$	$\left(s_1, s_2, s_3, \frac{d-b_1s_2s_3}{b_3s_3+b_2s_2}\right)$	$\frac{\sqrt{d}\sqrt{(b_3x_1+b_2x_2-b_1x_3)^2-4b_3b_2x_1x_2}}{i\sqrt{b_1}\sqrt{b_2}\sqrt{b_3}}$
K.7.11	$G(u_1,u_2,u_3)-u$	$\left(s_1, s_2, \frac{i\sqrt{\frac{a_2}{a_3}}R}{a_2}, \frac{-R}{a_3}\right)$, $R=\frac{a_1s_2^2+a_3s_3^2-s_1}{2s_3}$	$\frac{1}{4}\left(\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\frac{x_3^2}{a_3}\right)$
K.7.29	$\sum_{i=1}^{3}a_iu^iu_i-1$	$\left(s_1, \frac{-a_2s_1^2s_2^2-a_3s_1^3s_3^3+1}{a_1s_1}, s_2, s_3\right)$	$\sqrt{\frac{a_1x_2^2}{2a_2x_1}+\frac{4\sqrt{a_1}x_3^{3/2}}{3\sqrt{3}\sqrt{a_3}\sqrt{x_1}}+\frac{2x_1}{a_1}}$
K.7.31, $n\neq 1$	$u_1^n+u_2^n+u_3^n-1$	$\left(s_1, s_2, s_3, (-s_2^n-s_3^n+1)^{1/n}\right)$	$\left(x_1^{\frac{n}{n-1}}+x_2^{\frac{n}{n-1}}+x_3^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$
PZ.4.1.1.12	$u_1+\sum_{i=2}^{3}a_iu_i^2+b_iu_i$	$\left(s_1, -\sum_{i=2}^{3}(a_is_i^2+b_is_i), s_2, s_3\right)$	$\frac{a_2(b_3x_1-x_3)^2+a_3(b_2x_1-x_2)^2}{4a_2a_3x_1}$
PZ.4.1.2.3, $k\neq 2$	$G(u_1,u_2,u_3)-u^k$	$\left(s_1, s_2, s_3, \sqrt{\frac{-a_1s_2^2-a_2s_3^2+s_1^k}{a_3}}\right)$	$\left(\frac{(k-2)^2}{4}\left(\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\frac{x_3^2}{a_3}\right)\right)^{\frac{1}{2-k}}$
PZ.4.1.3.8	$u_1u_2+a_2u_1u_3+a_3u_2u_3-cu$	$\left(\frac{a_2s_1s_3+a_3s_3s_2+s_1s_2}{c}, s_1, s_2, s_3\right)$	$\frac{-c\left((-a_2x_2+a_3x_1-x_3)^2-4a_2x_2x_3\right)}{4a_2a_3}$
PZ.4.1.4.4	$(G+H)(u_1,u_2,u_3)-cu-d$	$\left(\frac{(G+H)(s_1,s_2,s_3)-d}{c}, s_1, s_2, s_3\right)$	$S(x_1,x_2,x_3)$

Table C.7.: Well-known APDEs in three variables with solutions found by Procedure 5.

Source	APDE, Transformation	Parametrization	Solution
K.6.2	$u_x^2 + u_y + u + x$		$S(x - c_1, y - c_2) - x - 1$
	$1, (1, 0), 1$		
	$u - 2u_x + u_x^2 + u_y$	$(s_1, s_2, -s_2^2 + 2s_2 - s_1)$	$S(x, y) = \mathrm{e}^{-y}(x + 2y) + \mathrm{e}^{-2y}$
K.6.33, PZ.2.1.2.12	$u_x(ku_y + ax + by + cu) - 1$		$\frac{S(x-c_1, y-c_2)-ax-by+\frac{bk}{c}}{c}$
	$c, (a, b), -\frac{bk}{c}$		
	$(u_x - a)(ku_y + cu) - c^2$	$\left(s_1, s_2, -\frac{c(as_1+c-s_1s_2)}{k(a-s_2)}\right)$	$S(x, y) = \frac{\mathrm{e}^{-\frac{cy}{k}}\left(a^2x - c\log\left(1-a\mathrm{e}^{\frac{cy}{k}}\right)\right)-ac}{a^2}$
PZ.1.1.1.23	$u_x + (au + bx)u_y$		$-\frac{-bc_1^2+bx^2+2c_2-2y}{2ax-2ac_1}$
	$1, (\frac{b}{a}, 0), c$		
	$u_x + a(u - c)u_y - \frac{b}{a}$	$\left(s_1, s_2, \frac{as_2-b}{a^2(c-s_1)}\right)$	$\frac{2ac(x-c_1)+b(x-c_1)^2+2(y-c_2)}{2a(x-c_1)}$
PZ.1.1.1.24	$u_x + (au + by)u_y$		$\frac{by\mathrm{e}^{bc_1}-bc_2\mathrm{e}^{bx}}{a\mathrm{e}^{bx}-a\mathrm{e}^{bc_1}}$
	$1, (0, \frac{b}{a}), c$		
	$b(c - u) + u_x + a(u - c)u_y$	$\left(s_1, s_2, \frac{bc-bs_1+s_2}{a(c-s_1)}\right)$	$\frac{b\mathrm{e}^{bx}(y-c_2)}{a\left(\mathrm{e}^{bx}-\mathrm{e}^{bc_1}\right)} + c$

Table C.8.: Non-autonomous APDEs solved by linear transformations and Procedure 5.

D. Method of Characteristics

Here we briefly describe the idea of the method of characteristics for solving quasilinear APDEs. For more details we refer to literature, for instance [73]. A generalization of the method of characteristics to arbitrary first-order partial differential equations was investigated by Lagrange and Charpit (compare [66]). Nevertheless, we describe it just for the quasilinear case. A first-order APDE is *quasilinear* iff it is linear in the derivatives (and possibly non-linear in the dependent function u). Let F define a quasilinear APDE, i. e.

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = -b(x_1, \dots, x_n, u) + \sum_{i=0}^{n} a_i(x_1, \dots, x_n, u) u_{x_i} = 0 \,.$$

A characteristic curve $(x_1(t), \dots, x_n(t), v(t))$ is defined by the following characteristic equations.

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= a_i(x_1, \dots, x_n, v) \,, && \text{for } i \in \{1, \dots, n\} \,, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= b(x_1, \dots, x_n, v) \,. \end{aligned}$$

Let us consider some initial data

$$u = g(x_1, \dots, x_n) = 0 \qquad \text{on } h(x_1, \dots, x_n) = 0 \,.$$

These can be written parametrically with parameters $k_2, \dots, k_n$ as $x_i = \sigma_i(k_2, \dots, k_n)$ and $u = v(k_2, \dots, k_n)$. These parametric expressions can be used as initial data (for $t = 0$) in solving the system of characteristic equations. Finally, a solution u of the above ODE still depends on $k_2, \dots, k_n$. Let $\chi_i(t, k_2, \dots, k_n)$ be solutions of the characteristic equations. Then we solve the system $x_i = \chi_i(t, k_2, \dots, k_n)$ for $k_2, \dots, k_n$. Note, that this might not always be possible. In case it is possible we get $k_i = \xi_i(x_1, \dots, x_n)$. Then, $u(x_1, \dots, x_n) = v(x_1, \xi_2(x_1, \dots, x_n), \dots, \xi_n(x_1, \dots, x_n))$ is a solution of the quasilinear APDE.

Note, that the method of characteristics does not necessarily result in a complete solution of suitable dimension. Compare for instance [66, Problem 3.12], where a complete solution of the eikonal equation is found by the method of characteristic. This solution is not of suitable dimension and envelope computations are needed to get the same solution of suitable dimension as in Example 3.11.

Example D.1. ([66, Problem 3.12])
Applying the generalized method of characteristics yields $u(x, y) = c_2x + \sqrt{a - c_2^2}y + c_1$ *as a solution of the eikonal equation* $u_x^2 + u_y^2 - 1$. *Then the Jacobians of the parametrization* $\mathcal{L}$ *induced by the solution with respect to* x, y *and* c_1, c_2 *respectively, are*

$$\mathcal{J}_{\mathcal{L}}^{x,y} = \begin{pmatrix} c_2 & \sqrt{1-c_2^2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} 1 & x - \frac{c_2 y}{\sqrt{1-c_2^2}} \\ 0 & 1 \\ 0 & -\frac{c_2 y}{\sqrt{1-c_2^2}} \end{pmatrix}.$$

Hence, the solution is complete, but not of suitable dimension (and not proper). The solution $\sqrt{x^2 + y^2}$ *which was found by Procedure 4 in Example 3.11 is complete of suitable dimension. It can be obtained from the one considered here by envelope computations (as shown in [66]).*

Index

Bibliography

[1] C. Andradas, T. Recio, J. R. Sendra, L.-F. Tabera, and C. Villarino. Reparametrizing swung surfaces over the reals. *Applicable Algebra in Engineering, Communication and Computing*, 25(1–2):39–65, 2014.

[2] C. Andradas, T. Recio, L. F. Tabera, J. R. Sendra, and C. Villarino. Proper real reparametrization of rational ruled surfaces. *Computer Aided Geometric Design*, 28(2):102–113, 2011.

[3] W. Arendt and K. Urban. *Partielle Differenzialgleichungen. Eine Einführung in analytische und numerische Methoden.* Spektrum Akademischer Verlag, Heidelberg, 2010.

[4] V. I. Arnold. *Lectures on Partial Differential Equations.* Springer-Verlag, Berlin Heidelberg, 2004.

[5] J. M. Aroca, J. Cano, R. Feng, and X.-S. Gao. Algebraic General Solutions of Algebraic Ordinary Differential Equations. In M. Kauers, editor, *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation*, ISSAC '05, pages 29–36, New York, 2005. ACM.

[6] C. L. Bajaj, R. L. Holt, and A. N. Netravali. Rational Parametrizations of Nonsingular Real Cubic Surfaces. *ACM Transactions on Graphics*, 17(1):1–31, 1998.

[7] A. Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2nd edition, 1996.

[8] M. C. Beltrametti, E. Carletti, D. Gallarati, and G. Monti Bragadin. *Lectures on curves, surfaces and projective varieties, A classical view of algebraic geometry.* European Mathematical Society (EMS), Zürich, 2009.

[9] M. Bronstein. *Symbolic integration I: Transcendental functions.* Springer, Berlin, 2nd edition, 2005.

[10] H.-J. Bungartz, S. Zimmer, M. Buchholz, and D. Pflüger. *Modellbildung und Simulation*. Springer-Verlag, Berlin Heidelberg, 2013.

[11] J. Cannon and Bosma W. (editors). *Handbook of Magma functions*. Sydney, 2.8 edition, 2001.

[12] J. Cannon and Bosma W. (editors). *Handbook of Magma functions*. Sydney, 2.15 edition, 2008.

[13] W. A. de Graaf, J. Pílniková, and J. Schicho. Parametrizing Del Pezzo surfaces of degree 8 using Lie algebras. *Journal of Symbolic Computation*, 44(1):1–14, 2009.

[14] S. B. Engelsman. Lagrange's early contributions to the theory of first-order partial differential equations. *Historia Mathematica*, 7:7–23, 1980.

[15] A. Eremenko. Rational solutions of first-order differential equations. *Annales Academiae Scientiarum Fennicae. Mathematica*, 23(1):181–190, 1998.

[16] H. Eves. *Elementary matrix theory*. Dover Publications, Inc., New York, 1980.

[17] R. Feng and X.-S. Gao. Rational General Solutions of Algebraic Ordinary Differential Equations. In J. Gutierrez, editor, *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, ISSAC '04, pages 155–162, New York, 2004. ACM.

[18] R. Feng and X.-S. Gao. Polynomial General Solutions for First Order Autonomous ODEs. In H. Li et al., editor, *Computer Algebra and Geometric Algebra with Applications*, volume 3519 of *Lecture Notes in Computer Science*, pages 5–7, Berlin Heidelberg, 2005. Springer.

[19] R. Feng and X.-S. Gao. A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs. *Journal of Symbolic Computation*, 41(7):739–762, 2006.

[20] J. González-Sánchez, M. Harrison, I. Polo-Blanco, and J. Schicho. Algorithms for Del Pezzo surfaces of degree 5 (construction, parametrization). *Journal of Symbolic Computation*, 47(3):342–353, 2012.

[21] G. Grasegger. Radical Solutions of Algebraic Ordinary Differential Equations. In K. Nabeshima, editor, *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC '14, pages 217–223, New York, 2014. ACM.

[22] G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. On Symbolic Solutions of Algebraic Partial Differential Equations. In V.P. Gerdt et al., editor, *Computer Algebra in Scientific Computing*, volume 8660 of *Lecture Notes in Computer Science*, pages 111–120. Springer International Publishing, 2014.

[23] G. Grasegger and F. Winkler. Symbolic solutions of first-order algebraic ODEs. In *Computer Algebra and Polynomials*, volume 8942 of *Lecture Notes in Computer Science*, pages 94–104. Springer International Publishing, 2015.

[24] M. Harrison. Explicit solution by radicals, gonal maps and plane models of algebraic curves of genus 5 or 6. *Journal of Symbolic Computation*, 51:3–21, 2013.

[25] M. Harrison and J. Schicho. Rational Parametrisation for Degree 6 Del Pezzo Surfaces Using Lie Algebras. In *Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation*, pages 132–137, New York, 2006. ACM.

[26] H. Hong and J. Schicho. *Journal of Symbolic Computation*, 26(3):279–300, 1998.

[27] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Trivariate Rational Systems of Autonomous ODEs. In *Proceedings of the Fourth International Conference on Mathematical Aspects of Computer and Information Sciences*, MACIS 2011, pages 93–100, 2011.

[28] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Trivariate Rational Differential Systems. *Mathematics in Computer Science*, 6(4):361–374, 2012.

[29] Y. Huang, L. X. C. Ngô, and F. Winkler. Rational General Solutions of Higher Order Algebraic ODEs. *Journal of Systems Science and Complexity*, 26(2):261–280, 2013.

[30] E. Hubert. The General Solution of an Ordinary Differential Equation. In Y. N. Lakshman, editor, *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, ISSAC '96, pages 189–195, New York, 1996. ACM.

[31] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen II.* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1965.

[32] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen.* B. G. Teubner, Stuttgart, 1997.

[33] E. R. Kolchin. *Differential algebra and algebraic groups*, volume 54 of *Pure and Applied Mathematics*. Academic Press, New York-London, 1973.

[34] W. Lü and H. Pottmann. Pipe surfaces with rational spine curve are rational. *Computer Aided Geometric Design*, 13(7):621–628, 1996.

[35] G. Landsmann, J. Schicho, and F. Winkler. The parametrization of canal surfaces and the decomposition of polynomials into a sum of two squares. *Journal of Symbolic Computation*, 32(1–2):119–132, 2001.

[36] G. Landsmann, J. Schicho, F. Winkler, and E. Hillgarter. Symbolic Parametrization of Pipe and Canal Surfaces. In *Proceedings of the 2000 International Symposium on Symbolic and Algebraic Computation*, ISSAC '00, pages 202–208, New York, 2000. ACM.

[37] A. Lastra, L. X. C. Ngô, J. R. Sendra, and F. Winkler. Rational General Solutions of Systems of Autonomous Ordinary Differential Equations of Algebro-Geometric Dimension One. *Publicationes Mathematicae Debrecen*, 86(1–2):49–69, 2015.

[38] J. Li, L.-Y. Shen, and X.-S. Gao. Proper Reparametrization of Rational Ruled Surface. *Journal of Computer Science and Technology*, 23(2):290–297, 2008.

[39] J. Liouville. Premier Mémoire Sur la détermination des Intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, 14:124–148, 1833.

[40] J. Liouville. Second Mémoire Sur la détermination des Intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, 14:149–193, 1833.

[41] J. Liouville. Mémoire sur l'intégration d'une classe des fonctions transcendantes. *Journal für die Reine und Angewandte Mathematik*, 13:93–118, 1835.

[42] L. X. C. Ngô, J. R. Sendra, and F. Winkler. Classification of algebraic ODEs with respect to rational solvability. In *Computational Algebraic and Analytic Geometry*, volume 572 of *Contemporary Mathematics*, pages 193–210. American Mathematical Society, Providence, RI, 2012.

[43] L. X. C. Ngô, J. R. Sendra, and F. Winkler. Birational transformations preserving rational solutions of algebraic ordinary differential equations. *Journal of Computational and Applied Mathematics*, 286:114–127, 2015.

[44] L. X. C. Ngô and F. Winkler. Rational general solutions of first order non-autonomous parametrizable ODEs. *Journal of Symbolic Computation*, 45(12):1426–1441, 2010.

[45] L. X. C. Ngô and F. Winkler. Rational general solutions of parametrizable AODEs. *Publicationes Mathematicae Debrecen*, 79(3–4):573–587, 2011.

[46] L. X. C. Ngô and F. Winkler. Rational general solutions of planar rational systems of autonomous ODEs. *Journal of Symbolic Computation*, 46(10):1173–1186, 2011.

[47] S. Pérez-Díaz, J. Sendra, and J. R. Sendra. Parametrization of approximate algebraic surfaces by lines. *Computer Aided Geometric Design*, 22(2):147–181, 2005.

[48] M. Peternell and H. Pottmann. Computing rational parametrizations of canal surfaces. *Journal of Symbolic Computation*, 23(2–3):255–266, 1997.

[49] A. D. Polyanin and V. F. Sajzew. *Sammlung gewöhnlicher Differentialgleichungen.* Verlag Harri Deutsch, Thun und Frankfurt am Main, 1996.

[50] A. D. Polyanin and V. F. Zaitsev. *Handbook of nonlinear partial differential equations.* CRC Press, Boca Raton, FL, 2nd edition, 2011.

[51] R. H. Risch. The problem of integration in finite terms. *Transactions of the American Mathematical Society*, 139:167–189, 1969.

[52] R. H. Risch. The solution of the problem of integration in finite terms. *Bulletin of the American Mathematical Society*, 76:605–608, 1970.

[53] J. F. Ritt. On algebraic functions, which can be expressed in terms of radicals. *Transactions of the American Mathematical Society*, 24:21–30, 1924.

[54] J. F. Ritt. *Integration in Finite Terms.* Columbia University Press, New York, 1948.

[55] J. F. Ritt. *Differential algebra*, volume 33 of *Colloquium Publications.* American Mathematical Society, New York, 1950.

[56] J. Schicho. Rational Parametrization of Surfaces. *Journal of Symbolic Computation*, 26(1):1–29, 1998.

[57] J. Schicho. Proper Parametrization of Real Tubular Surfaces. *Journal of Symbolic Computation*, 30(5):583–593, 2000.

[58] J. Schicho, F.-O. Schreyer, and M. Weimann. Computational aspects of gonal maps and radical parametrization of curves. *Applicable Algebra in Engineering, Communication and Computing*, 24(5):313–341, 2013.

[59] J. Schicho and D. Sevilla. Effective radical parametrization of trigonal curves. In *Computational Algebraic and Analytic Geometry*, volume 572 of *Contemporary Mathematics*, pages 221–231. American Mathematical Society, Providence, RI, 2012.

[60] J. R. Sendra and D. Sevilla. Radical parametrizations of algebraic curves by adjoint curves. *Journal of Symbolic Computation*, 46(9):1030–1038, 2011.

[61] J. R. Sendra and D. Sevilla. First steps towards radical parametrization of algebraic surfaces. *Computer Aided Geometric Design*, 30(4):374–388, 2013.

[62] J. R. Sendra and F. Winkler. Tracing index of rational curve parametrizations. *Computer Aided Geometric Design*, 18(8):771–795, 2001.

[63] J. R. Sendra, F. Winkler, and S. Pérez-Díaz. *Rational Algebraic Curves, A Computer Algebra Approach*, volume 22 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin Heidelberg, 2008.

[64] I. R. Shafarevich. *Basic Algebraic Geometry 1, Varieties in Projective Space*. Springer-Verlag, Berlin Heidelberg, third edition, 2013.

[65] L.-Y. Shen and S. Pérez-Díaz. Characterization of rational ruled surfaces. *Journal of Symbolic Computation*, 63:21–45, 2014.

[66] I. Shingareva and C. Lizárraga-Celaya. *Solving nonlinear partial differential equations with Maple and Mathematica*. Springer-Verlag, Wien, 2011.

[67] B. M. Trager. Integration of simple radical extensions. In E. W. Ng, editor, *Symbolic and algebraic computation, EUROSAM '79*, volume 72 of *Lecture Notes in Computer Science*, pages 408–414, Berlin Heidelberg, 1979. Springer.

[68] B. M. Trager. *On the integration of algebraic functions*. PhD thesis, Massachusetts Institute of Technology, 1984.

[69] B. L. van der Waerden. *Algebra*, volume II. Springer-Verlag, New York, 1991.

[70] M. van Hoeij. Rational Parametrizations of Algebraic Curves using a Canonical Divisor. *Journal of Symbolic Computation*, 23(2–3):209–227, 1997.

[71] O. Zariski. Sull' impossibilità di risolvere parametricamente per radicali un'equazione algebrica $f(x, y) = 0$ di genere $p > 6$ a moduli generali. *Atti della Accademia Nazionale dei Lincei, Rendiconti, VI. Serie*, 3:660–666, 1926.

[72] O. Zariski. *Collected papers*, volume III: Topology of curves and surfaces, and special topics in the theory of algebraic varieties. The MIT Press, Cambridge, Massachusetts, London, 1978.

[73] D. Zwillinger. *Handbook of Differential Equations.* Academic Press, San Diego, CA, third edition, 1998.

Curriculum Vitae

Personal Data

Full Name:	Georg Grasegger
Year of Birth:	1985
Place of Birth:	Linz
Nationality:	Austria

Education

2004	High School Diploma (Matura), with distinction
2005–2010	Bachelor Studies in Technical Mathematics at Johannes Kepler University Linz, Austria, Bakk.techn. 2010 (with distinction)
2008–2009	Exchange Student at the Norwegian University of Science and Technology, Trondheim, Norway
2010–2011	Master Studies in Computer Mathematics at Johannes Kepler University Linz, Austria, Dipl.-Ing. 2011 (with distinction)
2013 – 2014	Research Visit at University of Alcalá, Alcalá de Henares, Spain
2011–now	PhD Studies in Technical Sciences at Johannes Kepler University Linz, Austria

Career

2010 – 2011	Student assistant, Institute for Algebra, Johannes Kepler University Linz, Austria
2011	Research assistant, Research Institute for Symbolic Computation, Johannes Kepler University Linz, Austria
Feb. 2013, 2014, 2015	Course instructor, Projektwoche Angewandte Mathematik, Stiftung Talente
2011 – now	Research assistant, Doctoral Program Computational Mathematics, Johannes Kepler University Linz, Austria
2012 – now	Lecturer, Johannes Kepler University Linz, Austria

Publications

Refereed publications:

- G. Grasegger, G. Horváth, and K. A. Kearnes. Polynomial Equivalence of Finite Rings. Journal of the Australian Mathematical Society, 96(2):244–257, 2014
- G. Grasegger. Radical Solutions of Algebraic Ordinary Differential Equations. In K. Nabeshima, editor, Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation, pages 217–223, New York, 2014. ACM.
- G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. On Symbolic Solutions of Algebraic Partial Differential Equations. In V. P. Gerdt et al., editor, Computer Algebra in Scientific Computing, volume 8660 of Lecture Notes in Computer Science, pages 111–120. Springer International Publishing, 2014.
- G. Grasegger and F. Winkler. Symbolic solutions of first-order algebraic ODEs. In Computer Algebra and Polynomials, volume 8942 of Lecture Notes in Computer Science, pages 94–104. Springer International Publishing, 2015.

Technical reports:

- G. Grasegger. A procedure for solving autonomous AODEs. Technical Report: 2013-05, DK Computational Mathematics, Johannes Kepler University Linz, 2013. 2013-03, RISC, Johannes Kepler University Linz, 2013.

- G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. A solution method for autonomous first-order algebraic partial differential equations. Technical Report: 2014-07, DK Computational Mathematics, Johannes Kepler University Linz, 2014. 2014-03, RISC, Johannes Kepler University Linz, 2014.
- G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler. A solution method for autonomous first-order algebraic partial differential equations in several variables. Technical Report:
  2015-02, DK Computational Mathematics, Johannes Kepler University Linz, 2015.
  2015-02, RISC, Johannes Kepler University Linz, 2015.

Talks

At other universities:

- November 4, 2013: Seminario de física y matemáticas, Universidad de Alcalá
  Talk: Radical solutions of algebraic ordinary differential equations
- June 26, 2014: Séminaire Calcul Formel, Université de Limoges
  Talk: A solution method for algebraic partial differential equations
- September 2, 2014: University of Bath
  Talk: Solving first-order algebraic differential equations

At conferences:

- AAA84, 84th Workshop on General Algebra, June 8–10, 2012, Dresden, Germany
  Talk: An explicit polynomial equivalence of the rings $\mathbb{Z}_p^2$ and $\mathbb{Z}_p[t]/(t^2)$
- Conference on commutative rings, integer-valued polynomials and polynomial functions, December 19–22, 2012, Graz, Austria
  Talk: Polynomial equivalence of finite rings
- Workshop on Approximation Theory, CAGD, Numerical Analysis, and Symbolic Computation, August 25–30, 2013, Sozopol, Bulgaria
  Talk: Using parametrizations for solving algebraic ordinary differential equations
- AICA 2013, Aplicaciones Industriales del Álgebra Computacional, November 7–8, 2013, Universidad Complutense, Madrid, Spain
  Poster: Symbolic solutions of algebraic ordinary differential equations

- Computer algebra and polynomials (Workshop 3 of Special Semester on Applications of Algebra and Number Theory), November 25–29, 2013, RICAM, Linz, Austria
  Talk: Radical solutions of algebraic ordinary differential equations
- III Seminario sobre "Algoritmos y Aplicaciones en Geometría Algebraica", December 16–17, 2013, CIEM (Centro Internacional de Encuentros Matemáticos), Castro Urdiales, Spain
  Talk: Radical solutions of algebraic ordinary differential equations
- AAA87&CYA28, 87th Workshop on General Algebra and 28th Conference of Young Algebraists, February 7–9, 2014, Linz, Austria
  Talk: Rational and radical solutions of algebraic differential equations
- ISSAC 2014, 39th International Symposium on Symbolic and Algebraic Computation, July 23–25, Kobe, Japan
  Talk: Radical solutions of algebraic ordinary differential equations
- CASC 2014, 16th International Workshop on Computer Algebra in Scientific Computing, September 8–12, Warsaw, Poland
  Talk: On symbolic solutions of algebraic partial differential equations