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# Closed Linkages with Six Revolute Joints

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Linz, December 2015

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# Abstract

In this dissertation, we explore spatial overconstrained closed linkages with six revolute joints and a single-degree-of-freedom (6R linkages). The first 6R linkage was invented by Pierre Frédéric Sarrus in 1853. In the literature, a lot of 6R linkages were found by numerous methods. New 6R linkages are still being found by new methods, too. But the answer on the classification question of 6R linkages is open. The aim of this dissertation is to try to fill the gap. We will use two new methods: bond theory and factorization of motion polynomial to analyze 6R linkages. These two methods, which were invented by my supervisor Josef Schicho and his collaborators, are based on algebraic geometry and computer algebra.

In the first part, we will recall bond theory. Simultaneously, we give the genus bound for mobile 6R linkages. Using this new theory, we introduce a new technique for deriving equational conditions on the Denavit-Hartenberg parameters of 6R linkages that are necessary for movability. Several new families of 6R linkages are derived by this new technique. In the second part, we will recall the method of factorization of motion polynomials. There are cases where the factorization does not exist. But, even in these cases, we can do some reduction to the cases where the factorization does exist. Using the factorization method and bond theory, we construct several new 6R linkages. In the third part, we will give the sub classification of 6R linkages that have three equal pairs of opposite rotation angles (angle-symmetric 6R linkages). In the classification, there are three families. Two families are known and one family is new. This new 6R linkage has an additional parallel property, namely, three parallel pairs of joints. We also give the classification of the parallel 6R linkages. The new angle-symmetric family appears in both classifications. In addition, we find two other types. One has the translation property: three rotational axes can be obtained by a single translation from other three axes. The other one is a special case of the known family of angle-symmetric 6R linkages.

In the final part, we will give an overview of results and open questions on the classification of 6R linkages.

# Zusammenfassung

In dieser Dissertation werden räumliche überbestimmte geschlossene Gelenkmechanismen mit Drehgelenken und einem Freiheitsgrad der Bewegung (6R-Mechanismen) untersucht. Erstmals wurde ein solcher Mechanismus von Pierre Frédéric Sarrus 1853 gefunden. Seither wurden viele weitere Konstruktionen von 6R-Mechanismen in der Literatur angegeben, und bis heute werden noch bisher neue Mechanismen mit verschiedenen Methoden gefunden. Die Klassifikation aller 6R-Mechanismen ist ein offenes Problem. Diese Dissertation zielt auf diese offene Frage hin. Wir verwenden zwei neue Methoden zur Analyse von 6R-Mechanismen: Bond-Theorie und Faktorisierung von Bewegungspolynomen. Beide Methoden wurden von meinem Betreuer Josef Schicho und Ko-Autoren entwickelt und basieren auf algebraischer Geometrie und Computer-Algebra.

Im ersten Teil wird Bond-Theorie eingeführt und gleichzeitig ein Schranke für das Geschlecht der Bewegungskurve eines 6R-Mechanismus angegeben. Mithilfe der Bond-Theorie und einer neuen Technik leiten wir notwendige Bedingungen für die Beweglichkeit eines 6R-Mechanismus in Form von Gleichungen in den Denavit-Hartenberg-Parametern her. Mit der gleichen Technik werden auch noch mehrere bisher unbekannte Familien von 6R-Mechanismen hergeleitet.

Im zweiten Teil führen wir Bewegungspolynome und deren Faktorisierung ein. Zwar gibt es Fälle, in denen keine Faktorisierung existiert, die Methode also nicht anwendbar scheint. Jedoch gelingt in diesen Fällen eine Reduktion auf solche Fälle, für die eine Faktorisierung existiert. Mit diesem Trick und den beiden erwähnten neuen Methoden werden nun mehrere neue 6R-Mechanismen konstruiert.

Im dritten Teil geben wir die Teilklassifizierung aller 6R-Mechanismen an, deren Bewegungswinkel drei Paare von jeweils gleichen gegenüberliegenden Winkeln bilden (winkel-symmetrische 6R-Mechansimen). Diese lassen sich unterteilen in drei Familien. Davon sind zwei wohlbekannt, die dritte ist jedoch neu. Die dritte Familie besitzt eine zusätzliche Eigenschaft, nämlich drei Paare von jeweils parallelen Drehachsen. In der Folge geben wir auch die Klassifikation dieser "parallelen 6R-Mechanismen" an. Die oben erwähnte Familie findet man auch in dieser Klassifikation wieder. Darüber hinaus existieren noch zwei weitere. Eine ist durch folgende Translations-Eigenschaft charakterisiert: drei der Drehachsen erhält man durch eine

einzige Translation der anderen drei. Die andere Famile erweist sich als Spezialfall einer der beiden bekannten Familien von winkel-symmetrischen 6R-Mechanismen.

Im letzten Teil geben wir einen Überblick über die Ergebnisse und offene Fragen über die Klassifizierung von 6R-Mechanismen.

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Zijia Li
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# Contents

# Chapter 1

# Introduction

In kinematics, a *closed linkage* is a mechanical structure that consists of a finite number of rigid bodies – its *links* – and a finite number of *joints* that connect the links cyclically together, so that they possibly produce a self-motion. The *degree of freedom* (dof) of a closed linkage is the number of independent parameters that define its configuration. A self-motion means that the dof is bigger than zero. For the links, one could take different shapes without changing the self-motion property. But the joints do impact the self-motion. So we mainly focus on the joints in this dissertation. There will be four types of joints in the dissertation:

1. (R) revolute joints: allow rotations around a fixed axis (1-dof);

2. (P) prismatic joints: allow translations in a fixed direction (1-dof);

3. (C) cylindrical joints: allow rotations around a fixed axis and translations in the the direction of the axis (2-dof);

4. (H) helical joints: allow the motions of a cylindrical joint where the rotation angle and the translation length are coupled by a linear equation (1-dof).

Whenever two links are connected by a revolute joint, their relative position is constrained to a rotation around an axis. More precisely, the relative position is determined by the rotation angle about this joint with respect to a given reference configuration. The set of all tuples of possible rotation angles is called the linkage's *configuration space*. If it is of dimension one, we call it a *configuration curve* and say the linkage has *one degree of freedom.* If this is the case, we designate one link as fixed and another as moving and call them *fixed* and *moving frame,* respectively. We view the relative displacements of the moving frame with respect to the fixed frame as a curve in the special Euclidean group $SE_3$.

For a closed linkage, the mobility is defined as the dimension of the configuration space. If the links of the closed linkage move in three-dimensional space, then in general the mobility of a closed linkage with six revolute joints is 0 from the mobility formula (Chebychev-Grübler-Kutzbach criterion [52]) $6 - n = 0$ where $n$ is the number of 1-dof joints in the closed loop. A linkage is overconstrained if it has more mobility than the mobility formula predicted. An overconstrained 6R linkage is a closed 6R linkage that does produce a one-dimensional (at least) self-motion in three-dimensional space. It is known that it is mobile when the six joints are parallel (planar) or intersecting at one point (spherical). We mainly consider the closed linkage with six revolute joints (6R linkage) in three-dimensional space (non-planar or non-spherical). Furthermore, we will only focus on the mobility 1 case. When the mobility is more than 1, it can be treated as a 5R linkage with an extra redundant joint(s). The classification of mobile 5R linkage is known. There is only one spatial 5R linkage with mobility 1 [28, 36], i.e., the Goldberg 5R linkage, which was first introduced in [24]. It was constructed by merging two Bennett 4R linkages which have two adjacent common joints, where one of them has the same rotational speed and the other one does not. The Goldberg 5R linkage is the mechanism with removing the joint with the same speed. For the 6R case, the first overconstrained 6R linkage (6R linkage with mobility 1) was invented by Pierre Frédéric Sarrus in 1853 [56]. It is a mechanical linkage which can produce a linear motion from a limited circular motion. It is a spatial linkage which consists of two triples of parallel joints. Since that time, a lot of 6R linkages were found, e.g., Bricard 6R linkages, the Dietmaier 6R linkage, the Hooke linkage which as considered the generalized Sarrus linkage, the Wohlhart partial symmetric 6R linkage, the Waldron double Bennett linkage, etc. New 6R linkages are still being found until now with numerous methods. On these methods, we will not introduce them in this dissertation, because there is almost no relation with the methods used in the dissertation. We would like to suggest two review papers [17, 21] on the topic the 6R linkages which provide an overview on existing methods and further results.

As a rotational joint has a fixed axis, it is equivalent to consider the six rotational lines in the space. For the lines in the space, there are geometric invariant parameters which are also known as Denavit-Hartenberg parameters. A closed 6R linkage is uniquely determined by its set of Denavit-Hartenberg parameters [20], which contains 18 real geometric invariants: the six twist angles $\alpha_i$, the six normal distances $d_i$ and the six offsets $s_i$. where $i = 1, \ldots, 6$ as in Figure 1.1. A generic choice of these parameters leads to a rigid 6R linkage. From algebraic point of view, the set of geometric parameters of movable closed 6R linkages is an algebraic variety (affine set) which is one of the central objects in algebraic geometry, which is defined by algebraic equations (polynomials). But we do not know its equations, its dimensions or even the numbers of components (families). It is worth mentioning that the known families are just some subcomponents of this variety. For each genus ($\leq 5$), there exist mobile 6R linkages with a configuration curve of such genus.

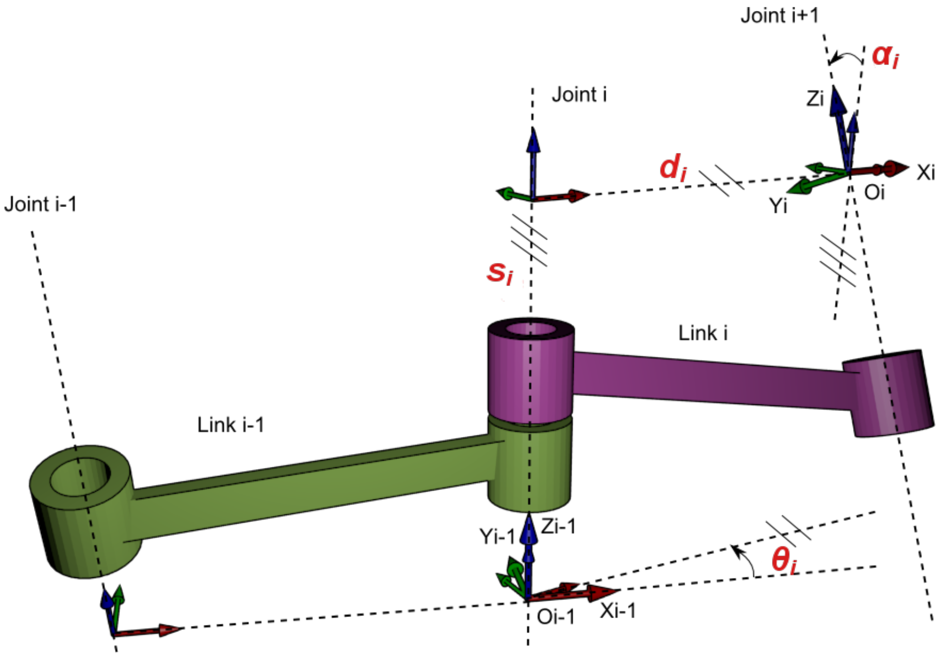



Figure 1.1: The four parameters of classic DH convention are shown in red text, which are $\theta_i, d_i, s_i, \alpha_i$.

## 1.1 Main Contributions

The main contributions of this dissertation include four parts.

The first part is devoted to exploring the main theory we used, i.e., *bond theory*. The bond theory was invented by my supervisor Josef Schicho and his collaborators basing on algebraic geometry and symbolic computation. As there are relative motions in the mobile closed linkages, the degrees of these motions can be calculated if we know the linkages. The bond theory could give you an explicit view of these degrees in items of bond diagrams, which have vertices (corresponding to links), edges (corresponding to joints) and connections between edges, where the number of connections between edges is the degree of their relative motion. For instance, we can see the bond diagram of Bricard's line symmetric 6R linkage as in Figure 1.2 (c), where the degrees of all relative motions can be read off by making the relative count of connections which go through the line that connected the two related vertices. Furthermore, we can say more on the variety of the geometric parameters of mobile 6R (1-dof) basing on the bond diagram. Using the bond theory and algebraic geometry, we proved that the genus of the configuration curve is at most 5. For each genus ($\leq 5$), there exist mobile 6R linkages with a configuration curve of such genus. The maximal genus was reached by several known examples, e.g., the Hooke linkage, the Bricard's orthogonal linkage, etc. Many families of 6R linkages with merging Bennett linkages have genus 0. We prove the following theorem which

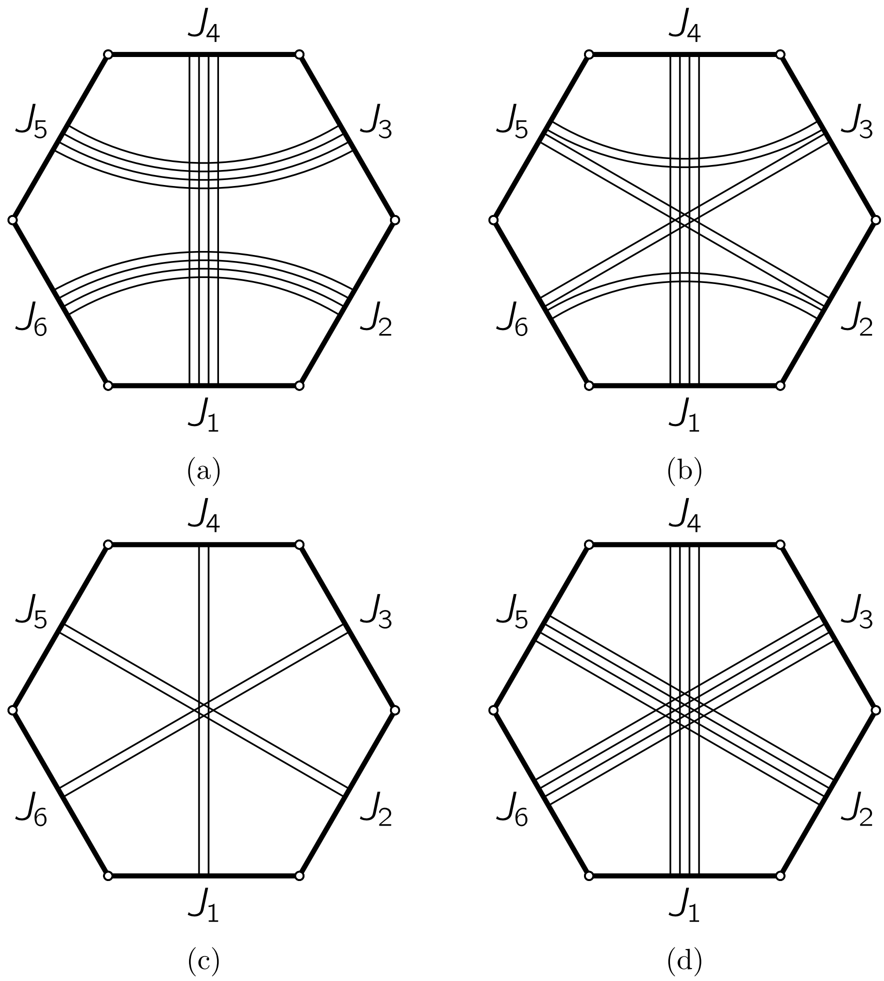



Figure 1.2: Bond diagrams for Hooke's double spherical linkage (a), Dietmaier's linkage (b), Bricard line symmetric linkage (c), and Bricard's orthogonal linkage (d).

is for classifying linkages with a genus 5 configuration curve.

**Theorem 1.1.** *If the bond diagram is different from the diagrams Figure 1.2(a), (b), and (d), then* $g(K) \leq 3$.

In principle, this is a glimpse of the classification of the components of the variety of mobile 6R linkages.

The second part is a technique , namely quad polynomial, to derive algebraic equations on the Denavit-Hartenberg parameters using bond connections. The aim of this technique is to make a step towards such a complete classification, by deriving necessary conditions for mobility; up to our knowledge, not a single necessary equation condition has been known up to now. If we number the joints cyclically, then a bond connects either joints $i$ and $i+2$ – then we speak of a *near connection*, – or it connects joints $i$ and $i+3$, and then we speak of a *far connection*. It was shown that the existence of a near connection implies the validity of a well-known condition which also arises frequently in many families, namely Bennett's condition: $s_i = 0$, and $\frac{d_i}{\sin(\phi_i)} = \frac{d_{i+1}}{\sin(\phi_{i+1})}$, where $s_i, d_i, \phi_i$ are Denavit-Hartenberg parameters. Bennett's condition is equivalent to a kinematic condition on three consecutive rotation axes, not all three parallel or intersecting, namely the existence of a fourth axis such that the closed 4R linkage with these four axes is movable (see [12]). However, there are many mobile 6R linkages without near connections, for instance Bricard's orthogonal linkage or Bricard's line symmetric linkage. So, the Bennett conditions are not necessary for mobility. The quad polynomial is a univariate polynomials of degree 2 with coefficients depending on the Denavit-Hartenberg parameters by an explicit formula.

$$Q_1^+(x) = \left(x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}\mathrm{i}\right)^2 + \frac{\mathrm{i}}{2}\left(b_1s_2 + b_3s_3 + s_2b_3c_2 + s_3b_1c_2\right) - \frac{b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2c_2^2}{4}.$$

For $i = 2, \ldots, 6$, we define the quad polynomial $Q_i^+(x)$ by a cyclic shift of indices that shifts 1 to $i$. Finally, we define $Q_i^-(x)$ by replacing the parameters $c_1, \ldots, c_6, b_1, \ldots, b_6$ and $s_2, s_4, s_6$ by their negatives, and leaving $s_1, s_3, s_5$ as they are. For instance,

$$Q_1^-(x) = \left(x + \frac{b_3c_3 - b_1c_1}{2} - \frac{s_1}{2}\mathrm{i}\right)^2 + \frac{\mathrm{i}}{2}\left(b_1s_2 - b_3s_3 - s_2b_3c_2 + s_3b_1c_2\right) - \frac{-b_1b_3c_2 - s_2s_3c_2}{2} + \frac{s_2^2 + s_3^2 - b_1^2 + b_2^2 - b_3^2 - b_2^2c_2^2}{4}.$$

The existence of a far connection implies a common root of two such quad polynomials, and this gives rise to necessary equational conditions.

**Theorem 1.2.** *Let $k$ be the number of bond connections of $J_1$ and $J_4$. Then*

$$k \leq \deg(\gcd(Q_1^+, Q_4^+)) + \deg(\gcd(Q_1^-, Q_4^-)).$$

Because every mobile linkage has either near or far connections (or both), it is then possible to write down necessary equational conditions for mobility (see Remark 3.6). However, the full system of equations is too big and complicated, and therefore it is better to follow the classification scheme suggested by bond theory and distinguish cases according to the bond diagram. For any bond diagram, we could write down the equations of the Bennett conditions for the near connections and quad conditions for the far connections. In some cases, the equations are even sufficient for mobility. For instance, we could classify the mobile 6R linkages with genus 5. These introduce two new families but also yield the exist families, e.g., the Dietmaier's linkage can be obtained in this way.

The third part focuses on the factorization of motion polynomials . In principle, the 6R linkage of genus 0 can be constructed in three steps, i.e., finding a motion polynomial, factoring it into two different factorizations (corresponding to two open chains) and combining the two open chains to a 6R linkages. It also needs the bond theory for finding the exact degree of the motion polynomial. We give all possible bond diagrams (at most 4 connections) of 6R linkages of genus 0. These contains 11 families of 6R linkages. There are only two bond diagrams with 3 bonds, Figure 1.3. There are only nine bond diagrams with 4 bonds, Figure 1.4. This is based on the

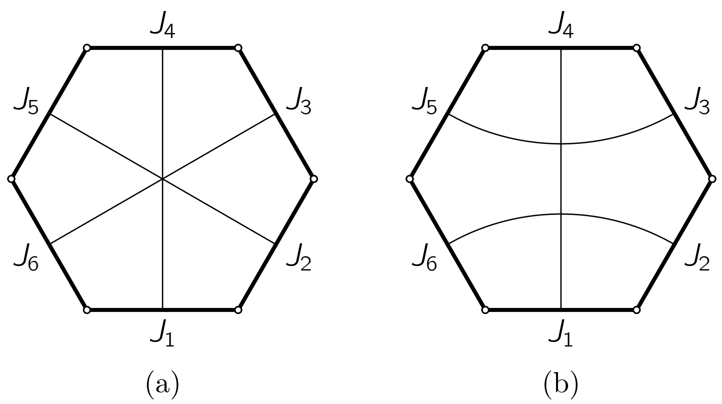



Figure 1.3: Bond diagrams for the cube linkage (a) and Waldrons double Bennett hybrid (b)

following lemma, where notations will be defined in Chapter 5.

**Lemma 1.3.** *For an nR (6R/5R) linkage* $L = [h_1, h_2, ..., h_n]$*, where* $n = 5$ *or* $6$*,* $h_i^2 = -1$ *and* $h_i \neq \pm h_{i+1}$*, if* $l_{123} = l_{234} = 6$*, then there is no bond which connects* $h_1$ *and* $h_4$*.*

Using the factorization of motion polynomials and the bond diagrams above, one can construct concrete example with each bond diagram (family). One can obtain them in terms of merging Bennett linkages too.

The fourth part is devoted to considering subclassification problems of 6R linkages. The classification of angle-symmetric 6R linkages which are linkages with the property that the rotation angles of the three opposite joints are equal was obtained. The classification of angle-symmetric 6R linkages contains three types of linkages. Type one is the Bricard line symmetry 6R linkage. Type two is new. Type three is the cube linkages which is constructed by the factorizations of a cubic motion polynomial. The main tool is a $\lambda$-matrix, denote by $\mathbf{M}^{\dagger}$, for an angle-symmetric linkage,

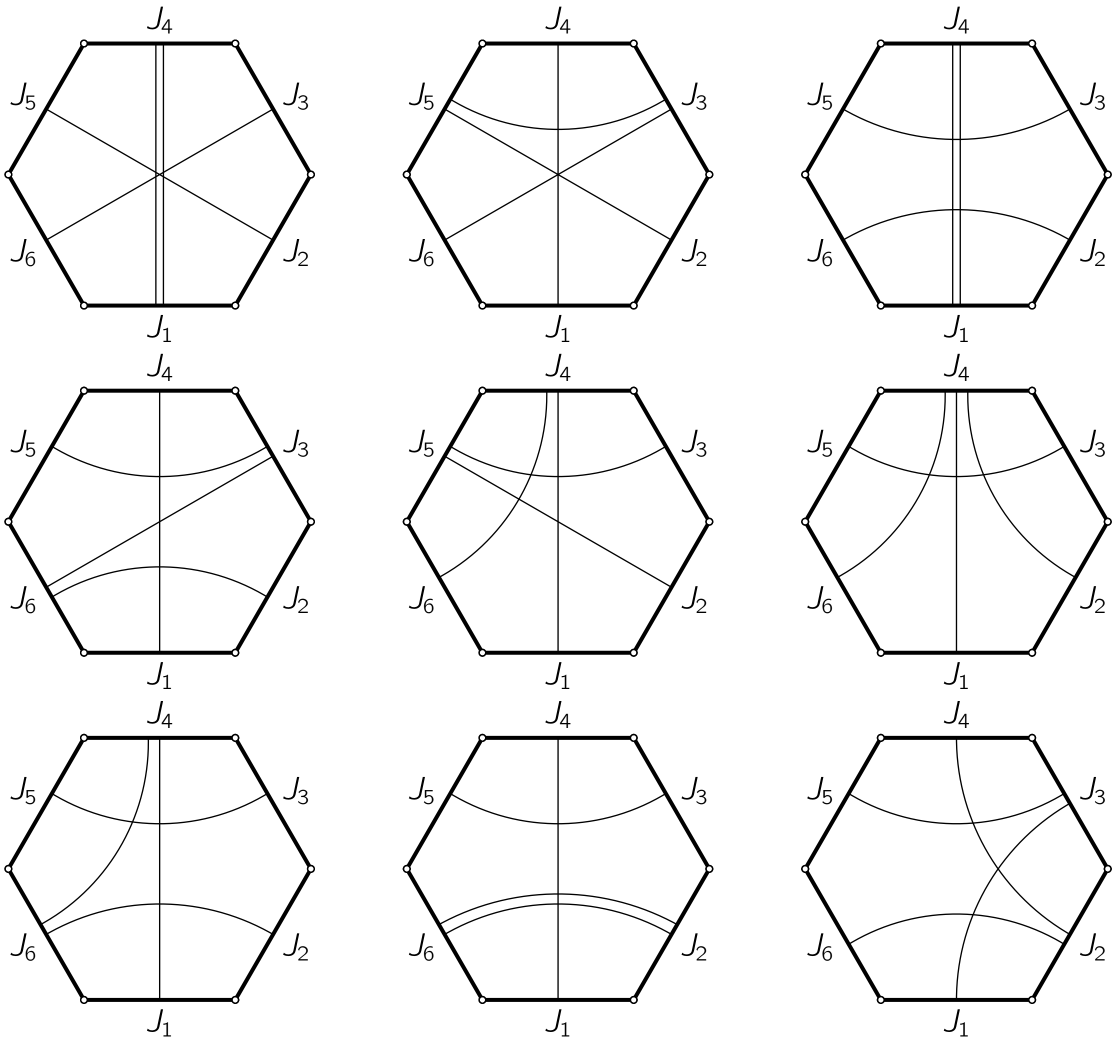



Figure 1.4: Bond diagrams of 6R linkages with four bonds

to be defined in Section 6.2, and its rank $r$. Intuitively speaking, the configuration set of an angle-symmetric linkage can be written as the vanishing set of $r$ equations in three variables, namely the cotangents of the half of the rotation angles. We show that the rank $r$ can be only 2, 3, or 4. For $r = 2$, the angle-symmetric linkage is line symmetric. For $r = 3$, we get the new linkage with three pairs of parallel axes. For $r = 4$, we obtain the cube linkage constructed in [29, 30] using motion polynomials. The new linkage has three pairs of parallel axes. Two pairs are neighbor axes. The other pair is opposite. We call the linkage with this property *parallel 6R linkage*. A classification of parallel 6R linkages was obtained. Namely, a parallel 6R linkage either has the translation property, or is angle-symmetric.

## 1.2 Aim and Open Questions

In the beginning of 2012, I came to Linz for joining DK (doctoral program). My supervisor Prof. Josef Schicho and colleagues, namely, Prof. Hans-Peter Schröcker and Dr. Gábor Hegedüs, just finished the conference paper of the bond theory. I am very happy to join them. At the same time, I got a good exercise on the calculation of bonds for a 6R linkage. This was a really good exercise for me. I decided to study 6R linkages using bond theory in my thesis. After three years living with bond theory, I understand the bond theory better, even derived advance results to refine the bond theory. This gives me quite a lot profit on understanding the classes of 6R linkages.

The aim of this dissertation is to try to fill the gap between the final classification and the known knowledge on 6R linkages. We mainly explore two algebraic methods: bond theory and factorization of motion polynomial to analyze 6R linkages. These two new methods which were invented by my supervisor Josef Schicho and his collaborators are based on algebraic geometry and symbolic computation. We use the language of dual quaternions (see also [14, 16, 26, 33, 34, 58, 59]). The use of algebra and geometry for studying linkages is very classical and goes back to Sylvester, Kempe, Cayley and Chebyshev. We also hope that the reader will be inspired to use these two methods in kinematics by this process of exploring.

Concerning that the readers are from different area, we can only try our best to introducing everything as simple as possible. We use a lot of computation with the software Maple. It is not necessary to rebuilt every computation if the reader trusts us. Linear algebra is necessary. Especially, knowledge on polynomials and computational algebraic geometry are quite helpful. Even less, basics of computer algebra should suffice. General Lie group theory is quite enough. For a general reference for the starting point of this topic we suggest some chapters from [58, Chapter 2-4]. The overlap between existing methods [5, 8, 10, 11, 21, 55, 62, 65] and these new methods is minor.

For the open questions, we have to say the classification of 6R linkage is not solved by the dissertation. We are not near to the final gap. We would like to suggest several subclassification problems which are useful for the final classification.

First one is the classification of the mobile 6R linkages of genus 3. Using quad polynomial, we already can get several families of such linkages that have genus 3. But the explicit formulas on Denavit-Hartenberg parameters are not yet well done. Namely, we knew the formulas for the known families, but there are new families which we do not know.

Second one is to find an example of mobile 6R linkage of genus 2 with the bond diagram 1.5. One can find a special case of Bricard line symmetric 6R linkage such that it has the bond diagram 1.5. But the genus of that is 1.

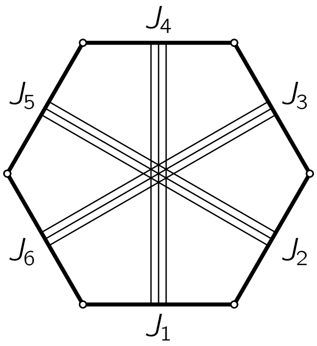



Figure 1.5: Bond diagrams for the mobile 6R linkage of genus 2.

Third one is the classification of the mobile 6R linkages of genus 1. This might be the most difficult one. There are several families known, but the quad polynomial helps quite little. We have no suggestion on this one.

Forth one is concerning the genus 0 case. One problem is to find all possible bond diagrams. The other one is formulas for each bond diagram. The quad polynomial can only give several equations, which are not sufficient for mobility. Other methods might be helpful for this case.

## 1.3 Structure of the Thesis

The remaining part of this thesis is set up as follows.

Chapter 2 gives the framework of the first method, namely, bond theory which is based on dual quaternions. Simultaneously, we give the genus bound (maximal 5) for mobile 6R linkages (1-dof). On the proof (Section 2.4), it is quite related to algebraic geometry. If one is not interested in algebraic geometry, one can just accept the results instead of going through all the proofs. The maximal genus cases are reached by several known families. The classifications of maximal genus cases are also given with bond diagrams.

Chapter 3 introduces a technique for deriving equational conditions on the Denavit-Hartenberg parameters of mobile 6R linkages that are necessary for mobility. This new technique is based on the bond theory. Furthermore, the classifications of maximal genus cases in Denavit-Hartenberg parameters equational conditions are derived by the quad polynomial. The classification contains two new families.

Chapter 4 gives the framework of the second method, namely, factorization of motion polynomials. The main part is the first factorization algorithm. We also fix the gap arising when the algorithm does not work. Namely, we can find one factorization on the motion polynomial by multiplying a real polynomial which does not change the motion.

Chapter 5 presents a combinatorial classification on the bond diagram with less than or equal to four bonds for mobile 6R linkages. All of them have a rational configuration curve. We also give the construction for each.

Chapter 6 is dedicated to the classification of angle-symmetric 6R linkages which is very related to the factorization of motion polynomials. These are linkages with the property that the rotation angles of the three pairs of opposite joints are equal for all possible configurations, or at least for infinitely many configurations (it could be that a certain linkage has two components, where only one of them is angle-symmetric). A full classification of these linkages is obtained. The classification of angle-symmetric 6R linkages contains three types of linkages. Type one is the Bricard line symmetry 6R linkage. Type two is new. Type three is the cube linkages which were constructed by the factorizations of a cubic motion polynomial.

In Chapter 7, three types of parallel 6R linkages are presented and the classification theorem is also included. A parallel linkage either has the translation property, or is angle-symmetric. The translation property means that there is a translation relation between three continuous axes and the other three continuous axes from the 6R linkages.

Chapter 8 focuses on the conclusion of the classification obtained and gives an overview about future work.

At several places, we used the computer algebra system Maple for more elaborate computations: examples, discussions of systems of equations, the derivation of the quad polynomials, construction of new 6R linkages etc. Because of the length of these computations, it is not reasonable to reproduce all them in the dissertation. We only show results with skipping details of the computation. These can be found at `http://people.ricam.oeaw.ac.at/z.li/softwares.html`. They can be read with any text editor and verified using Maple 16 or a later version.

# Chapter 2

# Bond Theory

In this chapter, we will recall some elementary definitions and properties of the bond theory. These definitions and properties are recollected from [27, 28, 31]. The bond theory was introduced for the first time in the conference of Latest Advances in Robot Kinematics in 2012 [28]. It was used to prove the classification of the overconstrained 5R linkages in [31]. The first proof [36] of the classification needs the aid of a computer algebra system. Using bond theory, one can produce a proof without computer aid. It is also worth mentioning that one should not be confused with the exist concept of a "kinematic bond" [3, Chapter 5]. For this dissertation, the bond theory helps us quite a lot on understanding 6R linkages. I hope one can get more impression from this chapter in contrast to the individual paper [27, 28, 31]. The results presented below evolved from a collaboration with Gábor Hegedüs, Josef Schicho, Hans-Peter Schröcker and have recently been published in [27].

**Structure of the chapter** The remaining part of this chapter is set up as follows. In Section 2.1, we introduce all preliminary definitions we need. In Section 2.2, we give the definition of the *bond*. Its main properties are introduced in Section 2.3. Section 2.4 contains a result on bounding the genus of mobile 6R linkages (1-dof). It also contains the classification on the highest genus (genus 5) case.

## 2.1 Preliminary Notations

In the beginning, we first recall several classical concepts and definitions: dual quaternions, the Study quadric, linkages, their configuration set and coupler maps.

We denote by $SE_3$ the group of Euclidean displacements, i.e., the group of maps from $\mathbb{R}^3$ to itself that preserve Euclidean distances and orientation. It is well-known that

$SE_3$ is a semidirect product of the translation subgroup and the orthogonal group $SO_3$, which may be identified with the stabilizer of a single point.

We denote by $\mathbb{D} := \mathbb{R} + \epsilon\mathbb{R}$ the ring of dual numbers, with multiplication defined by $\epsilon^2 = 0$. The algebra $\mathbb{H}$ is the non-commutative algebra of quaternions, and $\mathbb{DH} := \mathbb{D} \otimes_{\mathbb{R}} \mathbb{H}$. The conjugate dual quaternion $\overline{h}$ of $h$ is obtained by multiplying the vectorial part of $h$ by $-1$. The dual numbers $\mathrm{N}(h) = h\overline{h}$ and $h + \overline{h}$ are called the *norm* and *trace* of $h$, respectively.

By projectivizing $\mathbb{DH}$ as a real 8-dimensional vector space, we obtain $\mathbb{P}^7$. The condition that $\mathrm{N}(h)$ is strictly real, i.e., its dual part is zero, is a homogeneous quadratic equation. Its zero set, denoted by $S$, is called the Study quadric. The linear 3-space represented by all dual quaternions with zero primal part is denoted by $E$. It is contained in the Study quadric. The complement $S - E$ is closed under multiplication and multiplicative inverse and therefore forms a group, which is isomorphic to $SE_3$ (see [34, Section 2.4]).

A nonzero dual quaternion represents a rotation if and only if its norm and trace are strictly real and its primal vectorial part is nonzero. It represents a translation if and only if its norm and trace are strictly real and its primal vectorial part is zero. The 1-parameter rotation subgroups with fixed axis and the 1-parameter translation subgroups with fixed direction can be geometrically characterized as the lines on $S$ through the identity element 1. Among them, translations are those lines that meet the exceptional 3-plane $E$.

Let $n \geq 4$. For the analysis of the configurations of a closed $n$R linkage with links $o_1, \ldots, o_n$, the actual shape of links is irrelevant; it is enough to know the position of the rotation axes. Exploiting the fact that there is a bijection between lines in $\mathbb{R}^3$ and involutions in $SE_3$, we describe a closed $n$R linkage by a sequence $L = (h_1, \ldots, h_n)$ of dual quaternions $h_1, \ldots, h_n$ such that $h_i^2 = -1$ and $h_i \neq \pm h_{i+1}$ for $i = 1, \ldots, n$ (we set $h_{i+kn} = h_i$ and $o_{i+kn} = o_i$ for all $k \in \mathbb{Z}$). The line $h_i$ specifies the joint connecting the links $o_{i-1}$ and $o_i$. The subgroup of rotations with axis $h_i$ is parametrized by $(t - h_i)_{t \in \mathbb{P}^1}$. The pose of $o_i$ with respect to $o_n$ is then given by a product $(t_1 - h_1)(t_2 - h_2) \cdots (t_i - h_i)$, with $t_1, \ldots, t_i \in \mathbb{P}^1$. Setting $i := n$, we get the closure condition

$$(t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) \in \mathbb{R}^*. \tag{2.1}$$

The set $K$ of all $n$-tuples $(t_1, \ldots, t_n) \in (\mathbb{P}^1)^n$ fulfilling equation (2.1) is called the *configuration set* of the linkage $L$.

The dimension of the configuration set is called the *mobility* of the linkage. We are mostly interested in linkages of mobility one. Let $L = (h_1, \ldots, h_n)$ be such a linkage. Let $K$ be its configuration curve. For any pair $(o_i, o_j)$ of links, there is a map

$$f_{i,j} : K \to \mathbb{P}^7, (t_1, \ldots, t_n) \to (t_{i+1} - h_{i+1}) \cdots (t_j - h_j)$$

parametrizing the motion of $o_j$ with respect to $o_i$. This map is is called coupler map, and the image $C_{i,j}$ is the coupler curve. The *algebraic degree* of the coupler curve is defined as $\deg(C_{i,j}) \deg(f_{i,j})$, where $\deg(C_{i,j})$ is the degree of $C_{i,j}$ as a projective curve, and $\deg(f_{i,j})$ is the degree of $f_{i,j}$ as a rational map $K \to C_{i,j}$.

For a given $n$R linkage $L$ with $n$ lines $h_1, h_2, \ldots, h_n$, we compute the *configuration set* by Algorithm 1. The main idea is following. We expand the left hand side of the closure equation (2.1). The coordinates 2,...,8 have to be zero, this gives 7 polynomial equations in $t_1, \ldots, t_6$. One of these equations is redundant, namely the 5th (the coefficient of $\epsilon$). The reason is that the left hand side is always in the Study quadric $S$, and if an element of the form $a + b\epsilon$ is in $S$ then it follows that $b = 0$. In order to exclude unwanted solutions, we add the inequality $(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1)(t_6^2 + 1) \neq 0$. In order to solve this system with the computer program Maple, we introduce an extra variable $u$ and add the equation $(t_1^2+1)(t_2^2+1)(t_3^2+1)(t_4^2+1)(t_5^2+1)(t_6^2+1)u - 1 = 0$, and compute a Gröbner basis that eliminates $u$ again.

**Algorithm 1** *configuration set*

**Input:** An $n$R linkage $L$ with one initial configuration, namely, $n$ lines $h_1$, $h_2$, ..., $h_n$.

**Output:** A list of equations in $t_1, t_2, \ldots, t_n$ such that its solution set equals the *configuration set* of $L$.

1: Write $F := (t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) \in \mathbb{DH}$.
2: Take the seven coefficients of polynomial $F$ except the first coefficient and add them to list $E$.
3: Supplement $E$ by one more polynomial, namely, $(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1)(t_6^2 + 1)u - 1$.
4: Compute the Gröbner basis $G$ of the elimination ideal of $E$ with respect to $u$.
5: **Return** $G$ –the elimination ideal.

One can find a Maple code for this computation in [1]. We add one example to support our algorithm. As we mainly focus on 6R linkages, we will only take care the case of when $n = 6$. It is worth mentioning that it also works for the cases of $4R$ and $5R$ linkages.

The solution set has some zero-dimensional components, which are not interesting, and a one-dimensional component:

**Example 2.1.**

$$h_1 = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} + \epsilon\mathbf{j}, \quad h_2 = k + \epsilon\mathbf{i}, \qquad h_3 = \frac{4}{5}\mathbf{j} + \frac{3}{5}\mathbf{k} + 2\epsilon\mathbf{j},$$
$$h_4 = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k} - \epsilon\mathbf{j}, \quad h_5 = -k + \epsilon\mathbf{i}, \quad h_6 = -\frac{4}{5}\mathbf{j} - \frac{3}{5}\mathbf{k} - 2\epsilon\mathbf{j}.$$

$$\begin{aligned} t_1 - t_4, \quad t_2 - t_5, \quad t_3 - t_6, \quad 15t_2t_6 + 16t_2 + 20t_4 - 9,\\ -375t_4t_6^2 + 1000t_4^2 - 400t_4t_6 + 375t_6^2 - 750t_4 + 820t_6 + 583,\\ 50t_2t_4 + 25t_4t_6 - 15t_2 - 25t_6 - 28. \end{aligned}$$

We call the case when we have lines ($h$s) as homogeneous case. One can ask the question what can we do if we do not have such input, namely, if we do not know the lines? The later case, when we do not have the lines, we call inhomogeneous case. One well-known technique for treating the inhomogeneous case is using Denavit-Hartenberg parameters [20]. For the detail of using this technique, we put it in the next chapter. Till end of this chapter, we only consider the homogeneous case.

## 2.2 Definition

The fundamentals of bond theory will be recalled from [31]. Let $n \geq 4$ be an integer. Let $L = (h_1, \ldots, h_n)$ be a closed $n$R linkage with mobility 1. We assume, for simplicity, that the configuration curve $K \subset (\mathbb{P}^1_{\mathbb{R}})^n$ (defined by equation (2.1)) has only one component of dimension 1 (see Remark 2.19 for a comment on the reducible case).

Let $K_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$ be the Zariski closure of $K$. We set

$$B := \{(t_1, \ldots, t_n) \in K_{\mathbb{C}} \mid (t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) = 0\}. \tag{2.2}$$

The set $B$ is a finite set of conjugate complex points on the configuration curve's Zariski closure. If $K$ is a nonsingular curve, then we define a bond as a point of $B$. If $K$ has singularities, then it is necessary to pass to the normalization $N(K)$ of $K$ as a complex algebraic curve, and a bond is then a point on $N(K)$ lying over $B$. Zero-dimensional components of $K$ never fulfill the equation above and so they have no effect on bonds.

Let $\beta$ be a bond lying over $(t_1, \ldots, t_n)$. By Theorem 2 in [31], there exist indices $i, j \in [n]$, $i < j$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$. If there are exactly two coordinates of $\beta$ with values $\pm$i, then we say that $\beta$ *connects* joints $i$ and $j$. In general, the situation, is more complicated. Let $\beta \in N(K)$ be a bond; we assume, for simplicity, that it lies over a point $(t_1, \ldots, t_n)$ such that no $t_i$ is the infinite point in $\mathbb{P}^1$. For $i, j \in \{1, \ldots, n\}$, we define

$$F_{i,j}(\beta) = (t_i(\beta) - h_i) \cdots (t_j(\beta) - h_j) \in \mathbb{DH}, \tag{2.3}$$

The distinction between $F_{i,j}$ and $f_{i,j}$ is necessary because $F_{i,j}$ may vanish at the bonds, and then it does not give a well-defined pose in $\mathbb{P}^7$. We define $v_\tau(i,j)$ as the vanishing order of $F_{i,j}$ at $\tau$. We define the connection number

$$k_\beta(i,j) := v_\beta(i,j) + v_\beta(i+1, j-1) - v_\beta(i, j-1) - v_\beta(i+1, j). \tag{2.4}$$

We visualize bonds and their connection numbers by *bond diagrams.* We start with the link diagram, where vertices correspond to links and edges correspond to joints. Then we draw $k_\beta(i,j)$ connecting lines between the edges $h_i$ and $h_j$ for each set $\{\beta, \overline{\beta}\}$ of conjugate complex bonds. Since we cannot exclude that $k_\beta(i,j) < 0$, we visualize negative connection numbers by drawing the appropriate number of dashed connecting lines (because the dash resembles a “minus” sign). No linkage in this dissertation has a negative connection number. Actually, the authors do not know if closed 6R linkages may or may not have bonds with negative connection numbers.

For a given $n$R linkage $L$ with $n$ lines $h_1, h_2, \ldots, h_n$, we compute the bond connections (bond diagram) for the linkage $L$ by Algorithm 2. The main idea is as follow: We calculate the connection numbers for all bonds together instead of calculating one by one. This is based on the formula

$$k_B(i,j) := v_B(i,j) + v_B(i+1, j-1) - v_B(i, j-1) - v_B(i+1, j),$$

where we can get $v_B(i,j)$ by counting the solutions of an intersection of $K$ and $F_{i,j} = 0$. This counting includes the multiplicity. In order to catch the case $t_i = \infty$, one might need to do linear transformations of $t_i$ in $\mathbb{P}^1$ such that this counting is proper.

One can find a Maple code for this computation in [1]. We treat the same example (Example 2.1) as in the computation of configuration set. The bond connection for this Bricard line symmetric 6R linkage is

$$D := [[1,4,2],[2,5,2],[3,6,2]],$$

where the three integers $[a, b, c]$ means that there are $c$ bonds which connect $a$ and $b$ (e.g. Figure 2.1).

## 2.3 Main Properties

We start to introduce the main properties from [27, 31].

**Algorithm 2** `BondConnectionsI`

**Input:** $L := [h_1, h_2, \ldots, h_n]$, an $n$R linkage with one configuration at $h_1, h_2, \ldots, h_n$.
**Output:** A list $D = [D_1, D_2, \ldots, D_m]$, where $D_s = [p, q, r]$ for $s = 1, \ldots, m$, $p = 1, 2, \ldots, n$, $|q - p| = 2, \ldots, n-2$, $r$ is a nonzero integer.

Use Algorithm 1 to get a list of polynomials $G$ which give the configuration set.
Set $P := \{1, 2, \ldots, n-2\}$.
**repeat**
  Take $p \in P$ and set $P \leftarrow P - \{p\}$, $Q := \{p+2, \ldots, n\}$.
  **repeat**
    Take $q \in Q$ and set $Q \leftarrow Q - \{q\}$.
    Compute the vanishing orders of
    $v_1 := v_B(p, q)$, $v_2 := v_B(p+1, q-1)$,
    $v_3 := v_B(p, q-1)$, $v_4 := v_B(p+1, q)$.
    Set $r := \frac{v_1+v_2-v_3-v_4}{2}$.
    If $r \neq 0$, then $D \leftarrow D + [p, q, r]$.
  **until** $Q = NULL$.
**until** $P = NULL$.
**Return** $D = [D_1, D_2, \ldots, D_m]$ for some integer $m$.

**Theorem 2.2.** *The algebraic degree of the coupler curve $C_{i,j}$ can be read off from the bond diagram as follows: Cut the bond diagram at the vertices $o_i$ and $o_j$ to obtain two chains with endpoints $o_i$ and $o_j$; the algebraic degree of $C_{i,j}$ is the sum of all connections that are drawn between these two components (dashed connections counted negatively).*

*Proof.* This is a consequence of Theorem 5 in [31]. Note that here we give a different definition of connection numbers, but Lemma 2 in [31] shows that the definitions are equivalent.

The basic idea of the proof is that the algebraic degree of $C_{i,j}$ is $\frac{1}{2}$ times the number of points $\tau$ in the configuration curve such that $\mathrm{N}(f_{i,j}(\tau)) = 0$. All these points are bonds, and a closer investigation leads to the statement above. $\square$

**Example 2.3.** *We illustrate the procedure for computing the degrees in Figure 2.2. In order to determine the algebraic degree of the coupler curve $C_{4,6}$, we cut the bond diagram along the line through $o_4$ and $o_6$ and count the connections between the two chain graphs. There are precisely two of them. Thus, the algebraic degree $d(4,6)$ of $C_{4,6}$ is two.*

For a sequence $h_i, h_{i+1}, \ldots, h_j$ of consecutive joints, we define the *coupling space* $L_{i,i+1,\ldots,j}$ as the linear subspace of $\mathbb{R}^8$ generated by all products $h_{k_1} \cdots h_{k_s}$, where

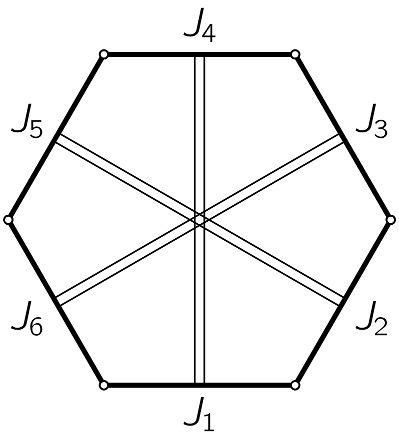



Figure 2.1: Bond diagrams for the Bricard line symmetric linkage.

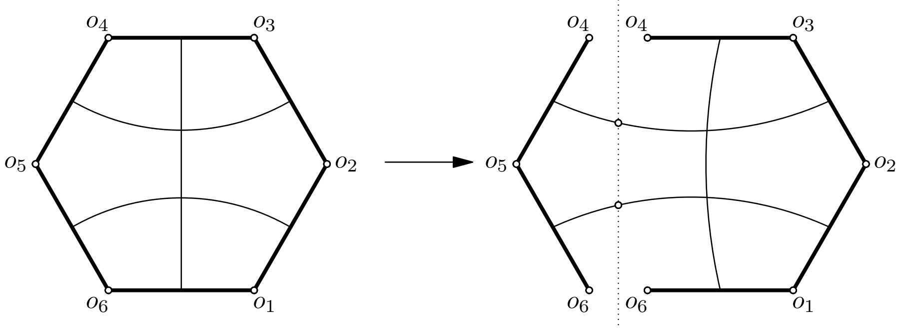



Figure 2.2: Computing the degree of coupler curves by counting connections in the bond-diagram. There are two connections between the two chains, hence the algebraic degree of the coupler curve $C_{4,6}$ is two.

$s \geq 0$ and $k_1, \ldots, k_s$ are integers such that $i \leq k_1 < \cdots < k_s \leq j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product corresponding to $s = 0$ is included, its value is 1. The *coupling dimension* $l_{i,i+1,\ldots,j}$ is the dimension of $L_{i,i+1,\ldots,j}$ and the *coupling variety* $X_{i,i+1,\ldots,j} \subset \mathbb{P}^7$ is the set of all products $(t_i - h_i) \cdots (t_j - h_j)$ with $t_k \in \mathbb{P}^1$ for $k = i, \ldots, j$ or, more precisely, the set of all equivalence classes of these products in the projective space.

The coupling variety is a subset of the projectivization of the coupling space. The relation between the coupler curve and the coupling variety is described by the "coupler equality" $C_{i,j} = X_{i+1,\ldots,j} \cap X_{i,\ldots,-n+j+1}$.

The relation between bonds and coupling dimensions is described in the following

**Theorem 2.4.** *All coupling dimensions* $l_{1,\ldots,i}$ *with* $1 \leq i \leq n$ *are even. We have* $l_{1,2} = 4$ *and* $k_\beta(1,2) = 0$ *for every bond* $\beta$*. If* $k_\beta(1,3) \neq 0$ *for some* $\beta$*, then* $l_{1,2,3} \leq 6$*. If* $l_{1,2,3} = 4$*, then the lines* $h_1, h_2, h_3$ *are parallel or have a common point.*

*Proof.* This is part of Theorem 1, Theorem 3, and Corollary 3 in [31]. The first statement is a consequence of the fact that the coupling spaces can be given the structure of a complex vector space, because they are closed under multiplication by $h_1$ from the left. □

We will also use a more precise description of the coupling varieties in each of the three possible cases, which is interesting in itself.

**Theorem 2.5.** *If $l_{1,2,3} = 4$, then $X_{1,2,3}$ is a linear projective 3-space, and its parametrization by $t_1, t_2, t_3$ is a 2:1 map branched along two quadrics in this 3-space.*

*If $l_{1,2,3} = 6$, then $X_{1,2,3}$ is a complete intersection of two quadrics in a 5-space and its parametrization by $t_1, t_2, t_3$ is birational.*

*If $l_{1,2,3} = 8$, then $X_{1,2,3}$ is a Segre embedding of $(\mathbb{P}^1)^3$ in $\mathbb{P}^7$, and its parametrization by $t_1, t_2, t_3$ is an isomorphism.*

*Proof.* The first statement is well-known in kinematics. For non-parallel axes it is, for example, implicit in the exposition of [58, Section 5]. Branching occurs for co-planar joint axes. There are two components of the branching surface, and each of the component is the image of a subset of $(\mathbb{P}^1)^3$ in which $t_2$ is constant.

If $l_{1,2,3} = 6$, then $(\mathrm{i} - h_1)(s_2 - h_2)(\pm\mathrm{i} - h_3) = 0$ for some $s_2 \in \mathbb{P}^1_{\mathbb{C}}$, by the proof of Theorem 1 in [31]; we may assume that the third factor is $(+\mathrm{i} - h_3)$. Clearly there is also a complex conjugate relation $(-\mathrm{i} - h_1)(\bar{s_2} - h_2)(-\mathrm{i} - h_3) = 0$. The parametrization $p : (\mathbb{P}^1)^3 \to X_{1,2,3}$ has two base points $(\mathrm{i}, s_2, \mathrm{i})$ and $(-\mathrm{i}, \overline{s_2}, -\mathrm{i})$. We distinguish two cases.

If $s_2 \neq \overline{s_2}$, then we apply projective transformations moving the base points to $(0, 0, 0)$ and $(\infty, \infty, \infty)$. The transformed parametrization is

$$(\mathbb{P}^1)^3 \to \mathbb{P}^5, (y_1, y_2, y_3) \mapsto (x_0{:}x_1{:}x_2{:}x_3{:}x_4{:}x_5) = (y_1{:}y_2{:}y_3{:}y_1y_2{:}y_1y_3{:}y_2y_3),$$

which is birational to the quartic three-fold defined by $x_0x_5 = x_1x_4 = x_2x_3$.

If $s_2 = \overline{s_2}$, then we apply projective transformations moving the base points to $(0, \infty, 0)$ and $(\infty, \infty, \infty)$. The transformed parametrization is

$$(\mathbb{P}^1)^3 \to \mathbb{P}^5, (y_1, y_2, y_3) \mapsto (x_0{:}x_1{:}x_2{:}x_3{:}x_4{:}x_5) = (1{:}y_1{:}y_3{:}y_1y_2{:}y_1y_3{:}y_2y_3),$$

which is birational to the quartic three-fold defined by $x_0x_4 = x_1x_2$, $x_1x_5 = x_2x_4$.

If $l_{1,2,3} = 8$, then the eight products generating $L_{1,2,3}$ are linearly independent, and it follows that the parametrization is the Segre embedding in the projective coordinate system induced by this basis. □

All bonds connecting $h_1$ and $h_4$ satisfy $t_1, t_4 \in \{+\mathrm{i}, -\mathrm{i}\}$. We will prove a lemma that is useful to give an upper bound for the number of bonds in some situations; before that, we need an algebraic lemma.

**Lemma 2.6.** *Let $h_1, h_2 \in \mathbb{DH}$ be dual quaternions representing lines (i.e. $h_1^2 = h_2^2 = -1$). Let $\mathbb{D}_\mathbb{C} := \mathbb{D} \otimes_\mathbb{R} \mathbb{C}$ and $\mathbb{DH}_\mathbb{C} := \mathbb{DH} \otimes_\mathbb{R} \mathbb{C}$ be the extensions of the dual numbers/quaternions to $\mathbb{C}$.*

*(a) The left annihilator of $(\mathrm{i} - h_1)$ is equal to the left ideal $\mathbb{DH}_\mathbb{C}(\mathrm{i} + h_1)$.*

*(b) The intersection of this left ideal and the right ideal $(\mathrm{i} - h_2)\mathbb{DH}_\mathbb{C}$ is a free $\mathbb{D}_\mathbb{C}$-module of rank 1.*

*(c) The set of all complex dual quaternions $x$ such that $(\mathrm{i} - h_2)x(\mathrm{i} - h_1) = 0$ is a free $\mathbb{D}_\mathbb{C}$-module of rank 3.*

*Proof.* For $h_1 = h_2 = \mathbf{i}$, the proofs for all three statements are straightforward.

The group of unit dual quaternions acts transitively on lines by conjugation, so there exist invertible $g_1, g_2 \in \mathbb{DH}$ such that $h_1 = g_1 \mathbf{i} g_1^{-1}$ and $h_2 = g_2 \mathbf{i} g_2^{-1}$. Then

$$\{q \mid q(\mathrm{i} - h_1) = 0\} = \{q \mid qg_1(\mathrm{i} - \mathbf{i})g_1^{-1} = 0\} = \{q \mid qg_1(\mathrm{i} - \mathbf{i}) = 0\} =$$

$$\mathbb{DH}_\mathbb{C}(\mathrm{i} + \mathbf{i})g_1^{-1} = \mathbb{DH}_\mathbb{C} g_1^{-1}(\mathrm{i} + h_1) = \mathbb{DH}_\mathbb{C}(\mathrm{i} + h_1),$$

which shows (a). The $\mathbb{D}_\mathbb{C}$-linear bijective map $\mathbb{DH}_\mathbb{C} \to \mathbb{DH}_\mathbb{C}$, $q \mapsto g_2^{-1} q g_1$ maps the left ideal $\mathbb{DH}_\mathbb{C}(\mathrm{i} + h_1)$ to the left ideal $\mathbb{DH}_\mathbb{C}(\mathrm{i} + \mathbf{i})$ and the right ideal $(\mathrm{i} - h_2)\mathbb{DH}_\mathbb{C}$ to the right ideal $(\mathrm{i} - \mathbf{i})\mathbb{DH}_\mathbb{C}$, which shows (b). The same map also maps the set $\{x \mid (\mathrm{i} - h_2)x(\mathrm{i} - h_1) = 0\}$ to the set $\{x \mid (\mathrm{i} - \mathbf{i})x(\mathrm{i} - \mathbf{i}) = 0\}$, which shows (c). □

**Lemma 2.7.** *Assume that $l_{1,2,3} = l_{4,5,6} = 8$. Then there are at most 2 bonds $\beta := (t_1, \ldots, t_6)$ connecting $h_1, h_4$ for fixed values of $t_1$ and $t_4$ in $\{+\mathrm{i}, -\mathrm{i}\}$ (counted with multiplicity).*

*Proof.* Without loss of generality, we may assume $t_1 = t_4 = +\mathrm{i}$; the other situations can be reduced to this case by replacing $h_1$ or $h_4$ or both by its negative.

By the algebraic lemma above, the intersection of the left annihilator of $(t_4 - h_4)$ and the right ideal $(t_1 - h_1)\mathbb{DH}_C$ is a 2-dimensional $\mathbb{C}$-linear subspace. Let $G \subset \mathbb{P}^7$ be its projectivization. Let $q := f_{3,6}(\beta)$ be image of a bond $\beta = (t_1, \ldots, t_6)$ connecting $h_1$ and $h_4$ with $t_1 = t_4 = +\mathrm{i}$. Then we have $q = (t_1 - h_1)(t_2 - h_2)(t_3 - h_3)$, hence $q$ is in the right ideal $(t_1 - h_1)\mathbb{DH}_\mathbb{C}$. Since $\beta$ connects $h_1$ and $h_4$, we have $q(t_4 - h_4) = F_{3,6}(\beta) = 0$, $q$ is in the left annihilator of $(t_4 - h_4)$, and therefore $q \in G$.

There exist no two bonds $\beta_1, \beta_2$ with the same bond image $q$, because the parameterization of $X_{1,2,3}$ by $t_1, t_2, t_3$ is an isomorphism, hence $q$ determines the first three coordinates, and the parametrization of $X_{6,5,4}$ by $t_4, t_5, t_6$ is also an isomorphism, hence $q$ determines the second three coordinates. This shows that the number of bonds connecting $h_1$ and $h_4$ with $t_1 = t_4 = +\mathrm{i}$ is equal to the number of intersections of $G$ and $C_{3,6}$; tangential intersections give rise to higher connection numbers.

On the other hand, $C_{3,6}$ is generated by quadrics, so it does not have any tritangents, so the number of such bonds is at most 2. □

## 2.4 Bounding the Genus

In this section, we prove that the genus of the configuration curve of a closed 6R linkage is at most 5.

Let $L = (h_1, \ldots, h_6)$ be a closed 6R linkage with mobility 1. We use the notation of the previous section. As before, we assume that the configuration curve $K$ has only one irreducible one-dimensional component. We write $g(K)$ for the genus of this component.

Here is an auxiliary Lemma.

**Lemma 2.8.** *Let $C_1, C_2$ be two curves of genus at most 1. Let $C \subset C_1 \times C_2$ be an irreducible curve such that the two projections restricted to $C$ are either birational or 2:1 maps to $C_1$ resp. $C_2$. Then $g(C) \leq 5$, with equality only if $g(C_1) = g(C_2) = 1$ and both projections being 2:1.*

*Proof.* If one of the two projections is birational, say the projection to $C_1$, then $g(C) = g(C_1) \leq 1$. So we may assume both projections are 2:1 maps.

If $C_1$ and $C_2$ are isomorphic to $\mathbb{P}^1$, then $C$ is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree 2, which has arithmetic genus 1. The geometric genus is 1 in the nonsingular case and 0 if $C$ has a double point.

If $C_1 = \mathbb{P}^1$ and $C_2$ is elliptic, then the numerical class group is generated by the two fibers $F_1 \cong C_2$ and $F_2 \cong C_1$ of the two projections. The class of $C$ is $2F_1 + 2F_2$, and the canonical class is $-2F_2$. Hence the arithmetic genus of $C$ is $\frac{C(C+K)}{2} + 1 = 2(F_1 + F_2)F_1 + 1 = 3$.

If $C_1$ and $C_2$ are elliptic, then the canonical class of $C_1 \times C_2$ is zero. If $F_1, F_2$ are fibers of the projections, then $F_1C = F_2C = 2$ and $(F_1 + F_2)^2 = 2$. By the Hodge index theorem, $(C - 2F_1 - 2F_2)^2 \leq 0$, which is equivalent to $C^2 \leq 8$.

Hence the arithmetic genus of $C$ is at most $\frac{C^2}{2} + 1 = 5$. □

**Lemma 2.9.** *If* $l_{1,2,3} = 4$*, then* $g(K) \leq 5$*.*

*Proof.* Let $C_1, C_2 \subset (\mathbb{P}^1)^3$ be the projections of $K$ to $(t_1, t_2, t_3)$ and $(t_4, t_5, t_6)$, respectively. Let $p_1 : K \to C_1$ and $p_2 : K \to C_2$ be the projection maps. The coupler curve $C_{3,6}$ is a common image of $C_1$ and $C_2$, by the two sides of the closure equation

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3) \equiv (t_6 - h_6)(t_5 - h_5)(t_4 - h_4),$$

where we write $\equiv$ for equality in the projective sense, modulo scalar multiplication. Let $f_1 : C_1 \to C_{3,6}$ and $f_2 : C_2 \to C_{3,6}$ be these two maps. Then $K$ is a component of the pullback of $f_1, f_2$. We distinguish several cases.

Case 1: $l_{6,5,4} = 4$. Then $C_{3,6}$ is the intersection of two linear subspaces, hence a line by the mobility 1 assumption. One can introduce an additional joint, rotational or translational, between links $o_3$ and $o_6$, and the linkage decomposes into two 4-bar linkages which are planar or spherical. The configuration curves of these two linkages are isomorphic to $C_1$ and $C_2$. The maps $f_1, f_2$ are restrictions of the 2:1 parametrizations of $X_{1,2,3}$ and $X_{6,5,4}$, hence they are either 2:1 or birational to the line $C_{3,6}$. Therefore $p_1$ and $p_2$ are also either 2:1 or birational. The configuration curve of a planar or spherical 4-bar linkage is the intersection curve of two quadrics (see [15, Chapter 11, § 8] for the planar and [53, § 21] for the spherical case). Hence its genus is at most 1. By Lemma 2.8, $g(K) \leq 5$.

Case 2: $l_{6,5,4} = 6$. Then $X_{6,5,4}$ is an intersection of quadrics in a 5-space and $X_{1,2,3}$ is a linear 3-space contained in the Study quadric. The intersection of both linear spaces is either a line or a plane, because the vector space $L_{6,5,4}$ does not contain any 4-dimensional subalgebras. Hence the intersection $C_{3,6}$ is either a line or a plane conic. If $C_{3,6}$ is a line, then we have a similar situation as before: the linkage decomposes into two 4-bar linkages, one planar or spherical and the second being a Bennett linkage. In any Bennett linkage, the maps from $K$ to $\mathbb{P}^1$ parametrizing the 4 rotations are isomorphisms. Therefore $K$ is isomorphic to the configuration curve of the planar or spherical component, hence $g(K) \leq 1$. If $C_{3,6}$ is a plane conic, then we decompose it into two rotational linear motions with coplanar axes. These two axes form together with $h_4, h_5, h_6$ a closed 5R linkage, which is known as the Goldberg 5R linkage (see [24]). Its configuration curve is rational, more precisely the coupling map to the plane conic is an isomorphism (see [31]). Hence $K$ is isomorphic to $C_1$. Now $f_1 : C_1 \to C_{3,6}$ has 8 branching points (counted with multiplicity), namely the intersections of $C_{3,6}$ with the branching surface. By the Hurwitz genus formula, it follows that $g(K) = 3$; the genus may drop in case of singularities.

Case 3: $l_{6,5,4} = 8$. Then $C_{3,6}$ is a curve in a Segre embedding of $(\mathbb{P}^1)^3$ in $\mathbb{P}^7$ cut out by four hyperplane sections. This is only possible if $C_{3,6}$ is a twisted cubic. Then the lines $h_4$, $h_5$, $h_6$ could be re-covered from $C_{3,6}$ by factoring the cubic motion parametrized by $C_{3,6}$ described in [30]. On the other hand, $C_{3,6}$ is either a planar or spherical motion, hence the whole linkage is either planar or spherical, and this contradicts the mobility 1 assumption, as planar and spherical 6R linkages have mobility 3. So this case is impossible. □

**Remark 2.10.** *If $g(K) \geq 4$, then we are in Case 1, and the linkage is a composite of two planar or spherical 4-bar linkages with one common joint, which is removed from the 6-loop. The most general linkage of this type is Hooke's linkage [7], using two spherical linkages. The genus of its configuration curve is generically 5, but it may drop in the presence of singularities. If we take two planar RRRP linkages and remove the common translational joint, then we obtain the Sarrus linkage [56] with two triples of parallel consecutive axes. The bond diagrams of both linkages can be seen in Figure 2.3(a).*

**Lemma 2.11.** *If $l_{1,2,3} = l_{6,5,4} = 6$, then $g(K) \leq 5$.*

*Proof.* Let $V := L_{1,2,3} \cap L_{6,5,4}$. Then $4 \leq \dim(V) \leq 5$. The $\dim(V) = 6$ case is not possible by Lemma 6 in [31]. If $\dim(V) = 4$, then $C_{6,3}$ is embedded into a three dimensional projective space $\mathbb{P}^3$. The coupler varieties are defined by quadrics in $\mathbb{P}^5$, therefore the ideal of $C_{6,3}$ is generated by linear forms and quadrics, and so its genus is at most 1. The coupler map $f_{6,3}$ is birational, therefore $g(K) \leq 1$. So we may assume $\dim(V) = 5$.

By Theorem 2.5, the varieties $X_{1,2,3}$ and $X_{6,5,4}$ are complete intersections of two quadrics. We may assume in each case that one of the defining equations is the equation of the Study quadric. Then the coupler curve $C_{6,3} = X_{1,2,3} \cap X_{6,5,4}$ is defined by three quadratic equations and the linear forms defining $V$. It follows that $C_{6,3}$ is a complete intersection of three quadrics in $\mathbb{P}^4$, which implies $g(K) \leq 5$, with equality in the case that there are no singularities. □

**Remark 2.12.** *In [21], Dietmaier found a new linkage by a computer-supported numerical search. It turns out, by comparing the geometric parameters, that his family is exactly the family of linkages with $l_{1,2,3} = l_{4,5,6} = 6$ and $\dim(L_{1,2,3} \cap L_{6,5,4}) = 5$. See Figure 2.3(b) for the bond diagram of the Dietmaier linkage.*

**Lemma 2.13.** *If $l_{1,2,3} = 6$ and $l_{6,5,4} = 8$, then $g(K) \leq 3$.*

*Proof.* If $Y := X_{6,5,4} \cap L_{1,2,3}$ has dimension 1, then its Betti table coincides with the Betti table of $X_{6,5,4}$ and it follows that $Y$ is a union of curves with genus at most 1 (the genus 1 case occurs only if $Y$ is irreducible). Since $C_{6,3} \subseteq Y$, it follows that $g(C_{6,3}) \leq 1$, and by birationality of $f_{6,3}$ we get $g(K) \leq 1$.

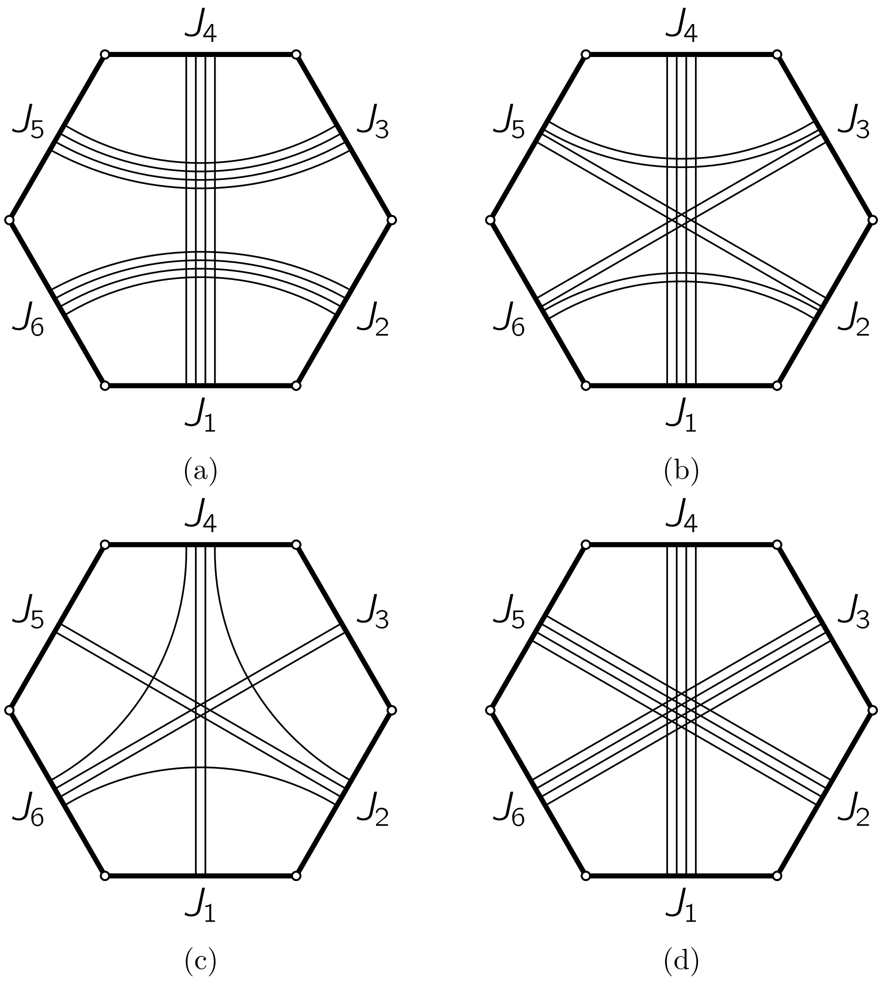



Figure 2.3: Bond diagrams for Hooke's double spherical linkage (a), Dietmaier's linkage (b), Wohlhart's partially symmetric linkage (c), and Bricard's orthogonal linkage (d).

Assume $Y$ is a surface. The preimage $Z$ of $Y$ under the parametrization $p : (\mathbb{P}^1)^3 \to X_{6,5,4}$ is defined by two equations of tri-degree $(1,1,1)$, and because $Y$ is a surface, the two equations must have a common divisor $F$ which defines $Z$. Up to permutation of coordinates, the tri-degree of $F$ is either $(1,0,0)$ or $(1,1,0)$. In the first case, one of the angles would be constant throughout the motion. Hence the 6R linkage is actually a 5R linkage with an extra immobile axis somewhere; then $g(K) = 0$ by the classification of 5R linkages (see [31] ) (if one does not want to exclude this degenerate case). In the second case, we consider the preimage $C'$ of $C_{6,3}$ under $p$. It is defined by $F$ and the pullback of the quadric equations which defines $X_{1,2,3}$. Hence $C'$ is a component of the complete intersection of two equations, with tri-degree $(1,1,0)$ and $(2,2,2)$. Using the first equation, we can express the first variable by the second, and so we get an isomorphic image of $C'$ in $(\mathbb{P}^1)^2$ of bi-degree $(4,2)$, which has arithmetic genus 3. But $p$ is is an isomorphism by Theorem 2.5, hence

$g(C_{6,3}) \leq 3$ and $g(K) \leq 3$. □

**Remark 2.14.** *An example of a linkage where $Y$ is a surface is Wohlhart's partially symmetric linkage [66] (see Figure 2.3(c) for the bond diagram). We do not know if there exist also other linkages with $l_{1,2,3} = 6$ and $l_{6,5,4} = 8$ and $g(K) = 3$ in the literature.*

**Lemma 2.15.** *If $l_{i,i+1,i+2} = 8$ for $i = 1, \ldots, 6$, then $g(K) \leq 5$.*

*Proof.* By Theorem 2.4, all bonds connect opposite joints: the bond diagram consists of $b_1$ connections between $h_1$ and $h_4$, $b_2$ connections between $h_2$ and $h_5$, and $b_3$ connections between $h_3$ and $h_6$. By Theorem 2.2, the degree of $f_{6,1}$ and the degree of $f_{3,4}$ are both equal to $b_1$. Note that $f_{6,1}$ and $f_{3,4}$ are the projections from $K$ to the first and to the fourth coordinate, respectively, up to isomorphic parameterization of the line describing rotations around $h_1$ and $h_4$, respectively. Similarly, the projections to $t_2, t_5, t_3, t_6$ have (respective) degrees $b_2, b_2, b_3, b_3$.

Let $b_1^+$ be the number of pairs of complex conjugate bonds connecting $h_1$ and $h_4$ such that $t_1 = t_4$ and $b_1^-$ be the number of pairs such that $t_1 = -t_4$ (recall that $t_1^2 = t_4^2 = -1$). The numbers $b_2^+, b_2^-, b_3^+, b_3^-$ are defined analogously. By Lemma 2.7, we have $b_1^+, \ldots, b_3^- \leq 2$.

We consider the projection $q_{1,4} : K \to (\mathbb{P}^1)^2, (t_1, \ldots, t_6) \mapsto (t_1, t_4)$. The image of this curve has bi-degree $(r_1, r_1)$, with $r_1 \deg(q_{1,4}) = b_1$. The preimage of $(\pm\mathrm{i}, \pm\mathrm{i})$ consists entirely of bonds; moreover, if one of the coordinates of a point on $C_{1,4}$ is equal to $\pm\mathrm{i}$, then it must already be a bond. If, say, $b_1^+ = b_2^+ = 2$, and $q_{1,4}$ is birational, then $+\mathrm{i}$ is a branching point for both projections, hence it must be a double point. If $q_{1,4}$ is not birational, then it is a 2:1 map, because the preimage of any of the 4 points $(\pm\mathrm{i}, \pm\mathrm{i})$ is at most 2. In this case, the numbers $b_1^+$ and $b_1^-$ are either 0 or 2, and the bi-degree of $C_{1,4}$ is $(1, 1)$ or $(2, 2)$. It follows that, in the 2:1 case, the curve $C_{1,4}$ has genus 0 or 1. We now have to sort out several cases.

Case 1: the three maps $q_{1,4}$, $q_{2,5}$ and $q_{3,6}$ are 2:1 maps. It is not possible that all three maps factor through the same 2:1 quotient, because $K$ is contained in the product $C_{1,4} \times C_{2,5} \times C_{3,6}$. Assume, without loss of generality, that $q_{1,4}$ and $q_{2,5}$ do not factor by the same 2:1 quotient. Then $(q_{1,4}, q_{2,5}) : K \to C_{1,4} \times C_{2,5}$ is birational. By Lemma 2.8, the image has genus at most five, and therefore $g(K) \leq 5$.

For the remaining cases, we may assume that $q_{1,4}$ is birational.

Case 2: $b_1 = 3$. Then the arithmetic genus of $C_{1,4}$ is $(b_1 - 1)^2 = 4$. Since $b_1^+ + b_1^- = 3$, at least one of the two numbers is equal to two; assume, without loss of generality, that $b_1^+ = 2$ and $b_1^- = 1$. Then $(+\mathrm{i}, +\mathrm{i})$ and $(-\mathrm{i}, -\mathrm{i})$ are double points of $C_{1,4}$, and therefore $g(K) \leq 4 - 2 = 2$.

Case 3: $b_1 \leq 2$. Then the arithmetic genus of $C_{1,4}$ is $(b_1 - 1)^2 \leq 1$.

Case 4: $b_1 = 4$, hence $b_1^+ = b_1^- = 2$. Then the arithmetic genus of $C_{1,4}$ is $(b_1-1)^2 = 9$, and all four points $(\pm\mathrm{i}, \pm\mathrm{i})$ are double points. Then $g(C_{1,4}) \leq 9 - 4 = 5$, and therefore $g(K) \leq 5$. □

**Corollary 2.16.** *The maximal genus 5 is reached in Case 1 when all $C_{1,4}$, $C_{2,5}$, and $C_{3,6}$ are elliptic and have bi-degree $(2, 2)$, and in Case 4; in both cases, we have $b_1 = b_2 = b_3 = 4$.*

As a consequence of Lemma 2.9, Lemma 2.11, Lemma 2.13, and Lemma 2.15 above, we finally obtain our bound for the genus.

**Theorem 2.17.** *The genus of the configuration curve of a closed 6R linkage is at most 5.*

By re-examining the proof of Lemma 2.15 more closely, we can prove the following theorem which will be useful later for classifying linkages with a genus 5 configuration curve.

**Theorem 2.18.** *If the bond diagram is different from the diagrams Figure 2.3(a), (b), and (d), then $g(K) \leq 3$.*

*Proof.* In view of Remark 2.10, Lemma 2.13, and the proofs of Lemmas 2.15 and 2.11, we just need to consider the case where $l_{i,i+1,i+2} = 8$ for $i = 1, \ldots, 6$. Assume indirectly that $b_1 < 4$ (using the notation as in the proof of Lemma 2.15). If $q_{1,4}$ is birational, then it follows $g(K) \leq 3$, hence we may assume that $q_{1,4}$ is a 2:1 map. Hence $b_1 = 2$ and $C_{1,4}$ is a curve of bi-degree $(1, 1)$, which is rational. Consequently $K$ is hyperelliptic (or $g(K) \leq 1$ and the proof is finished).

If the other two maps $q_{2,5}$, $q_{3,6}$ are also 2:1 maps, then we have a 2:1 map from $K$ to a rational curve and another 2:1 map to a curve of genus at most 1; by Lemma 2.8, we obtain $g(K) \leq 3$.

So we may assume there is another map, say $q_{2,5} : K \to C_{2,5} \subset (\mathbb{P}^1)^2$, which is birational. Its image has then bi-degree $(b_2, b_2)$, and if $b_2 \leq 3$ then we again get $g(K) \leq 3$. So we assume $b_2 = 4$. Then $C_{2,5}$ has bi-degree $(4, 4)$ and 4 double points $(\pm\mathrm{i}, \pm\mathrm{i})$. The canonical map of $C_{1,4}$ is defined by the polynomials of bi-degree $(2, 2)$ passing to all $m$-fold singular points with order $m - 1$. If there is at most one double point, then it would just pass to the 4 double points $(\pm\mathrm{i}, \pm\mathrm{i})$ and maybe one additional double point, but this map maps $(\mathbb{P}^1)^2$ birational to a rational surface, and this contradicts to the fact that $C_{1,4}$ is hyperelliptic, because the canonical map of a hyperelliptic curve is 2:1. Hence there must be at least two more double points or a triple point on $C_{2,5}$, and so $g(K) \leq 3$. □

**Remark 2.19.** *If the configuration curve has more than one one-dimensional component, then one can define bonds for the individual components. These bonds add up to a diagram which satisfies the same conditions we just proved for bond diagrams of irreducible configuration curves. We conclude that the genus of any component is at most 3 in a linkage with more than one component.*

# Chapter 3

# Quad Polynomials

In this chapter, we will introduce a powerful technique, namely, quad polynomial, which is quite related to the bond theory. It is used to derive necessary equation condition for mobile 6R linkages. These equations are defined over the Denavit-Hartenberg parameters [20]. A closed 6R (six revolute joints) linkage is uniquely determined by its set of Denavit-Hartenberg parameters, which contains 18 real geometric invariants: the twist angles, the normal distances and the offsets. A generic choice of these parameters leads to a rigid 6R linkage. In the literature, there are many families of special choices of parameters such that the linkage is mobile, or in other words, numerous sufficient conditions for mobility are known [5, 21]. An overview on known families of 6R linkages can be found in [17, 21]. However, we are still far away from a complete classification of all mobile 6R linkages.

The aim of this technique is to make a step towards such a complete classification, by deriving necessary conditions for mobility; up to our knowledge, not a single necessary equational condition has been known up to now. These results presented below evolved from a collaboration with Hamid Ahmadinezhad, Josef Schicho and have recently been published in [2, 44].

**Structure of the chapter** The remaining part of the chapter is set up as follows. Section 3.1 introduces the overview relation between the bond theory and the quad polynomial. In section 3.2, we introduce all preliminary definitions for a new definition of bond. In section 3.3, we give the definition of the *quad polynomial* and its main property. Section 3.4 contains some known examples. Section 3.5 focuses on new examples which are derived by the quad polynomial. In the section 3.6, we introduce one technique for constructing 6-bar linkages with helical (screw) joints.

## 3.1 Relationship with Bond Theory

The bond theory already provides a *classification scheme* for 6R linkages: for any mobile 6R linkage, one can calculate a certain combinatorial structure describing algebraic relations between the joints, namely, the bond diagram. This diagram consists of *bonds*, which are connections between two joints of the linkage. Any joint is connected to at least one other joint, and adjacent joints are never connected. If we number the joints cyclically, then a bond connects either joints $i$ and $i+2$ – then we speak of a *near connection*, – or it connects joints $i$ and $i+3$, and then we speak of a *far connection*.

It was shown that the existence of a near connection implies the validity of a well-known condition which also arises frequently in many families, namely Bennett's condition: $s_i = 0$, and $\frac{d_i}{\sin(\phi_i)} = \frac{d_{i+1}}{\sin(\phi_{i+1})}$, where the $s_i, d_i, \phi_i$ are Denavit-Hartenberg parameters (see section 3.2 for the precise definitions). Bennett's condition is equivalent to a kinematic condition on three consecutive rotation axes, not all three parallel or intersecting, namely the existence of a fourth axis such that the closed 4R linkage with these four axes is movable (see [12]). However, there are many mobile 6R linkages without near connections, for instance Bricard's orthogonal linkage or Bricard's line symmetric linkage. So, the Bennett conditions are not necessary for mobility.

Chapter 2 contains no equational condition implied by the existence for a far connection. The main aim of this chapter is to fill this gap by introducing the *quad polynomials*: these are univariate polynomials of degree 2 with coefficients depending on the Denavit-Hartenberg parameters by an explicit formula. The existence of a far connection implies a common root of two such quad polynomials, and this gives rise to necessary equational conditions.

Because every mobile linkage has either near or far connections (or both), it is then possible to write down equational conditions for movability (see Remark 3.6). However, the full system of equations is too big and complicated, and therefore it is better to follow the classification scheme suggested by bond theory and distinguish cases according to the bond diagram. For any bond diagram, we will derive a non-trivial system of algebraic equations, consisting of Bennett conditions for the near connections and quad polynomial conditions for the far connections. In some cases, the equations are even sufficient for movability, hence the equations characterize all linkages with this particular bond diagram.

In section 3.4, we illustrate the method by deriving the equational conditions for various well-known linkages. The bond diagram studied in section 3.5 leads to a new movable 6R linkage $L$: we show that for every known family, there is an algebraic condition which is not satisfied by the set of Denavit-Hartenberg parameters of $L$.

In order to show that $L$ is indeed movable, we calculate the configuration set by solving the corresponding algebraic system of equations and observe that it is one-dimensional (there is no geometric proof for mobility for this example). We find this new family of linkages especially remarkable because two of its R-joints can be replaced by H-joints (helical joints), and the linkage remains movable.

Our main motivation is not to invent new families of linkages but to make progress in the complete classification of mobile 6R linkages. It should also be pointed out the scope of bond theory is much larger than the technique of quad polynomials. While bond theory is applicable for a large class if linkages (e.g. multiply closed, linkages with different types of joints), quad polynomials can only be defined for simply closed linkages with 6 joints/links.

## 3.2 New Bonds

In this section we recall another method of computing the configuration space of a closed 6R linkage using dual quaternions and Denavit-Hartenberg parameters. This is different to the Algorithm 1 in Chapter 2.

First, we start by introducing the set of Denavit-Hartenberg parameters of a closed 6R linkage. For $i = 1, \ldots, 6$, let $l_i$ be the rotation axis of the $i$-th joint. The angle $\phi_i$ is defined as the angle of the direction vectors of $l_i$ and $l_{i+1}$ (with some choice of orientation). We also set $c_i := \cos(\phi_i)$ and $w_i = \cot(\frac{\phi_i}{2}) = \frac{\cos(\phi_i)+1}{\sin(\phi_i)}$. The number $d_i$ is defined as the orthogonal distance of the lines $l_i$ and $l_{i+1}$. Note that $d_i$ may be negative; this depends on some choice of orientation of the common normal, which we denote by $n_i$.

If we assume that there are no parallel adjacent lines, which means that the angles $\phi_1, \ldots, \phi_6$ are not equal to 0 or $\pi$, then we may set $b_i := \frac{d_i}{\sin(\phi_i)}$ (Bennett ratios [50, 51] (inverse)) as an abbreviation. Finally, we define the offset $s_i$ as the signed distance of the intersections of the common normals $n_{i-1}$ and $n_i$ with $l_i$.

The Denavit-Hartenberg parameters $\phi_i, d_i, s_i$ are invariant when the linkage is moving. Moreover, it is well-known that they form a complete system of invariants for all closed 6R linkages without adjacent parallel lines: if two such linkages share all parameters, then there is a configuration that moves the first into the second. (A description of invariant parameters for 6-bar linkages with adjacent parallel lines may be found in [8]). We will treat one subclass of this type of 6R linkages in Chapter 7.

We give a formulation in the language of dual quaternions, based on the fact that

$SE_3$ is isomorphic to the multiplicative group of dual quaternions with nonzero real norm modulo multiplication by nonzero real scalars.

In the isomorphism described in [34, Section 2.4], the rotation with axis determined by $\mathbf{i}$ and angle $\phi$ corresponds to the dual quaternion $(\cos(\frac{\phi}{2}) - \sin(\frac{\phi}{2})\mathbf{i})$, which is projectively equivalent to $(\cot(\frac{\phi}{2}) - \mathbf{i})$. The translation parallel to $\mathbf{i}$ by a distance $d$ corresponds to the dual quaternion $\left(1 - \frac{d}{2}\epsilon\mathbf{i}\right)$. So the closure equation is

$$(t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2 \cdots (t_6 - \mathbf{i})g_6 \in \mathbb{R}^*, \tag{3.1}$$

where

$$g_i = \left(1 - \frac{s_i}{2}\epsilon\mathbf{i}\right)(w_i - \mathbf{k})\left(1 - \frac{d_i}{2}\epsilon\mathbf{k}\right), \tag{3.2}$$

for $i = 1, \ldots, 6$. This is just the reformulation of the well-known closure equations [20] in terms of dual quaternions.

**Remark 3.1.** *In* (2.1)*, we used a different formulation of the closure equation, namely*

$$(t_1 - h_1)(t_2 - h_2) \cdots (t_6 - h_6) \in \mathbb{R}^*,$$

*where $h_1, \ldots, h_6$ are dual quaternions specifying the rotation axes in some initial position of the linkage.*

The set $K$ of all 6-tuples $(t_1, \ldots, t_6)$ fulfilling (3.1) is the *configuration set* of the linkage $L$ with respect to 18 Denavit-Hartenberg parameters, namely, six orthogonal distances $(d_1, \ldots, d_6)$, six cotangent of the half of the twisted angles $(w_1, \ldots, w_6)$ and six offsets $(s_1, \ldots, s_6)$. We want to mention that we are mostly interested in linkages of mobility one again.

We compute the *configuration set* by Algorithm 3 with 18 given Denavit - Hartenberg parameters, $(d_1, \ldots, d_6)$, $(w_1, \ldots, w_6)$, and $(s_1, \ldots, s_6)$. The idea of Algorithm 3 is same as the idea of Algorithm 1.

It is worth pointing out that it also works for the cases of $4R$ and $5R$ linkages. For a fixed set of Denavit-Hartenberg parameters, the configuration set can be computed with the help of the computer algebra system Maple. For example, let $L$ be the Bricard line symmetric linkage with parameters

**Input:**

$$\begin{aligned} (d_1, \ldots, d_6) &= \left(\frac{3}{5}, \frac{24}{13}, \frac{72}{25}, \frac{3}{5}, \frac{24}{13}, \frac{72}{25}\right) \\ (w_1, \ldots, w_6) &= \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right), \\ (s_1, \ldots, s_6) &= (4, 5, 1, 4, 5, 1) \end{aligned}$$

**Algorithm 3** *configuration set*

**Input:** Denavit-Hartenberg parameters, $(d_1, \ldots, d_6)$, $(w_1, \ldots, w_6)$ and $(s_1, \ldots, s_6)$.
**Output:** A list of equations in $t_1, t_2, \ldots, t_n$ such that its solution set equals the *configuration set* of (3.1).

1: Write $F := (t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2 \cdots (t_6 - \mathbf{i})g_6$, in $\mathbb{DH}$ for $g_i = \left(1 - \frac{s_i}{2}\epsilon\mathbf{i}\right)(w_i - \mathbf{k})\left(1 - \frac{d_i}{2}\epsilon\mathbf{k}\right)$, where $i = 1, \ldots, 6$.
2: Take the seven coefficients of $F$ except the first coefficient and add them to list $E$.
3: Supplement $E$ by one more polynomial, namely, $(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1)(t_6^2 + 1)u - 1$.
4: Compute the Gröbner basis $G$ of the elimination ideal of $E$ with respect to $u$.
5: **Return** $G$ –the elimination ideal.

**Output:** Then all $g_i$s of this 6R linkage are

$$\begin{aligned}
g_1 &= \frac{1}{3} - \frac{3}{10}\epsilon - \frac{2}{3}\epsilon\mathbf{i} - 2\epsilon\mathbf{j} - \left(\frac{1}{10}\epsilon + 1\right)\mathbf{k},\\
g_2 &= \frac{2}{3} - \frac{12}{13}\epsilon - \frac{5}{3}\epsilon\mathbf{i} - \frac{5}{2}\epsilon\mathbf{j} - \left(\frac{8}{13}\epsilon + 1\right)\mathbf{k},\\
g_3 &= \frac{3}{4} - \frac{36}{25}\epsilon - \frac{3}{8}\epsilon\mathbf{i} - \frac{1}{2}\epsilon\mathbf{j} - \left(\frac{27}{25}\epsilon + 1\right)\mathbf{k},\\
g_4 &= g_1,\ g_5 = g_2,\ g_6 = g_3.
\end{aligned}$$

The solution set has some zero-dimensional components, which are not interesting, and a one-dimensional component:

$$171t_1^2t_2^2 - 134t_1^2t_2 + 40t_1t_2^2 + 49t_1^2 - 160t_1t_2 - 5t_2^2 - 24t_1 + 90t_2 - 255,$$

$$171t_1t_2^2 + 19t_2^2t_3 - 134t_1t_2 + 40t_2^2 - 222t_2t_3 + 49t_1 - 288t_2 + 105t_3,$$

$$171t_1t_2 - 133t_1t_3 + 19t_2t_3 - 134t_1 + 40t_2 - 222t_3 - 323, t_1 - t_4, t_2 - t_5, t_3 - t_6.$$

**Remark 3.2.** *The configuration set we computed is just with respect to some initial position of the linkage.*

Let $L$ be a closed 6R linkage with mobility 1. Let $K_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^6$ be the Zariski closure of the configuration set $K$, that is, the zero set of all polynomial equations that vanish on $K$, including complex points and points at infinity. The set of bonds is defined as

$$B := \{(t_1, \ldots, t_6) \in K_{\mathbb{C}} \mid (t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2 \cdots (t_6 - \mathbf{i})g_6 = 0\}. \tag{3.3}$$

Let $\beta$ be a bond with coordinates $(t_1, \ldots, t_6)$. By Theorem 2 in [31], there exist indices $1 \leq i < j \leq 6$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$. If there are exactly two coordinates of $\beta$ with values $\pm$i (where i denotes the imaginary unit in the field of complex numbers $\mathbb{C}$), then we say that $\beta$ *connects* joints $i$ and $j$. By [31, Corollary 12], we have

$$(t_i - \mathbf{i})g_i(t_{i+1} - \mathbf{i})g_{i+1} \cdots (t_j - \mathbf{i}) = 0. \tag{3.4}$$

In general, the situation is more complicated: a bond may connect several pairs of joints, or it may connect a single pair of joints with higher connection multiplicity; we refer to [31] for the technical details in these cases.

In order to draw the bond diagram for a given linkage, we first compute its configuration space as in the previous chapter 2. If $K$ is a Gröbner bases for the configuration space, then we calculate for any pair $(i, j)$ of indices a Gröbner bases $K_{ij}$ of the union of $K$ and $\{t_i^2 + 1, t_j^2 + 1\}$. The number of bonds connecting joints $J_i, J_j$ is then the degree of the ideal $K_{ij}$.

For a given 6R linkage $L$ with 18 given Denavit-Hartenberg parameters, $(d_1, \ldots, d_6)$, $(w_1, \ldots, w_6)$ and $(s_1, \ldots, s_6)$, we compute the bond connections (bond diagram) for the linkage $L$ by Algorithm 4. The main idea is as follows. We calculate the connection numbers for all bonds together instead of calculating them one by one. This is based on the same formula as in Chapter 2

$$k_B(i, j) := v_B(i, j) + v_B(i + 1, j - 1) - v_B(i, j - 1) - v_B(i + 1, j),$$

where we can get $v_B(i, j)$ by counting the solutions of a intersection of $K$ and $F_{i,j} = 0$. This counting includes the multiplicity. One might need to do linear transformations of $t_i$ in $\mathbb{P}$ such that this counting is proper.

One can find a Maple code (using Maple 16 or later) for this computation in [1]. We treat the same example 3.2 as in the computation of the configuration set. The bond connection for this Bricard line symmetric 6R linkage is

$$D := [[1, 4, 2], [2, 5, 2], [3, 6, 2]].$$

In Figure 3.1, we show some known examples with bond diagrams.

We number the joints cyclically by $J_1, \ldots, J_6$. By [31, Theorem 3(c)], a bond cannot connect $J_i$ and $J_{i+1}$ (modulo 6). We speak of a *near connection* if a bond connects joints $J_i$ and $J_{i+2}$, and of a *far connection* if a bond connects $J_i$ and $J_{i+3}$. For instance, the bond diagram in Figure 3.1(d) has 3 near connections and 6 far connections.

**Theorem 3.3.** *A near connection implies a Bennett condition: if $J_i$ and $J_{i+2}$ are connected, then $b_i = \pm b_{i+1}$ and $s_{i+1} = 0$.*

**Algorithm 4** `BondConnectionsII`

**Input:** Denavit-Hartenberg parameters, $(d_1, \ldots, d_6)$, $(w_1, \ldots, w_6)$ and $(s_1, \ldots, s_6)$.
**Output:** A list $D = [D_1, D_2, \ldots, D_m]$, where $D_s = [p, q, r]$ for $s = 1, \ldots, m$, $p = 1, 2, \ldots, n$, $|q - p| = 2, \ldots, n - 2$, $r$ is an nonzero integer.

Use Algorithm 3 to get a list of polynomials $G$ which give the configuration set.
Set $P := \{1, 2, \ldots, n-2\}$.
**repeat**
    Take $p \in P$ and set $P \leftarrow P - \{p\}$, $Q := \{p+2, \ldots, n\}$.
    **repeat**
        Take $q \in Q$ and set $Q \leftarrow Q - \{q\}$.
        Compute the vanishing orders of
        $v_1 := v_B(p, q)$, $v_2 := v_B(p+1, q-1)$,
        $v_3 := v_B(p, q-1)$, $v_4 := v_B(p+1, q)$.
        Set $r := \frac{v_1+v_2-v_3-v_4}{2}$.
        If $r \neq 0$, then $D \leftarrow D + [p, q, r]$.
    **until** $Q = NULL$.
**until** $P = NULL$.
**Return** $D = [D_1, D_2, \ldots, D_m]$ for some integer $m$.

*Proof.* This is an immediate consequence of Theorem 1 and Corollary 2 of [31]. □

For any two links, their relative motion can be described by a curve in the Study quadric. The degree of this curve can be read off from the bond diagram, using [31, Theorem 5]. We will not use this theorem in its full generality. The only consequence which we will use is that every joint is connected to at least one other joint. Otherwise the relative motion of the two adjacent links would be of degree 0. Hence, this joint would be frozen throughout the motion of the linkage and can be deleted from the linkage.

## 3.3 Quad Polynomials

In this section, we introduce a technique to derive algebraic equations on the Denavit-Hartenberg parameters which are necessary for the existence of a far connection. Because any mobile linkage has either a near or a far connection, this allows to deduce necessary conditions for movability.

Assume that $\beta = (\mathrm{i}, \alpha, \beta, \mathrm{i}, \alpha', \beta')$ is a bond connecting $J_1$ and $J_4$ (note that the first and fourth coordinate must be $\pm\mathrm{i}$ by the definition of a connection). We define

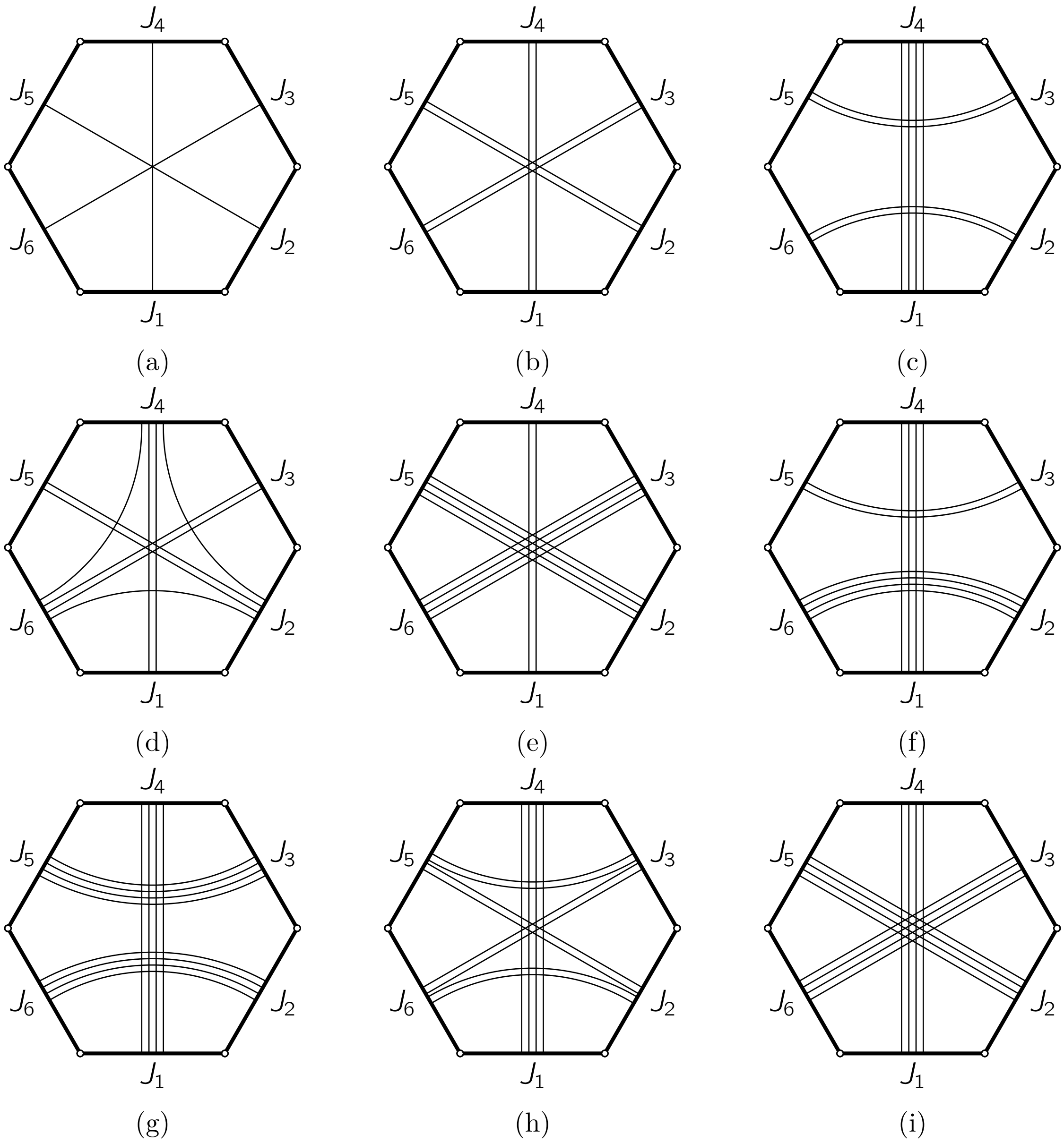



Figure 3.1: Bond diagrams for the Cube linkage (a), the Bricard line symmetric linkage (b), the Bricard plane symmetric linkage (c), the Wohlharts partially symmetric linkage (d), the first new linkage with genus 3 (e), the second new linkage with genus 3 (f), the Hooke linkage (g), the Dietmaier linkage (h), and the Orthogonal Bricard linkage (i). The joints are labeled by $J_1, \ldots, J_6$. Each bond connects two joints.

$q := (\mathrm{i} - \mathbf{i})g_1(\alpha - \mathbf{i})g_2(\beta - \mathbf{i})g_3$. Then we have

$$(\mathrm{i} + \mathbf{i})q = 0 = q(\mathrm{i} - \mathbf{i}),$$

where the first equation follows from $\mathrm{i}^2 - \mathbf{i}^2 = 0$ and the second equation is a consequence of bond theory. All solutions of the two equations above are scalar multiples of $\mathbf{j}+\mathrm{i}\mathbf{k}$, the scalar being an arbitrary dual number. Hence we may write $q=(a+b\epsilon)(\mathbf{j}+\mathrm{i}\mathbf{k})$ with unique numbers $a,b\in\mathbb{C}$. If $a\neq 0$ (this is the case when both $\alpha$ and $\beta$ are not equal to $\pm\mathrm{i}$), then $q$ is projectively equivalent (that is, up to multiplication by a nonzero complex number) to $(1+x_0\epsilon)(\mathbf{j}+\mathrm{i}\mathbf{k})$ for some $x_0\in\mathbb{C}$. Our goal is to define a polynomial $Q_1^+\in\mathbb{C}[x]$, with coefficients depending on the Denavit-Hartenberg parameters, such that $Q_1^+(x_0)=0$.

The triple of complex numbers $(\alpha,\beta,x_0)$ satisfies the following three equations:

1. the first coordinate (the coefficient of 1) of $q$ is zero;

2. the fifth coordinate (the coefficient of $\epsilon$) of $q$ is zero;

3. the product of the third coordinate (the coefficient of $\mathbf{j}$) of $q$ and $x$ is equal to the seventh coordinate (the coefficient of $\epsilon\mathbf{j}$) of $q$.

Conversely, if a triple $(\alpha,\beta,x_0)$ satisfies these three equations, it is straightforward to show that $q$ as defined above is projectively equivalent to $(1+x_0\epsilon)(\mathbf{j}+\mathrm{i}\mathbf{k})$. We define now the quad polynomial $Q_1^+(x)$ as the resultant of the three polynomial equations above with respect to the variables $\alpha,\beta$. In order to achieve uniqueness, we assume that $Q_1^+$ is normed, that is, its leading coefficient is 1. Using the computer algebra system Maple, the computation of the resultant can be done with symbolic expressions for the Denavit-Hartenberg parameters. The result is the quadric polynomial

$$Q_1^+(x)=\left(x+\frac{b_3c_3-b_1c_1}{2}-\frac{s_1}{2}\mathrm{i}\right)^2+\frac{\mathrm{i}}{2}\left(b_1s_2+b_3s_3+s_2b_3c_2+s_3b_1c_2\right)-$$
$$\frac{b_1b_3c_2-s_2s_3c_2}{2}+\frac{s_2^2+s_3^2-b_1^2+b_2^2-b_3^2-b_2^2c_2^2}{4}.$$

For $i=2,\dots,6$, we define the quad polynomial $Q_i^+(x)$ by a cyclic shift of indices that shifts 1 to $i$. Finally, we define $Q_i^-(x)$ by replacing the parameters $c_1,\dots,c_6,b_1,\dots,b_6$ and $s_2,s_4,s_6$ by their negatives, and leaving $s_1,s_3,s_5$ as they are. For instance,

$$Q_1^-(x)=\left(x+\frac{b_3c_3-b_1c_1}{2}-\frac{s_1}{2}\mathrm{i}\right)^2+\frac{\mathrm{i}}{2}\left(b_1s_2-b_3s_3-s_2b_3c_2+s_3b_1c_2\right)-$$
$$\frac{-b_1b_3c_2-s_2s_3c_2}{2}+\frac{s_2^2+s_3^2-b_1^2+b_2^2-b_3^2-b_2^2c_2^2}{4}.$$

**Theorem 3.4.** *Let $k$ be the number of bond connections of $J_1$ and $J_4$. Then*

$$k\leq\deg(\gcd(Q_1^+,Q_4^+))+\deg(\gcd(Q_1^-,Q_4^-)).$$

*Proof.* Assume that $\beta = (\pm \mathrm{i}, \alpha, \pm\beta, \pm \mathrm{i}, \alpha', \beta')$ is a bond connecting $J_1$ and $J_4$. We may assume that its first coordinate is i; otherwise, we replace $\beta$ by its complex conjugate. As above, we define $q := (\mathrm{i}-\mathbf{i})g_1(\alpha-\mathbf{i})g_2(\beta-\mathbf{i})g_3$. By the construction of the quad polynomial $Q_1^+$, there exists $x_0 \in \mathbb{C}$ such that $q$ is projectively equivalent to $(1+x_0\epsilon)(\mathbf{j}+\mathrm{i}\mathbf{k})$ and $Q_1^+(x_0) = 0$. Now we apply a cyclic shift and obtain, in the same way, an $x_1 \in \mathbb{C}$ such that $q' := (\mathrm{i}-\mathbf{i})g_4(\alpha'-\mathbf{i})g_5(\beta'-\mathbf{i})g_6$ is projectively equivalent to $(1+x_1\epsilon)(\mathbf{j}+\mathrm{i}\mathbf{k})$ and $Q_4^+(x_1) = 0$. Now $\beta$ satisfies all algebraic equations that are valid in the configuration set, in particular the equation expressing that $(t_1-\mathbf{i})g_1(t_2-\mathbf{i})g_2(t_3-\mathbf{i})g_4$ is projectively equivalent to the quaternion conjugate of $(t_4-\mathbf{i})g_4(t_5-\mathbf{i})g_5(t_6-\mathbf{i})g_6$. Hence $q$ and $q'$ are conjugate as dual quaternions, up to complex scalar multiplication. But the scalar parts of both $q$ and $q'$ vanish, hence $q$ and $q'$ are projectively equivalent. Hence $x_0 = x_1$, and we have derived the existence of a common zero of $Q_1^+$ and $Q_4^+$, under the assumption of the existence of a bond with $t_1 = t_4 = \mathrm{i}$. Hence $\deg(\gcd(Q_1^+, Q_4^+))$ is an upper bound for the number of bond connections of $J_1$ and $J_4$ by bonds with $t_1 = t_4 = \mathrm{i}$. Similarly, one shows that $\deg(\gcd(Q_1^-, Q_4^-))$ is an upper bound for the number of bond connections of $J_1$ and $J_4$ by bonds with $t_1 = -t_4 = \mathrm{i}$. □

**Remark 3.5.** *The argument of the proof of Theorem 3.4 can be partially reversed: a common root of the quad polynomials $Q_1$ and $Q_4$ implies a common point of $X_{1,2,3}$ and $X_{6,5,4}$ with norm zero. Its preimage $\alpha \in (\mathbb{P}^1)^6$ satisfies the equation $(t_1-h_1)\dots(t_6-h_6) = 0$. But $\alpha$ is not necessarily a bond, because it also could be an isolated intersection point of $X_{1,2,3}$ and $X_{6,5,4}$, and then it is not an element in the Zariski closure of $K$.*

**Remark 3.6.** *It is well-known that two univariate polynomials have a greatest common divisor of positive degree if and only their resultant is zero. The resultant of two quad polynomials is a polynomial expression in the Denavit-Hartenberg parameters. Its vanishing gives rise to two equations, because the resultant has a real and an imaginary part. If the product of all these resultants times, say, the product of all offsets is not zero, then no bond can exist and the linkage is rigid.*

The polynomial conditions obtained in the way described above are big and difficult to solve. It is therefore more promising to go through some case distinctions on the bond diagram. In order to obtain the strongest possible algebraic conditions, one should make the following assumptions on the bond structure.

- For each of the six pairs $(J_i, J_{i+2})$ of near joints, we make an assumption whether they are connected or not.
- For each pair of the three pairs $(J_i, J_{i+3})$ of far joints, we make an assumption on the number of connections by bonds with $t_i = t_{i+3} = \mathrm{i}$ and on the number of connections by bonds with $t_i = -t_{i+3} = \mathrm{i}$. In both cases, this number is in the set $\{0, 1, 2\}$.

- The assumptions must be consistent with the condition that every joint is attached to at least one bond. This condition is a consequence of [31, Theorem 5], assuming that every joint is moving.

In the case when the number of connections of $(J_i, J_{i+3})$ by bonds with $t_i = t_{i+3} = \mathrm{i}$ is equal to 2, then the two polynomials $Q_i^+(x)$ and $Q_{i+3}^+(x)$ must be equal, because they are both quadratic and normed and have a quadratic greatest common divisor. This is equivalent to the vanishing of four polynomials in the Denavit-Hartenberg parameters, namely the real and the complex part of the linear and the constant coefficient of the difference polynomial.

## 3.4 Some Known Examples

In this section, we apply the method of quad polynomials to several well-known families of mobile 6R linkages. The main purpose of this section is to show that our method is another way to "explain" already known equations with a unifying method.

As in the section 3.2, we use the Bennett ratios $b_1, \dots, b_6$, the angle cosines $c_1, \dots, c_6$ and the offsets $s_1, \dots, s_6$. In addition, we also use the values $f_k = c_k b_k$, $k = 1, \dots, 6$, as abbreviations; this leads to shorter formulas.

### 3.4.1 Bricard's Line Symmetric Linkage

If $b_i = b_{i+3}$, $w_i = w_{i+3}$, and $s_i = s_{i+3}$ for $i = 1, 2, 3$, then there is a one-dimensional set of line symmetric positions which allow the link to move. Apparently, we also have

$$Q_1^+ = Q_4^+,\ Q_2^+ = Q_5^+,\ Q_3^+ = Q_6^+,$$

which means that the necessary conditions for a double connection between each pair of far joints are satisfied. As we saw in Figure 3.1(b), the bond diagram does indeed have these three double connections.

Conversely, it is not true that the 12 equations (obtained by $Q_i^+ = Q_{i+3}^+$ for $i = 1, 2, 3$) imply that the linkage is line symmetric. A counterexample is Bricard's orthogonal linkage, see subsection 3.4.4 below.

### 3.4.2 Hooke's Double Spherical Linkage

By combining two spherical linkages with one joint in common, and then removing the common joint, we obtain a movable 6R linkage that has two triples of three joint axes meeting in a point (say the axes of $J_6, J_1, J_2$ and the axes of $J_3, J_4, J_5$). In this case, it is easy to see that $b_1 = b_3 = b_4 = b_6 = s_1 = s_4 = 0$. Another equation, namely

$$s_2^2 + s_3^2 + b_2^2 - f_2^2 + 2s_2s_3c_2 = s_5^2 + s_6^2 + b_5^2 - f_5^2 + 2s_5s_6c_5$$

can be derived by geometric considerations (see [21]) or an algebraic method (see [18]). Alternatively, we consider the bond diagram of this linkage, which is shown in Figure 3.1(g). As we have a fourfold connection of $J_1$ and $J_4$, we get $Q_1^+ = Q_4^+$ and $Q_1^- = Q_4^-$, and under the assumption that $b_1 = b_3 = b_4 = b_6 = s_1 = s_4 = 0$, this is equivalent to the above condition.

### 3.4.3 Dietmaier's Linkage

In [21], Dietmaier describes a family of mobile 6R linkages, which he found by a computer-supported numerical search. It can be characterised by the equations

$$b_6 = b_1, b_3 = b_4, b_2 = b_5, c_2 = c_5, f_6 + f_1 = f_3 + f_4,$$

$$s_6 = s_2, s_3 = s_5, s_1 = s_4 = 0.$$

Its bond diagram is shown in Figure 3.1(h).

Starting from the assumption on the bond structure, we first obtain the conditions $b_6 = b_1$, $b_3 = b_4$, $s_1 = s_4 = 0$ as consequences of Bennett conditions implied by the existence of short connections. Since we have again a fourfold connection of $J_1$ and $J_4$, we again get $Q_1^+ = Q_4^+$ and $Q_1^- = Q_4^-$. We added the inequality condition $b_1b_4 \neq 0$ and did a computer-supported analysis of the solution set using Maple. It turns out that there are two components. The first one is Dietmaier's family. The second is given by the equations

$$b_6 = b_1, b_3 = b_4, b_2 = -b_5, c_2 = c_5, f_6 + f_1 = f_3 + f_4,$$

$$s_6 = s_2, s_3 = s_5, s_1 = s_4 = 0.$$

We computed the configuration set of a random instance of the second component. It appeared to be finite, hence the second component is not a family of mobile 6R linkages. However, the subset of solutions that also fulfill the condition $f_1 + f_6 = 0$ is a well-known family, namely the Bricard plane symmetric linkage. Its bond diagram is shown in Figure 3.1(c).

### 3.4.4 Bricard's Orthogonal Linkage

The well-known family (see [4]) of orthogonal linkages can be described by the conditions

$$s_1 = \cdots = s_6 = 0,\ c_1 = \cdots = c_6 = 0,$$

$$b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2.$$

(The name of this family already tells the twist angles are right angles.) It is easy to prove that these equations imply

$$Q_1^+ = Q_4^+, Q_2^+ = Q_5^+, Q_3^+ = Q_6^+, Q_1^- = Q_4^-, Q_2^- = Q_5^-, Q_3^- = Q_6^-, \tag{3.5}$$

which means that the necessary conditions for the existence of the maximal number of far connections is fulfilled. Indeed, the bond diagram has all these far connections; this can be seen in Figure 3.1(i).

The system of equations (3.5) has more solutions, leading to other linkages with the same bond diagram. This is studied in [27].

## 3.5 Several New Examples

In this section, we use the method to obtain several new family of mobile 6R linkages. These families are remarkable because they have maximal genus.

### 3.5.1 Linkages with Maximal Genus

In Chapter 2, we give a classification of all closed 6R linkages with a configuration curve of genus at least four that do not have links with parallel joint axes, in terms of bond diagrams in Theorem 2.18. Now we will see the classification in terms of their Denavit–Hartenberg parameters. It turns out there are four irreducible families; two of them are well-known, the other two are new.

As in Section 3.4, we use the angle cosines $c_1, \dots, c_6$, the Bennett ratios $b_1, \dots, b_6$ and the offsets $s_1, \dots, s_6$. In addition, we also use the f-values $f_k = c_k b_k$, $k = 1, \dots, 6$; this leads to shorter formulas.

Let $L$ be a linkage such that no adjacent axes are parallel, and assume that the genus of its configuration curve at least four. By Theorem 2.18, its bond diagram is Figure 2.3(a), (b), or (d). Cases (a) and (b) are well-known and have been described in the Lemmas 2.9 and 2.11: these are the Hooke linkage in Section 3.4.2 and the Dietmaier linkage in Section 3.4.3, respectively. Both of them have short connections in their bond diagrams. This yields that there is a coupling dimension less than 8 (4 or 6).

From now on, we assume that $l_{k,k+1,k+2} = 8$ for $k = 1, \ldots, 6$; consequently, the bond diagram is Figure 2.3(d). The number of bonds is maximal, for $k = 1, 2, 3$, and for any choice of $t_k, t_{k+3}$ in $\{+\mathrm{i}, -\mathrm{i}\}$, there exist 2 bonds connecting $h_k$ and $h_{k+3}$. By Theorem 3.4, we get the following equalities of polynomials in $\mathbb{C}[x]$:

$$Q_1^+ = Q_4^+, Q_2^+ = Q_5^+, Q_3^+ = Q_6^+, Q_1^- = Q_4^-, Q_2^- = Q_5^-, Q_3^- = Q_6^-. \tag{3.6}$$

Each equality of polynomials gives rise to four scalar equations, namely the real and imaginary part of the linear and the constant coefficient.

**Lemma 3.7.** *The zero set of the 24 equations above is the union of two irreducible components. For both, we have $s_1 = \cdots = s_6 = 0$ and the three equations*

$$b_1c_2b_3 = b_4c_5b_6, b_2c_3b_4 = b_5c_6b_1, b_3c_4b_5 = b_6c_1b_2. \tag{3.7}$$

*The two components are*

1. $f_1 = f_4, f_2 = f_5, f_3 = f_6, b_1b_3b_5 = b_2b_4b_6,$
   $b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2$

2. $f_1 = f_3 = f_5, f_2 = f_4 = f_6, b_1b_3b_5f_2 = b_2b_4b_6f_1,$
   $b_1^2 + b_3^2 + b_5^2 + f_2^2 = b_2^2 + b_4^2 + b_6^2 + f_1^2.$

*If no Bennett ratio is zero, then the three equations (3.7) are redundant.*

*Proof.* By comparing the imaginary parts of the linear coefficients, it follows immediately that $s_1 = \cdots = s_6 = 0$. For the simplified system, we obtained the decomposition above by Gröbner basis computation using the computer algebra system Maple. □

**Theorem 3.8.** *There are two irreducible families of 6R linkages with coupling dimensions 8 such that the configuration curve has genus 5 generically. They are characterized by cases 1 and 2 in Lemma 3.7.*

*Proof.* The validity of the equations (3.5) implies the existence of 24 points in the intersection of $X_{1,2,3}$ and $X_{6,5,4}$, by Remark 3.5. Intersection theory predicts an intersection of only 16 points (see [58, Section 11.5.1]), therefore the intersection is infinite and the linkage moves.

Since the genus is a lower semicontinuous function in a family of curves, and 5 is the largest possible value, it suffices to exhibit a single example with a configuration curve of genus 5 for each of the two families in order to prove that the genus is 5 in the generic case. Here is an example that works for both, because it is in the intersection of the two families:

$$b_1 = 0, b_2 = 40, b_3 = 32, b_4 = 0, b_5 = 25, b_6 = 7, c_1 = \cdots = c_6 = 0.$$

□

**Remark 3.9.** *A special case of the second family is Bricard's orthogonal linkage (see [4]). It can be characterized by the condition* $s_1 = \cdots = s_6 = c_1 = \cdots = c_6 = 0$ *and* $b_1^2+b_3^2+b_5^2 = b_2^2+b_4^2+b_6^2$. *The example in the proof of Theorem 3.4 is actually an instance of Bricard's orthogonal linkage. Therefore we can conclude that the genus of the configuration curve of Bricard's orthogonal linkage is 5 generically.*

**Remark 3.10.** *The linkages with a configuration curve of genus 4 are contained in the 4 families described in this section as special cases. A concrete example is the Bricard orthogonal linkage with* $(b_1, \ldots, b_6) = (4, 3, 5, 7, 9, 8)$.

### 3.5.2 The first new linkage with Genus 3

We first assume that there are no near connections, four connections of $J_2$ and $J_5$, four connections of $J_3$ and $J_6$, and two connections of $J_1$ and $J_4$ by bonds with $t_1 = t_4$. The bond diagram can be seen in Figure 3.1(e). Then we get the following equalities of polynomials in $\mathbb{C}[x]$:

$$Q_1^+ = Q_4^+, Q_2^+ = Q_5^+, Q_3^+ = Q_6^+, Q_3^- = Q_6^-, Q_2^- = Q_5^-. \tag{3.8}$$

Using the computer algebra system Maple, we obtained the following equivalent system of solutions:

$$\begin{aligned} &b_1^2 + b_3^2 + b_5^2 + f_6^2 = b_2^2 + b_4^2 + b_6^2 + f_3^2, f_2 + f_3 = f_5 + f_6, \\ &b_2c_1 - b_3 = b_2c_3 - b_1 = b_5c_4 - b_6 = b_5c_6 - b_4 = 0, \\ &s_2 = s_3 = s_5 = s_6 = 0, s_1 = s_4. \end{aligned} \tag{3.9}$$

The solution set is irreducible. Here is a random numerical example:

$$\begin{aligned} &b_1 = -\frac{1}{3}, b_2 = -\frac{61}{33}, b_3 = \frac{305}{429}, b_4 = \frac{2000}{1001}, b_5 = -\frac{2900}{1001}, b_6 = \frac{1740}{1001}, \\ &w_1 = \frac{2}{3}, w_2 = -4, w_3 = \frac{6}{5}, w_4 = \frac{1}{2}, w_5 = \frac{\sqrt{54083849}}{6619}, w_6 = \frac{3}{7}, \\ &s_2 = s_3 = s_5 = s_6 = 0, s_1 = s_4 = \frac{2}{3}. \end{aligned} \tag{3.10}$$

(a) (b) (c)

Figure 3.2: Three configurations of the new linkage (3.10).

**Theorem 3.11.** *The solution of the system of equations (3.9) is a set of Denavit - Hartenberg parameters for a mobile linkage. This linkage is different from all linkages listed in [17, 21].*

*Proof.* We calculate the configuration space using the method described in Section 3.2. The Gröbner basis consists of 32 polynomials (1194 terms in total) in $t_1, \ldots, t_6$ and the radical number $w_5$, which are too long to be reproduced here. The dimension of the ideal can be calculated from the Gröbner bases, and it is indeed 1 and contains infinite real solutions. This shows that the linkage moves. Some configurations are shown in Figure 3.2.

In order to show that the linkage is different from the known linkages in lists [5, 17, 21], one could compute all bond diagrams of the known linkages and see that the diagram in Figure 3.1(e) is not among them. (Indeed, this was our first proof.) The disadvantage of this approach is that we would have to include the bond diagrams of all known linkages. However, there is a shortcut based on the observation that almost all linkages in [5, 17, 21] fulfill at least one Bennett condition, while our Example (3.9) does not satisfy Bennett conditions. We just need to check against the known examples that do not fulfill the Bennett conditions. There are the Bricard line symmetric linkage, the Bricard orthogonal linkage, and the cube linkage. For these four cases, the bond diagrams are Figure 3.1(a),(b), and (i), and these are clearly different from Figure 3.1(e). □

One can observe, in addition, that all points in the configuration space satisfy the equation $t_1 = t_4$. When we vary $s_1 = s_4$, we get a configuration set of a CRRCRR linkage (2 cylindrical joints). This set is an infinite union of curves, so its dimension is 2. Let $a \in \mathbb{R}$. Then we may pose a new constraint like $t_1 = \cot(as_1)$, and still have dimension 1; the second constraint $t_4 = \cot(as_4)$ is implied because $t_1 = t_4$

and $s_1 = s_4$. This defines an HRRHRR linkage (2 helical joints) with movability 1. This technique is introduced in next section.

### 3.5.3 The second new linkage with Genus 3

We assume that there are near connections, four connections of $J_1$ and $J_4$, four connections of $J_2$ and $J_6$ (Spherical condition), and two connections of $J_3$ and $J_5$ (Bennett condition). The bond diagram can be seen in Figure 3.1(f). Then we get the following equalities of polynomials in $\mathbb{C}[x]$:

$$Q_1^+ = Q_4^+, Q_1^- = Q_4^-. \tag{3.11}$$

Using the computer algebra system Maple, we obtained the following equivalent system of solutions:

$$\begin{aligned} & f_2^2 + c_2^2 s_2^2 - b_2^2 - s_2^2 = f_5^2 + c_5^2 s_6^2 - b_5^2 - s_6^2, \\ & c_3 + c_4 = c_2 s_2 - c_5 s_6 + s_3 - s_5 = 0, \\ & b_3 = b_4,\ b_1 = b_6 = s_1 = s_4 = 0. \end{aligned} \tag{3.12}$$

It is worth mentioning that the 6R linkage is still mobile with genus 3 when one replaces the spherical 3R linkage by a planar 3R linkage.

The solution set is irreducible. Here is a random numerical example:

$$\begin{aligned} & b_1 = 0, b_2 = -\frac{12}{5}, b_3 = 5, b_4 = 5, b_5 = 3, b_6 = 0, \\ & w_1 = 1, w_2 = 1, w_3 = 3, w_4 = \frac{1}{3}, w_5 = 1, w_6 = 2, \\ & s_1 = s_4 = s_6 = 0, s_2 = \frac{9}{5}, s_3 = s_5 = \frac{5}{3}. \end{aligned} \tag{3.13}$$

One can observe, in addition, that all points in the configuration space satisfy the equation $t_3 = t_5$. When we vary $s_3 = s_5$, we get a configuration set of a RRCRCR linkage (2 cylindrical joints). This set is an infinite union of curves, so its dimension is 2. Let $a \in \mathbb{R}$. Then we may pose a new constraint like $t_3 = \cot(as_3)$, and still have dimension 1; the second constraint $t_5 = \cot(as_5)$ is implied because $t_3 = t_5$ and $s_3 = s_5$. This defines an RRHRHR linkage (2 helical joints) with movability 1. This technique is introduced in next section.

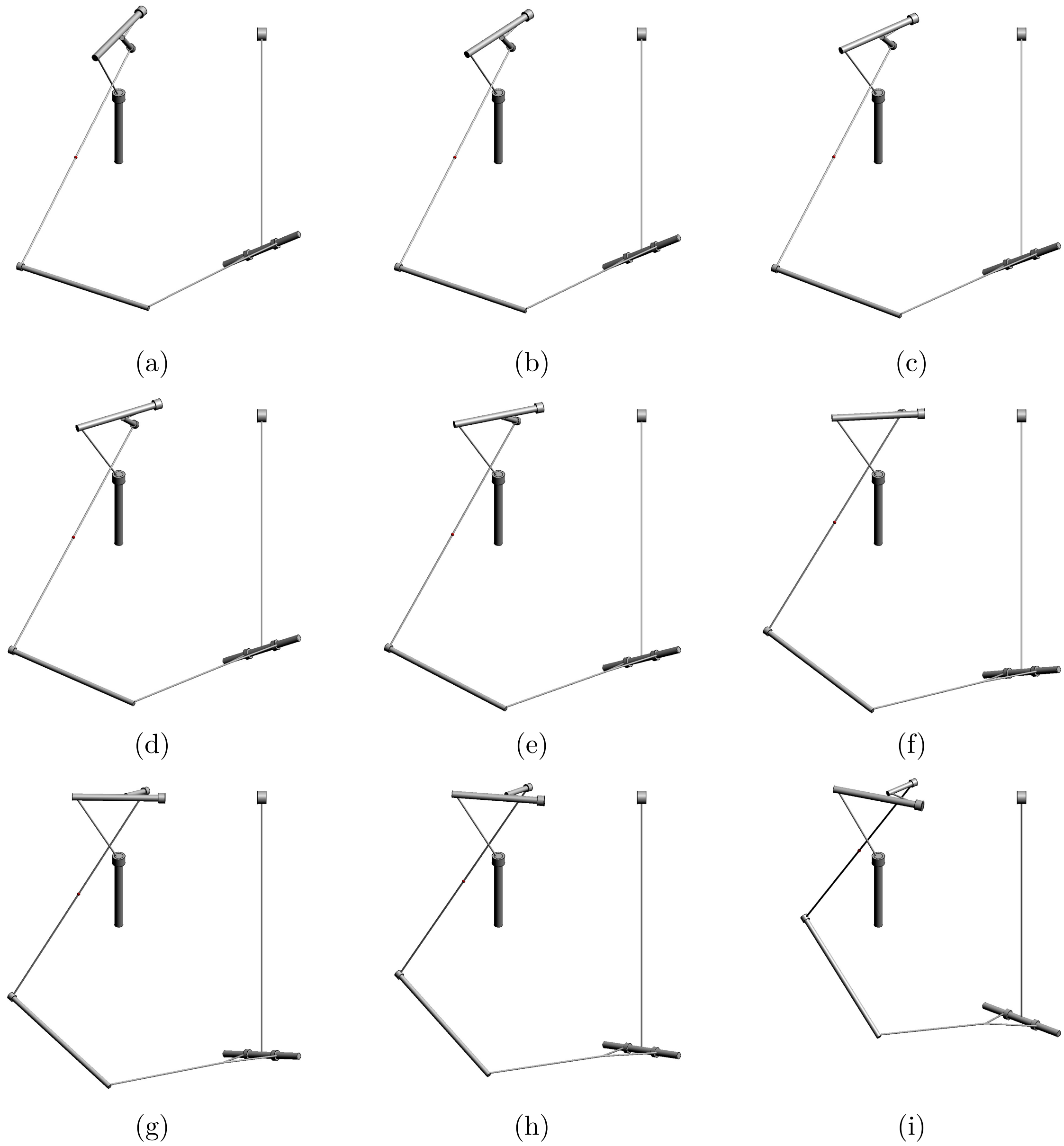



Figure 3.3: Nine configurations of the new linkage (3.13).

## 3.6 Construction of Linkages with Helical Joints

In this section we give a construction that produces mobile linkages with H-joints from linkages with C-, P-, and R-joints. We illustrate the construction by several well-known examples and one example which is new.

We start with a simple construction: take a linkage with $r$ C-joints that has mobility at least $r+1$. For each C-joint $j_k$, impose the additional restriction $t_k = \cot(\frac{s_k}{2g_k})$

on its joint parameters $(s_k, t_k)$, where $g_k$ is a nonzero real constant. Any additional equation reduces the mobility at most by 1, so we get a mobile linkage where every C-joint $j_k$ is replaced by an H-joint with pitch $g_k$.

We can extend this simple construction using the observation that $\mathbb{Q}$-linear relations between the angles imply algebraic relations between their tangents. For the general construction, which we call *screw carving* , we need the following ingredients.

1. a linkage $L$ with $m$ C-joints $j_{k_1}, \ldots, j_{k_m}$ and an undetermined number of R- and P-joints;

2. an irreducible analytic subspace $K_0$ of the configuration space of $L$;

3. an integer matrix $A$ with $m$ columns that annihilates the vector of analytic functions $(\alpha_{k_1}, \ldots, \alpha_{k_m})^t \in \mathbb{C}(K_0)^m$ such that $\cot(\frac{\alpha_k}{2}) = t_k$;

4. an $m$-tuple $(g_{k_1}, \ldots, g_{k_m})$ of nonzero real numbers, so that $A$ also annihilates the vector of functions $(a_{k_1}, \ldots, a_{k_m})^t$, where $a_k : K_0 \to \mathbb{C}$ is the function $(s_*, t_*) \mapsto \frac{s_k}{g_k}$.

As before, the linkage $L'$ with H-joints instead of C-joints is obtained by imposing the additional restriction $t_k = \cot(\frac{s_k}{2g_k})$ on its joint parameters $(s_k, t_k)$, for each C-joint $t_k$. To obtain linkages with large mobility, the integer matrix $A$ should have the largest possible rank, which means that all integral relations between the analytic angle functions are linear combination of matrix rows. (In the next section, we will indeed always choose such matrices of maximal rank.) The empty matrix with zero rows is allowed, then we just get the simple construction above.

**Lemma 3.12.** *Let $d := \dim(K_0)$ and $\ell := \operatorname{rank}(A)$. Then the mobility of the linkage produced by screw carving is at least $d - m + \ell$.*

*Proof.* The subset $K'$ of $K_0$ that satisfies the additional restrictions $t_k = \cot(\frac{s_k}{2g_k})$ is contained in the configuration space of $L'$. Since the codimension of an analytic subset is never bigger than the number of defining equations, we see that $\dim(K') \geq d - m$. We claim that $K'$ can be defined (as a subset of $K'$) by only $m - \ell$ equations.

Let $\alpha_{k_1}, \ldots, \alpha_{k_m} \in \mathbb{C}(K_0)$ be as above. The $\mathbb{Q}$-vector space generated by these $m$ functions has dimension at most $m - \ell$. Without loss of generality, we assume that $\{\alpha_{k_1}, \ldots, \alpha_{k_{m-\ell}}\}$ is a generating set. Any other $\alpha_k$ can be expressed as a $\mathbb{Q}$-linear combination

$$\alpha_k = q_1 \alpha_{k_1} + \cdots + q_{k_{m-\ell}} \alpha_{k_{m-\ell}},$$

with rational coefficients depending on the matrix $A$. But then we also have

$$\frac{s_k}{g_k} = q_1 \frac{s_{k_1}}{g_{k_1}} + \cdots + q_{k_{m-\ell}} \frac{s_{k_{m-\ell}}}{g_{k_{m-\ell}}}.$$

It follows that the equations $t_{k_1} = \cot(\frac{s_{k_1}}{2g_{k_1}}), \ldots, t_{k_{m-\ell}} = \cot(\frac{s_{k_{m-\ell}}}{2g_{k_{m-\ell}}})$ imply all other equations. □

**Example 3.13.** *Let $L$ be a 4-linkage with 4 cylindrical joints with parallel axes. Its mobility is 4. For all configurations $(t_1 = \cot(\frac{\alpha_1}{2}), s_1, \ldots, t_4 = \cot(\frac{\alpha_4}{2}), s_4)$, we have $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ and $s_1 + s_2 + s_3 + s_4 = 0$. So we take $K_0$ as the full configuration set, $A$ as the $1 \times 4$ matrix $(1, 1, 1, 1)$, and $g_1 = g_2 = g_3 = g_4$, and apply screw carving. We obtain a 4-linkage with 4 helical joints and mobility 4-4+1=1. One can find more examples of such linkages in [19, 38].*

*Similarly, one can obtain an $n$-linkage with $n$ H-joints with parallel axes with mobility $n-3$, $n \geq 4$.*

**Example 3.14.** *Here is a variation of the previous example. Set*

$$h_1 = \mathbf{k} - \epsilon\mathbf{i}, h_2 = \mathbf{k} + \epsilon\mathbf{i}, h_3 = h_5 = \mathbf{k}, h_4 = \mathbf{k} + 2\epsilon\mathbf{j}$$

*and let $L$ be the CCRRR linkage with C-joint axes $h_1, h_2$ and R-joint axes $h_3, h_4, h_5$. Its mobility is 3, and all configurations satisfy $s_1 + s_2 = 0$. We define $K_0$ as the subvariety defined by $\tan(17 \operatorname{arccot}(t_1) - 11 \operatorname{arccot}(t_2)) = 0$ (this is a rational function in $t_1, t_2$). Its dimension is 2. We set as the $1 \times 2$ matrix $A = (1, 1)$ and $g_1 = \frac{1}{17}$, $g_2 = \frac{-1}{11}$. By screw carving we get an HHRRR linkage with mobility 1. Figure 3.4 shows the trace of the joint $j_4$ when the link with the two H-joints $j_1, j_2$ is fixed.*

**Example 3.15.** *Let $h_1, h_2, h_3$ be lines. Reflecting them by the coordinate axes represented by $\mathbf{i}$, we get $h_4 = \mathbf{i}h_1\mathbf{i}$, $h_5 = \mathbf{i}h_2\mathbf{i}$, $h_6 = \mathbf{i}h_3\mathbf{i}$. Let $L$ be the 6C-linkage with axes $h_1, \ldots, h_6$. The zero set of the closure equation*

$$(t_1 - h_1)(1 - \epsilon s_1 h_1) \cdots (t_6 - h_6)(1 - \epsilon s_6 h_6) \equiv 1$$

*has a component of dimension 4, given by the equations*

$$t_1 = t_4, t_2 = t_5, t_3 = t_6, s_1 = s_4, s_2 = s_5, s_3 = s_6, x\mathbf{i} + \mathbf{i}x = 0,$$

$$\text{where } x = (t_1 - h_1)(1 - \epsilon s_1 h_1)(t_2 - h_2)(1 - \epsilon s_2 h_2)(t_3 - h_3)(1 - \epsilon s_3 h_3).$$

*With $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ and $g_1 = g_4$, $g_2 = g_5$, $g_3 = g_6$, the screw carving procedure gives a line symmetric 6H linkage with mobility 1.*

*Similarly, one can construct a plane symmetric RHHRHH linkage with mobility 1. Both linkages are well-known, see [6].*

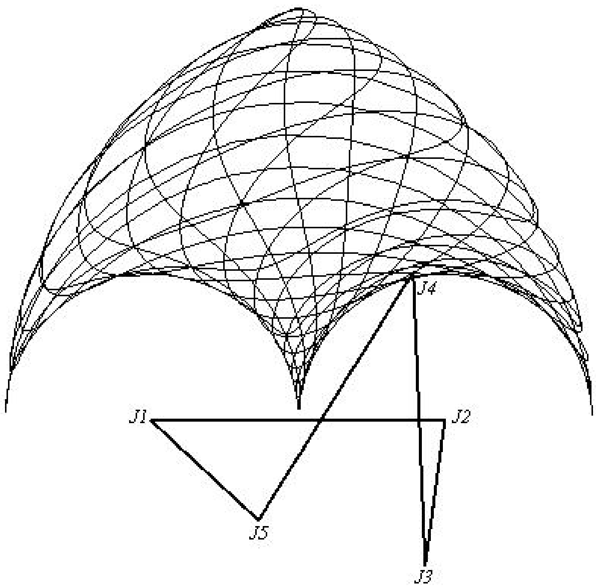



Figure 3.4: Planar projection of an HHRRR linkage with 5 parallel axes to the plane orthogonal to the axes (Example 3.14). The helical joints are at $j_1$ and $j_2$. The ratio of the pitches at the two helical joints $j_1, j_2$ is 11:17. The curve shown is the trace of the joint $j_4$. It is an algebraic curve of large degree.

**Example 3.16.** *Let $h_1$, $h_2$, $h_3$ be lines with linear independent primal parts that do not intersect pairwise, such that two offsets $s(h_1, h_2, h_3) = s(h_3, h_1, h_2) = 0$. Let $L$ be the RRCRRC linkage with axes $h_1, h_2, h_3, h_2, h_1, h_3$. The zero set of the closure equation*

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(1 - \epsilon s_1 h_3)(t_4 - h_2)(t_5 - h_1)(t_6 - h_3)(1 - \epsilon s_6 h_3) \equiv 1$$

*has two components of dimension 2. The first is given by $t_1 = -t_5$, $t_2 = -t_4$, $t_3 = t_6 = \infty$, $s_3 = s_6 = 0$; this is a degenerate motion which does not separate the pairs of axes at joints $(j_1, j_5)$ and at $(j_2, j_4)$. The equations of the second component $K_0$ can be computed by computer algebra. Two of them are $s_3 = s_6$ and $t_3 = t_6$; the remaining are more complicated. With $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $g_3 = g_6$, the screw carving procedure gives an RRHRRH linkage with mobility 1. In contrast to all families of mobile 6-linkages with H-joints that have been known up to now, this linkage has no parallel axes or apparent geometric symmetries. A distinctive property is the existence of a starting position with three pairs of coinciding axes.*

# Chapter 4

# Factorization of Motion Polynomials

In this chapter, we will recall some elementary definitions and properties of the factorization of motion polynomials. These definitions and properties are recollected from [30].

The chapter recalls a factorization algorithm for generic motion polynomials (Algorithm 5) from [30]. For the non-generic case, we give two algorithms (Algorithms 7 and 8). The results presented below evolved from a collaboration with Josef Schicho and Hans-Peter Schröcker [46].

Factorizations of polynomials over non-commutative ring (e.g. quaternions) have been studied by many authors. The most recent review can be found in [61]. Especially concerning quaternion ring, we would like to mention [25, 32, 54] where one can find the results on the number of roots and explicit solution formulas for quadratic quaternion polynomials.

One can construct 6R linkages by combining different factorizations of a motion polynomial which we will define in next section. In other words, every 6R linkage whose configuration set contains a non-degenerate rational curve can be constructed by this method. There are lots of such families of 6R linkages. Namely, all the linkages constructed using Bennett linkages belong to this rational families, e.g., Waldrons Double Bennett Hybrid.

**Structure of the chapter** The remaining part of the chapter is set up as follows. In Section 4.1, we introduce all preliminary definitions on motion polynomials. In Section 4.2, we recall the first factorization algorithm from [30]. It only treats the generic case where there is no real polynomial factor in the primal part of the motion polynomial. Section 4.3 contains two new algorithms on fixing the gap. Namely, we can factor the non-generic case under the assumption of bounded motion polynomial which only admits quadratic irreducible real polynomials factor in the primal part.

In fact, a linear real polynomial factor can be replaced by quadratic irreducible real polynomial factor by substitution. In Section 4.4.3, we mention that the general algorithm can be used for the planar motion. Even more, we can get non-planar factorization for planar motions.

## 4.1 Preliminary Preparations

We are still with the dual quaternion model of rigid body displacements. In particular, we focus on one degree of freedom rational motions that can be parameterized by motion polynomials [30, 41, 45, 48].

Denote by $\mathbb{DH}[t]$ the ring of polynomials in $t$ with dual quaternion coefficients where multiplication is defined by the convention that the indeterminate $t$ commutes with all coefficients. We follow the convention to write the coefficients to the left of the indeterminate $t$. Similarly, we denote by $\mathbb{H}[t]$ the sub-ring of polynomials with coefficients in $\mathbb{H}$. The *conjugate polynomial* to $C = \sum_{i=0}^{n} c_i t_i \in \mathbb{DH}[t]$ is $\overline{C} = \sum_{i=0}^{n} \overline{c}_i t^i$ and the *norm polynomial* is $C\overline{C}$. Its coefficients are dual numbers. If $C = \sum_{i=0}^{n} c_i t^i$, the value $C(h)$ of $C$ at $h \in \mathbb{DH}$ is defined as $C(h) = \sum_{i=0}^{n} c_i h^i$. We also define $C(\infty) := c_n$.

A polynomial $M = P + \epsilon Q \in \mathbb{DH}[t]$ is called a *motion polynomial* if $P\overline{Q} + Q\overline{P} = 0$ and its leading coefficient is invertible. Usually we will even assume that the leading coefficient is one (the polynomial is *monic*). This can be accomplished by left-multiplying $M$ with the inverse of the leading coefficient and often constitutes no loss of generality. The defining conditions of a motion polynomial ensure that its norm polynomial has real coefficients.

A motion polynomial $M = P + \epsilon Q$ acts on a point $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ according to

$$x \mapsto \frac{Px\overline{P} + 2P\overline{Q}}{P\overline{P}}. \tag{4.1}$$

This equation defines a rigid body displacement for all values $t \in \mathbb{R} \cup \{\infty\}$ that are not zeros of $P$. Any map of the shape (4.1) with a motion polynomial $M = P + \epsilon Q$ is called a *rational motion.* We also say that the motion polynomial *parameterizes* the rational motion. The motion's trajectories (orbits of points for $t \in \mathbb{R} \cup \{\infty\}$) are rational curves. It is known that any motion with only rational trajectories is parameterized by a motion polynomial [35].

The simplest motion polynomials are of degree one and can be written as $M = t - h$ where $h - \overline{h} \in \mathbb{R}$ and $h\overline{h} \in \mathbb{R}$. They parameterize either rotations about a fixed axis or translations in a fixed direction. We speak of the *rotation* or *translation quaternion* $h$ and the *rotation* or *translation polynomial* $t-h$, respectively. In this chapter we are concerned with the factorization of motion polynomials into the product of rotation polynomials. These are distinguished from translation polynomials by having a primal part not in $\mathbb{R}[t]$.

## 4.2 Factorization of Generic Cases

In [30] it has been shown that a generic monic motion polynomial $M = P + \epsilon D$ of degree $n$ admits factorizations of the shape

$$M = (t - h_1) \cdots (t - h_n) \tag{4.2}$$

with rotation polynomials $t - h_1, \ldots, t - h_n$. Here, the term "generic" means that the primal part $P$ of $M$ has no real factors. The factorization (4.2) can be computed by the non-deterministic Algorithm 5. The details of this algorithm are explained in [30] but some comments are appropriate at this place.

- In all our algorithms, we denote concatenation of lists by the operator symbol "+". List concatenation is not commutative: The list $L_1 + L_2$ starts with the elements of $L_1$ and ends with the elements of $L_2$.

- By genericity of $M$, the norm polynomial $P\overline{P}$ is real and positive. Hence, it is the product of $n$ quadratic, real factors which are irreducible over $\mathbb{R}$.

- The choice of a quadratic factor in Line 5 is arbitrary. Different choices result in different factorizations. In general, there are $n!$ factorizations of the shape (4.2), each corresponding to a permutation of the quadratic factors of $P\overline{P}$.

- For left polynomials with dual quaternion coefficients in our sense, right division is possible: Given two polynomials $M$, $N \in \mathbb{H}[t]$ with $N$ monic, there exist unique polynomials $Q$, $R \in \mathbb{H}[t]$ with $M = QN + R$ and $\deg R < \deg N$.

- The dual quaternion $h_i$ in Line 6 can be computed as zero of the linear polynomial $R_i$ obtained by writing $M = QM_i + R_i$ (polynomial division). The assumptions on $M$ guarantee existence of a unique zero over the dual quaternions but the algorithm may fail at this point if these assumptions are not met.

- We may exit the algorithm after just one iteration to find a linear right factor of $M$, that is, write the motion polynomial as $M = M'(t - h)$. This we will often do in our factorization algorithm for non-generic motion polynomials.

**Algorithm 5** `GFactor`

**Input:** $M = P + \epsilon D \in \mathbb{DH}[t]$, a monic, generic motion polynomial of degree $n$.
**Output:** A list $L = [L_1, \dots, L_n]$ such that $M = L_1 \cdots L_n$.

$L \leftarrow []$ ▷ (empty list)
$F \leftarrow [M_1, \dots, M_n]$ ▷ Each $M_i \in \mathbb{R}[t]$, $i = 1, \dots, n$ is a quadratic, irreducible factor of $P\overline{P} \in \mathbb{R}[t]$.
**repeat**
  Choose $M_i \in F$ and set $F \leftarrow F - [M_i]$.
  Compute $h_i$ such that $M_i(h_i) = M(h_i) = 0$.
  $L \leftarrow [t - h_i] + L$ ▷ (add $t - h_i$ to start of list)
  $M \leftarrow M/(t - h_i)$ ▷ (polynomial division)
**until** $\deg M = 0$.
**Return** $L = [L_1, L_2, \dots, L_n]$.

For later reference, we state the result of [30, Theorem 3] as a lemma. We do this in a form that highlights the dependence of the factorization on an ordering of the norm polynomial's quadratic factors.

**Lemma 4.1.** *Given a generic, monic motion polynomial $M$ of degree $n$ with $M\overline{M} = M_1 \cdots M_n$ and monic, quadratic and irreducible real polynomials $M_1, \dots, M_n$, there exist rotation quaternions $h_1, \dots, h_n$ such that $M = (t - h_1) \cdots (t - h_n)$ and $M_i = (t - h_i)(t - \overline{h_i})$ for $i = 1, \dots, n$. Different labeling of the quadratic factors of $M\overline{M}$ give different factorizations.*

Here are examples of non-generic motion polynomials with exceptional factorizations.

**Example 4.2.** *The motion polynomial $M := t^2 + 1 + \epsilon\mathbf{i}$ is not generic. A straightforward computation shows that no linear motion polynomials $t - h_1$ and $t - h_2$ in $\mathbb{DH}[t]$ with $M = (t - h_1)(t - h_2)$ exist. The motion parameterized by $M$ is a translation with constant direction.*

**Example 4.3.** *Non-generic motion polynomials with infinitely many factorizations exist. One example is $M := t^2 + 1 - \epsilon t(\mathbf{i}t - \mathbf{j})$. It can be factorized as $M = (t - h_1)(t - h_2)$ where*

$$h_1 = \mathbf{k} - \epsilon(a\mathbf{i} + (b - 1)\mathbf{j}), \quad h_2 = -\mathbf{k} + \epsilon(a\mathbf{i} + b\mathbf{j})$$

*and $a$, $b$ are arbitrary real numbers. The motion parameterized by $M$ is a circular translation. Any of the infinitely many factorizations of $M$ corresponds two one leg of a parallelogram linkage that can generate this motion.*

**Example 4.4.** *The motion polynomial* $M := t^2 - (1+\mathbf{j})t + \mathbf{j} - \epsilon((\mathbf{i}+\mathbf{k})t - 2\mathbf{k})$ *can be factored as*

$$M = (t - 1 - \epsilon\mathbf{i})(t - \mathbf{j} - \epsilon\mathbf{k}) = (t - \mathbf{j} - \epsilon(\mathbf{i} + 2\mathbf{k}))(t - 1 + \epsilon\mathbf{k}).$$

*The polynomial factors* $t - 1 - \epsilon\mathbf{i}$ *and* $t - 1 + \epsilon\mathbf{k}$ *parameterize, however, translations, not rotations. The reason for this is the possibility to factor the primal part of* $M$ *as* $t^2 - (1+\mathbf{j})t + \mathbf{j} = (t-1)(t-\mathbf{j})$. *For* $t = 1$, *the motion parameterization becomes singular and the trajectories pass through infinite points.*

We will present a method to factor even the motion polynomials of these examples into products of linear rotation polynomials. This will be made possible by allowing alterations of the given motion polynomial that change its kinematic and algebraic properties in an "admissible" way. This alterations are:

1. Multiplication of $M$ with a strictly positive real polynomial $Q$ and factorization of $QM$ instead of $M$. This is an admissible change because $M$ and $QM$ parameterize the same motion. This "multiplication trick" has already been used in [22] for the factorization of planar motion polynomials.

2. Substitution of a rational expression $R/Q$ with $R$, $Q \in \mathbb{R}[t]$ for the indeterminate $t$ in $M$ and factorization of $Q^{\deg M} M(R/Q)$ instead of $M$. This amounts to a not necessarily invertible re-parameterization of the motion. In particular, it is possible to parameterize only one part of the original motion.

Multiplication with real polynomials does not change kinematic properties but gives additional flexibility to find factorizations in otherwise nonfactorizable cases. In order to explain the meaning and necessity of substitution of real polynomials, we first give an important definition.

**Definition 4.5.** *A motion polynomial* $M = P + \epsilon D$ *is called* bounded, *if its primal part* $P$ *has no real zeros.*

Generic motion polynomials are bounded. Bounded motion polynomials parameterize precisely the rational motions with only bounded trajectories. If the motion polynomial is not bounded, zeros of the primal part belong to infinite points on the trajectories. For this reason, unbounded motion polynomials can never be written as the product of linear rotation polynomials. For example, we can never succeed in finding a factorization $(t - h_1)(t - h_2)$ with rotation quaternions $h_1$, $h_2$ of the motion polynomial in Example 4.4 as it has unbounded trajectories.

Unbounded motion polynomials can always be turned into bounded ones by an appropriate substitution. This is the reason, why we henceforth restrict our attention to bounded motion polynomials. The kinematic meaning is that only a certain portion of the original trajectories is actually reached during the motion. Finally, we assume that our motion polynomials are monic. This is no loss of generality. If $M$ is bounded, the leading coefficient $c_n$ of $M$ is invertible and we may factor $c_n^{-1}M$ instead. This amounts to an admissible change of coordinates.

To summarize and give a precise problem statement: Given a bounded, monic motion polynomial $M$, we want to find a real polynomial $Q$ and a list of linear rotation polynomials $L = [t - h_1, \ldots, t - h_n]$ such that $QM = (t - h_1) \cdots (t - h_n)$. In this case we say that "$M$ admits a factorization". We will not only prove existence of $Q$ and $L$, we will also provide a simple algorithm for computing appropriate $Q$ and $L$, provide a bound on the degree of $Q$ (and hence also on the number of polynomials in $L$) and present a more elaborate algorithm that produces a polynomial $Q$ of minimal degree.

## 4.3 Factorization of Non-Generic Cases

On particular case for which existence of factorizations of non-generic motion polynomials has already been proved to exist is planar kinematics [22].

**Definition 4.6.** *A motion polynomial $M$ is called* planar, *if it parameterizes a planar motion (a subgroup consisting of all rotations around axes parallel to a fixed direction and translations orthogonal to that direction).*

Examples of planar motion polynomials are obtained by picking coefficients in $\langle 1, \mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k}\rangle$. In [22], the authors showed that for every monic, bounded, planar motion polynomial $M$ of degree $n$ a real polynomial $Q$ of degree $\deg Q \le n$ exists such that $QM$ admits a factorization of the shape (4.2). Input and output of this planar factorization algorithm are displayed in Algorithm 6. We list this algorithm only for the purpose of later reference. For details we refer to [22].

**Algorithm 6** `PFactor` (planar factorization algorithm of [22])

**Input:** $M = P + \epsilon D \in \mathbb{DH}[t]$, a planar, bounded, monic motion polynomial.
**Output:** $Q \in \mathbb{R}[t]$, list $L = [L_1, L_2, \ldots, L_n]$ of linear rotation polynomials such that $QM = L_1 L_2 \cdots L_n$.

The first factorization procedure we propose is of theoretical interest. It is displayed in Algorithm 7. It is based on the algorithm for factorization of planar motion polynomials and produces a real polynomial $Q$ and a factorization of $QM$ for a monic and bounded but not necessarily generic motion polynomial $M$. It is conceptually simpler than Algorithm 8 below but non optimal as far as minimality of $\deg Q$ is concerned. In its listing, we denote by $\mathrm{GRPF}(M)$ the greatest real polynomial factor of a quaternion polynomial $M \in \mathbb{H}[t]$. Lines 2 to 5 of Algorithm 7 are based on the factorization

$$MT\overline{T} = (R_1T + \epsilon D)T\overline{T} = (R_1T\overline{T} + \epsilon D\overline{T})T$$

of $MT\overline{T}$ into the product of a planar motion polynomial and a polynomial $T \in \mathbb{H}[t]$.

**Algorithm 7** `FactorI`

**Input:** $M = P + \epsilon D \in \mathbb{DH}[t]$, a monic, bounded motion polynomial with real quadratic factor in its primal part, $Q \in \mathbb{R}[t]$, list $L$ of linear motion polynomials. Initially, $Q = 1$ and $L = []$ (empty list).

**Output:** $Q$ and $L = [L_1, L_2, \ldots, L_n]$ such that $QM = L_1L_2 \cdots L_n$.

1: Write $P = R_1T$ where $R_1 = \mathrm{GRPF}(P)$.
2: **If** $\deg T \neq 0$ **Then**
3: $\quad L \leftarrow L + \texttt{GFactor}(T)$ ▷ Append linear factors of $T$ to $L$.
4: $\quad Q \leftarrow T\overline{T}$, $P \leftarrow R_1T\overline{T}$, $D \leftarrow D\overline{T}$, and $M \leftarrow P + \epsilon D$
5: **End If**
6: Factor $MP = (P + \epsilon(D_1\mathbf{i} + D_2\mathbf{j} + D_3\mathbf{k}))P = (P + \epsilon D_1\mathbf{i})(P + \epsilon D_2\mathbf{j} + \epsilon D_3\mathbf{k})$.
7: $Q_1, L_1 = \texttt{PFactor}(P + \epsilon D_1\mathbf{i})$ ▷ (planar factorization)
8: $Q_2, L_2 = \texttt{PFactor}(P + \epsilon D_2\mathbf{j} + \epsilon D_3\mathbf{k})$ ▷ (planar factorization)
9: $Q \leftarrow QQ_1Q_2 = QP^2$ ▷ (because $Q_1 = Q_2 = P$)
10: $L \leftarrow L_2 + L_1 + L$ ▷ Concatenate lists of linear factors.
11: **Return** $Q$, $L$

Together with [22], Algorithm 7 proves existence of a factorization:

**Theorem 4.7.** *Given a bounded, monic motion polynomial $M \in \mathbb{DH}[t]$ there always exists a real polynomial $Q$ such that $QM$ can be written as a product of linear rotation polynomials.*

## 4.4 Factorizations of Minimal Degree

Now we should further elaborate on the minimal possible degree of the real factor $Q$ that makes factorization possible. In the planar case, Algorithm 6 gives the bound $\deg Q \leq \deg M$ and this bound is known to be optimal [22]. The upper bound achievable with Algorithm 7 is worse. Let $m = \deg M$ and $r = \deg R_1$. Then, the degree of $Q$ in Line 4 is bounded by $2(m - r)$ and the degree of $P$ in Line 4 is bounded by $r+2(m-r) = 2m-r$. Hence, the degree of $Q$ at the end of Algorithm 7 is bounded by $2(m - r) + 2(2m - r) = 6m - 4r$. Because of $r \geq 2$, this gives the bound $\deg Q \le 6m - 8$. However, also in the spatial case the bound $\deg Q \le \deg M$ holds true. This is guaranteed by Algorithm 8.

**Algorithm 8** `FactorAll`

**Input:** $M = P + \epsilon D \in \mathbb{DH}[t]$, a monic, bounded motion polynomial of complexity $(\alpha, \beta, \gamma)$, $Q \in \mathbb{R}[t]$, lists $L_l$, $L_r$ of linear motion polynomials. Initially, $Q = 1$, $L_l = []$, $L_r = []$.

**Output:** $Q$, $L_l$, $L_r$ such that with $L_l + L_r = [L_1, L_2, \dots, L_n]$ we have $QM = L_1 L_2 \cdots L_n$.

**If** $P$ has no real factors **Then**<br>
  **Return** $Q$, $L_l$, $L_r +$ `GFactor`$(M)$.<br>
**End If**<br>
Let $R_1$ be the GRPF of $P$, i.e., $P = R_1 T$, $\deg P = \beta$.<br>
Let $\alpha := \deg(\gcd(P, \overline{P}, D\overline{D})) = \deg(\gcd(R_1, D\overline{D}))$. ▷ $\operatorname{comp}(M) = (\alpha, \beta, \gamma)$.<br>
**If** $\gcd(R_1, D\overline{D}) = 1$ $(\alpha = 0)$ **Then**<br>
  **If** $\gcd(R_1, T\overline{T}) = 1$ **Then**<br>
    **If** $T = 1$, i.e., $P$ is real **Then**<br>
      Let $P_1$ be a quadratic real divisor of $P$, i.e., $P = P_1 P'$.<br>
      Compute quaternion roots $h_r$, $h_l$ of $P_1$ such that<br>
      $h_l \ne \overline{h_r}$, $D(t - \overline{h_r}) = (t - h_l)D'$, ▷ (Lemma 4.1, Lemma 4.8)<br>
      $(t - h_l)D'(t - h_r) = DP_1$.<br>
      $Q \leftarrow QP_1$, $L_l \leftarrow L_l + [t - h_l]$, $L_r \leftarrow [t - h_r] + L_r$,<br>
      $M' \leftarrow P'(t - \overline{h_l})(t - \overline{h_r}) + \epsilon D'$. ▷ $\operatorname{comp}(M') = (0, \beta - 2, \gamma)$.<br>
      **Return** `FactorAll`$(M', Q, L_l, L_r)$<br>
    **Else**<br>
      Let $P_1$ be a quadratic real divisor of $T\overline{T}$.<br>
      Compute a common zero $h$ of $P_1$ and $M$ such that<br>
      $P_1 = (t - \overline{h})(t - h)$, $M = M'(t - h)$. ▷ $\operatorname{comp}(M') = (0, \beta, \gamma - 1)$.<br>
      **Return** `FactorAll`$(M', Q, L_l, [t - h] + L_r)$<br>
    **End If**<br>
  **Else**<br>
    Let $P_1$ be a quadratic real divisor of $\gcd(R_1, T\overline{T})$, i.e., $P = P'P_1$.<br>
    Compute quaternions roots $h_r$, $h_l$ of $P_1$ such that<br>
    $P_1(h_r) = 0$, $T(h_r) \ne 0$, $T(h_l) \ne 0$, ▷ (Lemma 4.1, Lemma 4.8)<br>
    $DP_1 = D(t - \overline{h_r})(t - h_r) = (t - h_l)D'(t - h_r)$. ▷ (Lemma 4.1, Lemma 4.8)<br>
    $Q \leftarrow QP_1$, $L_l \leftarrow L_l + [t - h_l]$, $L_r \leftarrow [t - h_r] + L_r$,<br>
    $M' \leftarrow (t - \overline{h_l})P'(t - \overline{h_r}) + \epsilon D'$. ▷ $\operatorname{comp}(M') = (0, \beta - 2, \gamma)$.<br>
    **Return** `FactorAll`$(M', Q, L_l, L_r)$<br>
  **End If**<br>
**Else** $(\alpha \ge 2)$<br>
  Let $P_1$ be a quadratic real divisor of $\gcd(R_1, D\overline{D})$.

33: Compute quaternion roots $h_r$, $h_l$ of $P_1$ such that ▷ (Lemma 4.1)
34: $D = (t - h_l)D_l = D_r(t - h_r)$ and $P = (t - h_l)P_l = P_r(t - h_r)$ ▷ (Lemma 4.1)
35: **If** $\deg \mathrm{GRPF}(P_l) \leq \deg \mathrm{GRPF}(P_r)$ **Then**
36: $L_l \leftarrow L_l + [t - h_l]$ , $M' \leftarrow P_l + \epsilon D_l$. ▷ $\mathrm{comp}(M') = (\alpha - 2, \beta - 2, \gamma - 1)$.
37: **Return** `FactorAll`$(M', Q, L_l, L_r)$
38: **Else**
39: $L_r \leftarrow [t - h_r] + L_r$ $M' \leftarrow P_r + \epsilon D_r$. ▷ $\mathrm{comp}(M') = (\alpha - 2, \beta, \gamma - 1)$
40: or $\mathrm{comp}(M') = (\alpha - 2, \beta - 2, \gamma - 1)$.
41: **Return** `FactorAll`$(M', Q, L_l, L_r)$
42: **End If**
43: **End If**

Here are a few remarks on Algorithm 8.

- In Algorithm 8, we mainly treat the case where the primal part $P$ of the motion polynomial $M = P + \epsilon D$ has a non-constant real factor $R_1 = \mathrm{GRPF}(P)$. Otherwise, we just resort to factorization of generic motion polynomials (Algorithm 5).

- The *complexity* of a monic bounded motion polynomial $M = P + \epsilon D \in \mathbb{DH}[t]$ in Algorithm 8 is a triple of integers

$$
\begin{aligned}
\mathrm{comp}(M) &:= (\alpha, \beta, \gamma), \\
\alpha &:= \deg(\gcd(P, \overline{P}, D\overline{D})), \\
\beta &:= \deg(\gcd(P, \overline{P})), \\
\gamma &:= \deg(P),
\end{aligned}
$$

  where $\deg(a)$ is the degree of the polynomial $a$ and $\gcd(a, b) \in \mathbb{R}[t]$ is the greatest real common factor of polynomials $a$ and $b$. With this definition, $\gcd(a, \overline{a})$ is the greatest real polynomial factor of $a$. In each step of the recursive Algorithm 8, we try to construct $M'$ such that $\mathrm{comp}(M') < \mathrm{comp}(M)$ with lexicographic order, e.g., $(4, 2, 5) < (4, 4, 3), (4, 2, 2) < (4, 2, 3)$. Then we recursively call `FactorAll` with $M'$ as argument. As soon as $\alpha = \beta = \gamma = 0$, Algorithm 8 terminates.

- The computation of $h_l$ and $h_r$ in Lines 33–34 is based on Lemma 4.1 and [32, Theorem 3.2]. One of this theorem's statements is that the set of quaternion roots of the irreducible quadratic polynomial $Q = t^2 + bt + c \in \mathbb{R}[t]$ is

$$
\left\{ \frac{1}{2}\Big( -b + \sqrt{4c - b^2}(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})\Big) \mid (x_1, x_1, x_3) \in S^2 \right\} \tag{4.3}
$$

  where $S^2$ is the unit 2-sphere in $\mathbb{R}^3$. In particular, for every unit vector $(x_1, x_2, x_3) \in S^2$, there is a quaternion root $q$ whose vector part is proportional to $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. Also note that $Q = (t-h)(t-\overline{h})$ if $h$ is a quaternion root of $Q$. In the algorithm, we can pick an arbitrary zero $h_r$ of $P_1$ and compute $D_r$ by polynomial division. Then we compute $\overline{h_l}$ as zero of the remainder polynomial $\tilde{R}$ in the division $\overline{D} = \tilde{Q}\tilde{M} + \tilde{R}$ with $\tilde{M} = (t-h_r)(t-\overline{h_r})$, as in one iteration of Algorithm 5, and $\overline{D_l}$ again by polynomial division.

- The computation of quaternions $h_l$ and $h_r$ in Lines 10–12 and Lines 24–26 of Algorithm 8 is again based on Lemma 4.1 but also on Lemma 4.8 below. Consider, for example, the situation in Lines 10–12. We may prescribe $h_r$ arbitrarily as a root of $P_1$, see (4.3). Then we use polynomial division (over $\mathbb{DH}$) to find $\tilde{Q}, \tilde{R} \in \mathbb{H}[t]$ such that $(t-h_r)\overline{D} = \tilde{Q}P_1 + \tilde{R}$ and compute $\overline{h_l}$ as unique zero of the linear remainder polynomial $\tilde{R}$. Using polynomial division once more, we then find $\overline{D'}$ such that $(t-h_r)\overline{D} = \overline{D'}(t-\overline{h_l})$.

**Lemma 4.8.** *Let $Q \in \mathbb{R}[t]$ be a quadratic polynomial that is irreducible over $\mathbb{R}$, $D \in \mathbb{H}[t]$ a polynomial with $\gcd(D\overline{D}, Q) = 1$ and $O$ the set of quaternion roots of $Q$. Then the map $f_{Q,D}\colon O \to O$, $h_l \mapsto h_r$ with $h_r$ being the common root of $(t-h_l)D$ and $Q$ is a well-defined bijection. Moreover, $f_{Q,D}(h) \neq h$ for all $h \in O$.*

*Proof.* Our proof is based on results of [29] that state that the quaternion roots of a polynomial $P \in \mathbb{H}[t]$ are also roots of the quadratic factors of $P\overline{P}$. Moreover, $h$ is a root of $P$ if and only if $t-h$ is a right factor of $P$ [29, Lemma 2].

By (4.3), the set $O$ is not empty. The norm polynomial of $(t-h_l)D$ has the quadratic factor $Q$. Hence, there exists a quaternion root $h_r \in O$ of $(t-h_l)D$. This root is unique because of $\gcd(D\overline{D}, Q) = 1$ and the map $f_{Q,D}$ is well-defined.

If $f_{Q,D}(h) = h$ for some $h \in O$, there exists $D' \in \mathbb{H}[t]$ with $D = (t-\overline{h})D'(t-h)$ and we get a contradiction to $\gcd(D\overline{D}, Q) = 1$:

$$D\overline{D} = (t-\overline{h})D'(t-h)(t-\overline{h})\overline{D'}(t-h) = Q(t-\overline{h})D'\overline{D'}(t-h).$$

By a linear parameter transformation $t \mapsto at + b$ with $a, b \in \mathbb{R}$ we can always achieve that $Q$ is a real multiple of $t^2+1$. Hence, it is no loss of generality to assume $Q = t^2 + 1$ when proving bijectivity of $f_{Q,D}$. Using polynomial division we find $K \in \mathbb{H}[t]$ and $a, b \in \mathbb{R}$ with $D = K(t^2+1) + at + b$. Then we have

$$\begin{aligned}(t-h_l)D &= (t-h_l)K(t^2+1) + (t-h_l)(at+b)\\ &= ((t-h_l)K + a)(t^2+1) + (b-h_la)t - a - h_1b.\end{aligned}$$

As already argued, there is $h_r = f_D(h_l) \in O$ such that

$$(b-h_la)h_r - a - h_lb = 0. \tag{4.4}$$

If there is $h_l' \neq h_l$ with $f_D(h_l') = h_r$ then we also have

$$(b - h_l'a)h_r - a - h_l'b = 0. \tag{4.5}$$

Subtracting Equations (4.4) and (4.5) yields

$$(h_l' - h_l)ah_r + (h_l' - h_l)b = 0. \tag{4.6}$$

As $h_l' - h_l \neq 0$, we have $ah_r + b = 0$ and this implies $D = K(t^2 + 1) + a(t - h_r)$. But then $\deg \gcd(D\overline{D}, t^2 + 1) > 0$ would contradict our assumptions. Hence $f_D$ is injective. To prove surjectivity, observe that for any $h_r \in O$, there is $h_l$ such that $(t - h_r)\overline{D} = \overline{D'}(t - h_l)$ by injectivity of $f_{\overline{D}}$. But then we have $f_D(h_l) = h_r$. □

The termination of Algorithm 8 is guaranteed by the following theorem.

**Theorem 4.9.** *Algorithm 8 terminates.*

*Proof.* The termination of the Algorithm 8 is based on the reduction of the complexity comp$(M)$. As one can see from the comments in the Algorithm 8, after each recursive step comp$(M')$ of the new motion polynomial $M'$ strictly decreases. Furthermore, Lines 17–20 can not happen continually because of $\beta \leq \gamma$ in each motion polynomial. Then in finitely many steps we can reduce $\alpha$ and $\beta$ to zero. After this the algorithm will terminate in one step using Algorithm 5. □

### 4.4.1 A Comprehensive Example

Now we illustrate Algorithm 8 by a comprehensive example where we really enter each sub-branch once. We wish to factor the motion polynomial $M = P + \epsilon D$ where

$$\begin{aligned} P &= (t^2 + 2t + 2)(t^2 + 1)^2, \\ D &= -(t^2 + 2t + 2)\mathbf{i} + (t^5 + t^4 + 2t^3 + t^2 - t - 1)\mathbf{j} + (t^4 + t^2 - 2t - 1)\mathbf{k}. \end{aligned} \tag{4.7}$$

**First iteration:** The input to Algorithm 8 is $M^{(1)} = P^{(1)} + \epsilon Q^{(1)}$ where $P^{(1)} = P$ and $D^{(1)} = D$ from (4.7). We compute

$$R_1 = \mathrm{GRPF}(P^{(1)}) = P^{(1)}, \quad T = 1, \quad \mathrm{comp}(M^{(1)}) = (2, 6, 6).$$

Thus, we have to use the branch in Lines 32–41 of Algorithm 8:

$$\begin{aligned} 2h_l &= -1 - \mathbf{i}, & h_r &= -1 + \mathbf{i}, \\ P_l &= (t^2 + 1)^2(t - \mathbf{i} + 1), & P_r &= (t^2 + 1)^2(t + \mathbf{i} + 1), \end{aligned}$$

$$\begin{aligned} D_l &= \mathbf{j}t^4 + 2\mathbf{j}t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t - 1 - \mathbf{i} - \mathbf{j}, \\ D_r &= \mathbf{j}t^4 + 2\mathbf{j}t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + 1 - \mathbf{i} - \mathbf{j}. \end{aligned} \tag{4.8}$$

Note on computation:

- We compute one quaternion root $h_l$ of $R_1$ by (4.3). We then have $R_1 = (t-h_l)(t-\overline{h_l})$ and use polynomial division to find $Q$ and $R$ with $D = Q(t-h_l)(t-\overline{h_l}) + R$. The dual quaternion $h_r$ is the zero of the linear remainder polynomial $R$.

- The polynomials $P_l$ and $P_r$ are also computed by polynomial division from

  $$P^{(1)} = P_r(t-h_r) \quad \text{and} \quad \overline{P^{(1)}} = \overline{P_l}(t-\overline{h_l}).$$

  A similar computation yields $D_l$ and $D_r$.

The updated values of $Q$, $L_l$ and $L_r$ are $Q = 1$, $L_l = [l_1]$, $L_r = [\,]$ where $l_1 = t+1+\mathbf{i}$.

**Second iteration:** The input to Algorithm 8 is $M^{(2)} = P^{(2)}+\epsilon Q^{(2)}$ where $P^{(2)} = P_l$, $D^{(2)} = D_l$ are taken from (4.8) and (4.8). We compute

$$R_1 = \mathrm{GRPF}(P^{(2)}) = (t^2+1)^2, \quad T = t - \mathbf{i} + 1, \quad \mathrm{comp}(M^{(2)}) = (0,4,5).$$

Because of $\gcd(R_1, D^{(2)}\overline{D^{(2)}}) = \gcd(R1, T\overline{T}) = 1$ and $T \neq 1$, we have to use the branch in Lines 17–20 of Algorithm 8. Using (4.3) and polynomial division, we find

$$\begin{aligned} P_1 &= t^2 + 2t + 2, \\ h &= -1 + \mathbf{i} - \tfrac{39}{25}\epsilon\mathbf{j} - \tfrac{2}{25}\epsilon\mathbf{k}, \\ M' &= t^4 - \tfrac{2}{25}\epsilon(7\mathbf{j}+\mathbf{k})t^3 + (2 + \tfrac{12}{25}\mathbf{j}\epsilon + \tfrac{16}{25}\epsilon\mathbf{k})t^2 \\ &\quad - \tfrac{8}{25}\epsilon(3\mathbf{j}+4\mathbf{k})t + 1 - \epsilon(\mathbf{i} + \tfrac{33}{25}\mathbf{j} - \tfrac{31}{25}\mathbf{k}). \end{aligned} \tag{4.9}$$

The updated values of $Q$, $L_l$, and $L_r$ are $Q = 1$, $L_l = [l_1]$, $L_r = [t-h]$ where $r_3 = t + h$ and $h$ is as in (4.9).

**Third iteration:** The input to Algorithm 8 is $M^{(3)} = P^{(3)}+\epsilon Q^{(3)}$ where $M^{(3)} = M'$ is taken from (4.9). We compute

$$R_1 = \mathrm{GRPF}(P^{(3)}) = (t^2+1)^2, \quad T = 1, \quad \mathrm{comp}(M^{(3)}) = (0,4,4).$$

Because of $\gcd(R_1, D^{(2)}\overline{D^{(2)}}) = 1$ and $T = 1$, we have to use the branch in Lines 9–15 of Algorithm 8. Similar to the first iteration we compute

$$\begin{aligned} P_1 = t^2 + 1, \quad P' = t^2 + 1, \quad h_l = \tfrac{3}{7}\mathbf{i} + \tfrac{6}{7}\mathbf{j} - \tfrac{2}{7}\mathbf{k}, \quad h_r = -\mathbf{i}, \\ D' = (-\tfrac{14}{25}\mathbf{j} - \tfrac{2}{25}\mathbf{k})t^3 + (\tfrac{16}{35} - \tfrac{8}{35}\mathbf{i} + \tfrac{104}{175}\mathbf{j} - \tfrac{4}{25}\mathbf{k})t^2 \\ -(\tfrac{16}{35} - \tfrac{8}{35}\mathbf{i} + \tfrac{188}{175}\mathbf{j} + \tfrac{12}{25}\mathbf{k})t + \tfrac{24}{35} - \tfrac{67}{35}\mathbf{i} - \tfrac{51}{175}\mathbf{j} - \tfrac{43}{175}\mathbf{k}. \end{aligned} \tag{4.10}$$

The updated values of $Q$, $L_l$, and $L_r$ are $Q = t^2 + 1$, $L_l = [l_1, l_2]$, $L_r = [r_2, r_3]$ where

$$l_2 = t - \tfrac{3}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{2}{7}\mathbf{k}, \quad r_2 = t + \mathbf{i}.$$

**Fourth iteration:** The input to Algorithm 8 is $M^{(4)} = P^{(4)} + \epsilon D^{(4)}$ where $P^{(4)} = P'(t - \overline{h_l})(t - \overline{h_r})$ and $D^{(4)} = D'$ are taken from (4.10). We compute

$$\begin{aligned} R_1 &= \mathrm{GRPF}(P^{(4)}) = t^2 + 1, \\ T &= t^2 - (\tfrac{4}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{2}{7}\mathbf{k})t + \tfrac{3}{7} + \tfrac{2}{7}\mathbf{j} + \tfrac{6}{7}\mathbf{k}, \\ \mathrm{comp}(M^{(4)}) &= (0, 2, 4). \end{aligned}$$

Because of $\gcd(R_1, D^{(4)}\overline{D^{(4)}}) = 1$ and $\gcd(R_1, T\overline{T}) = t^2 + 1$, we have to use the branch in Lines 23–29 of Algorithm 8. Similar to the first iteration we compute

$$\begin{gathered} P_1 = t^2 + 1, \quad P' = t^2 - (\tfrac{4}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{2}{7}\mathbf{k})t + \tfrac{3}{7} + \tfrac{2}{7}\mathbf{j} + \tfrac{6}{7}\mathbf{k}, \\ h_l = -\tfrac{158}{483}\mathbf{i} - \tfrac{218}{483}\mathbf{j} - \tfrac{401}{483}\mathbf{k} \quad h_r = -\mathbf{k}, \\ D' = (-\tfrac{14}{25}\mathbf{j} - \tfrac{2}{25}\mathbf{k})t^3 + (\tfrac{4}{69} - \tfrac{56}{575}\mathbf{i} + \tfrac{196}{345}\mathbf{j} + \tfrac{8}{345}\mathbf{k})t^2 \\ -(\tfrac{28}{75} - \tfrac{44}{575}\mathbf{i} + \tfrac{428}{345}\mathbf{j} + \tfrac{2096}{1725}\mathbf{k})t - \tfrac{2308}{1725} - \tfrac{2069}{1725}\mathbf{i} - \tfrac{613}{1725}\mathbf{j} + \tfrac{553}{575}\mathbf{k}. \end{gathered} \tag{4.11}$$

The updated values of $Q$, $L_l$, and $L_r$ are $Q = (t^2+1)^2$, $L_l = [l_1, l_2, l_3]$, $L_r = [r_3, r_2, r_1]$ where

$$l_3 = t + \tfrac{158}{483}\mathbf{i} + \tfrac{218}{483}\mathbf{j} + \tfrac{401}{483}\mathbf{k}, \quad r_1 = t + \mathbf{k}.$$

**Fifth iteration:** The input to Algorithm 8 is $M^{(5)} = P^{(5)} + \epsilon D^{(5)}$ where $P^{(5)} = (t - \overline{h_l})P'(t - \overline{h_r})$ and $D^{(5)} = D'$ are taken from (4.11). Because of $R_1 = 1$, we have to use Line 2 of Algorithm 8 and can compute a factorization of $M^{(5)}$ by means of Algorithm 5. Because of $M^{(5)}\overline{M^{(5)}} = (t^2 + 1)^4$, the factorization is unique. We find $M^{(5)} = f_1 f_2 f_3 f_4$ where

$$\begin{gathered} f_1 = t - \tfrac{158}{483}\mathbf{i} - \tfrac{218}{483}\mathbf{j} - \tfrac{401}{483}\mathbf{k} - \tfrac{29}{280}\epsilon\mathbf{i} - \tfrac{37}{56}\mathbf{j}\epsilon + \tfrac{2}{5}\epsilon\mathbf{k}, \\ f_2 = t + \tfrac{3}{7}\mathbf{i} + \tfrac{6}{7}\mathbf{j} - \tfrac{2}{7}\mathbf{k} + \tfrac{43}{35}\epsilon\mathbf{i} - \tfrac{48}{175}\mathbf{j}\epsilon + \tfrac{51}{50}\epsilon\mathbf{k}, \\ f_3 = t - \mathbf{i} - \tfrac{3}{2}\epsilon\mathbf{k}, \quad f_4 = t - \mathbf{k} - \tfrac{9}{8}\epsilon\mathbf{i} + \tfrac{3}{8}\epsilon\mathbf{j}. \end{gathered}$$

Algorithm 8 terminates and the polynomial $QM$ is the product of the ten linear factors $l_1$, $l_2$, $l_3$, $f_1$, $f_2$, $f_3$, $f_4$, $r_1$, $r_2$, $r_3$.

### 4.4.2 Degree bound of $Q$

An upper bound on the degree of $Q$ as returned by Algorithm 8 can be read from the following theorem. This degree bound is already know to be optimal. It is attained by certain planar motions [22].

**Theorem 4.10.** *The degree of $Q$ as returned by Algorithm 8 is less or equal to the degree of the* GRPF *of the primal part of $M$.*

*Proof.* The proof follows from a careful inspection of Algorithm 8. The increase of the degree of $Q$ happen either in Lines 12–13 or lines 26–27. Furthermore, the increase of the degree of $Q$ and the decrease of the degree of the GRPF are equal at these places. □

We illustrate Theorem 4.10 by one further example. One achieves the upper bound of Theorem 4.10, the other does not.

**Example 4.11.** *The first example is the general Darboux motion considered in [49]. Let $M = \xi P - \mathbf{i}\eta\epsilon P \in \mathbb{DH}[t]$ with*

$$\xi = t^2 + 1, \quad \eta = \frac{5}{2}t - \frac{3}{4}, \quad P = t - h \quad \text{and} \quad h = \frac{7}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}.$$

*As seen in [49], this give us the factorization $M = Q_1Q_2Q_3$, where*

$$\begin{aligned} Q_1 &= t - \frac{7}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} - \frac{5}{4}\epsilon\mathbf{i} + \frac{43}{64}\epsilon\mathbf{j} - \frac{97}{64}\epsilon\mathbf{k}, \\ Q_2 &= t + \frac{7}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}, \\ Q_3 &= t - \frac{7}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} - \frac{4}{9}\mathbf{k} - \frac{5}{4}\epsilon\mathbf{i} - \frac{43}{64}\epsilon\mathbf{j} + \frac{97}{64}\epsilon\mathbf{k}. \end{aligned}$$

*Here, no multiplication with a real polynomial is necessary.*

**Example 4.12.** *The second example is the vertical Darboux motion which is avoided in [49]. Let $M = \xi P - \mathbf{i}\eta\epsilon P \in \mathbb{DH}[t]$ with*

$$\xi = t^2 + 1, \quad \eta = \frac{5}{2}t - \frac{3}{4}, \quad P = t - \mathbf{i}.$$

*As seen in [49], no factorization of the shape $M = Q_1Q_2Q_3$ with linear motion polynomials $Q_1$, $Q_2$, $Q_3$ exists. However, we can find a factorization by multiplying with a real polynomial whose degree equals the degree of $\xi$, the greatest real polynomial factor of the primal part of $M$. We have $(t^2 + 1)M = Q_7Q_6^2Q_5Q_4$, where*

$$Q_7 = t - \mathbf{j} - \frac{3}{4}\epsilon\mathbf{k}, \quad Q_6 = t + \mathbf{j} - \frac{5}{4}\epsilon\mathbf{i} + \frac{3}{8}\epsilon\mathbf{k}, \quad Q_5 = t - \mathbf{j}, \quad Q_4 = P = t - \mathbf{i}.$$

It is worth to mentioning that the degree could be smaller if we are non only allowed to multiplying with real polynomials [47]. For this dissertation, we only consider multiplying with real polynomials.

### 4.4.3 Factorizations in Planar Motion Groups

Algorithm 8 can produce non-planar factorizations for planar motion polynomials. This is an interesting feature but may not always be desirable. If one wishes to find a factorization $(t - h_1) \cdots, (t - h_n)$ of a motion polynomial in a planar motion group, say $\langle 1, \mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k} \rangle$, with rotation quaternions $h_1, \ldots, h_n$ in that group, we have to pick suitable left and right factors $h_l$ and $h_r$ in Algorithm 8.

Note that for a planar motion in the subgroup $\langle 1, \mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k} \rangle$, the primal part and the dual part of a motion have a certain commutativity property. If $P$ is a polynomial with coefficients in $\langle 1, \mathbf{i} \rangle$ and $D$ is a polynomial with coefficients in $\langle \epsilon\mathbf{j}, \epsilon\mathbf{k} \rangle$, then $PD = D\overline{P}$, e.g., $(t-\mathbf{i})\epsilon\mathbf{j} = \epsilon\mathbf{j}(t+\mathbf{i})$ or $(t-\mathbf{i})\epsilon\mathbf{k} = \epsilon\mathbf{k}(t+\mathbf{i})$. This allows to transform right factors into left factors and vice versa. Moreover, from Equation (4.3) it follows that there are exactly two roots of a real irreducible quadratic polynomial $Q$ in the planar motion subgroup. We have, for example, $Q = t^2 + 1 = (t - \mathbf{i})(t + \mathbf{i}) = (t+\mathbf{i})(t-\mathbf{i})$. Thus, whenever we compute a quaternion root of a quadratic irreducible polynomial in Algorithm 8, we should select a solution in the planar motion group and whenever we transfer a left factor $h_l$ to a right factor $h_r$ we should do it in such a way that $h_r = \overline{h_l}$. This ensures that Algorithm 8 really returns a planar factorization.

# Chapter 5

# Construct 6R Linkages of Minimum Bonds

In the paper [30], the authors constructed a new 6R linkage by using the factorization of a cubic motion polynomial. It has bond diagram of Figure 3.1(a) which is one of the simplest bond diagrams. Also among the simplest bond diagram is Figure 3.1(b) which is known as the Waldrons double Bennett hybrid (see [21] 4.2.5). There is no other 6R linkages with bond diagrams of only three bonds. Using [31, Theorem 23], one can find the reason as an exercise. We consider diagrams with four bonds, e.g. Figure 3.1(c), (d). We will give all the possibilities of the bond diagrams with three or four bonds. Using the factorization of motion polynomials, we can construct concrete examples for each bond diagram.

**Structure of the chapter** The remaining part of the chapter is set up as follows. In Section 5.1, we give the classification on the possibilities of the bond diagrams with 3 or 4 bonds. This classification is based on an interesting lemma (Lemma 5.6) which is used for telling us the opposite connection numbers. In Section 5.2, we give the construction for all mobile 6R linkages with 3 bonds. There are two families. One is the cube linkage, and the other one is the Double Bennett 6R linkage. Section 5.3 focuses on the 4 bonds case, where we give constructions for each bond diagrams.

## 5.1 Minimum Bonds

We first show the minimum number of bonds for the bond diagram of a 6R linkage. As each joint has at least one bond connected, the minimum number of bonds is 3 for 6R linkages. There are two cases which have this number of bonds.

In Figure 5.1, we show two bond diagrams with 3 bonds.

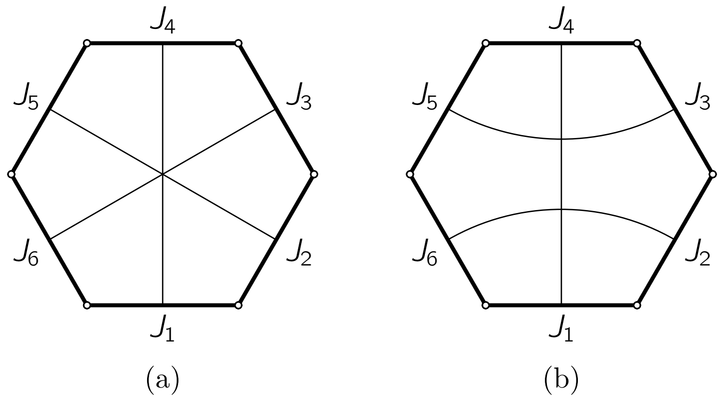



Figure 5.1: Bond diagrams for the cube linkage (a), the Waldrons double Bennett hybrid (b)

The next step is to consider 6R linkages with four bonds. Concerning this second minimum number of bonds, we show that there are 9 cases of bond diagrams as in Figure 5.2. For proving this, we need some lemmas.

First, let us recall a lemma in [31][Lemma 6].

**Lemma 5.1.** *Let $h_1, \ldots, h_6$ be six half-turns such that $L := L_{1,2,3} = L_{4,5,6}$ and $dim(L) = 6$. Then $h_1 = \pm h_4$ and $h_3 = \pm h_6$.*

**Lemma 5.2.** *For four lines $[h_1, h_2, h_3, h_4]$, where $h_i^2 = -1$ and no two lines are equal, if $l_{123} = l_{324} = 6$, then $l_{124} = 6$.*

*Proof.* Geometric proof. As $b_{12}^2 = b_{23}^2 = b_{42}^2$ and $o_{123} = o_{423} = 0$, then we have $b_{12}^2 = b_{42}^2$ and $o_{124} = 0$ which mean $l_{124} = 6$. □

**Lemma 5.3.** *For three lines $[h_1, h_2, h_3]$, where $h_i^2 = -1$, $h_1 \neq \pm h_3$ and $l_{123} = 6$, then we have $l_{132} = 8$.*

*Proof.* If $h_1 \neq \pm h_3$ and $l_{123} = 6$, then $h_3 \notin L_{12}$. Otherwise [31], it is either compatible with $h_1$ or $h_2$. This is impossible. Then we have $x + yh_2 + zh_3 = h_2h_3$ with unique $x, y, z \in L_1$ by [31]. Similarly, we have $x' + y'h_2 + z'h_3 = h_1h_2h_3$ with unique $x', y', z' \in L_1$ by multiplying $h_1$ from the left. Then the vectors $1, h_1, h_2, h_3, h_1h_2, h_1h_3$ are linearly independent (by the dimension) and $L_{1,2,3} \subset L_{1,3,2} = L$. We have $l_{132} \geq 6$. Using Lemma 5.1, assuming $l_{132} = 6$, we get $h_2 = \pm h_3$ which contradicts $l_{123} = 6$. As $l_{132}$ is an even number and $6 \leq l_{132} \leq 8$, we have $l_{132} = 8$. □

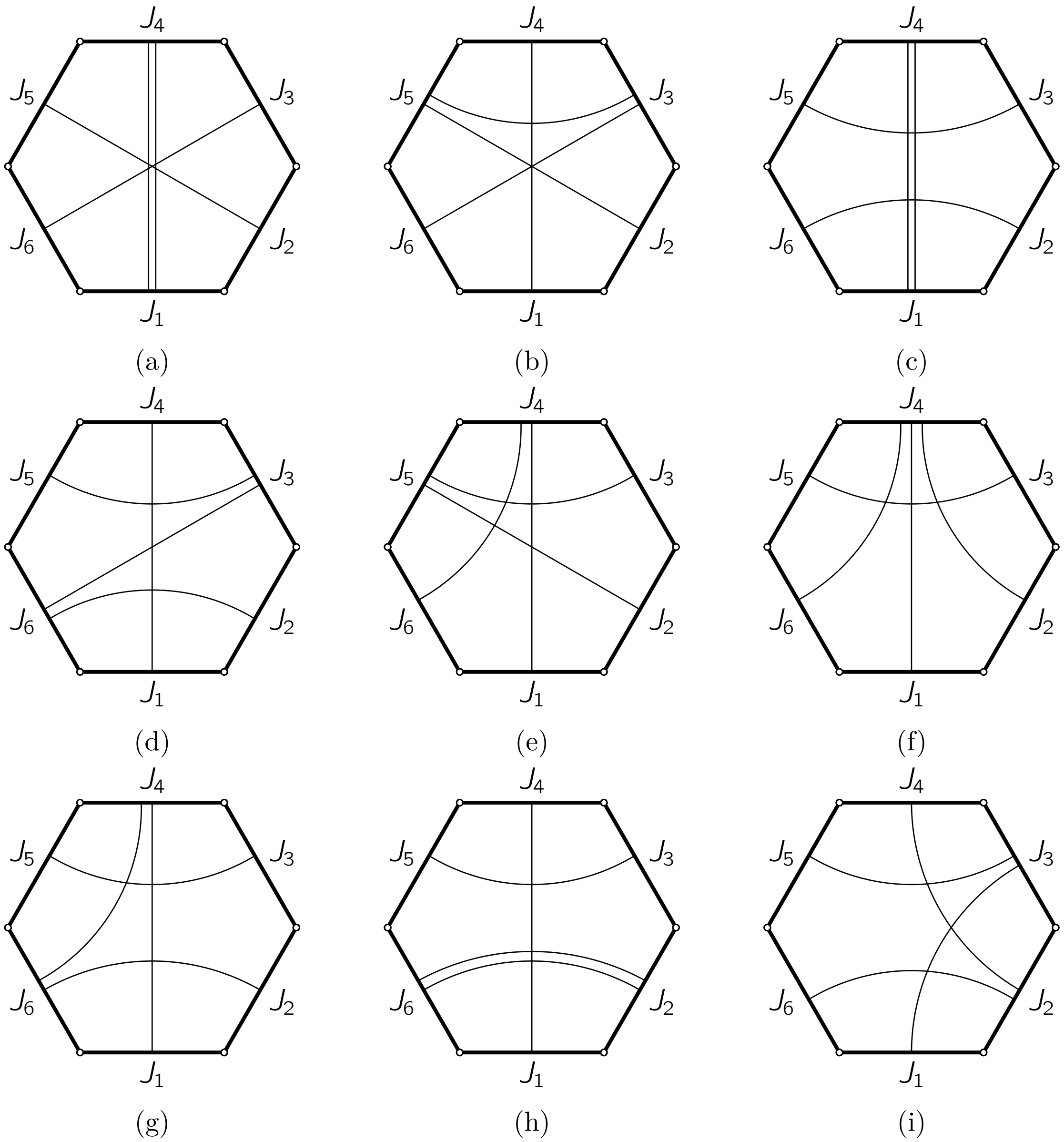



Figure 5.2: Bond diagrams of 6R linkages with four bonds

**Lemma 5.4.** *For three quaternions* $[p_1, p_2, p_3]$, *where* $p_i^2 = -1$. *If*

$$(\mathrm{i} - p_1)(\mathrm{i} - p_2)(\mathrm{i} - p_3) = 0$$

*where* i *is the imaginary unit in the field of complex numbers* $\mathbb{C}$, *then we have* $p_1 \neq -p_2$ *or* $p_2 \neq -p_3$.

*Proof.* Let $m := (t-p_1)(t-p_2)(t-p_3)$. As we have $(\mathrm{i}-p_1)(\mathrm{i}-p_2)(\mathrm{i}-p_3) = 0$, we have $t^2+1$ divide $m$. This means that we can write $m = (t^2+1)(t-q) = (t-q)(t^2+1)$, where $q \in \mathbb{H}$. Assume that $p_1 \neq -p_2$, we have $m_1 := m(t+p_3)/(t^2+1) =$

$(t - p_1)(t - p_2) = (t - q)(t + p_3)$. Then $m_1$ is a generic motion polynomial. The root of $m_1$ is unique defined because the norm of $m_1$ is $(t^2 + 1)^2$. Then we have $p_2 = -p_3$. □

**Lemma 5.5.** *For three lines $[h_1, h_2, h_3]$, where $h_i^2 = -1$ and $h_1 \neq \pm h_3$, if $l_{123} = 6$, then w.o.l.g we have $\alpha \neq \pm \mathrm{i}$ s.t.*

$$(\mathrm{i} - h_1)(\alpha - h_2)(\mathrm{i} - h_3) = 0$$

*where* i *is the imaginary unit in the field of complex numbers $\mathbb{C}$ and $\alpha \in \mathbb{C}$.*

*Proof.* As $l_{123} = 6$, then there exist three complex numbers $c_1, c_2, c_3$ such that

$$(c_1 - h_1)(c_2 - h_2)(c_3 - h_3) = 0.$$

If $c_1 \neq \pm \mathrm{i}$ (which means that $c_1 - h_1$ is invertible) , then we have $(c_2 - h_2)(c_3 - h_3) = 0$ after multiplying the $c_1 + h_1$ from the right. This is impossible, because it means that $h_2 = \pm h_3$ which contradicts our assumption of $l_{123} = 6$. Therefore, we have

$$(\mathrm{i} - h_1)(\alpha - h_2)(\mathrm{i} - h_3) = 0.$$

Assuming that $\alpha = \mathrm{i}$, then we have

$$(\mathrm{i} - h_1)(\mathrm{i} - h_2)(\mathrm{i} - h_3) = 0.$$

By Lemma 5.4, we have that the two lines $h_1$ and $h_2$ are parallel or the two lines $h_2$ and $h_3$ are parallel. This is impossible because of $l_{123} = 6$. □

**Lemma 5.6.** *For an nR (6R/5R) linkage $L = [h_1, h_2, ..., h_n]$, where $h_i^2 = -1$ ,$h_i \neq \pm h_{i+1}$ and $h_i \neq \pm h_{i+2}$, if $l_{123} = l_{234} = 6$, then there is no bond which connects $h_1$ and $h_4$.*

*Proof.* First, we assume there is at least one bond which connects the first and the fourth joints. Then we try to find a contradiction. Let $\beta = (\mathrm{i}, b_2, b_3, \mathrm{i}, b_5, b_6)$ be the bond which connects the first and the fourth joints. Furthermore, we assume that it fulfills the equation

$$(\mathrm{i} - h_1)(b_2 - h_2)(b_3 - h_3)(\mathrm{i} - h_4) = 0. \tag{5.1}$$

(If not, we just chose the complex conjugate solutions such that it fulfills Equation (5.1).) As $l_{123} = l_{234} = 6$, there exist $c_2, c_3 \in \mathbb{C}$ (if not, we just chose the complex conjugate solutions) such that

$$(\mathrm{i} - h_1)(c_2 - h_2)(\mathrm{i} - h_3) = 0, \quad (\mathrm{i} + h_4)(c_3 + h_3)(\mathrm{i} + h_2) = 0, \tag{5.2}$$

by Lemma 5.5. Now we reformulate the equation (5.1) as follows:

$$\begin{aligned} 0 &= (\mathrm{i}-h_1)(b_2-h_2)(b_3-h_3)(\mathrm{i}-h_4) \\ &= (\mathrm{i}-h_1)(c_2-h_2)(\mathrm{i}-h_3)(\mathrm{i}-h_4) + (\mathrm{i}-h_1)u(b_3-h_3)(\mathrm{i}-h_4) \\ &\quad +(\mathrm{i}-h_1)(c_2-h_2)v(\mathrm{i}-h_4) \\ &= (\mathrm{i}-h_1)u(b_3-h_3)(\mathrm{i}-h_4) + (\mathrm{i}-h_1)(c_2-h_2)v(\mathrm{i}-h_4) \\ &= (\mathrm{i}-h_1)(u(b_3-h_3)+v(c_2-h_2))(\mathrm{i}-h_4). \end{aligned}$$

where $u = b_2 - c_2$ and $v = b_3 - \mathrm{i}$, they are both complex numbers. We make a case distinction as follows:

Case I, $u = 0$ and $v \neq 0$. Then the equation (5.1) can be reduced to $(\mathrm{i}-h_1)(c_2-h_2)(\mathrm{i}-h_4) = 0$. This means that $l_{124} = 6$. As we also have $l_{123} = 6$, we can get $l_{423} = 6$ by Lemma 5.2. Furthermore, we have $l_{432} = l_{234} = 6$ by assumption. From Lemma 5.3, we get contradiction.

Case II, $v = 0$ and $u \neq 0$. The proof is similar to Case I.

Case III, $v \neq 0$ and $u \neq 0$. Then the equation (5.1) can be reduced to $(\mathrm{i}-h_1)(u(b_3-h_3)+v(c_2-h_2))(\mathrm{i}-h_4) = 0$. This means that $(\mathrm{i}-h_1)(u(b_3-h_3)+v(c_2-h_2))$ is in the left ideal $\mathbb{DH}(\mathrm{i}+h_4)$. Using equations (5.2), we have

$$(\mathrm{i}-h_1)(u(b_3-h_3)+v(c_2-h_2))(c_3+h_3)(\mathrm{i}+h_2) = 0. \tag{5.3}$$

Now we reformulate the equation (5.3) as following:

$$\begin{aligned} 0 &= (\mathrm{i}-h_1)(u(b_3-h_3)+v(c_2-h_2))(c_3+h_3)(\mathrm{i}+h_2) \\ &= u(\mathrm{i}-h_1)(b_3-h_3)(c_3+h_3)(\mathrm{i}+h_2) + v(\mathrm{i}-h_1)(c_2-h_2)(c_3+h_3)(\mathrm{i}+h_2) \\ &= u(\mathrm{i}-h_1)(b_3-h_3)(c_3+h_3)(\mathrm{i}+h_2) + vw(\mathrm{i}-h_1)(c_2-h_2)(\mathrm{i}+h_2) \\ &\quad -v(\mathrm{i}-h_1)(c_2-h_2)(\mathrm{i}-h_3)(\mathrm{i}+h_2) \\ &= u(\mathrm{i}-h_1)(b_3-h_3)(c_3+h_3)(\mathrm{i}+h_2) + vwr(\mathrm{i}-h_1)(\mathrm{i}+h_2) \\ &= uz(\mathrm{i}-h_1)(c_3+h_3)(\mathrm{i}+h_2) + u(c_3^2+1)(\mathrm{i}-h_1)(\mathrm{i}+h_2) + vwr(\mathrm{i}-h_1)(\mathrm{i}+h_2). \end{aligned}$$

where $z = b_3 - c_3$, $w = c_3 + \mathrm{i}$ and $r = c_2 - \mathrm{i}$. As $l_{123} = 6$, by Lemma 5.3, we have $l_{132} = 8$. Then the following equation

$$uz(\mathrm{i}-h_1)(c_3+h_3)(\mathrm{i}+h_2) + u(c_3^2+1)(\mathrm{i}-h_1)(\mathrm{i}+h_2) + vwr(\mathrm{i}-h_1)(\mathrm{i}+h_2) = 0$$

can be true if and only if all the coefficients of the eight independent vectors

$$[1, h_1, h_3, h_2, h_1h_3, h_1h_2, h_3h_2, h_1h_3h_2]$$

are zeros. This means that we have $uz = 0$ which gives us $z = 0$. Then we can find contradiction by treating the equation (4.3) as follows:

$$\begin{aligned} 0 &= (\mathrm{i} - h_1)(b_2 - h_2)(b_3 - h_3)(\mathrm{i} - h_4) \\ &= (\mathrm{i} - h_1)(\mathrm{i} - h_2)(c_3 - h_3)(\mathrm{i} - h_4) + (\mathrm{i} - h_1)y(b_3 - h_3)(\mathrm{i} - h_4) \\ &\quad +(\mathrm{i} - h_1)(\mathrm{i} - h_2)z(\mathrm{i} - h_4) \\ &= (\mathrm{i} - h_1)y(b_3 - h_3)(\mathrm{i} - h_4). \end{aligned}$$

where $y = b_2 - \mathrm{i}$ and $z = b_3 - c_3$. This means that we have $l_{134} = 6$. As $l_{234} = 6$, we have $l_{132} = l_{231} = 6$. This contradicts our assumption of $l_{123} = 6$ by Lemma 5.3.

Case IV, $u = 0$ and $v = 0$. Then we go to the other direction by taking $u' = b_3 - c_3$ and $v' = b_2 - \mathrm{i}$. We make the same case distinction as for $u$ and $v$. The other three cases when $|u'|^2 + |v'|^2 \neq 0$ are similar as these previous three cases on $u$ and $v$. If $u' = 0$ and $v' = 0$, then we have

$$(\mathrm{i} - h_1)(\mathrm{i} - h_2)(\mathrm{i} - h_3)(\mathrm{i} - h_4) = 0. \tag{5.4}$$

By Lemma 5.4, we get at least one parallel property of those $h_1 \| h_2$, $h_2 \| h_3$ and $h_3 \| h_4$. But each of them contradicts our assumption on $l_{123} = l_{234} = 6$. □

Using Lemma 5.6, we can classify that there are only nine possible bond diagrams with 4 bonds. All of them are listed in Figure 5.2.

## 5.2 Construct 6R Linkages with Three Bonds

The main idea for constructing 6R linkages by factorization of motion polynomials is from [30]. The first step is to find a rational motion, namely a motion polynomial. The second step is to do the factorization with respect to this motion polynomial. If one can get "different" factorizations, then one can combine them to get a closed linkage.

In [30], the first example of this implementation is a 4R linkage which is known as Bennett 4R linkage.

Here is the construction and a random example.

**Construction 5.7.** *(Bennett linkages)*

***I.*** *Choose two rotation axes $h_1$ and $h_2$, i.e. dual quaternions such that $h_1^2 = h_2^2 = -1$.*

***II.*** *Choose two random real numbers* $a, b$ *where* $b \neq 0$ *and* $b \neq \pm 1$ *(we usually take rational numbers).*

***III.*** *Compute the product* $m = (t - h_1)(t - a - bh_2)$.

***IV.*** *Compute another factorization (the one is different from* $(t-h_1)(t-a-bh_2)$*)* $m = (t - a - bh_4)(t - h_3)$ *by Algorithm 5.*

***V.*** *Our Bennett 4R Linkage is determined by* $L = [h_1, h_2, h_3, h_4]$. □

**Example 5.8.** *A random instance of the above construction is*

$$
\begin{aligned}
h_1 &= \mathbf{i},\\
h_2 &= -\mathbf{k} - \epsilon\mathbf{i},\\
a &= \frac{1}{3}, \quad b = \frac{3}{2},\\
h_3 &= \left(\frac{41}{121}\mathbf{i} - \frac{36}{121}\mathbf{j} + \frac{108}{121}\mathbf{k}\right) + \epsilon\left(\frac{17496}{14641}\mathbf{i} - \frac{3888}{14641}\mathbf{j} - \frac{7938}{14641}\mathbf{k}\right),\\
h_4 &= \left(\frac{108}{121}\mathbf{i} + \frac{24}{121}\mathbf{j} + \frac{49}{121}\mathbf{k}\right) + \epsilon\left(\frac{2977}{14641}\mathbf{i} + \frac{2592}{14641}\mathbf{j} + \frac{5292}{14641}\mathbf{k}\right).
\end{aligned}
$$

*Here we found that the configuration curve has one non-degenerate component with rational parametrization:*

$$(t_1, t_2, t_3, t_4) = \left(t, \frac{t-a}{b}, t, \frac{t-a}{b}\right).$$

*In Figure 5.3, we present ten configuration positions of this linkage.* □

This construction of Bennett 4R linkage is a basic item for constructing a 6R linkage with a rational configuration curve. We will show two constructions here. One is the cube linkage [30] with a relative cubic motion. The other one is the double Bennett linkage [21].

Let us see the motion polynomial construction for the cube linkage.

**Construction 5.9.** *(Cube linkages)*

***I.*** *Choose three rotation axes* $h_1$*,* $h_2$ *and* $h_3$*, i.e. dual quaternions such that* $h_1^2 = h_2^2 = h_3^2 = -1$.

***II.*** *Choose four random real numbers* $a, b, c, d$ *where* $bd \neq 0$*,* $(t-a)^2 + b^2 \neq t^2 + 1$*,* $(t-c)^2 + d^2 \neq t^2 + 1$ *and* $(t-a)^2 + b^2 \neq (t-c)^2 + d^2$ *(we usually take rational numbers).*

***III.*** *Compute the product* $m = (t - h_1)(t - a - bh_2)(t - c - dh_3)$.

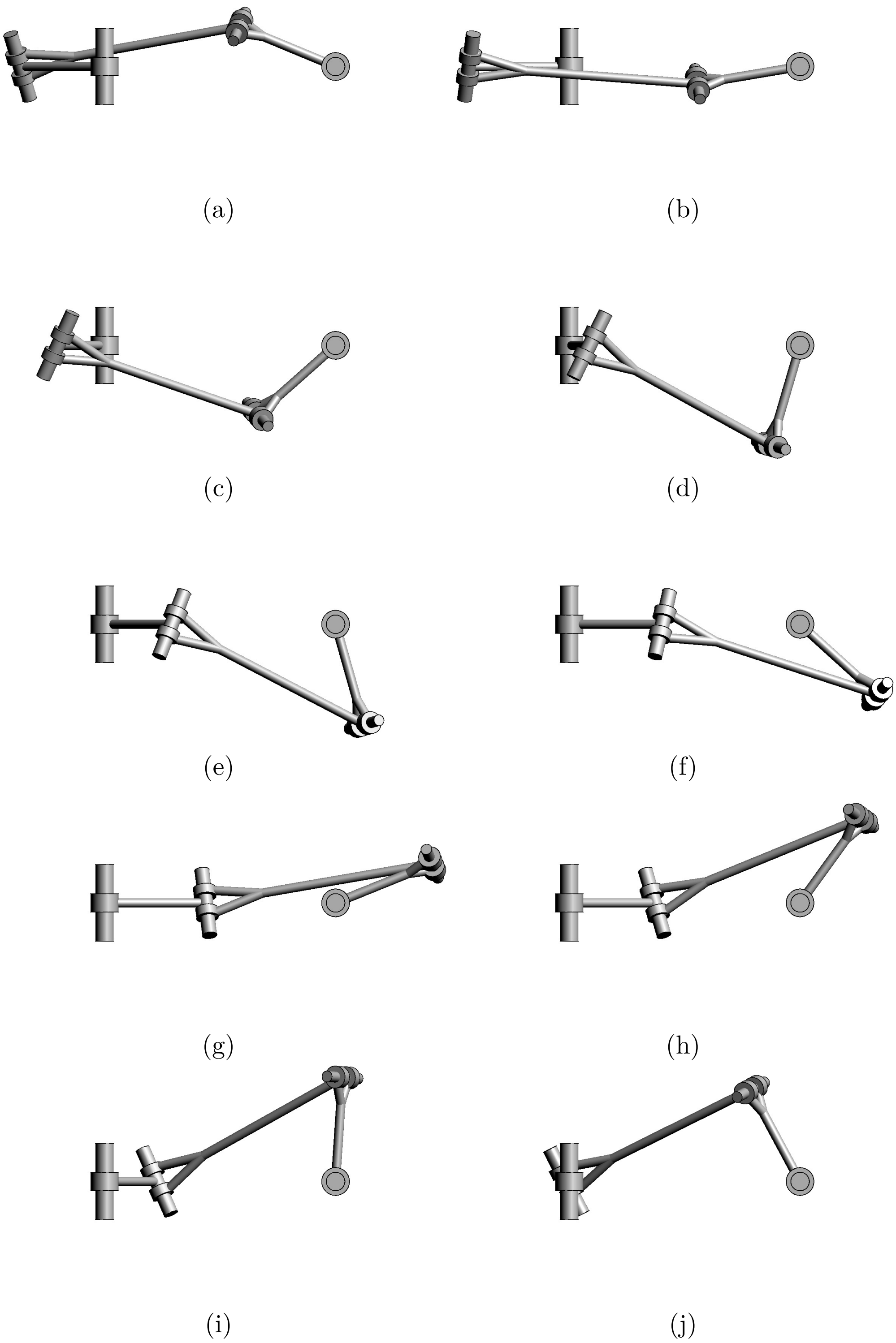



Figure 5.3: These are ten postures of the Bennett linkage in Example 5.8

***IV.*** *Compute another factorization (the one is different from* $(t-h_1)(t-a-bh_2)(t-c-dh_3)$*)* $m=(t-c-dh_6)(t-a-bh_5)(t-h_4)$ *by Algorithm 5.*

***V.*** *Our cube 6R Linkage is determined by* $L=[h_1,h_2,h_3,h_4,h_5,h_6]$. □

**Example 5.10.** *A random instance of the above construction is*

$$
\begin{aligned}
h_1 &= \mathbf{i},\\
h_2 &= \left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)+\epsilon\left(\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}\right),\\
h_3 &= \mathbf{k}+\epsilon(\mathbf{i}-\mathbf{j}),\\
a &= 1;\quad b=2;\quad c=2;\quad d=2;\\
h_4 &= -\left(\frac{149}{221}\mathbf{i}-\frac{48}{221}\mathbf{j}+\frac{12}{17}\mathbf{k}\right)-\epsilon\left(\frac{19464}{18785}\mathbf{i}-\frac{25478}{18785}\mathbf{j}-\frac{5286}{3757}\mathbf{k}\right),\\
h_5 &= -\left(\frac{89931}{482885}\mathbf{i}+\frac{76308}{482885}\mathbf{j}+\frac{7204}{7429}\mathbf{k}\right)\\
&\quad -\epsilon\left(\frac{2091960164}{3587352665}\mathbf{i}-\frac{2465911798}{3587352665}\mathbf{j}+\frac{15687}{717470533}\mathbf{k}\right),\\
h_6 &= -\left(\frac{252}{437}\mathbf{i}+\frac{328}{437}\mathbf{j}-\frac{141}{437}\mathbf{k}\right)-\epsilon\left(\frac{133445}{190969}\mathbf{i}-\frac{44775}{190969}\mathbf{j}+\frac{134340}{190969}\mathbf{k}\right),
\end{aligned}
$$

*Here we found that the configuration curve has one non-degenerate component with rational parametrization:*

$$(t_1,t_2,t_3,t_4,t_5,t_6)=\left(t,\frac{t-a}{b},\frac{t-c}{d},t,\frac{t-a}{b},\frac{t-c}{d}\right).$$

*In Figure 5.4, we present nine configuration positions of this linkage.* □

**Remark 5.11.** *The condition of* $bd\neq 0$ *is necessary. Because* $b$ *and* $d$ *appear in the denominator of the configuration curve's parametrization. The conditions of* $(t-a)^2+b^2\neq t^2+1$, $(t-c)^2+d^2\neq t^2+1$ *and* $(t-a)^2+b^2\neq(t-c)^2+d^2$ *are also necessary. Otherwise, one can not find another useful factorization (if* $(t-a)^2+b^2\neq t^2+1$*) or one will get a Double Bennett 6R linkage.*

Let us see the motion polynomial construction for the Double Bennett 6R linkage.

**Construction 5.12.** *(Double Bennett 6R linkage)*

***I.*** *Choose three rotation axes* $h_1$, $h_2$ *and* $h_3$, *i.e. dual quaternions such that* $h_1^2=h_2^2=h_3^2=-1$.

***II.*** *Choose four random real numbers* $a,b,c,d$ *where* $bd\neq 0$, $(t-a)^2+b^2\neq t^2+1$, $(t-c)^2+d^2\neq t^2+1$ *(we usually take rational numbers).*

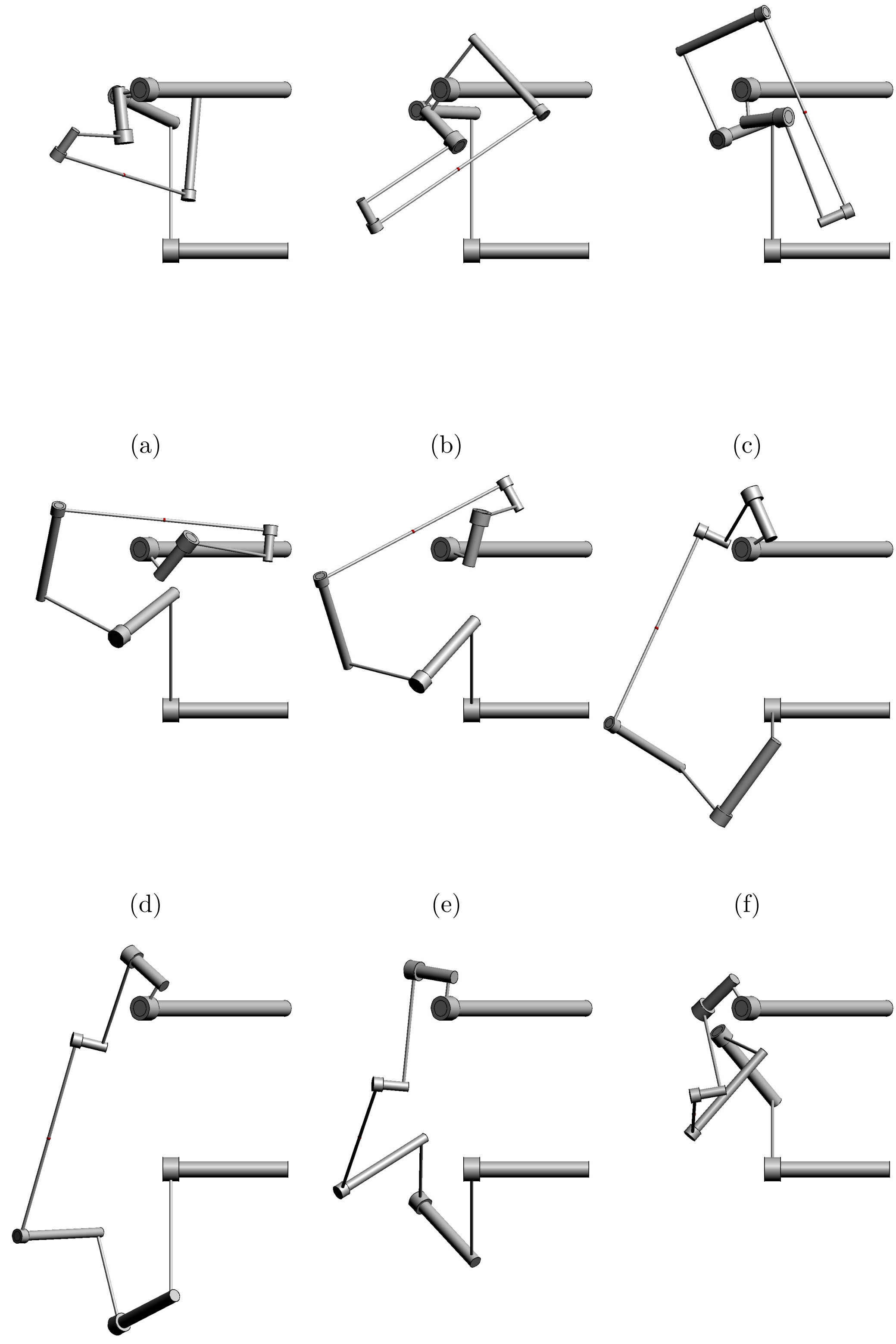



Figure 5.4: These are nine postures of the cube linkage in Example 5.10

***III.*** *Compute the product* $m = (t-h_1)(t-a-bh_2)(t-c-dh_3)$.

***IV.*** *Compute another factorization (the one is different from* $(t-h_1)(t-a-bh_2)(t-c-dh_3)$*)* $m = (t-a-bh_6)(t-c-dh_5)(t-h_4)$ *by Algorithm 5.*

***V.*** *The Double Bennett 6R Linkage is determined by* $L = [h_1, h_2, h_3, h_4, h_5, h_6]$. □

**Example 5.13.** *A random instance of the above construction is*

$$\begin{aligned}
h_1 &= \mathbf{i},\\
h_2 &= \left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)+\epsilon\left(\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}\right),\\
h_3 &= \mathbf{k}+\epsilon(\mathbf{i}-\mathbf{j}),\\
a &= 1;\quad b=2;\quad c=2;\quad d=2;\\
h_4 &= -\left(\frac{149}{221}\mathbf{i}-\frac{48}{221}\mathbf{j}+\frac{12}{17}\mathbf{k}\right)-\epsilon\left(\frac{19464}{18785}\mathbf{i}-\frac{25478}{18785}\mathbf{j}-\frac{5286}{3757}\mathbf{k}\right),\\
h_5 &= \left(\frac{3292}{23205}\mathbf{i}-\frac{12244}{23205}\mathbf{j}-\frac{299}{357}\mathbf{k}\right)-\epsilon\left(\frac{8741221}{8284185}\mathbf{i}-\frac{4067647}{8284185}\mathbf{j}+\frac{808648}{1656837}\mathbf{k}\right),\\
h_6 &= -\left(\frac{19}{21}\mathbf{i}+\frac{8}{21}\mathbf{j}-\frac{4}{21}\mathbf{k}\right)-\epsilon\left(\frac{100}{441}\mathbf{i}-\frac{190}{441}\mathbf{j}+\frac{95}{441}\mathbf{k}\right),
\end{aligned}$$

*Here we found that the configuration curve has one non-degenerate component with rational parametrization:*

$$(t_1,t_2,t_3,t_4,t_5,t_6) = \left(t, \frac{t-a}{b}, \frac{t-c}{d}, t, \frac{t-c}{d}, \frac{t-a}{b}\right).$$

*In Figure 5.5, we present nine configuration positions of this linkage.* □

**Remark 5.14.** *The condition of* $bd \neq 0$ *is necessary. Because* $b$ *and* $d$ *appear in the denominator of the configuration curve's parametrization. The conditions of* $(t-a)^2+b^2 \neq t^2+1$, $(t-c)^2+d^2 \neq t^2+1$ *and* $(t-a)^2+b^2 \neq (t-c)^2+d^2$ *are also necessary. Otherwise, one can not find another useful factorization (if* $(t-a)^2+b^2 \neq t^2+1$*).*

## 5.3 Construct 6R Linkages with Four Bonds

For the construction of 6R linkages with 4 bonds case, we introduce it as follows. First, let us make our purpose clear. We want to construct a monic quartic polynomial $Q$ in $\mathbb{DH}[t]$ such that $Q\bar{Q} \in \mathbb{R}[t]$. Furthermore, we can factor $Q$ in two different

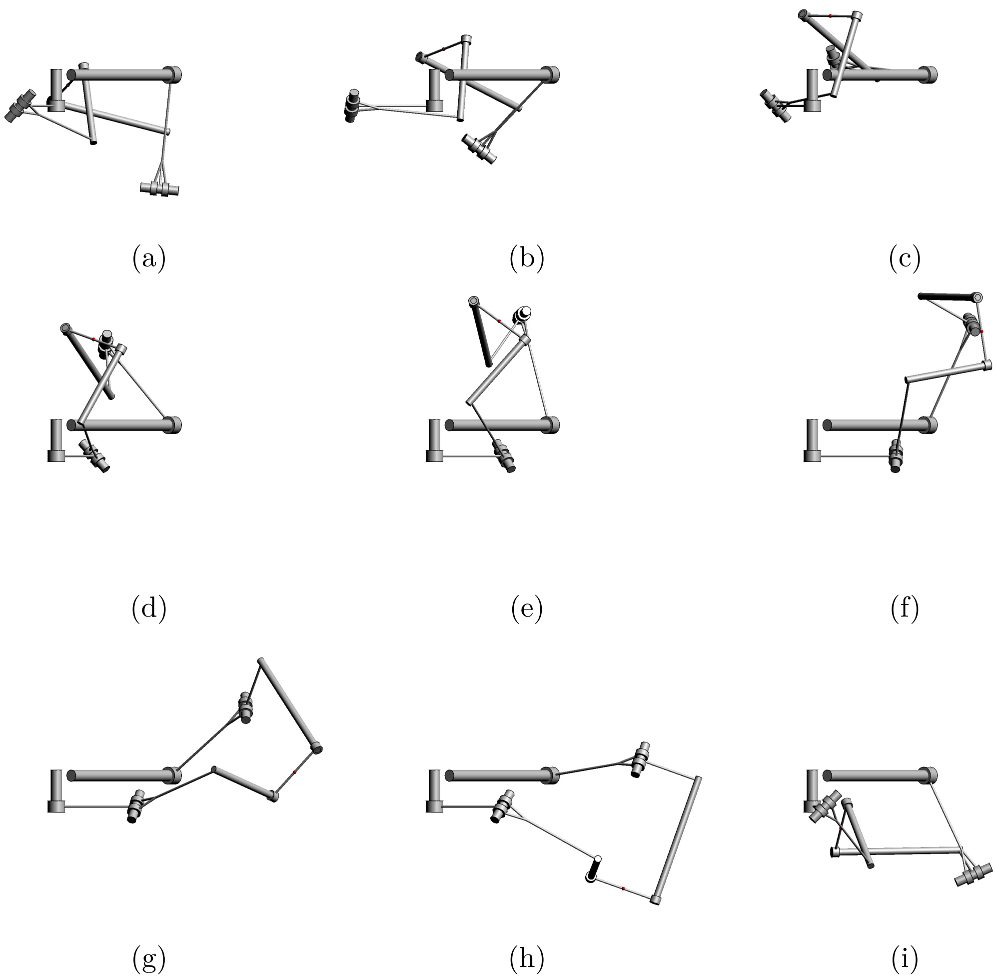



Figure 5.5: These are nine postures of the Double Bennett 6R linkage in Example 5.13

ways (at least) which both constitute a 3R open chain. Then we can construct a 6R linkage by combining these two factorizations.

For the nine possible bond diagrams in Figure 5.2, we only show the construction for first bond diagram. The others can be constructed similarly.

Now we introduce our procedure for finding a monic quartic motion polynomial for constructing a 6R linkage with bond diagram as in Figure 5.2 (a).

**Construction 5.15.** *(6R linkages with 4 bonds)*

***I.*** *We choose four lines with two different bond connections (3.4) as following*

$$\begin{aligned}(\mathrm{i}-h_1)(\alpha-h_2)(\beta-h_3)(\mathrm{i}-h_4) &= 0,\\ (\mathrm{i}-h_1)(\alpha'-h_2)(\beta'-h_3)(\mathrm{i}+h_4) &= 0,\end{aligned} \tag{5.5}$$

*where* i *is the imaginary unit, complex numbers* $\alpha$ *and* $\beta$ *have the same linear relation as* $\alpha'$ *and* $\beta'$ *i.e.*

$$\beta = a\alpha + b, \quad \beta' = a\alpha' + b.$$

***II.*** *Use these two bond conditions from Equations (5.5) to calculate quartic (degree 4) motion polynomials*

$$(t_1(t)-h_1)(t-h_2)(at+b-h_3)(t_4(t)-h_4).$$

***III.*** *Use the generic factorization Algorithm 5 to compute another factorization of the motion polynomial*

$$(t_1(t)-h_1)(t-h_2)(at+b-h_3).$$

*This procedure contributes the other two lines* $h_5$ *and* $h_6$ *which we want.*

***IV.*** *Return the 6R linkage* $[h_1, h_2, h_3, h_4, h_5, h_6]$. □

**Remark 5.16.** *There are two options in procedure III (the norm polynomials of second and third can be exchange), which contribute two kinds of* 6*R linkage with bond diagrams 5.2(a) and (c).*

As the first step is the most important step, we show the details in the following subroutine.

**I.a** Choose $h_2$ and $h_3$ as two random lines with $h_2^2 = h_3^2 = -1$.

**I.b** Choose two complex number $\alpha$ and $\alpha'$ where $\alpha \neq \pm \mathrm{i}$ and $\alpha' \neq \pm \mathrm{i}$.

**I.c** Choose two random real numbers $a, b$ with $a \neq 0$.

**I.d** Assume that the other two lines have the following formula

$$
\begin{aligned}
h_1 &= (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) + (y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})\epsilon, \\
h_4 &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) + (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\epsilon.
\end{aligned}
$$

**I.e** Solve the following system for unknowns $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$, $u_1$, $u_2$, $u_3$, $v_1$, $v_2$, $v_3$

$$
\begin{cases}
(\mathrm{i} - h_1)(\alpha - h_2)(\beta - h_3)(\mathrm{i} - h_4) = 0, \\
(\mathrm{i} - h_1)(\alpha' - h_2)(\beta' - h_3)(\mathrm{i} + h_4) = 0, \\
h_1^2 = -1, \qquad h_4^2 = -1.
\end{cases}
$$

**I.f** Choose one real solution for the next steps.

We add one example to support our procedure. This is a particularly easy example which we found by our procedure.

**Input:** I.a, I.b, I.c

$$
\begin{aligned}
h_2 &= \left(-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) - \frac{6}{5}\mathbf{k}\epsilon, \\
h_3 &= \left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) + \left(\frac{76}{49}\mathbf{i} + \frac{24}{49}\mathbf{j} - \frac{30}{49}\mathbf{k}\right)\epsilon, \\
\alpha &= -\frac{1}{5} - \frac{4}{3}\mathrm{i}, \quad \alpha' = \frac{4}{5} - \frac{1}{2}\mathrm{i}, \\
a &= \frac{5}{2}, \qquad\qquad b = -\frac{3}{4}.
\end{aligned}
$$

**Output:** Then one can get a numerical solution with 10 digits as following

$$
\begin{aligned}
&x_1 = 0.4058453976, \quad x_2 = -0.9139192147, \quad x_3 = -0.0064173294, \\
&y_1 = 1.244931364, \quad y_2 = 0.5535129673, \quad y_3 = -0.09606363509, \\
&u_1 = -0.6219669897, \quad u_2 = -0.3316117352, \quad u_3 = 0.7093593733, \\
&v_1 = -0.5417103337, \quad v_2 = -1.024569908, \quad v_3 = -0.9539386886.
\end{aligned}
$$

Then the next two steps are for calculating the factorization. We assume that $t_1(t)$ and $t_4(t)$ are quadratic rational functions of $t$, and we also assume that

$$
t_1(\alpha) = \mathrm{i}, \quad t_1(\alpha') = \mathrm{i}, \quad t_4(\alpha) = \mathrm{i}, \quad t_4(\alpha') = -\mathrm{i}. \tag{5.6}
$$

The quartic motion polynomial is $(t_1(t) - h_1)(t - h_2)(at + b - h_3)$. The other factorization is obtained by multiplying $(t_4(t) - h_4)$ from the right. Then $(t_1(t) - h_1)(t-h_2)(at+b-h_3)(t_4(t)-h_4)$ is a quadratic motion polynomial when we remove the real denominators and factors. The next step is to factor this quadratic motion polynomial. We show all these details in the following:

**Assumption:**

$$t_1(t) = \frac{t^2 + p_2 t + p_3}{p_4 t + p_5}, \qquad t_4(t) = \frac{t^2 + p_2' t + p_3'}{p_4' t + p_5'},$$
$$\alpha = -\frac{1}{5} - \frac{4}{3}\mathrm{i}, \qquad \alpha' = \frac{4}{5} - \frac{1}{2}\mathrm{i}.$$

**Do:** Solve the linear system (5.6) for unknowns $p_2, p_3, p_4, p_5, p_2', p_3', p_4', p_5'$.

**Output:** Then one can get a solution of $t_1(t)$ and $t_4(t)$ as following

$$t_1(t) = \frac{t^2 - \frac{3}{5}t - \frac{62}{75}}{-\frac{11}{6}t + \frac{29}{30}}, \qquad t_4(t) = \frac{t^2 - \frac{3}{5}t + \frac{38}{75}}{-\frac{5}{6}t + \frac{7}{6}}.$$

After substituting $t_1(t)$ and $t_4(t)$ into

$$(t_1(t) - h_1)(t - h_2)(at + b - h_3)(t_4(t) - h_4),$$

we have a numeric quadratic motion polynomial in 10 digits (replacing the real denominators and factors)

$$\begin{aligned} t^2 + (&-0.3000000000 + 0.6543154994\mathbf{i} - 1.037575959\mathbf{j} + 0.2365105645\mathbf{k} + \\ &1.210540727\mathbf{i}\epsilon - 0.0349528507\mathbf{j}\epsilon + 0.4738323880\epsilon\mathbf{k})t - \\ &0.2003149450 - 0.0160185109\mathbf{i} + 0.3911798525\mathbf{j} + .2378984092\mathbf{k} - \\ &0.9404081633\epsilon - 1.436504834\mathbf{i}\epsilon - 0.5526215606\mathbf{j}\epsilon + 0.0201175896\epsilon\mathbf{k}. \end{aligned}$$

As the norm of this quadratic motion polynomial is $(t^2 + 1)(t^2 - \frac{3}{5}t + \frac{1}{4})$, we can construct two 6R linkages $L_c = [h_1^c, h_2^c, h_3^c, h_4^c, h_5^c, h_6^c]$ and $L_d = [h_1^d, h_2^d, h_3^d, h_4^d, h_5^d, h_6^d]$ (with bond diagram 3.1(c) and (d)) basing on these two factorization as following (numerically in 10 digits).

**Example 5.17.**

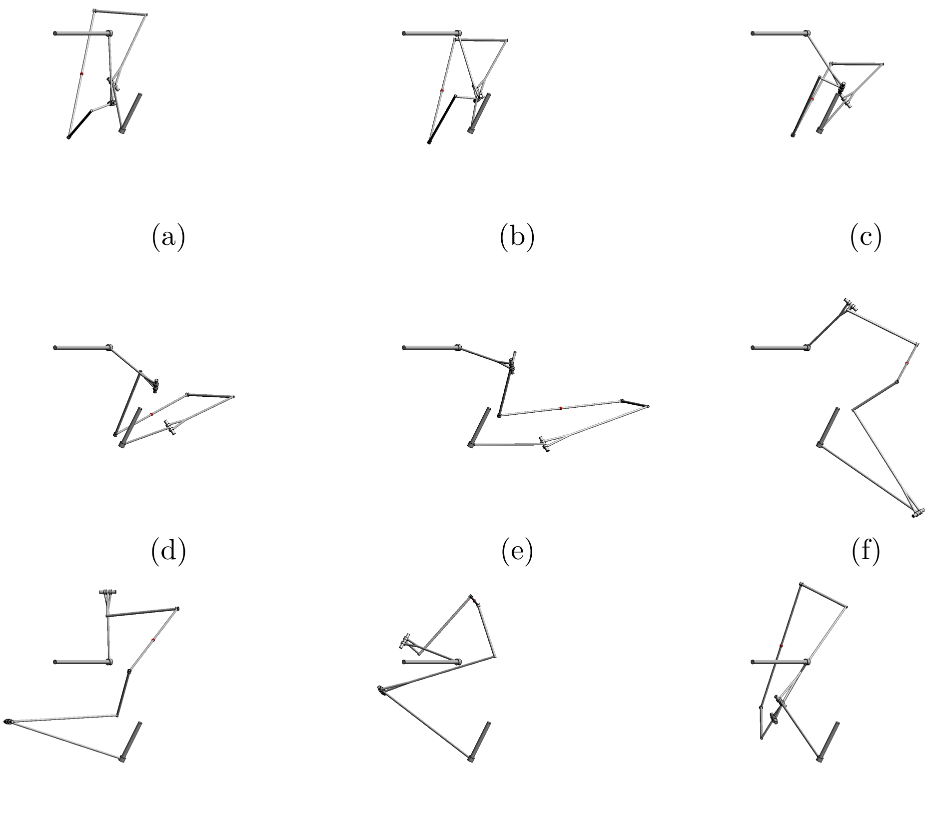
$$
\begin{aligned}
h_1^c &= (0.4058453976\mathbf{i} - 0.9139192147\mathbf{j} - 0.0064173294\mathbf{k}) + \\
&\quad (1.244931364\mathbf{i} + 0.5535129673\mathbf{j} - 0.09606363509\mathbf{k})\,\epsilon, \\
h_2^c &= \left(-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) - \frac{6}{5}\mathbf{k}\epsilon, \\
h_3^c &= \left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) + \left(\frac{76}{49}\mathbf{i} + \frac{24}{49}\mathbf{j} - \frac{30}{49}\mathbf{k}\right)\epsilon, \\
h_4^c &= (-0.6219669897\mathbf{i} - 0.3316117352\mathbf{j} + 0.7093593733\mathbf{k}) + \\
&\quad (-0.5417103337\mathbf{i} - 1.024569908\mathbf{j} - 0.9539386883\mathbf{k})\,\epsilon, \\
h_5^c &= (0.9529670102)\mathbf{i} - 0.2884245020\mathbf{j} - 0.0930869702\mathbf{k}) + \\
&\quad (0.1459988817\mathbf{i} - 0.4419436106\mathbf{j} + 2.863982166\epsilon\mathbf{k})\,\epsilon, \\
h_6^c &= (0.2731286954)\mathbf{i} - 0.9222061578\mathbf{j} + 0.2737453525\mathbf{k}) + \\
&\quad (1.152141200\mathbf{i} + 0.1418245937\mathbf{j} - 0.6717604788\mathbf{k})\,\epsilon.
\end{aligned}
$$

**Example 5.18.**

$$
\begin{aligned}
h_1^d &= h_1^c, \qquad h_2^d = h_2^c, \qquad h_3^d = h_3^c, \qquad h_4^d = h_4^c, \\
h_5^d &= (0.6843121346\mathbf{i} - 0.7290081982\mathbf{j} - 0.0162465108\mathbf{k}) + \\
&\quad (0.7852041130\mathbf{i} + 0.7074301081\mathbf{j} + 1.329661169\mathbf{k})\,\epsilon, \\
h_6^d &= (-0.0749915882\mathbf{i} - 0.7714194013\mathbf{j} + 0.6318926880\mathbf{k}) + \\
&\quad (1.063341534\mathbf{i} - 1.855957397\mathbf{j} - 2.139571953\mathbf{k})\,\epsilon.
\end{aligned}
$$

In Figure 5.6 and Figure 5.7, we show 9 configurations for these two sharp linkages [40].

**Remark 5.19.** *One can use the new technique, namely, quad polynomials which is introduced in Chapter 3, to derive some equational conditions on the Denavit-Hartenberg parameters. But they are not enough for fully constraining these 6R linkages.*

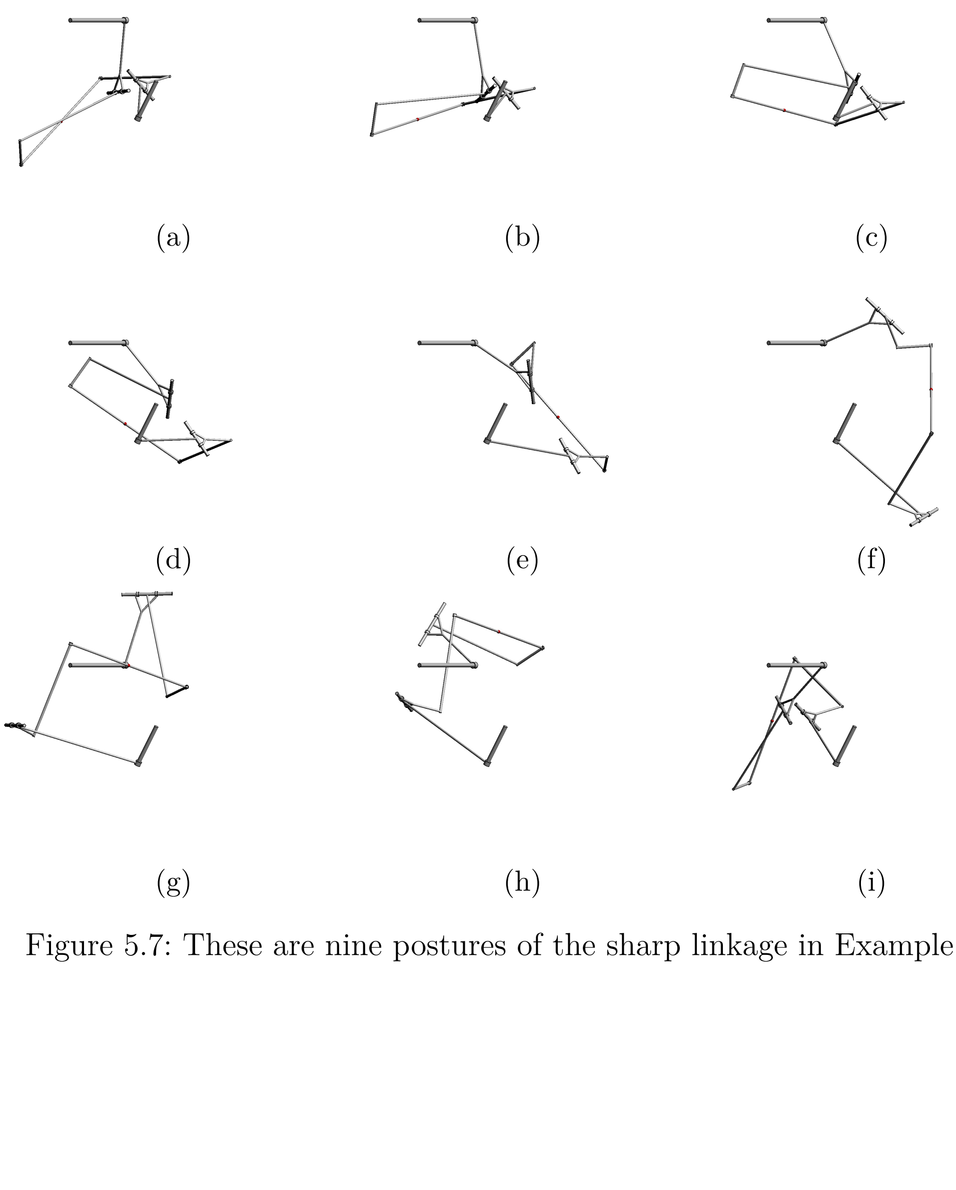


(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 5.6: These are nine postures of the sharp linkage in Example 5.17



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 5.7: These are nine postures of the sharp linkage in Example 5.18

# Chapter 6

# Angle-Symmetric 6R Linkages

In this chapter we want to handle a subclassification problem of 6R linkages. We name it classification of angle-symmetric 6R linkages which are linkages with the property that the rotation angles of the three opposite joints are equal. The results presented below evolved from a collaboration with Josef Schicho and have recently been published in [42].

**Structure of the chapter** The remaining part of the chapter is set up as follows. Section 6.1 gives us the motivation of doing such classification for angle-symmetric 6R linkages. In Section 6.2, we give one important tool –the $\lambda$-matrix. We also show that the rank of this matrix is 2, 3, or 4. Section 6.3 contains the main result (classification) and examples (new 6R linkages).

## 6.1 Motivation

In this section, we first give the definition of an angle-symmetric 6R linkage. These are linkages with the property that the rotation angles of the three pairs of opposite joints are equal for all possible configurations, or at least for infinitely many configurations (it could be that a certain linkage has two components, where only one of them is angle-symmetric). A full classification of these linkages is obtained. The classification of angle-symmetric 6R linkages contains three types of linkages. Type one is the Bricard line symmetry 6R linkage. Type two is new. Type three is the cube linkages which constructed by the factorizations of a cubic motion polynomial.

The motivation is to find all angle-symmetric 6R linkages. The angle-symmetric property of the famous Bricard line symmetric 6R linkage is easy to get. This is because of the line symmetric property. One combination of factorizations of a motion polynomial could generate a closed linkage with angle-symmetric property.

For example, the Bennett linkage is angle-symmetric. The new 6R linkage [30] (cube linkage) also has the angle-symmetric property. Thus, it is natural to ask whether these are all angle-symmetric 6R linkages or not. The answer is positive. There is another type of 6R linkage with angle-symmetric property. In addition, it fulfills another property which we name parallel property. It is worth mentioning that this type and several new 6-bar linkages [9, 37, 39, 43] fill a gap in [8, Section 3.8]. An exhaustive study of these linkages with the parallel property will be include in the next chapter.

Our main tool is a $\lambda$-matrix for an angle-symmetric linkage, to be defined in Section 6.2, and its rank $r$. Intuitively speaking, the configuration set of an angle-symmetric linkage can be written as the vanishing set of $r$ equations in three variables, namely the cotangents of the half of the rotation angles. We will show that the rank $r$ can be only 2, 3, or 4. For $r = 2$, the angle-symmetric linkage is line symmetric. For $r = 3$, we get the new linkage with three pairs of parallel axes. For $r = 4$, we obtain the cube linkage constructed in [29, 30] using motion polynomials.

## 6.2 The $\lambda$-matrix

In this section we define, for a given linkage, a matrix whose rows are related to an algebraic system defining the configuration space. In the next section, we will see that the rank of this matrix is the basic criterion for classifying angle-symmetric linkages.

The set of all possible motions of a closed 6R linkage is determined by the position of the six rotation axes in some fixed initial configuration. (The choice of the initial configuration among all possible configurations is arbitrary. In some later steps in the classification, we will occasionally change the initial configuration.)

The algebra $\mathbb{DH}$ of dual quaternions is the 8-dimensional real vector space generated by $1, \epsilon, \mathbf{i}, \mathbf{j}, \mathbf{k}, \epsilon\mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k}$ (see [29, 30]). Following [29, 30], we can represent a rotation by a dual quaternion of the form $\left(\cot\left(\frac{\phi}{2}\right) - h\right)$, where $\phi$ is the rotation angle and $h$ is a dual quaternion such that $h^2 = -1$ depending only on the rotation axis. We use projective representations, which means that two dual quaternions represent the same Euclidean displacement if only if one is a real scalar multiple of the other.

Let $L$ be a 6R linkage given by 6 lines, represented by dual quaternions $h_1, \ldots, h_6$ such that $h_i^2 = -1$ for $i = 1, \ldots, 6$. A configuration (see [29, 30]) is a 6-tuple $(t_1, \ldots, t_6)$, such that the closure condition

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)(t_5 - h_5)(t_6 - h_6) \in \mathbb{R}\backslash\{0\}$$

holds. The configuration parameters $t_i$ – the cotangents of the rotation angles – may be real numbers or $\infty$, and in the second case we evaluate the expression $(t_i - h_i)$ to 1, the rotation with angle 0. The set of all configurations of $L$ is denoted by $K_L$.

There is a subset of $K_L$, denoted by $K_{sym}$, defined by the additional restrictions $t_1 = t_4, t_2 = t_5, t_3 = t_6$. We assume that $K_{sym}$ is a one-dimensional set, i.e. the linkage has an angle-symmetric motion. Mostly, we will assume, slightly stronger, that there exists an irreducible one-dimensional set for which none of the $t_i$ is fixed. Such a component is called a non-degenerate component. We also exclude the case $\dim_{\mathbb{C}} K_{sym} \geq 2$. Linkages with mobility $\geq 2$ do exist, but they are well understood.

The closure condition is equivalent to

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3) = \lambda(t_3 + h_6)(t_2 + h_5)(t_1 + h_4),$$

where $\lambda$ is a nonzero real value depending on $t_1, t_2, t_3$. By taking norm on both sides, we get $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. By multiplying both sides with $(t_1 + h_1)$ from the left and with $(t_1 - h_4)$ from the right, and afterwords dividing by $(t_1^2 + 1)$, we obtain the equation

$$(t_2 - h_2)(t_3 - h_3)(t_1 - h_4) = \lambda(t_1 + h_1)(t_3 + h_6)(t_2 + h_5).$$

Similarly, we obtain

$$(t_3 - h_3)(t_1 - h_4)(t_2 - h_5) = \lambda(t_2 + h_2)(t_1 + h_1)(t_3 + h_6),$$

$$(t_1 - h_4)(t_2 - h_5)(t_3 - h_6) = \lambda(t_3 + h_3)(t_2 + h_2)(t_1 + h_1),$$
$$(t_2 - h_5)(t_3 - h_6)(t_1 - h_1) = \lambda(t_1 + h_4)(t_3 + h_3)(t_2 + h_2),$$
$$(t_3 - h_6)(t_1 - h_1)(t_2 - h_2) = \lambda(t_2 + h_5)(t_1 + h_4)(t_3 + h_3).$$

We may divide $K_{sym}$ into two disjoint subsets $K^+_{sym}$ and $K^-_{sym}$, according to whether $\lambda$ is equal to $+1$ or $-1$ in the equations above. Any irreducible component of $K_{sym}$ is either contained in $K^+_{sym}$ or in $K^-_{sym}$. Note that $(t_1, t_2, t_3) = (\infty, \infty, \infty)$ is an element of $K^+_{sym}$.

**Remark 6.1.** *When we want to study some component $K_0 \subset K^-_{sym}$, we may proceed in the following way: we take a configuration $\tau \in K_0$, which defines a set of rotations around the joint axes. Then we apply these rotations, obtaining new positions for the 6 lines. In the transformed linkage, the component corresponding to $K_0$ contains $(\infty, \infty, \infty)$. So we will always assume that $\lambda = 1$.*

When $\lambda = 1$, after moving the right parts of the above equations to the left, we get an equation

$$\mathbf{M}^\dagger \mathbf{X} = \mathbf{0},$$

where $\mathbf{X} = [t_1t_2, t_1t_3, t_2t_3, t_3, t_2, t_1, 1]^T$. If we denote $h_6 + h_3, h_5 + h_2, h_4 + h_1$ by $g_3, g_2, g_1$ respectively, then the coefficient matrix $\mathbf{M}^\dagger$ is

$$\begin{bmatrix} g_3, g_2, g_1, & h_5h_4 - h_1h_2, & h_6h_4 - h_1h_3, & h_6h_5 - h_2h_3, & h_6h_5h_4 + h_1h_2h_3 \\ g_3, g_2, g_1, & h_1h_5 - h_2h_4, & h_1h_6 - h_3h_4, & h_6h_5 - h_2h_3, & h_1h_6h_5 + h_2h_3h_4 \\ g_3, g_2, g_1, & h_2h_1 - h_4h_5, & h_1h_6 - h_3h_4, & h_2h_6 - h_3h_5, & h_2h_1h_6 + h_3h_4h_5 \\ g_3, g_2, g_1, & h_2h_1 - h_4h_5, & h_3h_1 - h_4h_6, & h_3h_2 - h_5h_6, & h_3h_2h_1 + h_4h_5h_6 \\ g_3, g_2, g_1, & h_4h_2 - h_5h_1, & h_4h_3 - h_6h_1, & h_3h_2 - h_5h_6, & h_4h_3h_2 + h_5h_6h_1 \\ g_3, g_2, g_1, & h_5h_4 - h_1h_2, & h_4h_3 - h_6h_1, & h_5h_3 - h_6h_2, & h_5h_4h_3 + h_6h_1h_2 \end{bmatrix}.$$

Note that $\mathbf{M}^\dagger$ is a $6 \times 7$ matrix with entries in dual quaternions. We also consider $\mathbf{M}^\dagger$ to be a $48 \times 7$ matrix with real entries. It can be decomposed into submatrices $M_1^\dagger, \cdots, M_6^\dagger$, where $M_i^\dagger$ is the real $8 \times 7$ matrix – or the row vector with 7 dual quaternion entries – corresponding to the $i-th$ equivalent formulation of the closure condition above, for $i = 1, \ldots, 6$.

Our classification is based on the following theorem which gives the bounds for the rank of $\mathbf{M}^\dagger$.

**Theorem 6.2.** *Assume that $K_{sym}$ contains a non-degenerate component of dimension 1. Then $r := \operatorname{rank}(\mathbf{M}^\dagger) \in \{2, 3, 4\}$.*

Before we prove Theorem 6.2, we give a lemma.

**Lemma 6.3.** *Assume that $K_{sym}$ contains a non-degenerate component $K_0$ of dimension 1 such that $\infty^3 \in K_0$, and $r \geq 4$. Then there exists a polynomial of the form*

$$bt_1 + ct_2 + d,$$

*where $b, c, d \in \mathbb{R}$ and $bc \neq 0$, which vanishes on $K_{sym}$, maybe after some permutation of the variables $t_1, t_2, t_3$. Moreover, we can define a matrix $\mathbf{N}^\dagger$ of rank $\geq r - 2$ such that the projection of $K_{sym}$ to $(t_1, t_3)$ is defined by*

$$\mathbf{N}^\dagger \mathbf{X}' = \mathbf{0}, \tag{6.1}$$

*where $\mathbf{X}' = [t_1^2, t_1t_3, t_1, t_3, 1]^T$.*

*Proof.* As $r \geq 4$, we have at least four independent equations in three variables $(t_1, t_2, t_3)$ of tridegree at most $(1, 1, 1)$. We denote four of them by $F_1, F_2, F_3, F_4$.

First, we assume that the $F_1$ is irreducible. The resultants of $F_1$ and $F_i$, $i = 2, 3, 4$ with respect to the last variable $t_3$ are denoted by $F_{12}, F_{13}, F_{14}$. The bidegrees of them are at most $(2, 2)$. All these polynomials vanish on $K_{sym}$. If one of them is 0, such as $F_{12} = 0$, then $F_1$ and $F_2$ must have a non-trivial common factor. This can only be $F_1$, since $F_1$ is irreducible. Then the tridegree of $F_1$ is less then

$(1,1,1)$. Because $F_1$ vanishes on the non-degenerate component $K_0$, it must contain at least two variables, and so $F_1$ is a polynomial of degree $(1,1,0)$, maybe after some permutation of variables.

If none of the three resultants vanishes, then let $G = gcd(F_{12}, F_{13}, F_{14})$. The bidegree of $G$ is in the set $\{(2,2),\ (2,1),\ (1,1)\}$, up to permutation of variables $t_1, t_2$. If it is $(1,1)$, then $G$ can be considered as a polynomial of tridegree $(1,1,0)$ that vanishes on $K_0$. If the bidegree of $G$ is $(2,2)$ or $(2,1)$, then we write $F_{12} = GU_2, F_{13} = GU_3, F_{14} = GU_4$ with suitable polynomials $U_2, U_3, U_4$. The bidegrees of $U_2, U_3, U_4$ are at most $(0,1)$, hence $U_2,\ U_3,\ U_4$ are linearly dependent, which means that there are three real number $\lambda_2,\ \lambda_3,\ \lambda_4$ such that

$$\lambda_2 F_{12} + \lambda_3 F_{13} + \lambda_4 F_{14} = 0.$$

As a consequence, we have

$$Res(F_1, \lambda_2 F_2 + \lambda_3 F_3 + \lambda_4 F_4) = 0,$$

where $Res$ denotes the resultant. Then we can continue as in the case $F_{12} = 0$ above. Again we get a polynomial of degree $(1,1,0)$, maybe after some permutation of variables.

Second, if $F_1$ is reducible, then it has two factors with degree $(1,1,0)$ and $(0,0,1)$, up to permutation of variables $t_1, t_2, t_3$. Again, $F_1$ vanishes on the non-degenerate component $K_0$, and so it must contain at least two variables, and so it is a polynomial of degree $(1,1,0)$, maybe after some permutation of variables.

In all cases above, we have a polynomial of tridegree $(1,1,0)$ vanishing on $K_0$. Since $\infty^3$ is in $K_{sym}$, it is of the form $bt_1 + ct_2 + d = 0$, with $b, c, d \in \mathbb{R}$ and $bc \neq 0$, as stated in the lemma. We can use it to eliminate $t_2$: on $K_0$, we have $t_2 = -\frac{bt_1+d}{c}$.

The equations for the projection of $K_0$ to the $(t_1, t_3)$-plane can be obtained by substituting. We get the equation $\mathbf{N}^\dagger \mathbf{X}' = \mathbf{0}$, where $\mathbf{N}^\dagger := \mathbf{M}^\dagger \mathbf{L}$, and

$$\mathbf{L} = \begin{bmatrix} \frac{-b}{c} & 0 & \frac{-d}{c} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-b}{c} & 0 & \frac{-d}{c} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-b}{c} & 0 & \frac{-d}{c} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This follows from the fact that on $K_0$, we can replace $\mathbf{X}$ by $\mathbf{L}\mathbf{X}'$. Because $\text{rank}(L) = 5$, we also get $\text{rank}(\mathbf{N}^\dagger) \geq \text{rank}(\mathbf{M}^\dagger) - 2$. $\square$

*Proof.* Proof of Theorem 6.2 $r \geq 2$: Assume, indirectly, that $r \leq 1$. Then the system $\mathbf{M}^\dagger\mathbf{X} = \mathbf{0}$ is equivalent to zero or only one single equation in three variables, and it will have at least a two-dimensional complex configuration set, which contradicts our assumption.

$r \leq 4$: Assume, indirectly, that $r \geq 5$. Then from Lemma 6.3, the projection of $K_{sym}$ to $(t_1, t_3)$ is defined by

$$\mathbf{N}^\dagger\mathbf{X}' = \mathbf{0}, \tag{6.2}$$

where $r_1 := \operatorname{rank}(\mathbf{N}^\dagger) \geq r - 2 \geq 3$. The equation (6.2) is equivalent to a system of $r_1$ polynomial equations of bidegree at most $(2, 1)$. Because $K_{sym}$ is a curve and has non-degenerate components, the $r_1$ polynomials have a common factor with bidegree at least $(1, 1)$. Then $r_1 \leq 2$ which contradicts to $r_1 \geq 3$. □

## 6.3 Classification

This section contains three parts. First, we show that the existence of a line symmetry implies $r = 2$. Second, we show that $r = 2$ or $r = 3$ implies a line symmetry or another geometric consequence which we call the "parallel property". Third, we relate the case $r = 4$ to a linkage described in [29, 30].

### 6.3.1 Line Symmetric Linkages

We now describe line symmetric 6R linkages in terms of dual quaternions. A 6R linkage $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ is line symmetric if and only if there is a line represented by a dual quaternion $l$ such that $l^2 = -1$ and

$$h_1 = lh_4l^{-1}, \quad h_2 = lh_5l^{-1}, \quad h_3 = lh_6l^{-1}. \tag{6.3}$$

Geometrically, the rotation around $l$ by the angle $\pi$ takes $h_i$ to $h_{i+3}$ for $i = 1, 2, 3$.

**Lemma 6.4.** *If $L$ is line symmetric, then $r = 2$.*

*Proof.* As the norm of $l$ is equal to 1, it follows $l^{-1} = -l$ and we write (6.3) as

$$h_1 = -lh_4l, \quad h_2 = -lh_5l, \quad h_3 = -lh_6l. \tag{6.4}$$

We define a map $\alpha$ from the set of dual quaternions to itself as

$$\alpha : \mathbb{DH} \longrightarrow \mathbb{DH}, \qquad h \longmapsto h + l\bar{h}l,$$

where $\bar{h}$ denotes the dual quaternion conjugate of $h$. It is true that all entries of $M_1^\dagger$ are in $\mathrm{Im}(\alpha)$. For instance, we have $\alpha(h_1) = h_1 - lh_1l = h_1 + h_4 = g_1, \alpha(h_5h_4) = h_5h_4 + lh_4h_5l = h_5h_4 - (lh_4l)(-lh_5l) = h_5h_4 - h_1h_2, \alpha(h_6h_5h_4) = h_6h_5h_4 - lh_4h_5h_6l = h_6h_5h_4 + (-lh_4l)(-lh_5l)(-lh_6l) = h_6h_5h_4 + h_1h_2h_3$. It is not difficult to prove that $\alpha$ is a $\mathbb{R}$-linear map. If we consider $M_1^\dagger$ to be an $8 \times 7$ matrix with real entries, then $r_2 := \mathrm{rank}(M_1^\dagger)$ is less or equal to the dimension of $\mathrm{Im}(\alpha)$. W.l.o.g. we assume $l = \mathbf{i}$. We compute $\mathrm{Im}(\alpha)$ as $\alpha(1) = 1 + \mathbf{ii} = 1 - 1 = 0, \alpha(\epsilon) = \epsilon + \epsilon\mathbf{ii} = 0, \alpha(\mathbf{i}) = \mathbf{i} - \mathbf{iii} = 2\mathbf{i}, \alpha(\mathbf{j}) = \mathbf{j} - \mathbf{iji} = 0, \alpha(\mathbf{k}) = \mathbf{k} - \mathbf{iki} = 0, \alpha(\epsilon\mathbf{i}) = \epsilon\mathbf{i} - \epsilon\mathbf{iii} = 2\epsilon\mathbf{i}, \alpha(\epsilon\mathbf{j}) = \epsilon\mathbf{j} - \epsilon\mathbf{iji} = 0, \alpha(\epsilon\mathbf{k}) = \epsilon\mathbf{k} - \epsilon\mathbf{iki} = 0$. Therefore, the dimension of $\mathrm{Im}(\alpha)$ is 2. So we have $r_2 \leq 2$.

The next step is to prove that all $M_i^\dagger$ for $i = 1, ..., 6$ are equal. It is true that the first three columns are equal in all $M_i^\dagger$ for $i = 1, ..., 6$. As $\mathrm{Im}(\alpha)$ is equal to $\langle \mathbf{i}, \epsilon\mathbf{i} \rangle_{\mathbb{R}}$ and $g_1, g_2, g_3, h_6h_5 - h_2h_3 \in \mathrm{Im}(\alpha)$, we obtain

$$g_1 \times g_2 = g_1 \times g_3 = g_2 \times g_3 = (h_6h_5 - h_2h_3) \times g_1 = 0, \tag{6.5}$$

where $g \times h$ denotes the cross product of purely vectorial dual quaternions $g, h$. The equalities $M_1^\dagger = \cdots = M_6^\dagger$ can be shown from (6.5). For instance, $h_5h_4 - h_1h_2 - (h_1h_5 - h_2h_4) = h_5 \times h_4 - h_1 \times h_2 - h_1 \times h_5 + h_2 \times h_4 = g_2 \times h_4 - h_1 \times g_2 = g_2 \times g_1 = 0, h_1h_5 - h_2h_4 - (h_4h_2 - h_5h_1) = h_1h_5 - h_2h_4 + (\overline{h_1h_5 - h_2h_4}) = 0$ or $h_6h_5h_4 + h_1h_2h_3 - (h_1h_6h_5 + h_2h_3h_4) = -\langle h_6, h_5 \rangle h_4 + \langle h_2, h_3 \rangle h_4 - \langle h_2, h_3 \rangle h_1 + \langle h_6, h_5 \rangle h_1 + (h_6 \times h_5) \times h_4 + h_1 \times (h_2 \times h_3) - h_1 \times (h_6 \times h_5) - (h_2 \times h_3) \times h_4 = (h_6 \times h_5 + h_3 \times h_2) \times g_1 = (h_6h_5 - h_2h_3) \times g_1 = 0$, where $\langle g, h \rangle$ denotes the inner product of purely vectorial dual quaternions $g, h$. As a consequence, we have $r = r_2 \leq 2$. But we have $r \geq 2$ by Theorem 6.2, so $r = 2$. □

**Remark 6.5.** *The well-known fact that line symmetric linkages are movable can also be obtained as a corollary from Theorem 6.2. When $r = 2$, then the configuration set is defined by 2 equations in 3 variables.*

### 6.3.2 Linkages with Rank 2 and 3

In this section, we show that $r = 2$ or 3 implies either a line symmetry or another property, defined as follows. We say that $L = [h_1, \ldots, h_6]$ has the parallel property if $h_1 \parallel h_4$, $h_2 \parallel h_3$, $h_5 \parallel h_6$, maybe after some cyclic permutation of indices. In this section, we always assume that the rank of the $\lambda$-matrix of $L$ is 2 or 3.

In the following, we use the technique of generic points of algebraic curves. This simplifies the analysis a lot. Let $C$ be an irreducible algebraic curve. Let $F$ be a field such $C$ can be defined by equations over $F$ (for instance $F = \mathbb{Q}$). Following [63, Section 93], we say that some point $p \in C$ is generic if it fulfills no algebraic conditions defined by polynomials with coefficients in $F$, excerpt those that are a consequence of the equations of $C$. The existence of generic points is shown in [63, Section 93]; typically, the coordinates of a generic point are transcendental numbers.

Let $K_0 \subset K^+_{sym}$ be an irreducible non-degenerate component of the linkage $L = [h_1, \ldots, h_6]$, and let $\tau_0 = (t'_1, t'_2, t'_3)$ be a generic point of $K_0$. The configuration $\tau_0$ corresponds to a set of rotations around the joint axes. When we apply these rotations, we get new positions for the 6 lines, and we define the transformed linkage by $L' = [h'_1, h'_2, h'_3, h'_4, h'_5, h'_6]$. Note that $L$ and $L'$ represent really the same linkage, just in different initial positions.

**Lemma 6.6.** *If* $\mathrm{primal}(g'_1) = 0$*, then* $L$ *has the parallel property. Here* $\mathrm{primal}(h)$ *denotes the primal part of the dual quaternion* $h$*. More precisely, we will have* $h_1 \parallel h_4$*,* $h_2 \parallel h_3$*,* $h_5 \parallel h_6$*, in all configurations in* $K_0$*.*

*Proof.* Assume that $\mathrm{primal}(g'_1) = 0$. The parallelity of the first and fourth axis can be expressed as a set of polynomial equations in the configuration parameters $(t_1, t_2, t_3)$. These equations are fulfilled for the generic point $\tau_0$. By a well-known property of generic points it follows that they are fulfilled for all points in $K_0$. For this reason, the first and fourth axis are parallel at all position.

Let $S = [p_1, p_2, p_3, p_4, p_5, p_6]$, where $p_i = \mathrm{primal}(h'_i)$ for $i = 1, \ldots, 6$. Then $S$ is a spherical linkage with the first and fourth axis coinciding at all positions. We can separate $S$ into two 3R linkages $S_1 = [p_1, p_2, p_3]$ and $S_2 = [p_4, p_5, p_6]$. A 3R linkage is necessarily degenerate: either some angles are constant or some axes coincide. Since $t_2$ is not a constant in $K_0$, we obtain $p_2 = \pm p_3$ or $p_1 = \pm p_2$. Since $t_3$ is not a constant in $K_0$, we obtain $p_2 = \pm p_3$ or $p_1 = \pm p_3$. If $p_2 \neq \pm p_3$, then we have $p_1 = \pm p_2$ and $p_1 = \pm p_3$, a contradiction. So we obtain $p_2 = \pm p_3$. Similarly, we also have $p_5 = \pm p_6$.

Therefore, we get a linkage with $h'_1 \parallel h'_4$, $h'_2 \parallel h'_3$, $h'_5 \parallel h'_6$. Since the parallel property is fulfilled for the generic point of the configuration curve, it is fulfilled for all points in $K_0$. In particular, the original linkage $L$ has the parallel property. □

There is no $i$ such that $g'_i = 0$ for $i = 1, 2, 3$, because if $g'_i = 0$ would be true, then the lines $h'_i$ and $h'_{i+3}$ would be equal; the initial configuration was chosen generically, so the lines $h_i$ and $h_{i+3}$ would be equal for all configurations in $K_0$, and this is not possible. Moreover, it is not possible that two of $g_i$ for $i = 1, 2, 3$ have 0 primal parts. In order to prove this, we assume indirectly $\mathrm{primal}(g'_2) = 0$ and $\mathrm{primal}(g'_3) = 0$. By Lemma 6.6, we get $h_2 \parallel h_5$, $h_3 \parallel h_4$, $h_1 \parallel h_6$ and $h_3 \parallel h_6$, $h_4 \parallel h_5$, $h_1 \parallel h_2$. It follows that $L$ is a planar 6R Linkage which has mobility more than one.

Before the main theorem, we give several lemmas in the following.

**Lemma 6.7.** *Let $a, b$ be two purely vectorial dual quaternions. If $a \times b = 0$, then there is a dual number $\alpha$ such that $b = \alpha a$ or $a = \alpha b$, or the primal parts of $a$ and $b$ both vanish.*

*Proof.* Straightforward. □

In the next two proofs, we use the following argument from linear algebra. Let $1 \leq i_1 < \cdots < i_r < i_{r+1} < \cdots < i_s \leq 7$ be integers. Let $A := a_1 M_1^\dagger + \cdots + a_6 M_6^\dagger$ be some linear combination of the matrices $M_1^\dagger, \ldots, M_6^\dagger$, where $a_1, \ldots, a_6 \in \mathbb{R}$. If the vector space generated by the columns $(i_1, \ldots, i_s)$ of $M^\dagger$ is already generated by the columns $(i_1, \ldots, i_r)$ of $M^\dagger$, then the vector space generated by the columns $(i_1, \ldots, i_s)$ of $A$ is also generated by the columns $(i_1, \ldots, i_r)$ of $A$.

**Lemma 6.8.** *If $g_3' \times g_1' = g_2' \times g_1' = 0$, then $g_2' \times g_3' = 0$.*

*Proof.* We distinguish two cases.

Case I: $\mathrm{primal}(g_1') \neq 0$. By Lemma 6.7, there exist $\alpha_2, \alpha_3 \in \mathbb{D}$ such that $g_2' = \alpha_2 g_1'$ and $g_3' = \alpha_3 g_1'$, and it follows that $g_2' \times g_3' = 0$.

Case II: $\mathrm{primal}(g_1') = 0$. Then $\mathrm{primal}(g_2') \neq 0$ and $\mathrm{primal}(g_3') \neq 0$. If there exists $\alpha \in \mathbb{D}$ such that $g_3' = \alpha g_2'$, then $g_2' \times g_3' = 0$. Otherwise, $g_1'$ is a dual multiple of $g_2'$ but $g_3'$ is not, so $g_1', g_2', g_3'$ are linearly independent. Then the first three columns generate the column space of $M^\dagger$. By linear algebra, the first three columns of $A := M_1^\dagger + M_4^\dagger - M_3^\dagger - M_6^\dagger$ also generate the column space of $A$. But

$$A = [0, 0, 0, 0, 2g_3' \times g_1', 2g_3' \times g_2', *] \tag{6.6}$$

(we do not care about the last entry denoted by $*$), and it follows that $g_2' \times g_3' = 0$. □

**Lemma 6.9.** *We have $g_3' \times g_1' = g_2' \times g_1' = g_2' \times g_3' = 0$.*

*Proof.* Let $r_3$ be the dimension of the vector space generated by $g_1', g_2', g_3'$. If $r_3 = 1$, then it follows that $g_3' \times g_1' = g_2' \times g_1' = g_2' \times g_3' = 0$. If $r_3 = 2$ or $r_3 = 3$, then the vector space $V$ generated by the first 6 columns of $M^\dagger$ is already generated by the first three and one of the other three columns.

Assume, for instance, that $V$ is generated by columns $(1, 2, 3, 6)$. By linear algebra, the corresponding columns also generate the space of the first six columns of

$$M_1^\dagger + M_4^\dagger - M_2^\dagger - M_5^\dagger = [0, 0, 0, 2g_2' \times g_1', 2g_3' \times g_1', 0, *].$$

This implies $g_3' \times g_1' = g_2' \times g_1' = 0$, and by Lemma 6.8, we also get $g_2' \times g_3' = 0$.

If $V$ is generated by columns $(1,2,3,4)$, then the above linear algebra argument shows $g_1' \times g_3' = g_2' \times g_3' = 0$. The equality $g_2' \times g_1' = 0$ follows again from by Lemma 6.8, applied to the linkage $[h_3, h_4, h_5, h_6, h_1, h_2]$. The third case, when $V$ is generated by columns $(1,2,3,5)$, is also similar. □

**Lemma 6.10.** *If* primal$(g_i') \neq 0$ *for* $i = 1,2,3$*, then* $L'$ *is line symmetric.*

*Proof.* By Lemma 6.7, there exists a dual quaternion $u$ and invertible dual numbers $\alpha_1, \alpha_2, \alpha_3$ such that $g_i' = \alpha_i u$ for $i = 1,2,3$. Let $\beta := u\bar{u} \in \mathbb{D}$. Because the primal part of $u$ is nonzero, the primal part of $\beta$ is positive and $\frac{1}{\sqrt{\beta}}$ is defined. We set $l' := \frac{1}{\sqrt{\beta}}u$. Then $l'^2 = -1$ and $g_i'h_i' = h_i'^2 + h_{i+3}'h_i' = {h_{i+3}'}^2 + h_{i+3}'h_1' = h_{i+3}'g_i'$, hence $h_{i+3}' = g_i'h_i'g_i'^{-1} = l'h_i'l'^{-1}$ for $i = 1,2,3$. □

**Theorem 6.11.** *If* $r = 2$ or $3$*, then* $L$ *has a line symmetry or the parallel property.*

*Proof.* Let $K_0 \subset K_{sym}^+$ be an irreducible non-degenerate component and $\tau_0 = (t_1, t_2, t_3, t_1, t_2, t_3)$ be a generic point of $K_0$. We get $L' = [h_1', h_2', h_3', h_4', h_5', h_6']$ by applying the rotations specified in $\tau$. By Lemmas 6.8, 6.9, and 6.10, we conclude that $L'$ has a line symmetry or the parallel property. If a line symmetric linkage moves in an angle-symmetric way, then the transformed linkage is also angle-symmetric. This implies that when $L'$ is line symmetric, then $L$ is also line symmetric. On the other hand, if $L'$ has the parallel property, then parallelity holds for all points in $K_0$, in particular $L$ has the parallel property. □

**Theorem 6.12.** *If* $r = 2$*, then* $L$ *is line symmetric.*

*Proof.* By Lemma 6.4 and Theorem 6.11, we may assume that $L$ has the parallel property and $r = 2$. Let $L' = [h_1', h_2', h_3', h_4', h_5', h_6']$ be the linkage transformed to a generic position. We may assume $h_1' \parallel h_4'$, $h_2' \parallel h_3'$, $h_5' \parallel h_6'$. The primal part of $g_1'$ is 0 and the primal parts $g_2'$ and $g_3'$ are not. We define $l'$ as $\frac{1}{\sqrt{g_2'\overline{g_2'}}}g_2'$. Then $l'^2 = -1$. By Lemma 6.9, we also get $h_2' = -l'h_5'l'$ and $h_3' = -l'h_6'l'$ (see also the proof of Lemma 6.10). Moreover, $g_1'$ is a real multiple of $\epsilon l'$, and $g_1'h_1' = h_4'g_1'$. By the last equation, the primal part of $h_1' + l'h_4'l'$ is zero. The dual part of $h_1' + l'h_4'l'$ is equal to $u := g_1' - h_4' + l'h_4'l'$. The vectorial part of $ul' = g_1'l' - h_4'l' - lh_4'$ vanishes, so $u$ is a multiple of $l'$. On the other hand, the scalar product of $u$ with $l'$ also vanishes, hence $u = 0$ and $h_1' = -l'h_4'l'$. It follows that $L'$ and $L$ are both line symmetric. □

In the end of this section, we give a construction of angle-symmetric 6R linkages with the parallel property. The construction is based on the fact that we have a partially line symmetry taking $h_2$ to $h_5$ and $h_3$ to $h_6$ (see Lemma 6.7 and Lemma 6.9 above).

**Construction 6.13.** *(Angle-Symmetric 6R Linkage with Parallel Property)*

**I.** *Choose a rotation axis* $u$ *such that* $u^2 = -1$.

**II.** *Choose another rotation axis* $h_1$ *such that* $h_1^2 = -1$ *and it is perpendicular to* $u$.

**III.** *Choose two parallel rotation axes* $h_2$ *and* $h_3$ *which are not perpendicular to* $u$ *such that* $h_2^2 = h_3^2 = -1$.

**IV.** *Set* $h_4 = -uh_1u + r\epsilon u$, *where* $r$ *is a random real number.*

**V.** *Set* $h_5 = -uh_2u$ *and* $h_6 = -uh_3u$.

**VI.** *Our angle-symmetric 6R Linkage with parallel property is* $L = [h_1, h_2, h_3, h_4, h_5, h_6]$. □

**Example 6.14.** *(Angle-Symmetric 6R Linkage with Parallel Property) We set*

$$
\begin{aligned}
u &= \mathbf{i},\\
h_1 &= -\frac{7}{11}\epsilon\mathbf{i} + \mathbf{j},\\
h_2 &= \left(2\epsilon - \frac{3}{5}\right)\mathbf{i} - \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} - \epsilon\mathbf{k},\\
h_3 &= \left(-2\epsilon + \frac{3}{5}\right)\mathbf{i} + \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} + 2\epsilon\mathbf{k},\\
r &= \frac{14}{11},\\
h_4 &= \frac{7}{11}\epsilon\mathbf{i} - \mathbf{j},\\
h_5 &= \left(2\epsilon - \frac{3}{5}\right)\mathbf{i} + \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} + \epsilon\mathbf{k},\\
h_6 &= \left(-2\epsilon + \frac{3}{5}\right)\mathbf{i} - \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} - 2\epsilon\mathbf{k}.
\end{aligned}
$$

*It can be seen that the axes of* $h_1$, $h_4$ *are parallel, and the axes of* $h_2$, $h_3$ *and* $h_5$, $h_6$, *respectively, are parallel. Furthermore, the configuration curve contains a non-degenerate component:*

$$
(t_1, t_2, t_3, t_4, t_5, t_6) = \left(\frac{5}{4}t, t, t, \frac{5}{4}t, t, t\right).
$$

*Thus, we have an example of angle-symmetric 6R linkage with parallel property. The rank of* $\mathbf{M}^\dagger$ *is 3. In Figure 6.1, we present nine configuration positions of this linkage.* □

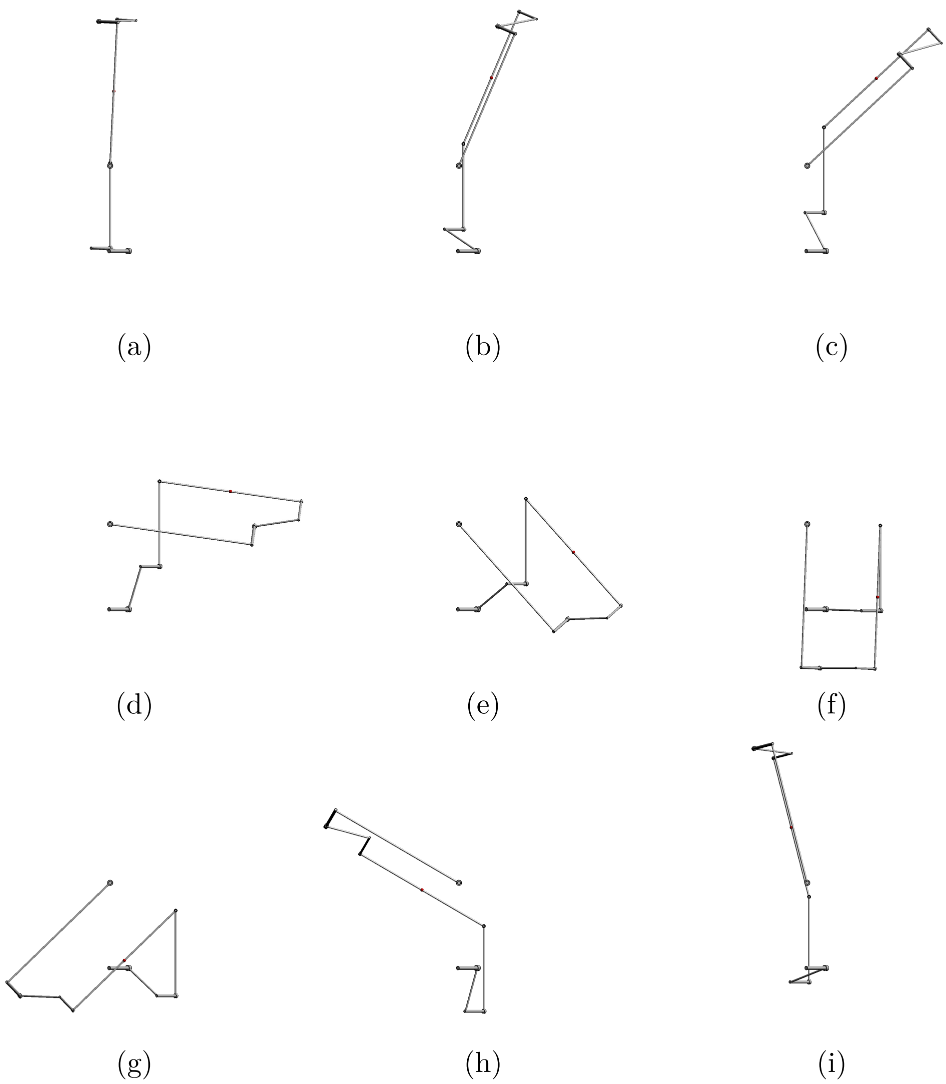


Figure 6.1: These nine pictures are different positions of the linkage in Example 6.14.

**Remark 6.15.** *A random instance of Construction 6.13 produces a linkage where $t_1$ is parametrized by a quadratic function in $t = t_2 = t_3$. This example is special because $t_1$ is linear in $t$. (There is a degenerate component of the configuration curve that is responsible for this drop of the degree.)*

### 6.3.3 Linkages with Rank 4

In this section, we show that the angle-symmetric linkages with Rank 4 are exactly those that have been constructed in [30, Example 3] by factorization of cubic motion polynomials.

Recall that a motion polynomial $P$ is a polynomial in one variable $t$ with coefficients in $\mathbb{DH}$ such that $P\bar{P}$ is a real polynomial that does not vanish identically and the leading coefficient is invertible. (Multiplication in $\mathbb{DH}[t]$ is defined by requiring that $t$ commutes with the coefficients in $\mathbb{DH}$.) Motion polynomials parametrize motions: by substituting a real number for $t$, we obtain an element in the Study quadric.

We give a brief sketch of the construction in [29, 30]. Linear motion polynomials of the form $(t - a - bh)$, $a, b \in \mathbb{R}$, $b \neq 0$, $h \in \mathbb{DH}$, $h^2 = -1$ parametrize revolutions. When we multiply three such polynomials $R_1, R_2, R_3$, we get a cubic motion polynomial $Q$. Generically, there are 6 different factorizations into linear monic polynomials, and there is one of the form $R_6R_5R_4$ such that the equations $R_1\bar{R}_1 = R_4\bar{R}_4$, $R_2\bar{R}_2 = R_5\bar{R}_5$, $R_3\bar{R}_3 = R_6\bar{R}_6$ hold. The three linear factors $R_4, R_5, R_6$ are again motion polynomials parametrizing revolutions. The six axes of $R_1, \dots, R_6$ define a closed 6R linkage; let us call it a linkage of cubic polynomial type.

We set $R_i(t) = t - a_i - b_ih_i$ for $i = 1, \dots, 6$, $a_i, b_i \in \mathbb{R}$, $b_i \neq 0$, $h_i \in \mathbb{DH}$, $h_i^2 = -1$. The equations above are equivalent to $a_i = a_{i+3}$ and $b_i^2 = b_{i+3}^2$ for $i = 1, 2, 3$. We may even assume $b_i = -b_{i+3}$; if not, we replace $h_{i+3}$ and $b_{i+3}$ by $-h_{i+3}$ and $-b_{i+3}$. We multiply $R_1R_2R_3 = R_6R_5R_4$ by $\bar{R}_4\bar{R}_5\bar{R}_6$ and get that

$$(t-a_1-b_1h_1)(t-a_2-b_2h_2)(t-a_3-b_3h_3)(t-a_1-b_1h_4)(t-a_2-b_2h_5)(t-a_3-b_3h_6)$$

is a real polynomial. This shows that the configuration curve is parametrized by

$$(t_1, t_2, t_3, t_4, t_5, t_6) = \left(\frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3}, \frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3}\right).$$

In particular, the linkage of cubic polynomial type is angle-symmetric.

Here is a converse of the above statement.

**Theorem 6.16.** *If $L$ is an angle-symmetric linkage such that the $\lambda$-matrix has rank $r = 4$, then $L$ is of cubic polynomial type.*

*Proof.* By Lemma 6.3, there exist a polynomial of the form $bt_1+ct_2+d$ that vanishes on $K_{sym}$, $b, c, d \in \mathbb{R}$, $bc \neq 0$, and the projection of $K_{sym}$ to $(t_1, t_3)$ is in the common zero set of two linear independent polynomials of bidegree $(2, 1)$. The equation of the projection is therefore a common factor of these two equations and must have bidegree smaller than $(2, 1)$. Since $K_{sym}$ has a non-degenerate component, the common factor cannot be constant in $t_1$ or $t_3$, hence it has bidegree $(1, 1)$. Because $(\infty, \infty)$ is contained in the projection, the common factor has the form $b't_1+c't_2+d'$ for $b', c', d' \in \mathbb{R}$, $b'c' \neq 0$. This allows to parametrize $K_{sym}$ with linear functions

$$(t_1, t_2, t_3) = \left(\frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3}\right)$$

for $a_1, \ldots, b_3 \in \mathbb{R}$, $b_1b_2b_3 \neq 0$. Now the linkage can be reconstructed from the two factorizations of the cubic motion polynomial

$$(t-a_1-b_1h_1)(t-a_2-b_2h_2)(t-a_3-b_3h_3) = (t-a_3+b_3h_6)(t-a_2+b_2h_5)(t-a_1+b_1h_4),$$

so it is of cubic polynomial type. □

In the analysis of the case $r = 3$, we obtained a new type of linkages (with parallel property $h_1 \parallel h_4$, $h_2 \parallel h_3$, $h_5 \parallel h_6$). It is not clear from the chapter if every linkage with parallel property is angle-symmetric. This is not the case: examples of linkages with parallel property that are not angle-symmetric can be found in [60, 23]. A complete classification of linkages with parallel property can be found in the next chapter.

# Chapter 7

# Parallel 6R Linkages

In this chapter we want to handle a subclassification problem of 6R linkages. We name it classification of parallel 6R linkages. The results presented below evolved from a collaboration with Josef Schicho and have recently been published in [43].

**Structure of the chapter** The remaining part of the chapter is set up as follows. Section 7.1 gives us the motivation for the classification of parallel 6R linkages. In Section 7.2, we give the classification of all parallel 6R linkages which contains three families. Section 7.3 contains the construction for the first family of parallel 6R linkages which has the translation property. Section 7.4 contains the second and the third families of parallel 6R linkages which have the angle-symmetric property.

## 7.1 Motivation

In this section, we first introduce the definition of a parallel 6R linkage. As always, let $L$ be a 6R linkage given by 6 lines, represented by dual quaternions $h_1, \ldots, h_6$ such that $h_i^2 = -1$ for $i = 1, \ldots, 6$. Let $K_L$ denote the set of all configurations of $L$.

If $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ is a 6R linkage with mobility 1, then we say that $L$ is a parallel 6R linkage if the axes $h_1$, $h_6$ are parallel and the axes $h_3$, $h_4$ are parallel, and the non-adjacent axes $h_2$, $h_5$ are parallel for infinitely many configurations in $K_L$. The parallelity conditions in the initial configuration can be expressed as:

$$\begin{aligned} h_1 = p_1 + \epsilon q_1, \quad h_2 = p_2 + \epsilon q_2, \quad h_3 = p_3 + \epsilon q_3, \\ h_6 = -p_1 + \epsilon q_6, \quad h_5 = -p_2 + \epsilon q_5, \quad h_4 = -p_3 + \epsilon q_4, \end{aligned} \tag{7.1}$$

where $p_i$ are the primal parts of $h_i$ and $h_{7-i}$ for $i = 1, 2, 3$, and $q_j$ are the dual parts of $h_j$ for $j = 1, \ldots, 6$.

There is a subset of $K_L$, denoted by $K_{qsym}$ which we call *quasi-angle-symmetric configuration*, defined by the additional restrictions $t_1 = t_6, t_2 = t_5, t_3 = t_4$. For all configurations in $\tau \in K_{qsym}$, the transformed lines $h_2^\tau$ and $h_5^\tau$ are again parallel. Conversely, if $K_0 \subseteq K_L$ is an irreducible component of dimension 1 that contains the initial configuration $\infty^6$ and that preserves the parallelity of the second and the fifth axis, then $K_0 \subseteq K_{qsym}$.

**Remark 7.1.** *There exist 6R linkages, with a one dimensional $K_0 \subseteq K_{qsym}$, but they are not necessary parallel 6R linkages. One possible construction of such linkage is the special case of the Double Bennett 6R linkage with additional constraint of $a = c, b = d$ in Construction 5.12.*

## 7.2 Classification

Before introducing the following lemma, we recall the definition of *coupling space* and its dimension (see Chapter 2). For a sequence $h_i, h_{i+1}, \ldots, h_j$ of consecutive joints, we define the coupling space $L_{i,i+1,\ldots,j}$ as the linear subspace of $\mathbb{R}^8$ generated by all products $h_{k_1} \cdots h_{k_s}, i \le k_1 < \cdots < k_s \le j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product is allowed, its value is 1. The *coupling dimension* $l_{i,i+1,\ldots,j}$ is the dimension of $L_{i,i+1,\ldots,j}$.

For a parallel 6R linkage $L$ in (7.1), we make a special transformation as following:

$$h_1' := P_1 h_1 \overline{P_1},\ h_6' := P_1 h_6 \overline{P_1},\ h_3' := P_2 h_3 \overline{P_2},\ h_4' := P_2 h_4 \overline{P_2},$$

where $\overline{P_i}$ denote the conjugations of $P_i$ for $i = 1, 2$, and $P_1$ and $P_2$ are translations such that $h_1', h_2' = h_2, h_3'$ meet in a common point. This is equivalent to the statement that the dimension of coupling space $L_{123}'$ is 4. Furthermore, we have $(t_1 - h_6)(t_1 - h_1) = (t_1 - h_6')(t_1 - h_1')$ and $(t_3 - h_3)(t_3 - h_4) = (t_3 - h_3')(t_3 - h_4')$, and we get the following.

We recall the definition of the *translation property* in [43]. Let $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ be a 6R linkage. There exists a translation taking $h_1$ to $h_6$, $h_2$ to $h_5$, and $h_3$ to $h_6$. Then we call that the linkage $L$ has translation property. The classification is in the following theorem.

**Theorem 7.2.** *If L is a parallel linkage, then it either has the translation property or is angle-symmetric.*

*Proof.* First, it is straightforward to show that a parallel 6R linkage $L$ and its transformed linkage $L'$ as above have the same quasi-angle-symmetric configuration space $K_{qsym}$. By Lemma 2.9, we have the dimension of the linear coupling space $l'_{654} = 4$ or 6 for the transformed parallel linkage $L'$.

If $l'_{654} = 4$, then the lines $h'_4$, $h'_5$, and $h'_6$ also meet in a common point. There is an unique translation $P$ that maps the common point of $h'_1$, $h_2$, $h'_3$ to the common point of $h'_4$, $h_5$, $h'_6$. So, $P$ maps $h'_1$ to $h'_6$, $h_2$ to $h_5$, and $h'_3$ to $h'_4$. But then, $P$ also maps $h_1$ to $h_6$ and $h_3$ to $h_4$. Conversely, assume that for six lines $h_1, \ldots, h_6$, there exists a translation taking $h_1$ to $h_6$, $h_2$ to $h_5$, and $h_3$ to $h_6$. Then the linkage $L = [h_1, \ldots, h_6]$ is mobile.

If $l'_{654} = 6$, then two cases are possible: either $L'$ is a composition of a spherical linkage $[h'_1, h_2, h'_3, h_7]$ and a Bennett linkage $[h'_6, h_5, h'_4, h_7]$, with a suitable line $h_7$, or $L'$ is a composition of a spherical linkage $[h'_1, h_2, h'_3, h_7, h_8]$ and a Goldberg 5R linkage $[h'_6, h_5, h'_4, h_7, h_8]$, with suitable lines $h_7$, $h_8$ passing through the common point of $h'_1$, $h_2$, $h'_3$. In both cases, we get $t_1 = t_3$, so the linkage $L'$ – therefore also $L$ – is angle-symmetric in the sense of Chapter 6. The first case coincides with the "rank 3" case in Chapter 6, and the second case is subsumed by the "rank 4" case in Chapter 6. □

## 7.3 Translation Property

Here is a construction of parallel 6R linkage with translation property.

All constructions in this section are given in algebraic terms, using dual quaternions. The examples have been produced by an implementation of the constructions in Maple™.

**Construction 7.3.** *(Parallel 6R Linkage with Translation Property)*

- **I.** *Choose three rotation axes $h_1, h_2, h_3$, i.e. dual quaternions such that $h_i^2 = -1$.*

- **II.** *Choose a translation $P = 1 + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, with $a, b, c$ in the set of real numbers.*

- **III.** *Set $h_4 = -Ph_3\overline{P}$, $h_5 = -Ph_2\overline{P}$ and $h_6 = -Ph_1\overline{P}$.*

- **IV.** *Our parallel 6R Linkage with translation property is $L = [h_1, h_2, \ldots, h_6]$.* □

**Example 7.4.** *A random instance of the above construction is*

$$\begin{aligned}
h_1 &= \left(\frac{7}{9} - \frac{80}{81}\epsilon\right)\mathbf{i} - \left(\frac{4}{9} + \frac{34}{81}\epsilon\right)\mathbf{j} + \left(\frac{4}{9} + \frac{106}{81}\epsilon\right)\mathbf{k},\\
h_2 &= \left(\frac{3}{5} + \frac{8}{25}\epsilon\right)\mathbf{i} - \frac{8}{5}\epsilon\mathbf{j} - \left(\frac{4}{5} - \frac{6}{25}\epsilon\right)\mathbf{k},\\
h_3 &= -\left(\frac{1}{3} - \frac{4}{9}\epsilon\right)\mathbf{i} - \left(\frac{2}{3} + \frac{4}{9}\epsilon\right)\mathbf{j} - \left(\frac{2}{3} - \frac{2}{9}\epsilon\right)\mathbf{k},\\
P &= 1 - \frac{16}{27}\epsilon\mathbf{i} - \frac{20}{27}\epsilon\mathbf{j} + \frac{8}{27}\epsilon\mathbf{k},\\
h_4 &= \left(\frac{1}{3} - \frac{148}{81}\epsilon\right)\mathbf{i} + \left(\frac{2}{3} + \frac{116}{81}\epsilon\right)\mathbf{j} + \left(\frac{2}{3} - \frac{14}{27}\epsilon\right)\mathbf{k},\\
h_5 &= -\left(\frac{3}{5} + \frac{1016}{675}\epsilon\right)\mathbf{i} + \frac{296}{135}\epsilon\mathbf{j} + \left(\frac{4}{5} - \frac{254}{225}\epsilon\right)\mathbf{k},\\
h_6 &= -\left(\frac{7}{9} - \frac{112}{81}\epsilon\right)\mathbf{i} + \left(\frac{4}{9} - \frac{46}{81}\epsilon\right)\mathbf{j} - \left(\frac{4}{9} + \frac{242}{81}\epsilon\right)\mathbf{k}.
\end{aligned}$$

*Its configuration curve is irreducible of genus 1. Its equations are:*

$$\begin{aligned}
-21t_1^2 + 9t_1^2t_2 + 25t_2^2t_1 + 6t_1t_2 - 9t_1 + 6 - 9t_2 - 15t_2^2 = 0,\\
-21 + 63t_1 + 5t_2 - 27t_1t_2 - 6t_3 + 72t_3t_2 = 0.
\end{aligned}$$

Here are the Denavit-Hartenberg parameters [20] of the above linkage. These are the orthogonal distance between two adjacent joint axes $a_{ij}$, the distance $d_i$ between the two footpoints of the two neighboring axes on the $i-th$ axis, and the twist angle between two adjacent joint axes $\alpha_{ij}$, for $i = 1, \ldots, 6$ and $j = i + 1$ (modulo 6). For any parallel linkage with translation property, the parameters fulfill the conditions

$$a_{12} = a_{56},\ a_{23} = a_{45},$$

$$d_1 = d_4 = 0,\ d_2 = d_5,\ d_3^2 + a_{34}^2 = d_6^2 + a_{61}^2,$$

$$\alpha_{34} = \alpha_{61} = 0,\ \alpha_{23} = \alpha_{45},\ \alpha_{56} = \alpha_{12}.$$

In the example, the values are

$$a_{12} = a_{56} = \frac{58\sqrt{5}}{225},\ a_{23} = a_{45} = \frac{2\sqrt{2}}{3},\ a_{34} = \frac{8\sqrt{305}}{81},\ a_{61} = \frac{8\sqrt{5}}{9},$$

$$\alpha_{34} = \alpha_{61} = 0,\ \alpha_{23} = \alpha_{45} = \arccos\left(\frac{1}{3}\right),\ \alpha_{56} = \alpha_{12} = \arccos\left(\frac{1}{9}\right),$$

$$d_1 = d_4 = 0,\ d_2 = d_5 = \frac{11}{25},\ d_3 = \frac{80}{81},\ d_6 = 0.$$

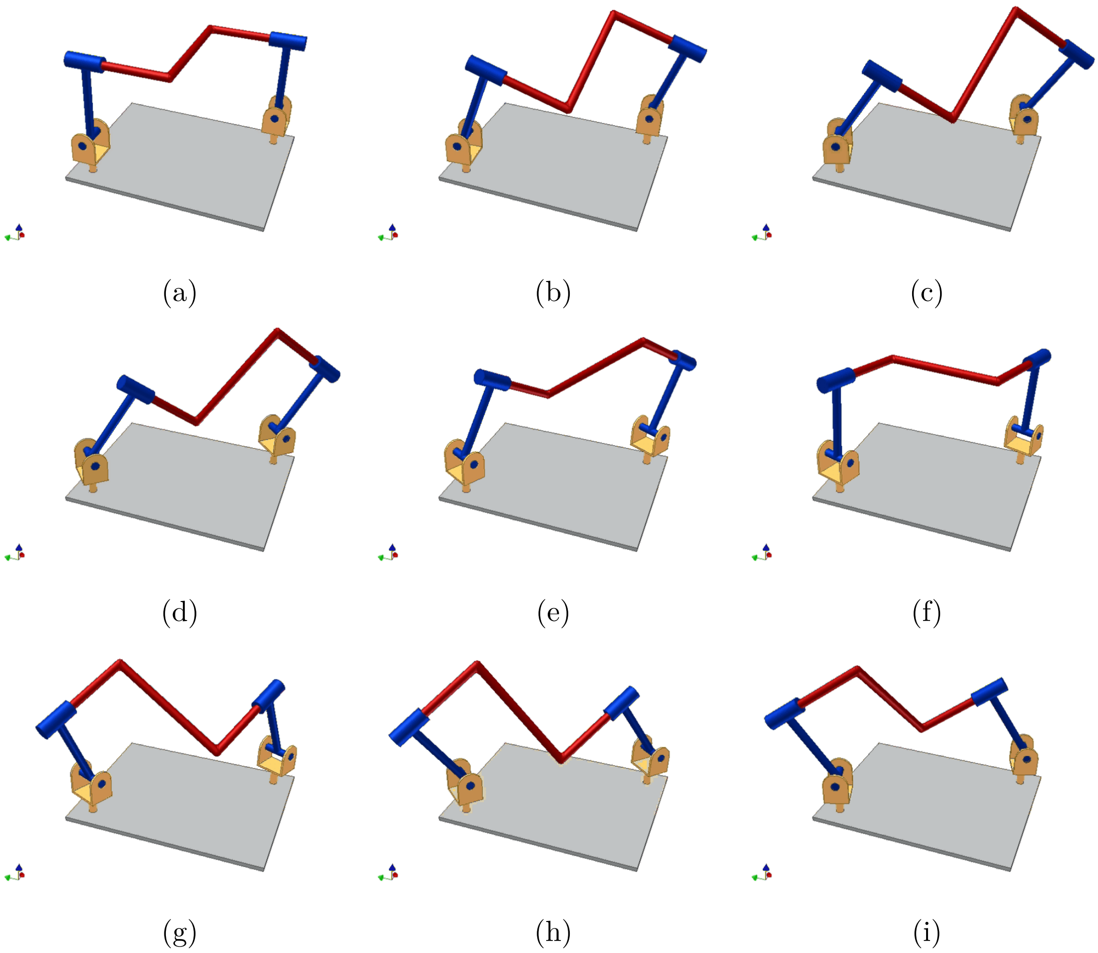


Figure 7.1: These are nine postures of a parallel linkage with translation property constructed from [23]

## 7.4 Parallel 6R Linkage with Angle-Symmetric Property

There are two constructions, corresponding to the two sub cases of angle-symmetric parallel linkages. The first one appeared in [42] and gives Parallel 6R Linkage with angle-symmetric property (type 1). Here is the second construction.

**Construction 7.5.** *(Parallel 6R Linkage with angle-symmetric property, type 2)*

***I.*** *Choose two rotation axes $h_1$ and $h_2$, i.e. dual quaternions such that $h_1^2 = h_2^2 = -1$.*

***II.*** *Choose another rotation axis $h_6$ parallel to $h_1$; the primal part of $h_6$ should be the primal part of $h_1$ times $-1$.*

***III.*** *Compute two rotation axes $m_1$ and $m_2$ such that $h_1, h_2, m_1, m_2$ form a Bennett 4R linkage. One way to do this is to use the factorization algorithm for motion polynomials in Chapter 4.*

***IV.*** *Compute two rotation axes $m_3$ and $h_5$ such that $h_6, m_2, m_3, h_5$ form a Bennett 4R linkage, and such that the configuration curve is equal to the one in step III. Again, this can be done by factorizing a motion polynomial.*

***V.*** *Choose a translation $P = 1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $b, c, d$ are real numbers.*

***VI.*** *Set $h_3 = -Pm_1\overline{P}$, $h_4 = -Pm_3\overline{P}$.*

***VII.*** *Our parallel 6R Linkage is $L = [h_1, h_2, h_3, h_4, h_5, h_6]$.* □

**Example 7.6.** *A random instance of the above construction is*

$$h_1 = \left(\frac{1}{3} - \frac{4}{9}\epsilon\right)\mathbf{i} - \left(\frac{2}{3} - \frac{2}{9}\epsilon\right)\mathbf{j} + \left(\frac{2}{3} + \frac{4}{9}\epsilon\right)\mathbf{k},$$
$$h_2 = -\left(\frac{1}{3} + \frac{8}{9}\epsilon\right)\mathbf{i} - \left(\frac{2}{3} - \frac{8}{9}\epsilon\right)\mathbf{j} + \left(\frac{2}{3} + \frac{4}{9}\epsilon\right)\mathbf{k},$$
$$h_6 = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k},$$
$$m_1 = \left(\frac{119}{411} + \frac{124340}{168921}\epsilon\right)\mathbf{i} + \left(\frac{226}{411} - \frac{172130}{168921}\epsilon\right)\mathbf{j} - \left(\frac{322}{411} + \frac{74860}{168921}\epsilon\right)\mathbf{k},$$
$$m_2 = -\left(\frac{119}{411} - \frac{100888}{168921}\epsilon\right)\mathbf{i} + \left(\frac{322}{411} - \frac{15560}{168921}\epsilon\right)\mathbf{j} - \left(\frac{226}{411} + \frac{75292}{168921}\epsilon\right)\mathbf{k},$$
$$m_3 = \left(\frac{11601824}{8614971}\epsilon - \frac{119}{411}\right)\mathbf{i} - \left(\frac{226}{411} - \frac{13771184}{8614971}\epsilon\right)\mathbf{j} + \left(\frac{322}{411} + \frac{4651040}{2871657}\epsilon\right)\mathbf{k},$$
$$h_5 = \left(\frac{1}{3} - \frac{344}{459}\epsilon\right)\mathbf{i} + \left(\frac{2}{3} - \frac{776}{459}\epsilon\right)\mathbf{j} - \left(\frac{2}{3} + \frac{316}{153}\epsilon\right)\mathbf{k},$$

$$P = 1 - \frac{2}{3}\epsilon\mathbf{i} - \frac{1}{2}\epsilon\mathbf{j} + \epsilon\mathbf{k},$$
$$h_3 = \left(\frac{119}{411} + \frac{177770}{168921}\epsilon\right)\mathbf{i} + \left(\frac{226}{411} - \frac{10388}{18769}\epsilon\right)\mathbf{j} - \left(\frac{322}{411} - \frac{79}{168921}\epsilon\right)\mathbf{k},$$
$$h_4 = -\left(\frac{119}{411} - \frac{8876894}{8614971}\epsilon\right)\mathbf{i} - \left(\frac{226}{411} - \frac{9760646}{8614971}\epsilon\right)\mathbf{j} + \left(\frac{322}{411} + \frac{3377077}{2871657}\epsilon\right)\mathbf{k}.$$

*Here we found that the configuration curve is reducible. It has one non-degenerate component in $K_{qsym}$, with rational parametrization:*

$$(t_1, t_2, t_3) = (t, t+1, t)\,.$$

*In Figure 7.2, we present twelve configuration positions of this linkage produced by Maple.* □

Here are the numeric values of the Denavit-Hartenberg parameters.

$$a_{61} = \frac{2}{3},\ a_{12} = \frac{\sqrt{2}}{3},\ a_{23} = \frac{4151\sqrt{34}}{41922},\ a_{34} = \frac{274\sqrt{17}}{459},\ a_{45} = \frac{6617\sqrt{34}}{41992},\ a_{56} = \frac{86\sqrt{2}}{153},$$
$$\alpha_{34} = \alpha_{61} = 0,\ \alpha_{23} = \alpha_{45} = \arccos\left(\frac{135}{137}\right),\ \alpha_{56} = \alpha_{12} = \arccos\left(\frac{7}{9}\right),$$
$$d_1 = d_4 = 0,\ d_2 = d_5 = \frac{923}{1224},\ d_3 = \frac{4795}{1836},\ d_6 = \frac{225}{68}.$$

We do not know the general conditions of the Denavit-Hartenberg parameters of a linkage obtained by this construction.

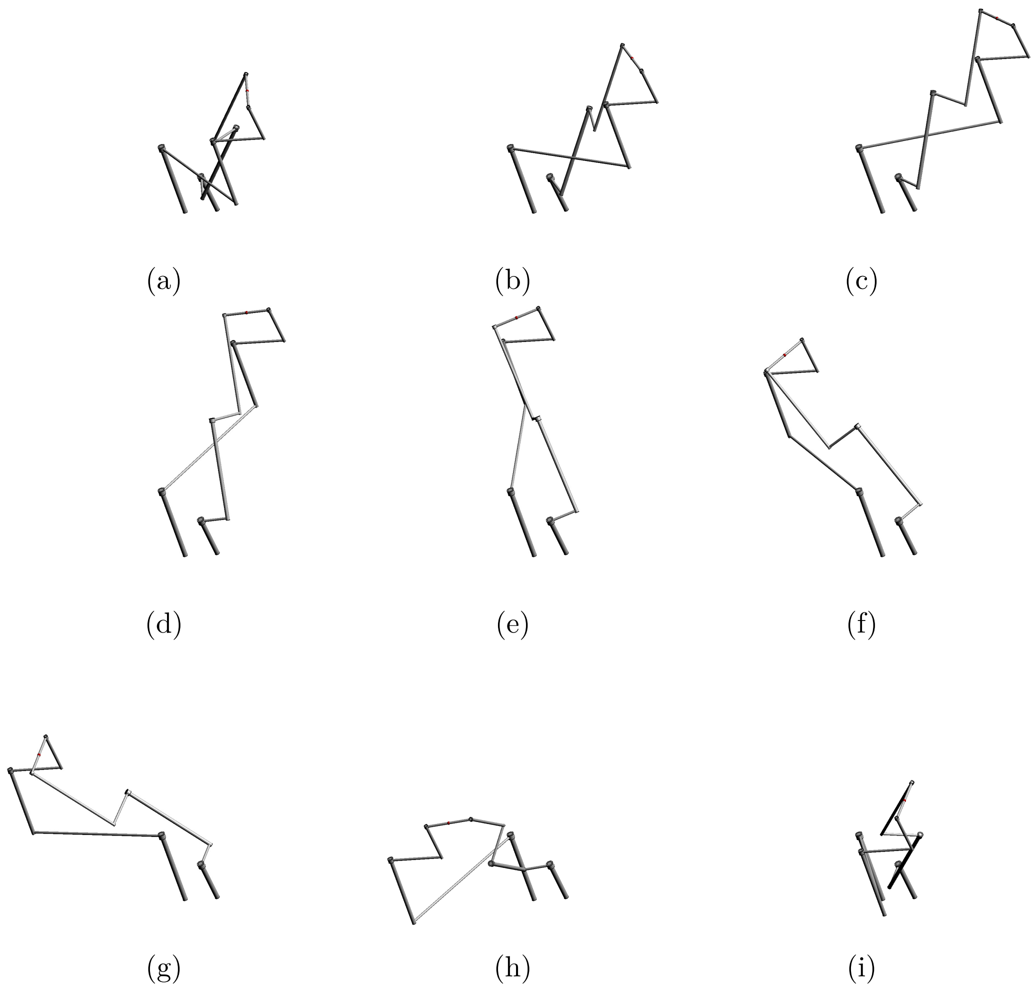



Figure 7.2: A parallel angle-symmetric linkage of type 2 (described in Example 7.6).

# Chapter 8

# Conspectuses

In this chapter, we try to make an overview of the known results and open question on mobile 6R linkages.

This is divided to two parts. The first part is concerning the mobile 6R linkages with genus bigger than zero. The second part is concerning the genus zero (rational) mobile 6R linkages. As always, the genus is the one we used in Chapter 2.

## 8.1 Genus $g > 0$

By Theorem 2.17 in Chapter 2, we get that the genus of the configuration curve of a closed 6R linkage is at most 5. We will go through all these five possible cases. We start from genus 5. If there is one twist angle of 90 degree in the Denavit-Hartenberg parameters, then we can not using the Bennett ratio to represent it. In this case, we use the orthogonal distance ($d$ between two axes) instead of the Bennett ratio. The cosines are still well defined in this situation. But the offsets are not well defined anymore. We set that the offsets of $h_1, h_2, h_3$ and $h_2, h_3, h_4$ are equal when $h_2$ and $h_3$ are parallel.

### 8.1.1 Genus $g = 4$ or $5$

By Theorem 2.18 in Chapter 2, there are only six families (only in three different bond diagrams) of closed 6R linkages with the property that the genus, in general, of their configuration curve is 5. For the genus 4 cases, they are all special cases of families of genus 5. A concrete example with genus 4 is the Bricard orthogonal linkage with $(b_1, \ldots, b_6) = (4, 3, 5, 7, 9, 8)$. We will only give the list of the genus 5 families in the following.

The first one is the Bennett's Planar Spherical Linkage [13] with bond diagram as in Figure 8.1(a). The Denavit-Hartenberg equations are

$$\begin{aligned} &b_1 = b_6 = s_1 = 0,\ c_3 = c_4 = 1 \\ &s_3 = s_4 = s_5, s_6c_5 + s_5 = s_2c_5. \end{aligned} \tag{8.1}$$

The second one is the Hooke's Double Spherical Linkage (Bennett's Planar Spherical Linkage) [13] with bond diagram as in Figure 8.1(a). The Denavit-Hartenberg equations are

$$\begin{aligned} &b_1 = b_3 = b_4 = b_6 = s_1 = s_4 = 0, \\ &s_2^2 + s_3^2 + b_2^2 - f_2^2 + 2s_2s_3c_2 = s_5^2 + s_6^2 + b_5^2 - f_5^2 + 2s_5s_6c_5. \end{aligned} \tag{8.2}$$

The third one is the Sarrus Linkage [13] with bond diagram as in Figure 8.1(a). The Denavit-Hartenberg equations are

$$c_1 = c_3 = c_4 = c_6 = 1,\ c_2 = c_5,\ s_1 = s_2 = s_6, s_3 = s_4 = s_5. \tag{8.3}$$

The fourth one is the Dietmaier's Linkage [21] with bond diagram as in Figure 8.1(b). The Denavit-Hartenberg equations are

$$\begin{aligned} &b_6 = b_1,\ b_3 = b_4,\ b_2 = b_5,\ c_2 = c_5,\ f_6 + f_1 = f_3 + f_4, \\ &s_6 = s_2,\ s_3 = s_5,\ s_1 = s_4 = 0. \end{aligned} \tag{8.4}$$

The fifth one and the sixth one are derived by the quad polynomial with maximal bonds (same bond diagrams as in Figure 8.1(b)). The Denavit-Hartenberg equations are

$$\begin{aligned} &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0, \\ &b_1c_2b_3 = b_4c_5b_6,\ b_2c_3b_4 = b_5c_6b_1,\ b_3c_4b_5 = b_6c_1b_2, \\ &f_1 = f_4,\ f_2 = f_5,\ f_3 = f_6, b_1b_3b_5 = b_2b_4b_6, \\ &b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2, \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0, \\ &b_1c_2b_3 = b_4c_5b_6,\ b_2c_3b_4 = b_5c_6b_1,\ b_3c_4b_5 = b_6c_1b_2, \\ &f_1 = f_3 = f_5,\ f_2 = f_4 = f_6,\ b_1b_3b_5f_2 = b_2b_4b_6f_1, \\ &b_1^2 + b_3^2 + b_5^2 + f_2^2 = b_2^2 + b_4^2 + b_6^2 + f_1^2. \end{aligned} \tag{8.6}$$

The well-known family (see [4]) of orthogonal linkages which can be described by the conditions

$$\begin{aligned} &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0, \\ &c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, \\ &b_1^2 + b_3^2 + b_5^2 = b_2^2 + b_4^2 + b_6^2, \end{aligned} \tag{8.7}$$

also has the property of genus 5. One can see that it is just a special case of first family in the previous two new families.

For the highest genus families, we already knew all them. Here one should also keep in mind that these six families could have special examples of lower genus.

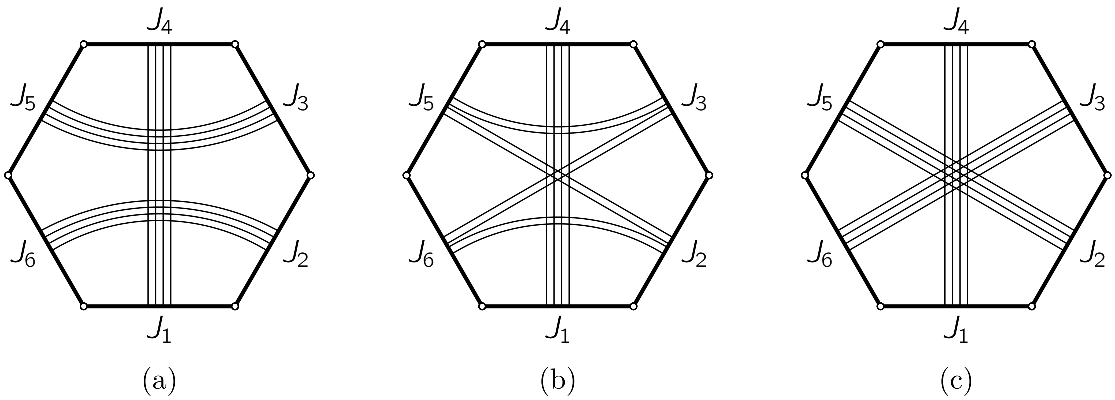



Figure 8.1: Bond diagrams for 6R linkages with genus 5, the Hooke linkage (a), the Dietmaier linkage (b), and the Orthogonal Bricard linkage (c). The joints are labeled by $J_1, \ldots, J_6$. Each bond connects two joints.

## 8.1.2 Genus $g = 3$

There are two known families of mobile 6R linkages with the property of genus equals to 3.

The first one is the Wohlhart's partially symmetric linkage [66] with bond diagram as in Figure 8.2(a). The Denavit-Hartenberg equations are

$$\begin{aligned} &b_1 = b_2,\ b_3 = b_4,\ b_5 = b_6, \\ &c_1 = -c_2,\ c_3 = -c_4,\ c_5 = c_6, \\ &s_1 + s_3 = s_5,\ s2 = s_4 = s_6 = 0. \end{aligned} \tag{8.8}$$

The second one is the Bennett-Spherical linkage with the property of genus equals to 3 with bond diagram as in Figure 8.2(b). They fulfill the following equational system derived by the quad polynomial:

$$Q_1^+ = Q_4^+, Q_1^- = Q_4^-.$$

Using Gröbner basis, we can get equational conditions:

$$\begin{aligned} &f_2^2 + c_2^2 s_2^2 - b_2^2 - s_2^2 = f_5^2 + c_5^2 s_6^2 - b_5^2 - s_6^2, \\ &c_3 + c_4 = c_2 s_2 - c_5 s_6 + s_3 - s_5 = 0, \\ &b_3 = b_4,\ b_1 = b_6 = s_1 = s_4 = 0. \end{aligned} \tag{8.9}$$

It is worth mentioning that the 6R linkage is still mobile with genus 3 when one replaces the spherical 3R linkage by a planar 3R linkage.

Using the quad polynomial, we obtained another family of mobile 6R linkages with the property of genus equals to 3 with bond diagram as in Figure 8.2(c). They fulfill the following equational system:

$$\begin{aligned}&b_1^2 + b_3^2 + b_5^2 + f_6^2 = b_2^2 + b_4^2 + b_6^2 + f_3^2,\ f_2 + f_3 = f_5 + f_6,\\ &b_2c_1 - b_3 = b_2c_3 - b_1 = b_5c_4 - b_6 = b_5c_6 - b_4 = 0,\\ &s_2 = s_3 = s_5 = s_6 = 0,\ s_1 = s_4.\end{aligned} \tag{8.10}$$

The complete classification of all mobile 6R linkages with the property of genus equals to 3 is not clear. This will be one of our future works.

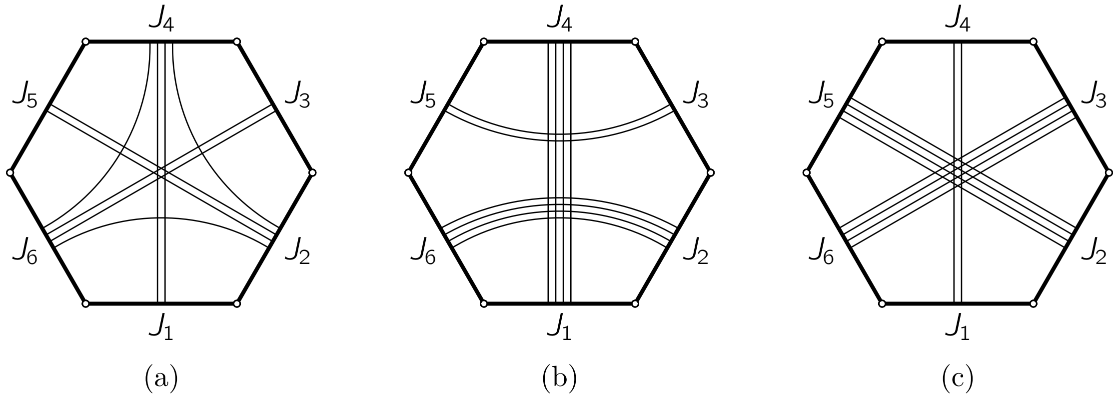



Figure 8.2: Bond diagrams for 6R linkages of genus 3, the Wohlhart's partially symmetric linkage (a), the new 6R linkage (b), and the Bennett-Spherical linkage (c). The joints are labeled by $J_1, \dots, J_6$. Each bond connects two joints.

### 8.1.3 Genus $g = 2$

In this case, we do not have a family of 6R linkages known. One could try to find special families from the higher genus cases. It is worth mentioning that the bond diagram in Figure 8.3(a) could be one possible family. But with our computer, we could not decompose the ideal to find the equational conditions which are generated by the quad polynomials.

We get the following equalities of polynomials in $\mathbb{C}[x]$ by the bond diagram in

Figure 8.3(a):

$$\begin{aligned} & Q_1^+ = Q_4^+, Q_2^+ = Q_5^+, Q_3^+ = Q_6^+, \\ & \text{Resultant}_{\text{x}}(Q_3^-, Q_6^-) = 0, \\ & \text{Resultant}_{\text{x}}(Q_3^-, Q_6^-) = 0, \\ & \text{Resultant}_{\text{x}}(Q_3^-, Q_6^-) = 0. \end{aligned} \tag{8.11}$$

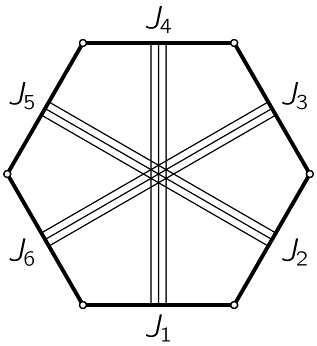



Figure 8.3: Bond diagrams for a possible mobile 6R linkages with genus 2. The joints are labeled by $J_1, \ldots, J_6$. Each bond connects two joints.

Finding the complete classification of all mobile 6R linkages with the property of genus equals to 2 will be one of main future works. Even a concrete example of genus 2 with bond diagram as in Figure 8.2 is not known. It is worth mentioning that there is a sub family of the Bricard line symmetric 6R linkage with such a bond diagram. But the genus is 1. This is constructed by taking the six lines $[h_1, h_2, h_3, h_4, h_5, h_6]$ with $h_4 = h_1, h_5 = h_2, h_6 = h_3$.

### 8.1.4 Genus $g = 1$

This case is the most interesting case for us. Because there are a lot of known families. The classification on this case is also open.

The first one is the Bricard line symmetric 6R linkage. We just list its equational conditions in Denavit-Hartenberg parameters as follows:

$$\begin{aligned} & b_1 = b_4,\ b_2 = b_5,\ b_3 = b_6, \\ & c_1 = c_4,\ c_2 = c_5,\ c_3 = c_6, \\ & s_1 = s_4,\ s_2 = s_5,\ s_3 = s_6. \end{aligned} \tag{8.12}$$

Its bond diagram is shown in Figure 8.4(a). As one might know, Bricard Octahedra can be considered as 6R linkages. There are three types of Bricard Octahedra [21]. The type I (line-symmetric) is a special case of the Bricard line symmetric 6R

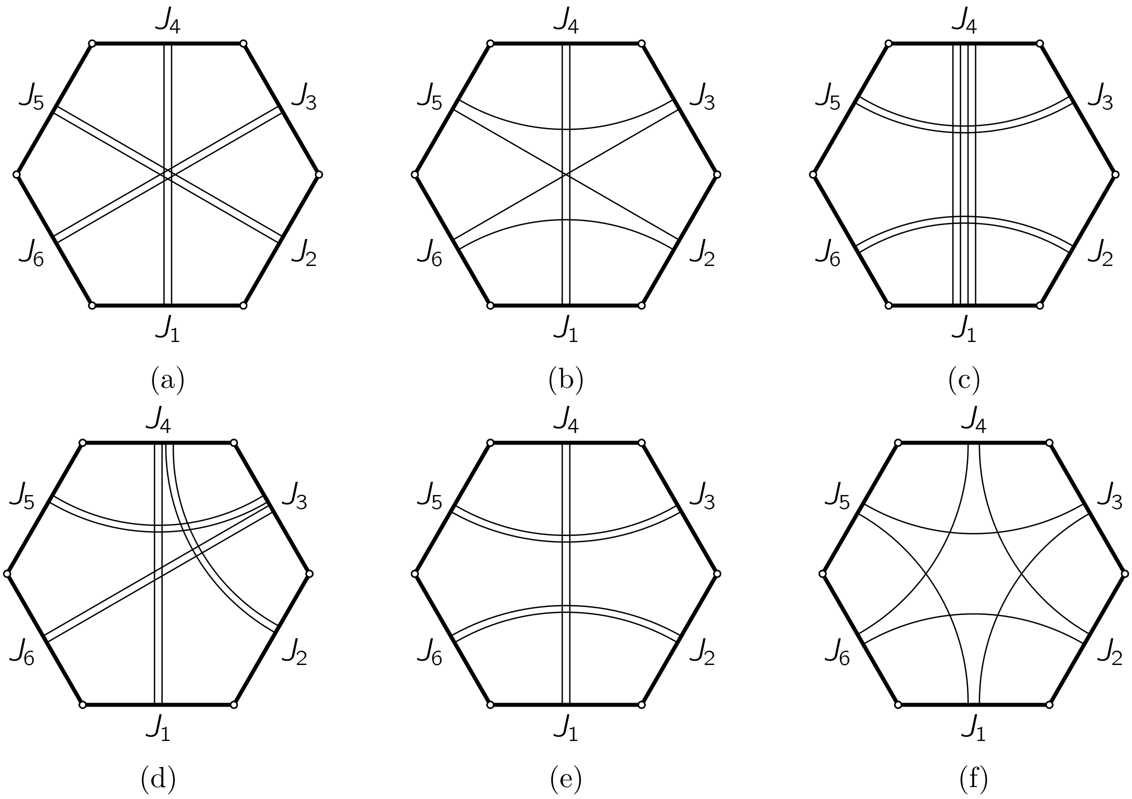



Figure 8.4: Bond diagrams for 6R linkages with genus 1, the Bricard line symmetric linkage (a), the isomerization of the Bricard line symmetric linkage with a Bennett condition (b), the Bricard plane symmetric linkage (c), the Schatz linkage (d), the Waldron hybrid (Bennett-Spherical) 6R linkage (e), the isomerization 6R linkage of the Wohlhart's partially symmetric linkage (f). The joints are labeled by $J_1, \ldots, J_6$. Each bond connects two joints.

linkage. The type III (two flat poses) is a special case of the cube linkage which has the bond diagram as in Figure 8.5(a). The type II has the same bond diagram as in Figure 8.4(a). We do not know whether it is special case of some familiy or not. We list its equational conditions in Denavit-Hartenberg parameters as follows:

$$\begin{aligned}
&b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0,\\
&2c_6s_1s_6 + s_1^2 + s_6^2 = s_5^2,\ 2c_3s_1s_3 + s_1^2 + s_3^2 = s_2^2,\\
&2c_4s_4s_5 + s_4^2 + s_5^2 = s_6^2,\ 2c_1s_1s_2 + s_1^2 + s_2^2 = s_3^2,\\
&s_1 = s_4,\ 2c_2s_2s_3 + s_2^2 + s_3^2 = 2c_5s_5s_6 + s_5^2 + s_6^2.
\end{aligned} \tag{8.13}$$

A subfamily of the Bricard line symmetric 6R linkage can be obtained by introducing a Bennett condition. By line symmetry, there is an opposite Bennett condition too. Its equational conditions in Denavit-Hartenberg parameters are changed as follows:

$$
\begin{aligned}
&b_1 = b_4 = b_3 = b_6,\ b_2 = b_5,\\
&c_1 = c_4,\ c_2 = c_5,\ c_3 = c_6,\\
&s_1 = s_4 = 0,\ s_2 = s_5,\ s_3 = s_6.
\end{aligned}
\tag{8.14}
$$

Its bond diagram is shown in Figure 8.4(b). Another family of 6R linkages with genus 1 can be obtained by a technique called "isomerization". It is introduced by Wohlhart in [68]. This is a technique for changing the equational conditions without changing the mobility for a mobile 6R linkage. The equational conditions of the isomered special Bricard line symmetric 6R linkages in Denavit-Hartenberg parameters are changed as follows:

$$
\begin{aligned}
&b_1 = b_4 = b_3 = b_6,\ b_2 = b_5,\\
&c_1 = c_3,\ c_2 = c_5,\ c_4 = c_6,\\
&s_1 = s_4 = 0,\ s_2 = s_5,\ s_3 = s_6.
\end{aligned}
\tag{8.15}
$$

Besides the isomerization technique, two more families of 6R linkages were introduced by Wohlhart in [67]. They are called Wohlharts Goldberg-Goldberg Hybrid 6R linkages. Both of them have the same bond diagram which is shown in Figure 8.4(b). The equational conditions in Denavit-Hartenberg parameters of one family are as follows:

$$
\begin{aligned}
&b_1 = b_4 = b_3 = b_6,\ b_2c_2 + b_2 = b_5c_5 + b_5,\\
&c_1 = c_3,\ \ c_4 = c_6,\\
&s_1 = s_4 = 0,\ s_2 = s_3,\ s_5 = s_6,\\
&s_5^2(1 + c_5) = (c_2 - c_5)(b_1^2 - b_2b_5) + s_2^2(1 + c_2)
\end{aligned}
\tag{8.16}
$$

Another family is as follows:

$$
\begin{aligned}
&b_1 = b_4 = b_3 = b_6,\ b_2c_2 + b_2 = b_5c_5 + b_5,\\
&c_1 = c_4,\ \ c_3 = c_6,\\
&s_1 = s_4 = 0,\ s_2 = s_3,\ s_5 = s_6,\\
&s_5^2(1 + c_5) = (c_2 - c_5)(b_1^2 - b_2b_5) + s_2^2(1 + c_2)
\end{aligned}
\tag{8.17}
$$

As one can use this technique to obtain more families from many other families (not necessary with genus 1), we will not include them all for reasons of brevity. One good review can be found in [21].

The next one is also from Bricard and it is called Bricard plane symmetric 6R linkage. The equational conditions in Denavit-Hartenberg parameters are as follows:

$$
\begin{aligned}
&b_6 = b_1,\ b_3 = b_4,\ b_2 = -b_5,\ c_2 = c_5,\ f_6 + f_1 = f_3 + f_4 = 0,\\
&s_6 = s_2,\ s_3 = s_5,\ s_1 = s_4 = 0.
\end{aligned}
\tag{8.18}
$$

Its bond diagram is shown in Figure 8.4(c). The isomerization trick does not work with this genus 1 family.

The next one is the Schatz linkage [57]. It is the first one which has been used in industry application for making the Turbula machine. There is one twist angle equals to 0 in the Denavit-Hartenberg parameters. Hence we can not using the Bennett ratio to represent it. We use the orthogonal distance ($d$). The equational conditions in Denavit-Hartenberg parameters are as follows:

$$\begin{aligned} &d_1 = \sqrt{3}d_3,\ d_2 = d_6 = 0,\ d_3 = d_4 = d_5, \\ &c_1 = 1,\ c_2 = c_3 = c_4 = c_5 = c_6 = 0, \\ &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0. \end{aligned} \tag{8.19}$$

Its bond diagram is shown in Figure 8.4(d).

The fourth one is the parallel 6R linkages with translation property in Section 7.3. There are two twist angles equal to 0 in the Denavit-Hartenberg parameters. The equational conditions in Denavit-Hartenberg parameters are as follows:

$$\begin{aligned} &d_1 = d_5,\ d_2 = d_4,\ d_3^2 + 4s_3^2 = d_6^2 + 4s_6^2, \\ &c_1 = c_5,\ c_2 = c_4,\ c_3 = c_6 = 0, \\ &s_1 = s_6,\ s_2 = s_5,\ s_3 = s_6. \end{aligned} \tag{8.20}$$

Its bond diagram is shown in Figure 8.4(a).

The next one is the Waldron's Bennett spherical (planar) hybrid 6R linkages [64, 65]. The equational conditions (planar case can be found in [21]) in Denavit-Hartenberg parameters are as follows:

$$\begin{aligned} &(1-c_1^2)(s_6^2 + 2s_5s_6c_5 + s_5^2c_5^2) = c_5^2[b_5^2(1-c_5^2) - b_1^2(1-c_1^2) + s_5^2(1-c_5^2)], \\ &s_3^2(1-c_1^2)(1-c_2^2) = [(b_5^2 + s_5^2)(1-c_6^2) - b_2^2(1-c_2^2)](1-c_5^2), \\ &s_2 + s_3c_1c_2 + s_5c_5c_6 + s_6c_6 = 0, \\ &b_1 = b_6,\ b_3 = b_4 = s_1 = s_4 = 0. \end{aligned} \tag{8.21}$$

Its bond diagram is shown in Figure 8.4(e).

There are some special cases of families with genus 1 which is obtained by specializing the families of higher genus. These are also interesting for classification if there is a Bennett condition in the equational conditions. Then we can use the isomerization trick to find new families. One interesting example is the 6R linkage with the bond diagram as in Figure 8.4(g). It is an isomerization of a special case of the Wohlhart's partially symmetric linkage (Section 4.6.7 in [21]). The equational conditions in Denavit-Hartenberg parameters are as follows:

$$\begin{aligned} &b_1 = b_3 = b_5 = -b_2 = -b_4 = -b_6, \\ &c_1 = -c_4,\ c_2 = -c_5,\ c_3 = c_6, \\ &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0. \end{aligned} \tag{8.22}$$

Its bond diagram is shown in Figure 8.4(f). The study of such examples will be another part of our future work for the classification.

## 8.2 Genus $g = 0$

In Chapter 5, we already introduced eleven families of 6R linkages which are constructed by the factorization of motion polynomials. In principle, all 6R linkages with genus 0 can be constructed by this technique. The only question is how to find a suitable motion polynomial which is a relative motion of a 6R linkage. The weakness of this technique is no equational conditions on Denavit-Hartenberg parameters. The final hope of the classification is to conclude all the equational conditions including the genus 0 case. There are already some equational conditions known in the literature [5, 17, 21]. We might need to combine other methods to obtain the equational condition. But it is clear that we are still far away from the classification even with knowing so many families! In Figure 8.5, we have listed all the possible bond diagrams for mobile 6R linkages with three or four bonds.

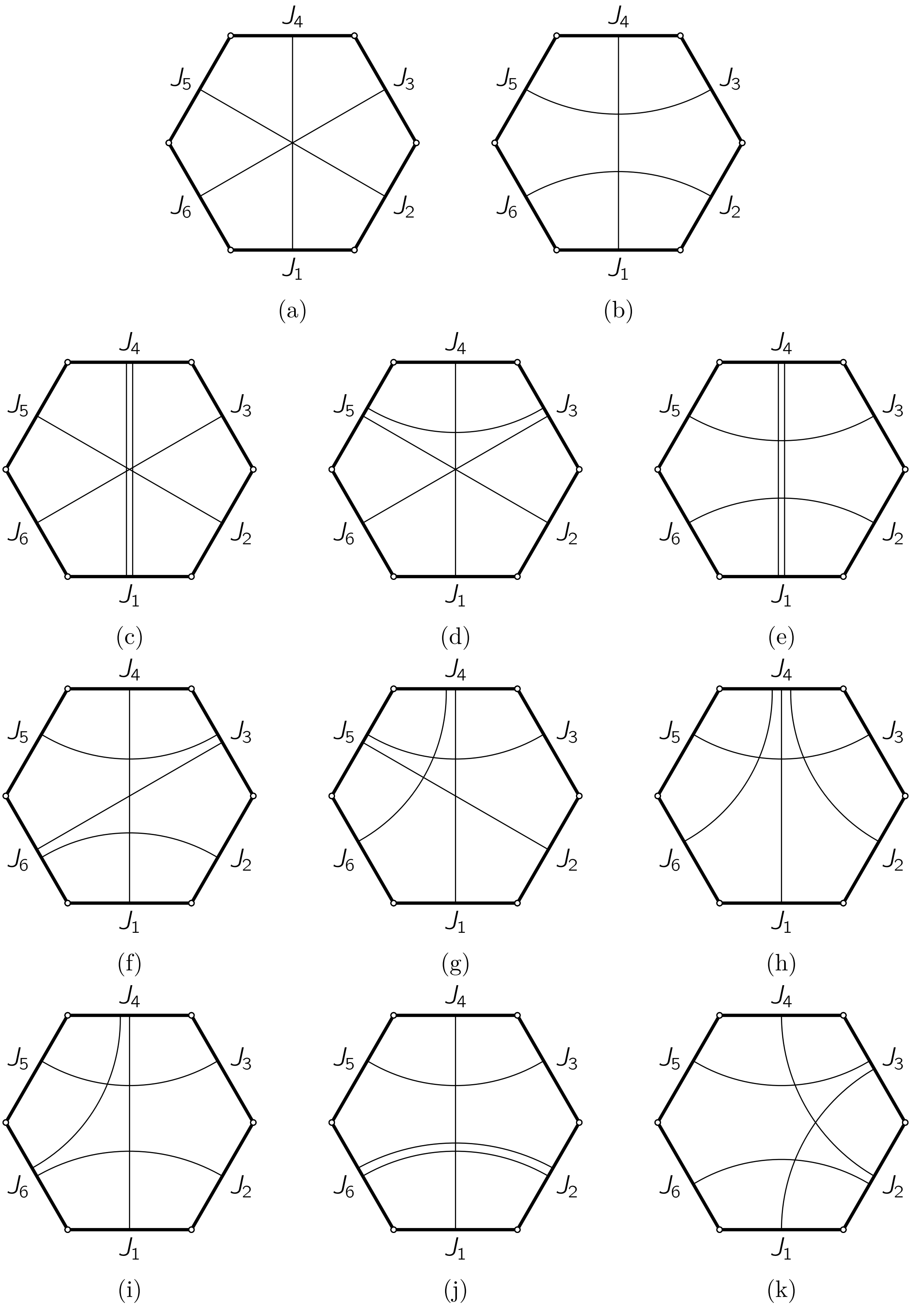



Figure 8.5: Bond diagrams of 6R linkages with three or four bonds

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# Index

# Curriculum Vitae

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**Nationality:** China

**Year of Birth:** 1984

**Place of Birth:** Hebei, China

## Education

| | |
|---|---|
| 2003–2007 | Studies in Mathematics and Applied Mathematics at Hebei University of Technology, Tianjin China. |
| 2008–2011 | Master-PhD Studies in Applied Mathematics at Key Laboratory of Mathematics Mechanization, Beijing China. |
| 2012–now | PhD Student in the Doctoral Program "Computational Mathematics", at JKU, Linz Austria. |

# Awards

| | |
|---|---|
| 2014 | Chinese government award for outstanding self-financed students abroad. |
| 2004–2007 | Scholarship of the second/third class of Hebut |
| 2006 | Scholarship of Hebei Province |

# Teaching experience

1. **Six hours lecture** in
   **Factorization of Motion Polynomials and Bonds Theory,**
   Joanneum Research Institute for Robotics and Mechatronics,
   October 12-13, 2015, Klagenfurt, Austria.

# Publications

17. *Spatial Straight Line Linkages by Factorization of Motion Polynomials* (2016), Z. Li, J. Schicho and H.-P. Schröcker. Journal of Mechanisms and Robotics, Vol 8.

16. *The Rational Motion of Minimal Dual Quaternion Degree With Prescribed Trajectory* (2016), Z. Li, J. Schicho and H.-P. Schröcker. Computer Aided Geometric Design, Vol 41, Pages 1-9.

15. *Kempe's Universality Theorem for Rational Space Curves* (2015), Z. Li, J. Schicho, H.-P. Schröcker. arXiv:1509.08690

14. *Factorization of Motion Polynomials* (2015), Z. Li, J. Schicho, H.-P. Schröcker. arXiv:1502.07600

13. *Planar Linkages Following a Prescribed Motion* (2015), M. Gallet, C. Koutschan, Z. Li G. Regensburger, J. Schicho and N.Villamizar arXiv:1502.05623

12. *7R Darboux Linkages by Factorization of Motion Polynomials* (2015), Z. Li, J. Schicho and H.-P. Schröcker. In Proceedings of the 14th IFToMM World Congress, Pages 841-847.

11. *Factorization of Rational Motions: A Survey with Examples and Applications* (2015), Z. Li, T. Rad, J. Schicho and H.-P. Schröcker. In Proceedings of the 14th IFToMM World Congress, Pages 833-840.

10. *A technique for deriving equational conditions on the Denavit-Hartenberg parameters of 6R linkages that are necessary for movability* (2015), Z. Li and J. Schicho Mechanism and Machine Theory, Vol 94, Pages 1-9.

9. *An algebraic study of linkages with helical joints* (2015), H. Ahmadinezhad, Z. Li and J. Schicho Journal of Pure and Applied Algebra, Vol 219, Pages 2245-2259.

8. *The Theory of Bonds II: Closed 6R Linkages with Maximal Genus* (2015), G. Hegedüs, Z. Li, J. Schicho, H.-P. Schröcker Journal of Symbolic Computation, Vol 68, Pages 167-180.

7. *Sharp Linkages* (2014), Z. Li in Advances in Robot Kinematics edited by J. Lenarčič and O. Khatib, Pages 131-138.

6. *Three Types of Parallel 6R Linkages* (Jan 2014) Z. Li and J. Schicho In Computational Kinematics, Pages 111-119.

5. *Classification of Angle-symmetric 6R Linkages* (Dec 2013) Z. Li and J. Schicho Mechanism and Machine Theory, Vol 70, Pages 372-379.

4. *A Coprime Blur Scheme for Data Security in Video Surveillance* (Dec 2013) C. Thorpe, F. Li, Z. Li, Z. Yu, D. Saunders, J. Yu IEEE Transactions on Software Engineering, Vol 35, Pages 3066-3072.

3. *Computing the Nearest Singular Univariate Polynomials with Given Root Multiplicities* (Apr 2012) Z. Li and L. Zhi Theoretical Computer Science, Vol 479, Pages 150-162.

2. *A Theory of Coprime Blurred Pairs* (Nov 2011) F. Li, Z. Li, D. Saunders, J. Yu Proceedings of the ICCV'11, Pages 217-224.

1. *Blind Image Deconvolution via Fast Approximate GCD* (Jul 2010) Z. Li and Z. Yang and L. Zhi Proceedings of the ISSAC '10, Pages 155-162.

## Recent conference and workshop talks

11. *A Collision Free Design for Planar Linkages Following A Prescribed Motion*, June 11, 2015 - CGTA, Kefermarkt, Austria.

10. *Closed Linkages with Six Revolute Joints*, November 06, 2014 - DK-Hearing, Vienna, Austria.
9. *Closed Linkages with Six Revolute Joints*, September 26, 2014 - DK-Hearing-Seminar, Strobl, Austria.
8. *Sharp Linkages*, June 29, 2014 - Advances in Robot Kinematics, Ljubljana, Slovenia.
7. *Sharp Linkages*, January 16, 2014 - Algebraic Geometry Seminar, Linz-Wien Workshop, Yspertal -Austria, Austria.
6. *RCCR Linkages*, November 25-29, 2013 - Special Semester 2013, Workshop 3, RICAM, Austria
5. *Movable Spatial 6R Linkages*, October 2, 2013 - DK-Hearing-Seminar, Strobl, Austria.
4. *Three Types of Parallel 6R Linkages*, May 12, 2013 - CK2013, Barcelona, Spain.
3. *Three Types of Parallel 6R Linkages*, February 27, 2013 - Algebraic Geometry Seminar, Linz-Wien Workshop, Laimbach am Ostrong, Austria.
2. *Classification of Angle-symmetric 6R Linkages*, October 3-5, 2012, DK Seminar, Strobl, Austria.
1. *Blind Image Deconvolution via Fast Approximate GCD*, July 25-28, 2010 - ISSAC10, Munich, Germany.

## Computer Skills

**Maple**, Proficient (6 Years)

## Referee activities

Mechanism and Machine Theory (MMT), Journal of Symbolic Computation (JSC), Theoretical Computer Science (TCS).