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# Nonstandard Sobolev Spaces for Preconditioning Mixed Methods and Optimal Control Problems

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# Abstract

The main focus of this thesis is on the construction of efficient solvers for two types of problems that fit into the class of PDE-constraint optimization:

- Distributed optimal control problems with a tracking-type cost functional and linear state equations
- Mixed methods for elliptic boundary value problems

A solution of an optimization problem can be computed via the first-order optimality conditions, also called the optimality system. For the type of problems considered here the optimality system is linear and has saddle point form. After discretization we end up with a large scale linear system (again in saddle point form) for which an efficient solver is required.

For the construction of an efficient solver we follow an approach which is called operator preconditioning. There efficient preconditioners are constructed based on the fact that the involved operator equation is well-posed in a (nonstandard) Sobolev space $X$.

We present two techniques for finding this space $X$ for a problem in saddle point form:

- Interpolation technique
- Lagrangian multiplier technique

This techniques are demonstrated for four model problems:

1. For the first biharmonic boundary value problem a well-posed continuous mixed variational formulation is derived, which is equivalent to a standard primal variational formulation on arbitrary polygonal domains. Based on a Helmholtz-like decomposition for an involved nonstandard Sobolev space it is shown that the biharmonic problem is equivalent to three second-order elliptic problems, which are to be solved consecutively. Two of them are Poisson problems, the remaining one is a planar linear elasticity problem with Poisson ratio 0. The Hellan-Herrmann-Johnson mixed method and a modified version are discussed within this framework. The unique feature of the proposed solution algorithm for the Hellan-Herrmann-Johnson method is that it is solely based on standard Lagrangian finite element spaces and standard multigrid methods for second-order elliptic problems. Therefore, it is of optimal complexity.

2. For the distributed optimal control problem with time-periodic Stokes equations a well-posed continuous mixed formulation of the corresponding optimality system is derived. Based on the involved parameter-dependent norms of the continuous problem, a practically efficient block-diagonal preconditioner is constructed, which is robust with respect to all model and mesh parameters. The theoretical results are illustrated by numerical experiments with the preconditioned minimal residual (PMINRES) method.

3. & 4. In addition we demonstrate the interpolation technique and Lagrangian multiplier technique for two further problems:

   – the Ciarlet-Raviart mixed method for the first biharmonic boundary problem

   – the distributed optimal control problem with time-periodic parabolic equations

## Zusammenfassung

Der Schwerpunkt dieser Arbeit liegt auf der Konstruktion von effizienten Lösern für zwei Problemtypen aus der Klasse der Optimierungsprobleme mit Nebenbedingung in Form von partiellen Differentialgleichungen:

- Optimale Steuerungsprobleme mit einem quadratischen Kostenfunktional und unbeschränkter Kontrolle
- Gemischte Methoden für elliptische Randwertprobleme

Eine Lösung des Optimierungsproblems kann über die Optimalitätsbedingungen erster Ordnung, auch als Optimalitätssystem bezeichnet, berechnet werden. Für die hier untersuchten Problemtypen ist das Optimalitätssystem linear und besitzt eine Sattelpunktsform. Nach der Diskretisierung erhalten wir ein großes lineares Gleichungssystem (wieder in Sattelpunktform), für welches ein effizienter Löser erforderlich ist.

Für die Konstruktion eines effizienten Lösers folgen wir einer Herangehensweise die als Operator-Präkonditionierung bezeichnet wird. Dabei werden effiziente Präkonditionierer basierend auf der Tatsache konstruiert, dass die involvierte Operatorgleichung in einem (Nicht-Standard) Sobolevraum $X$ gut gestellt ist.

Wir stellen zwei Techniken zur Bestimmung des Raumes $X$ für Probleme in Sattelpunktform vor:

- Interpolations-Technik
- Lagrange-Multiplikator-Technik

Diese Techniken werden anhand von vier Modellproblemen demonstriert:

1. Für das erste biharmonische Randwertproblem wird eine gut gestellte kontinuierliche gemischte variationelle Formulierung hergeleitet. Diese Formulierung besitzt weiters die Eigenschaft, dass sie auf polygonalen Bereichen äquivalent zu einer primalen standardmäßigen variationellen Formulierung ist. Basierend auf einer Helmholtz-artigen Zerlegung für den involvierten Sobolevraum $X$ lässt sich zeigen, dass das biharmonische Problem äquivalent zu drei (hintereinander zu lösenden) elliptischen Gleichungen zweiter Ordnung ist. Zwei dieser Probleme sind Poisson Probleme, das dritte Problem ist ein planares Elastizitätsproblem mit Poissonzahl 0. In Rahmen dessen

diskutieren wir die Hellan-Herrmann-Johnson gemischte Methode und eine modifizierte Version davon. Die einzigartige Eigenschaft der vorgestellten Lösungsmethode für die Hellan-Herrmann-Johnson Methode ist, dass sie ausschließlich auf herkömmliche Lagrange-Finite-Element-Räumen und standardmäßigen Multigrid Methoden beruht. Infolgedessen besitzt die Lösungsmethode optimale Komplexität.

2. Für das optimale Kontrollproblem für die Stokes Gleichungen mit unbeschränkter Kontrolle im zeitperiodischen Fall wird eine gut gestellte kontinuierliche gemischte variationelle Formulierung des entsprechenden Optimalitätssystems hergeleitet. Basierend auf den involvierten parameterabhängigen Normen des kontinuierlichen Problems konstruieren wir einen praktisch effizienten Block-Diagonal-Präkonditionierer, welcher robust bezüglich aller Modell- und Gitterparameter ist. Die theoretischen Resultate werden anhand von numerischen Experimenten mit dem präkonditionierten MINRES-Verfahren illustriert.

3. & 4. Abschließend demonstrieren wir die Interpolations-Technik und die Lagrange--Multiplikator-Technik anhand von zwei weiteren Problemen:

   - die Ciarlet-Raviart gemischte Methode für das erste biharmonische Randwertproblem
   - das Optimale Kontrollproblem für die parabolischen Gleichungen mit unbeschränkter Kontrolle im zeitperiodischen Fall

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# Contents

# Chapter 1

# Introduction

Various complex processes and systems in natural sciences, engineering and in many other areas can be described mathematically by partial differential equations (PDEs) or systems of PDEs. Often the focus is on the optimization or on the control of the underlying process. In this case the models lead to PDE-constrained optimization problems. In general, PDE-constrained optimization problems are characterised by an objective functional, that has to be minimized, and a constraint given by a PDE (or a system of PDEs), see, e.g., [18, 50, 89].

In this thesis we focus on two types of problems, which fit into the class of PDE-constrained optimization:

- Distributed optimal control problems with a quadratic cost functional, see, e.g., [58, 89]. The goal is to steer the state variable to some given desired state and control this by some cost term, i.e., a term that measures the costs of a control variable. The PDE-constraint (in general called the state equation), which models the underlying process to be controlled, couples the state and the control variable. In this thesis, we focus on optimal control problems with linear state equations. Some examples of optimal control problems are the optimal control of heating processes, fluid flows and deformation of media. For a wide range of applications we refer to [43, 89].

- Mixed methods: These problems fit also into the concept of PDE-constrained optimization: First the standard primal variational formulation is reformulated as an unconstrained optimization problem. In a second step one introduces an auxiliary variable, which leads to a constraint in form of a PDE and transforms the unconstrained optimization problem to a PDE-constrained optimization problem. Methods of this type are applied, for example, in linear elasticity [31] and in fluid mechanics [38, 90].

A solution of the optimization problem can be computed via the first order optimality conditions, also called the optimality system or Karush-Kuhn-Tucker (KKT) system. The optimality system for the considered type of problems is linear and in saddle point form. In order to handle the optimality system numerically there are two approaches available:

the optimize-then-discretize approach and the discretize-then-optimize approach. In the optimize-then-discretize approach, one first derives the optimality system on the continuous level and then discretize this system. For the discretize-then-optimize approach it is the other way round. In [33] it was shown that the first approach is strongly consistent, i.e., the discretized system is also satisfied if the discretized variables are replaced by the corresponding continuous ones. On the contrary to the first approach, in the second approach the linear system is not strongly consistent in general. In this thesis we use the optimize-then-discretize approach. As a result of the discretization process we obtain a large scale linear system in saddle point form. Subsequently efficient solvers for linear saddle point problems are required.

Many books and articles deal with efficient solvers for saddle point problems. Their efficient solution is a major challenge, because of their indefiniteness and poor spectral properties. For a detailed discussion of solution methods for saddle point problems we refer to [12].

Iterative methods, which have been specially constructed for saddle point problems are the Uzawa method and its variants, see, e.g., [3, 12]. In recent years multigrid techniques, which are well developed for elliptic problems, see, e.g., [19, 44], have been developed for saddle point problems as so-called all-at-once techniques, see, e.g., [15, 16, 17, 80, 84, 86, 87], whereby the most challenging part is the construction of appropriate smoothers.

Another class of methods are the Krylov subspace methods, see, e.g., [78]. Probably the most well-known and best understood Krylov subspace method is the conjugate gradient (CG) method, see, e.g., [49], developed for symmetric and positive definite problems. In [21] and [81] techniques for the reformulation of a saddle point system as a self-adjoint and positive definite problem were presented, for which CG can be applied, see also [76] for generalisations. Other well known Krylov subspace methods are the generalized minimal residual method (GMRES), see [79], designed for general nonsingular problems and the minimal residual method (MINRES), designed for symmetric and nonsingular problems, see [72].

For an efficient solution of the discrete saddle point systems with Krylov subspace methods, these methods are usually equipped with a preconditioning strategy that improves the spectral properties. One approach is to distribute the arising difficulties between the Krylov subspace method and the preconditioner, i.e., certain difficulties are handled by the (modified) Krylov subspace method and the remaining difficulties are handled by the preconditioner, see, e.g., [71]. We are considering Krylov subspace methods without any modification. Therefore, all difficulties should be treated by the construction of the preconditioner.

There are several construction techniques available for efficient preconditioners for saddle point problems, see, e.g., [12].

A very common preconditioning strategy is the Schur complement preconditioning, which can be applied on the algebraic level under certain requirements. Exact Schur complement preconditioners have very good spectral properties, see, e.g., [57, 66], but in general their practical usage is not recommended, because of the high computational costs for the application of their inverse. To circumvent this problem one replaces the exact Schur complement by an approximation, which keeps the nice spectral properties and whose inverse can be applied efficiently. Such approximations are used as building blocks for block-diagonal preconditioners, see, e.g., [77, 82], block-triangular preconditioners, see, e.g., [22, 35, 75] and symmetric indefinite preconditioners, see, e.g., [9, 34].

In this thesis we consider a very popular preconditioner construction technique, the so-called operator preconditioning, discussed in [51, 63] and used, e.g., in [51, 8, 48, 68]. There, symmetric and positive definite block-diagonal preconditioners are constructed based on the fact that the involved operator equation is well-posed in a Sobolev space $X$. Usually the construction of a proper Sobolev space $X$ is a big challenge.

The focus of this thesis is on construction techniques for a proper Sobolev space $X$ for saddle point problems. In particular we present two different techniques, the so-called interpolation technique and the Lagrangian multiplier technique.

The interpolation technique is based on two different continuous mixed variational formulations for the same system of PDEs: For the first formulation one assumes no smoothness for the primal variable and for the second formulation one assumes no smoothness for the dual variable. Then interpolation theory allows the computation of a new mixed formulation, for which the smoothness is evenly distributed for the primal variable and the dual variable. The new formulation is again well-posed. An analog technique on the algebraic level for the construction of preconditioners was used in [64, 92].

The Lagrangian multiplier technique can only be applied to mixed methods. Starting point is the above described reformulation of the corresponding primal variational formulation as a constrained optimization problem. The aim is now to find a proper Sobolev space for the Lagrangian multiplier such that the optimality system is well-posed.

We demonstrate the both techniques on four model problems:

1. For the first biharmonic boundary value problem a new well-posed continuous mixed variational formulation is derived on arbitrary polygonal domains. An additional feature of the mixed variational formulation is the equivalence to a standard primal variational formulation without any further assumptions on the domain, like convexity.

   A new Helmholtz-like decomposition for the involved nonstandard Sobolev space allows the new decomposition of the continuous problem in three second order elliptic problems, which are to be solved consecutively. The first and the last problem are Poisson problems with Dirichlet conditions and the second problem is a pure traction problem in planar linear elasticity with Poisson ratio 0.

   As discretization method we study the Hellan-Herrmann-Johnson (HHJ) finite element method (in this case a non-conforming method) see [46, 47, 53], which is strongly related to the non-conforming Morley finite element, see [65, 2]. Moreover, a new conforming modification of the HHJ method is presented.

Similar to the continuous problem, a new Helmholtz decomposition for the approximation space of the primal variable will be shown. This allows, as in the continuous case, to solve the system consecutively, by solving the discretized versions of the second order elliptic problems mentioned above. Therefore, the implementation requires only manipulations with standard conforming Lagrangian finite elements for second-order problems. The proposed preconditioners are standard multigrid preconditioners for second-order problems, which lead to mesh-independent convergence rates.

There are many alternative approaches for biharmonic problems discussed in literature. Finite element discretizations range from conforming and classical non-conforming finite element methods for fourth-order problems, discontinuous Galerkin methods for fourth-order problems to various mixed methods, see, e.g., [31, 36, 24, 10], and the references cited there. Solution techniques proposed for the linear systems, which show mesh-independent or nearly mesh-independent convergence rates are typically based on two-level or multilevel additive or multiplicative Schwarz methods (including multigrid methods), see, e.g., [74, 23, 91, 45], and the references cited there.

We are not aware of any other approach, which is based solely on standard components for second-order elliptic problems and for which optimal convergence behavior could be shown.

2. For the distributed optimal control problem with time-periodic Stokes equations a new continuous well-posed mixed variational formulation for the corresponding optimality system is derived. Based on the new parameter dependent norms for the continuous problem, a new practical efficient block-diagonal preconditioner is constructed. The preconditioner is robustness with respect to all model and mesh parameters. The theoretical results are illustrated by numerical experiments with the preconditioned minimal residual (PMINRES) method.

3. A well-posed continuous mixed variational formulation of the Ciarlet-Raviart mixed method for the first biharmonic boundary value problem was already presented in [92]. Here we give a new derivation of the occurring spaces, using the interpolation and Lagrangian multiplier technique.

4. For the distributed optimal control problem with time-periodic parabolic equations a new well-posed continuous variational formulation is derived.

### Organization of the thesis

In Chapter 2 we introduce abstract problems in saddle point form, including stability theory, and give a brief introduction to operator preconditioning.

Chapter 3 is the center part of this thesis. Here we give a detailed illustration of the interpolation technique and the Lagrangian multiplier technique for a mixed method for the first biharmonic boundary value problem. As a result of both techniques, we obtain the same well-posed mixed variational formulation. We close Chapter 3 with a further application of both techniques applied to the Ciarlet-Raviart mixed method for the first biharmonic boundary value problem.

In Chapter 4 we derive a well-posed continuous variational formulation for the corresponding optimality system for two model problems from optimal control, distributed time-periodic Stokes control and distributed time-periodic parabolic control.

In Chapter 5 a Helmholtz-like decomposition and a trace operator for the involved nonstandard Sobolev space $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ are derived.

At the beginning of Chapter 6 we discuss the HHJ mixed finite element method. Further a discrete version of the Helmholtz-like decomposition derived in Chapter 5, and a modified version of the HHJ method are presented. In the second part of this chapter we discuss the discretization of the distributed optimal control problem with the time-periodic Stoke equations, quite in the spirit of the corresponding continuous problems as presented in Chapter 4.

In the first part of Chapter 7 we illustrate the theoretical results for the HHJ discretization method and its conforming modified version. In the second part of this chapter we illustrate the theoretical results for the finite element method for the distributed optimal control problem with the time-periodic Stoke equations by some numerical examples with the preconditioned minimal residual method.

**Publications of the author**

Parts of this work have been published:

1. W. Krendl, V. Simoncini, and W. Zulehner. *Stability estimates and structural spectral properties of saddle point problems*, Numerische Mathematik, 124(1), pp. 183-213, 2013, see [54].

This article contain parts of the theoretical results on the optimal control problems.

2. W. Krendl, V. Simoncini, and W. Zulehner. *Efficient preconditioning for an optimal control problem with the time-periodic Stokes equations*, Numerical Mathematics and Advanced Applications, 2014, (to appear), see [55].

The focus of this article is on the computational aspects of the proposed preconditioner for the distributed optimal control problem with time-periodic Stokes equations.

3.

W. Krendl, and W. Zulehner. *The Hellan-Herrmann-Johnson method for Biharmonic problems: Mapping properties and preconditioning*, 2014, (submitted), see [56].

This article contains the results on the mixed method for the first biharmonic boundary value problem.

# Chapter 2

# Operator preconditioning

Here we follow essentially the ideas of [51] and [64].

In this thesis we consider problems of saddle point form: Find $u \in V$ and $p \in Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v \rangle \quad \text{for all } v \in V, \\ b(u,q) \ - c(p,q) &= \langle g, q \rangle \quad \text{for all } q \in Q, \end{aligned} \tag{2.1}$$

with Hilbert spaces $V$ and $Q$, $f \in V^\star$ and $g \in Q^\star$, and bounded bilinear forms $a : V \times V \to \mathbb{R}$, $b : V \times Q \to \mathbb{R}$, and $c : Q \times Q \to \mathbb{R}$. Here $H^*$ denotes the dual of a Hilbert space $H$ and $\langle \cdot, \cdot \rangle_{H^\star \times H}$ (in short $\langle \cdot, \cdot \rangle$) denotes the duality product. If $H = \mathbb{R}^n$, we use $\langle \cdot, \cdot \rangle$ for the Euclidean inner product.

Moreover, we assume that the bilinear forms $a$ and $c$ are symmetric, i.e.

$$a(u,v) = a(v,u) \quad \text{for all } u,v \in V \quad \text{and} \quad c(p,q) = c(q,p) \quad \text{for all } p,q \in Q. \tag{2.2}$$

Let $X = V \times Q$, equipped with the standard product norm

$$\|(v,q)\|_X = \left(\|v\|_V^2 + \|q\|_Q^2\right)^{1/2}$$

for $(v,q) \in X$, where $\|v\|_V$ and $\|q\|_Q$ denote the norms in $V$ and $Q$, respectively. If the linear operator $\mathcal{A}$: $X \to X^*$ is introduced by

$$\left\langle \mathcal{A} \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle = a(u,v) + b(v,p) + b(u,q) - c(p,q), \tag{2.3}$$

the mixed variational problem (2.1) can be rewritten as a linear operator equation

$$\mathcal{A} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{2.4}$$

Here and in the rest of this thesis, $(v,q)$ and $\begin{bmatrix} v \\ q \end{bmatrix}$ denote the same element of $V \times Q$. From (2.2) it follows immediately that the operator $\mathcal{A}$ is symmetric, i.e. $\langle \mathcal{A}w, z \rangle = \langle \mathcal{A}z, w \rangle$ for all $w, z \in X$.

Problem (2.4) is called well-posed iff $\mathcal{A}$ is an isomorphism, i.e., if there are constants $c$ and $C$ such that

$$\|\mathcal{A}\|_{\mathcal{L}(X,X^\star)} \leq C \quad \text{and} \quad \|\mathcal{A}^{-1}\|_{\mathcal{L}(X^\star,X)} \leq \frac{1}{c}.$$

Here $\mathcal{L}(Y,Z)$ denotes the space of all linear and continuous operators $M$ from Hilbert spaces $Y$ to $Z$ and $\|\cdot\|_{\mathcal{L}(Y,Z)}$ (in short $\|\cdot\|$) denotes the operator norm, which is defined by

$$\|M\|_{\mathcal{L}(Y,Z)} = \sup_{0\neq w\in Y} \frac{\|Mw\|_Z}{\|w\|_Y}.$$

In particular, for $\mathcal{A} \in \mathcal{L}(X,X^\star)$, we obtain the representation

$$\|\mathcal{A}\|_{\mathcal{L}(X,X^\star)} = \sup_{0\neq w\in X, 0\neq z\in X} \frac{\langle \mathcal{A}w, z\rangle}{\|w\|_X \|z\|_X}.$$

For the special case $c \equiv 0$ the following theorem provides conditions on the bilinear forms $a$ and $b$, which guarantee that $\mathcal{A}$ is an isomorphism (Babuška-Brezzi theory, see [6], [7], [26]).

**Theorem 2.1** (Brezzi's Theorem)**.** *The following three statements are equivalent:*

1. *$\mathcal{A}$ introduced by (2.3) is an isomorphism from $X$ to $X^*$.*

2. *There exist positive constants $c$ and $C$ such that:*

$$c\|x\|_X \leq \|\mathcal{A}x\|_{X^\star} \leq C\|x\|_X \quad \textit{for all } x \in X. \tag{2.5}$$

3. *The bilinear forms $a$ and $b$ satisfy the following conditions:*

   (a) *$a$ is bounded: There is a constant $\|a\| > 0$ such that*

   $$|a(u,v)| \leq \|a\| \, \|u\|_V \, \|v\|_V \quad \textit{for all } u, v \in V.$$

   (b) *$b$ is bounded: There is a constant $\|b\| > 0$ such that*

   $$|b(v,p)| \leq \|b\| \, \|v\|_V \|p\|_Q \quad \textit{for all } v \in V,\ p \in Q.$$

   (c) *$a$ satisfies an inf-sup condition: There is a constant $\alpha > 0$ such that*

   $$\inf_{0\neq u\in \ker B} \sup_{0\neq v\in\ker B} \frac{a(u,v)}{\|u\|_V \, \|v\|_V} \geq \alpha,$$

   *with* $\ker B = \{v \in V : b(v,q) = 0 \ \textit{ for all } q \in Q\}$.

(d) *b satisfies an inf-sup condition: There is a constant $\beta > 0$ such that*

$$\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \geq \beta.$$

We will refer to these conditions as Brezzi's conditions with constants $\|a\|$, $\|b\|$, $\alpha$, and $\beta$. (We tacitly assume that $\|a\|$ and $\|b\|$ are the smallest constants for estimating the bilinear forms $a$ and $b$. Then $\|a\|$ and $\|b\|$ match the standard notation for the norms of the bilinear forms $a$ and $b$.)

The following theorem (see [54, Theorem 1]) provides sharp estimates for the constants $c$ and $C$ in (2.5) in terms of the Brezzi constants.

**Theorem 2.2.** *Let all assumptions of Brezzi's Theorem 2.1 be satisfied. Then the operator $\mathcal{A}$ from $X$ to $X^\star$ satisfies*

$$c\|x\|_X \leq \|\mathcal{A}x\|_{X^\star} \leq C\|x\|_X \quad \textit{for all } x \in X$$

*with*

$$c = c_{opt}(\alpha, \beta, \|a\|) \quad \textit{and} \quad C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right). \tag{2.6}$$

*Here $c_{opt}(\alpha, \|a\|, \beta)$ is the smallest positive root of the cubic equation*

$$\eta^3 - (\|a\|^2 + \beta^2)\eta + \alpha\,\beta^2 = 0. \tag{2.7}$$

*Moreover, we have*

$$c_{opt}(\alpha, \|a\|, \beta) \geq \frac{\alpha}{1+\kappa^2}$$

*with $\kappa = \|a\|/\beta$.*

For the general case we have the following abstract result for general Hilbert spaces $V$ and $Q$, c.f. [92, Theorem 2.6].

**Theorem 2.3.** *Let additionally $a$ and $c$ be non-negative, i.e.*

$$a(v,v) \geq 0 \quad \textit{for all } v \in V \quad \textit{and} \quad c(p,p) \geq 0 \quad \textit{for all } p \in Q.$$

*The following statements are equivalent:*

1. *There exist positive constants $c$ and $C$ such that*

$$c\|x\|_X \leq \|\mathcal{A}x\|_{X^\star} \leq C\|x\|_X \quad \textit{for all } x \in X. \tag{2.8}$$

2. *The bilinear forms $a$, $b$ and $c$ satisfy the following conditions:*

*(a) There are positive constants $c_I$ and $C_I$ such that*

$$c_I \|v\|_V^2 \le a(v,v) + \|b(v,\cdot)\|_{Q^\star}^2 \le C_I \|v\|_V^2 \quad \text{for all } v \in V. \tag{2.9}$$

*(b) There are positive constants $c_{II}$ and $C_{II}$ such that*

$$c_{II} \|p\|_Q^2 \le c(p,p) + \|b(\cdot,p)\|_{V^\star}^2 \le C_{II} \|p\|_Q^2 \quad \text{for all } p \in Q. \tag{2.10}$$

*Moreover, the constants $c$ and $C$ depend only on the constants $c_I, C_I, c_{II}$, and $C_{II}$, and vice versa.*

From the proof of Theorem 2.3, we have the following explicit representation for the constants $c$ and $C$ in (2.8):

$$c = \frac{3-\sqrt{5}}{4} \frac{\min\left((\min(c_I, 1/2)c_I)^2, (\min(c_{II}, 1/2)c_{II})^2\right)}{\max\left(\sqrt{\max(1, C_I)C_I}, \sqrt{\max(1, C_{II})C_{II}}\right)} \tag{2.11}$$

and

$$C = \sqrt{2} \max\left(\sqrt{\max(1, C_I)C_I}, \sqrt{\max(1, C_{II})C_{II}}\right). \tag{2.12}$$

For the further considerations we introduce in a next step the operator $\mathcal{I}$.

**Definition 2.4.** *Let $\mathcal{I} : X \to X^*$ be given by*

$$\langle \mathcal{I}x, w \rangle = (x, w)_X \quad \text{for all } x, w \in X. \tag{2.13}$$

Observe that $\mathcal{I}$ is an isomorphism from $X$ to $X^\star$ and

$$\|\mathcal{I}x\|_{X^*} = \|x\|_X \quad \text{for all } x \in X. \tag{2.14}$$

The operator $\mathcal{R} = \mathcal{I}^{-1}$ is called the Riesz isomorphism. We have

$$\|\mathcal{R}f\|_X = \|f\|_{X^*} \quad \text{for all } f \in X^\star. \tag{2.15}$$

From Brezzi's Theorem 2.1 and (2.14) we immediately obtain the following result for the composition of the Riesz isomorphism and $\mathcal{A}$:

**Lemma 2.5.** *Let $\mathcal{A}$ be an isomorphism from $X$ to $X^\star$ with*

$$c\|x\|_X \le \|\mathcal{A}x\|_{X^\star} \le C\|x\|_X \quad \text{for all } x \in X \tag{2.16}$$

*for positive constants $c$ and $C$. Then $\mathcal{I}^{-1}\mathcal{A}$ is an isomorphism from $X$ to $X$ and*

$$c\|x\|_X \le \|\mathcal{I}^{-1}\mathcal{A}x\|_X \le C\|x\|_X \quad \text{for all } x \in X.$$

**Definition 2.6.** *Let $Y$ and $Z$ be Hilbert spaces and $M$ be an isomorphism from $Y$ to $Z$. The condition number $\kappa(M)$ is given by*

$$\kappa(M) = \|M\|\|M^{-1}\|.$$

A simple consequence of Lemma 2.5 is:

**Corollary 2.7.** *Let $\mathcal{A}$ be an isomorphism from $X$ to $X^\star$ with (2.16) for positive constants $c$ and $C$. Then*

$$\left\|(\mathcal{I}^{-1}\mathcal{A})^{-1}\right\| \leq \frac{1}{c} \quad \textit{and} \quad \|\mathcal{I}^{-1}\mathcal{A}\| \leq C$$

*and, therefore,*

$$\kappa(\mathcal{I}^{-1}\mathcal{A}) \leq \frac{C}{c}.$$

Sometimes the considered problem involves model and mesh parameters. If we have a space $X$ such that the operator $\mathcal{A}$ is an isomorphism from $X$ to $X^\star$ with bounds that are independent of the involved model and mesh parameters, i.e.

$$c\|x\|_X \leq \|\mathcal{A}x\|_{X^\star} \leq C\|x\|_X \quad \text{for all } x \in X$$

for positive constants $c$ and $C$ independent of the involved model and mesh parameters, then for the condition number we have the bound $\kappa(\mathcal{I}^{-1}\mathcal{A}) \leq C/c$, which is also independent of the involved model and mesh parameters.

In the discrete case this would provide an estimate $\kappa(\mathcal{I}_h^{-1}\mathcal{A}_h) \leq C/c$, where here the subscript $h$ denotes the matrix representation of the corresponding operators. This means that $\mathcal{I}_h$ would be suitable for preconditioning of iterative methods, provided the application of $\mathcal{I}_h^{-1}$ is efficient. In practise $\mathcal{I}_h$ is typically replaced by an easy-to-invert matrix $\tilde{\mathcal{I}}_h$, which is spectrally equivalent to $\mathcal{I}_h$, i.e.

$$\hat{c}\langle\hat{\mathcal{I}}_h\underline{x}, \underline{x}\rangle \leq \langle\mathcal{I}_h\underline{x}, \underline{x}\rangle \leq \hat{C}\langle\hat{\mathcal{I}}_h\underline{x}, \underline{x}\rangle \quad \text{for all } \underline{x} \in \mathbb{R}^n,$$

with positive constants $\hat{c}$ and $\hat{C}$ that are also independent of the involved model and mesh parameters. Then, for the condition number of $\hat{\mathcal{I}}^{-1}\mathcal{A}_h$, we obtain

$$\kappa(\hat{\mathcal{I}}_h^{-1}\mathcal{A}_h) \leq \frac{C}{c}\frac{\hat{C}}{\hat{c}}.$$

In order to construct a preconditioner for the discrete case, our goal is to find in a first step the space $X$ for the continuous problem, such that $\mathcal{A}$ is an isomorphism from $X$ to $X^\star$. Then in a second step we try to carry over the results to the discrete case.

In the next chapter we present two techniques for finding $X$ and its norm.

# Chapter 3

# Biharmonic model problems

In this chapter we present two techniques for finding the space $X$: the interpolation technique and the Lagrangian multiplier technique. Both techniques will be illustrated for a mixed method for the first biharmonic boundary value problem. Further applications will follow in the next chapter.

We consider the first biharmonic boundary value problem: For given $f$ find $w$ such that

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \partial_n w = 0 \quad \text{on } \Gamma, \tag{3.1}$$

where $\Omega$ is an open and bounded set in $\mathbb{R}^2$ with a polygonal Lipschitz boundary $\Gamma$, $\Delta$ and $\partial_n$ denote the Laplace operator and the derivative in the direction normal to the boundary, respectively. Problems of this type occur, e.g., in linear elasticity, where $w$ is the deflection of a clamped Kirchhoff plate under a vertical load with density $f$, see, e.g., [31], and in fluid mechanics, where $w$ is the stream function of a two-dimensional Stokes flow, see, e.g., [38].

A standard (primal) variational formulation of (3.1) reads as follows: For given $f \in H^{-1}(\Omega)$, find $w \in H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 w : \nabla^2 v \, dx = \langle f, v \rangle \quad \text{for all } v \in H_0^2(\Omega), \tag{3.2}$$

where $\nabla^2$ denotes the Hessian and $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^2 \boldsymbol{A}_{ij} \, \boldsymbol{B}_{ij}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{2\times 2}$. Here and throughout this thesis we use $L^2(\Omega)$, $H^m(\Omega)$, and $H_0^m(\Omega)$ with its dual space $H^{-m}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with corresponding norms $\|.\|_0$, $\|.\|_m$, $|.|_m$, and $\|.\|_{-m}$ for positive integers $m$, see, e.g., [1]. Existence and uniqueness of a solution to (3.2) is guaranteed even for more general right hand sides $f \in H^{-2}(\Omega)$ by the Lax-Milgram Theorem, see, e.g., [67, 59].

For the mixed method we introduce the auxiliary variable

$$\boldsymbol{\sigma} = \nabla^2 w,$$

whose elements are related to the bending moments in the context of linear elasticity. This allows to rewrite the biharmonic problem (3.1) as a boundary value problem of a system of two second-order equations

$$\nabla^2 w = \boldsymbol{\sigma}, \quad \operatorname{div} \mathbf{div}\, \boldsymbol{\sigma} = f \quad \text{in } \Omega, \quad w = \partial_n w = 0 \quad \text{on } \Gamma, \tag{3.3}$$

with the following notations for a matrix-valued function $\boldsymbol{\tau}$ and a vector-valued function $\phi$ in $\mathbb{R}^d$.

$$\mathbf{div}\, \boldsymbol{\tau} = \begin{bmatrix} \dfrac{\partial \boldsymbol{\tau}_{11}}{\partial x_1} + \cdots + \dfrac{\partial \boldsymbol{\tau}_{1d}}{\partial x_d} \\ \vdots \\ \dfrac{\partial \boldsymbol{\tau}_{d1}}{\partial x_1} + \cdots + \dfrac{\partial \boldsymbol{\tau}_{dd}}{\partial x_d} \end{bmatrix} \quad \text{and} \quad \operatorname{div} \phi = \frac{\partial \phi_1}{\partial x_1} + \cdots + \frac{\partial \phi_d}{\partial x_d}. \tag{3.4}$$

## 3.1 Interpolation technique

For deriving variational formulations of (3.3) we start in the usual way. We multiply the first and the second equation in (3.3) by arbitrary test functions $\boldsymbol{\tau}$ and $v$, respectively, and integrate over $\Omega$:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx - \int_\Omega \nabla^2 w : \boldsymbol{\tau}\ dx = \ 0, \quad \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\sigma})\, v\ dx \ = \langle f, v \rangle. \tag{3.5}$$

In the following two subsections we consider (3.5), where we reduce the smoothness assumptions either for $\boldsymbol{\sigma}$ or for $w$ by integration by parts.

### 3.1.1 A first variational formulation

To keep the smoothness assumptions for $\boldsymbol{\sigma}$ as low as possible, we apply integration by parts twice to the left-hand side of the second equation in (3.5):

$$\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\sigma})\, v\ dx = \int_\Gamma (\mathbf{div}\, \boldsymbol{\sigma} \cdot n) v\ ds - \int_\Gamma \boldsymbol{\sigma} n \cdot \nabla v\ ds + \int_\Omega \boldsymbol{\sigma} : \nabla^2 v\ dx. \tag{3.6}$$

Assuming $v = \partial_n v = 0$ on $\Gamma$ for the test functions $v$ the boundary integrals in (3.6) vanish. Together with the unchanged first equation from (3.5) we obtain a first mixed variational problem: For given $f \in H^{-2}(\Omega)$, find $\boldsymbol{\sigma} \in \mathbf{V}$ and $w \in Q$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx \qquad & - \int_\Omega \nabla^2 w : \boldsymbol{\tau}\ dx = 0 && \text{for all } \boldsymbol{\tau} \in \mathbf{V}, \\ - \int_\Omega \boldsymbol{\sigma} : \nabla^2 v\ dx \qquad & \qquad\qquad = -\langle f, v \rangle && \text{for all } v \in Q, \end{aligned} \tag{3.7}$$

with the natural choices $Q = H_0^2(\Omega)$ and $\mathbf{V} = \boldsymbol{L}^2(\Omega)_{\text{sym}}$, where

$$\boldsymbol{L}^2(\Omega)_{\text{sym}} = \{\boldsymbol{\tau}\colon \boldsymbol{\tau}_{ji} = \boldsymbol{\tau}_{ij} \in L^2(\Omega),\ i,j = 1,2\},$$

equipped with the standard $L^2$-norm $\|\boldsymbol{\tau}\|_0$ for matrix-valued functions $\boldsymbol{\tau}$.

In the next theorem we show that, for this choice for $\mathbf{V}$ and $Q$, Brezzi's conditions are satisfied for (3.7).

**Theorem 3.1.** *The bilinear forms*

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx \quad \textit{and} \quad b(\boldsymbol{\tau}, v) = -\int_\Omega \boldsymbol{\tau} : \nabla^2 v\ dx \tag{3.8}$$

*satisfy Brezzi's conditions on $\mathbf{V} = \boldsymbol{L}^2(\Omega)_{sym}$ and $Q = H_0^2(\Omega)$, equipped with the norms $\|\boldsymbol{\tau}\|_0$ and $|v|_2$, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = \beta = 1.$$

*Proof.* 1. Let $\boldsymbol{\sigma},\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$. Then

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \leq \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ and $v \in H_0^2(\Omega)$. Then

$$|b(\boldsymbol{\tau}, v)| = \left| \int_\Omega \boldsymbol{\tau} : \nabla^2 v\ dx \right| \leq \|\boldsymbol{\tau}\|_0\, |\nabla^2 v|_0 = \|\boldsymbol{\tau}\|_0\, |v|_2.$$

3. We have

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}.$$

4. Let $v \in H_0^2(\Omega)$. Then

$$\sup_{0 \neq \boldsymbol{\tau} \in \mathbf{V}} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_0} = \sup_{0 \neq \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}} \frac{\int_\Omega \boldsymbol{\tau} : \nabla^2 v\ dx}{\|\boldsymbol{\tau}\|_0} = |v|_2.$$

□

From Brezzi's Theorem 2.1 and Theorem 2.2 we obtain:

**Corollary 3.2.** *The operator $\mathcal{A}_0$ given by*

$$\left\langle \mathcal{A}_0 \begin{bmatrix} \boldsymbol{\sigma} \\ w \end{bmatrix}, \begin{bmatrix} \boldsymbol{\tau} \\ v \end{bmatrix} \right\rangle = a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) + b(\boldsymbol{\sigma}, v)$$

*with bilinear forms $a$ and $b$ defined in (3.8) is an isomorphism from $X_0$ to $X_0^*$ for $X_0 = \boldsymbol{L}^2(\Omega)_{sym} \times H_0^2(\Omega)$, whose natural norm is given by*

$$\|(\boldsymbol{\tau}, v)\|_{X_0} = \left(\|\boldsymbol{\tau}\|_0^2 + |v|_2^2\right)^{1/2}$$

*for* $(\boldsymbol{\tau}, v) \in X_0$. *Moreover, we have*

$$c\|x\|_{X_0} \leq \|\mathcal{A}_0 x\|_{X_0^*} \leq C\|x\|_{X_0} \quad \textit{for all } x \in X_0 \tag{3.9}$$

*with*

$$c = \frac{\sqrt{5}-1}{2} \approx 0.61803 \quad \textit{and} \quad C = \frac{1+\sqrt{5}}{2} \approx 1.6180.$$

*Proof.* $\mathcal{A}_0$ is an isomorphism from $X_0$ to $X_0^*$ since all conditions of Brezzi's Theorem 2.1 are satisfied with $\|a\| = \|b\| = \alpha = \beta = 1$.

Further we obtain from Theorem 2.2:

$$c\|x\|_{X_0} \leq \|\mathcal{A}_0 x\|_{X_0^*} \leq C\|x\|_{X_0} \quad \text{for all } x \in X_0,$$

with

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad c = \frac{\sqrt{5}-1}{2},$$

where $c$ is the smallest positive root $\eta$ for the cubic equation

$$\eta^3 - 2\eta + 1 = 0.$$

$\square$

### 3.1.2 A second variational formulation

To keep the smoothness assumptions for $w$ as low as possible, we apply integration by parts to the second term to the left-hand side of the first equation in (3.5):

$$\int_\Omega \nabla^2 w : \boldsymbol{\tau}\, dx = \int_\Gamma \nabla w \cdot \boldsymbol{\tau} n\, ds - \int_\Gamma w\, (\mathbf{div}\, \boldsymbol{\tau} \cdot n)\, ds + \int_\Omega w\, \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}\, dx.$$

Observe that the integrals over the boundary $\Gamma$ vanish, since $w = \partial_n w = 0$ on $\Gamma$. With the unchanged second equation from (3.5) this leads to a second mixed variational problem: For given $f \in L^2(\Omega)$, find $\boldsymbol{\sigma} \in \mathbf{V}$ and $w \in Q$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx && - \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, w\, dx = 0 && \text{for all } \boldsymbol{\tau} \in \mathbf{V}, \\ -&\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\sigma})\, v\, dx && = -\langle f, v\rangle\, dx && \text{for all } v \in Q. \end{aligned} \tag{3.10}$$

Natural choices for the Hilbert spaces are $Q = L^2(\Omega)$ and $\mathbf{V} = \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}}$, where

$$\boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} :\ \operatorname{div} \mathbf{div}\, \boldsymbol{\tau} \in L^2(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{\operatorname{div}\mathbf{div}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_0^2\right)^{1/2}.$$

Here $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ denotes the application of the $\operatorname{div}\mathbf{div}$ operator to $\boldsymbol{\tau}$ in the distributional sense, i.e., for $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ we have

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\, dx \quad \text{for all } v \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ denotes the space of all indefinitely differentiable functions with compact support in $\Omega$. It is easy to see that $\boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ is a Hilbert space.

The next lemma gives several representations of $\operatorname{div}\mathbf{div}$ under additional smoothness assumptions.

**Lemma 3.3** (Green's formulas)**.** *We have:*

- *For all $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{sym}$ and $v \in H_0^2(\Omega)$:*

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \ dx.$$

- *For all $\boldsymbol{\tau} \in \boldsymbol{H}^1(\Omega)_{sym}$ and $v \in H_0^1(\Omega)$:*

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = -\int_\Omega \mathbf{div}\,\boldsymbol{\tau} \cdot \nabla v \ dx.$$

- *For all $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{sym}$ and $v \in L^2(\Omega)$:*

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, v \ dx.$$

*Proof.* Let $v \in C_0^\infty(\Omega)$ and $\boldsymbol{\tau} \in \mathbf{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$, where $\mathbf{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$ denotes the space of functions in $\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ which are infinitely differentiable on $\overline{\Omega}$. Using integration by parts twice we obtain

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\, dx = -\int_\Omega \mathbf{div}\,\boldsymbol{\tau} \cdot \nabla v\, dx = \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, v\, dx. \tag{3.11}$$

The claim follows from the continuity of the second, third and fourth term in (3.11), and the density of $\mathbf{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$ and $C_0^\infty(\Omega)$ in the corresponding spaces (see appendix, Theorem 9.3). □

In the following theorem we show that Brezzi's conditions are satisfied for (3.10) for our choice for $\mathbf{V}$ and $Q$.

**Theorem 3.4.** *The bilinear forms*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \; dx \quad \textit{and} \quad b(\boldsymbol{\tau}, v) = -\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, v \; dx \tag{3.12}$$

*satisfy Brezzi's conditions on* $\mathbf{V} = \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{sym}$ *and* $Q = L^2(\Omega)$*, equipped with the norms* $\|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}}$ *and* $\|v\|_0$*, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = \frac{1}{\sqrt{1 + C_F^4}},$$

*where* $C_F$ *denotes the constant in Friedrichs' inequality:* $\|v\|_0 \le C_F\, |v|_1$ *for all* $v \in H_0^1(\Omega)$*.*

*Proof.* 1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. Then

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{\operatorname{div} \mathbf{div}} \|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ and $v \in L^2(\Omega)$. Then

$$|b(\boldsymbol{\tau}, v)| = \left| \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, v \; dx \right| \le \| \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}\|_0 \, \|v\|_0 \le \|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}} \, \|v\|_0.$$

3. Observe that $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} : \operatorname{div} \mathbf{div}\, \boldsymbol{\tau} = 0\}$. Therefore,

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B.$$

4. For $v \in L^2(\Omega)$, let $p \in H_0^2(\Omega)$ be the solution of the biharmonic problem

$$\int_\Omega \nabla^2 p : \nabla^2 q \, dx = \int_\Omega v\, q\, dx \quad \text{for all } q \in H_0^2(\Omega),$$

then, for $\hat{\boldsymbol{\tau}} = \nabla^2 p$, we have $\operatorname{div} \mathbf{div}\, \hat{\boldsymbol{\tau}} = v$ and

$$\|\hat{\boldsymbol{\tau}}\|_0 = |p|_2 = \sup_{0 \ne q \in H_0^2(\Omega)} \frac{\int_\Omega v\, q\, dx}{|q|_2} \le \sup_{0 \ne q \in L^2(\Omega)} \frac{\int_\Omega v\, q\, dx}{C_F^{-2} \|q\|_0} \le C_F^2 \|v\|_0$$

which implies

$$\|\hat{\boldsymbol{\tau}}\|_0 \le C_F^2 \|v\|_0 \quad \text{and} \quad \|\hat{\boldsymbol{\tau}}\|_{\operatorname{div} \mathbf{div}} \le \sqrt{1 + C_F^4} \|v\|_0.$$

Therefore

$$\begin{aligned} \sup_{0 \ne \boldsymbol{\tau} \in \mathbf{V}} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}}} &= \sup_{0 \ne \boldsymbol{\tau} \in V} \frac{\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, v \; dx}{\|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}}} \\ &\ge \frac{\int_\Omega (\operatorname{div} \mathbf{div}\, \hat{\boldsymbol{\tau}})\, v \; dx}{\|\hat{\boldsymbol{\tau}}\|_{\operatorname{div} \mathbf{div}}} = \frac{\|v\|_0^2}{\|\hat{\boldsymbol{\tau}}\|_{\operatorname{div} \mathbf{div}}} \\ &\ge \frac{1}{\sqrt{1 + C_F^4}} \|v\|_0. \end{aligned} \tag{3.13}$$

$\square$

From Brezzi's Theorem 2.1 and Theorem 2.2 we obtain:

**Corollary 3.5.** *The operator* $\mathcal{A}_1$ *given by*

$$\left\langle \mathcal{A}_1 \begin{bmatrix} \boldsymbol{\sigma} \\ w \end{bmatrix}, \begin{bmatrix} \boldsymbol{\tau} \\ v \end{bmatrix} \right\rangle = a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) + b(\boldsymbol{\sigma}, v)$$

*with bilinear forms* $a$ *and* $b$ *defined in (3.12) is an isomorphism from* $X_1$ *to* $X_1^*$ *for* $X_1 = \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{sym} \times L^2(\Omega)$, *whose natural norm is given by*

$$\|(\boldsymbol{\tau}, v)\|_{X_1} = \left(\|\boldsymbol{\tau}\|^2_{\operatorname{div} \mathbf{div}} + \|v\|_0^2\right)^{1/2}$$

*for* $(\boldsymbol{\tau}, v) \in X_1$. *Moreover, we have*

$$c\|x\|_{X_1} \leq \|\mathcal{A}_1 x\|_{X_1^*} \leq C\|x\|_{X_1} \quad \text{for all } x \in X_1 \tag{3.14}$$

*with*

$$c \geq \frac{1}{2 + C_F^4} \quad \text{and} \quad C = \frac{1 + \sqrt{5}}{2} \approx 1.6180.$$

*Proof.* $\mathcal{A}_1$ is an isomorphism from $X_1$ to $X_1^*$ since all conditions of Brezzi's Theorem 2.1 are satisfied with $\|a\| = \|b\| = \alpha = 1$ and $\beta = \frac{1}{\sqrt{1+C_F^4}}$.

Further we obtain from Theorem 2.2:

$$c\|x\|_{X_1} \leq \|\mathcal{A}_1 x\|_{X_1^*} \leq C\|x\|_{X_1} \quad \text{for all } x \in X_1,$$

with

$$c \geq \frac{\alpha}{1 + \|a\|^2/\beta^2} = \frac{1}{2 + C_F^4}$$

and

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1 + \sqrt{5}}{2}.$$

□

### 3.1.3 A new variational formulation by interpolation

To summarize, we have two spaces $X = X_0$ and $X = X_1$ for which the mixed variational problem (3.3) is well-posed. For the space $X_0$ the original unknown $w$ requires second-order smoothness and the auxiliary variable $\boldsymbol{\sigma}$ requires no smoothness. In the space $X_1$ we have the reverse situation, the original unknown $w$ requires no smoothness and the auxiliary variable $\boldsymbol{\sigma}$ requires second-order smoothness.

The idea is now to distribute the smoothness evenly for the original unknown and the auxiliary variable, by the use of interpolation theory.

We shortly recall here the definition and basic properties of interpolation spaces. For the following results and an introduction to interpolation, we refer to [88], [13] and [59].

Let $X$, $X_0$ and $X_1$ be Hilbert spaces, where $X_0$ and $X_1$ are subspaces of $X$. Then $X_0 \cap X_1$ and

$$X_0 + X_1 = \{x = x_1 + x_2 \ : \ x_0 \in X_0,\ x_1 \in X_1\} \tag{3.15}$$

are Hilbert spaces with respect to the norms

$$\begin{aligned}
\|x\|_{X_0 \cap X_1} &= \left(\|x\|_{X_0}^2 + \|x\|_{X_1}^2\right)^{1/2}, \\
\|x\|_{X_0 + X_1} &= \inf_{x = x_0 + x_1} \left(\|x_0\|_{X_0}^2 + \|x_1\|_{X_1}^2\right)^{1/2}.
\end{aligned}$$

In the following we call two norms $\|\cdot\|_W$ and $\|\cdot\|_V$ on a Hilbert spaces $X$ equivalent, if

$$c\|x\|_W \le \|x\|_V \le C\|x\|_W \quad \text{for all } x \in X \tag{3.16}$$

for some positive constants $c$ and $C$.

**Definition 3.6.** *The K-functional $K : \mathbb{R}^+ \times (X_0 + X_1) \to \mathbb{R}$ is given by*

$$K(t, x, X_0, X_1) = \inf_{x = x_0 + x_1} \left(\|x_0\|_{X_0}^2 + t^2\|x_1\|_{X_1}^2\right)^{1/2}.$$

**Definition 3.7.** *For $\theta \in (0,1)$ we define the interpolation norm*

$$\|x\|_\theta = \|x\|_{[X_0,X_1]_\theta} = \left(\int_0^\infty t^{-2\theta} K(t, x, X_0, X_1)^2 \, dt/t\right)^{1/2}$$

*and the interpolation space $[X_0, X_1]_\theta$ is given by*

$$[X_0, X_1]_\theta = \{x \in X_0 + X_1 : \ \|x\|_\theta < \infty\}.$$

**Example 3.8.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then*

$$[L^2(\Omega), H^2(\Omega)]_{1/2} = H^1(\Omega) \quad \textit{and} \quad [L^2(\Omega), H_0^2(\Omega)]_{1/2} = H_0^1(\Omega)$$

*with equivalent norms, see, e.g., [1] and [20, Theorem 2.1], respectively.*

We have the following properties for the interpolation spaces:

**Lemma 3.9.** *Let $\theta \in (0,1)$. Then*

1. *$[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$ with equal norms,*
2. *$X_0 \cap X_1 \subset [X_0, X_1]_\theta \subset X_0 + X_1$,*

*3.* $X_0 \cap X_1$ *is dense in* $[X_0, X_1]_\theta$.

**Lemma 3.10.** *We have*

$$[X_0, X_0 \cap X_1]_\theta = X_0 \cap [X_0, X_1]_\theta.$$

*for all* $\theta \in (0,1)$*, with equivalent norms.*

For a proof see, e.g., [87, Lemma 6.1].

In a next step we collect the most important properties of the interpolation spaces, which we need for the further discussion, see, e.g., [88], [13], [59] and [25].

**Theorem 3.11** (Duality Theorem). *Let* $X_0 \cap X_1$ *be dense in* $X_0$ *and* $X_1$*. Then*

$$X_0^\star + X_1^\star = (X_0 \cap X_1)^\star.$$

*Moreover, we have*

$$[X_0, X_1]_\theta^\star = [X_1^\star, X_0^\star]_{1-\theta}$$

*for all* $\theta \in (0,1)$*, with equivalent norms.*

**Example 3.12.** *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain. Then*

$$[H^{-2}(\Omega), L^2(\Omega)]_{1/2} = \left([H_0^2(\Omega), L^2(\Omega)]_{1/2}\right)^\star = H_0^1(\Omega)^\star = H^{-1}(\Omega)$$

*with equivalent norms.*

**Theorem 3.13** (Interpolation Theorem). *Let* $X_j, Y_j$*,* $j = 0, 1$*, be Hilbert spaces and* $T$ *a linear operator from* $X_0 + X_1$ *to* $Y_0 + Y_1$ *with*

$$\|Tx\|_{Y_0} \le C_0 \|x\|_{X_0} \text{ for all } x \in X_0 \quad \text{and} \quad \|Tx\|_{Y_1} \le C_1 \|x\|_{X_1} \text{ for all } x \in X_1,$$

*and* $0 < \theta < 1$*. Then, for* $X_\theta = [X_0, X_1]_\theta$ *and* $Y_\theta = [Y_0, Y_1]_\theta$*,*

$$\|Tx\|_{Y_\theta} \le C_0^{1-\theta} C_1^\theta \|x\|_{X_\theta}.$$

**Theorem 3.14.** *Let* $X_j, Y_j, Z_j$*,* $j = 0, 1$*, be Hilbert spaces such that* $X_0 \cap X_1$ *is dense in* $X_0$ *and* $X_1$*, and* $Z_0 \cap Z_1$ *is dense in* $Z_0$ *and* $Z_1$*. Suppose that* $Y_j$ *is dense in* $Z_j$*,* $j = 0, 1$ *and there exists a linear operator* $D$ *such that* $D : X_j \to Z_j$ *is bounded for* $j = 0, 1$*. Let* $X_j(D)$ *be given by*

$$X_j(D) = \{x \in X_j : \ Dx \in Y_j\}, \quad j = 0, 1, \tag{3.17}$$

*equipped with the graph norm, i.e.* $\|x\|_{X_j(D)} = \left(\|x\|_{X_j}^2 + \|Dx\|_{Y_j}^2\right)^{1/2}$*,* $j = 0, 1$*. For* $\theta \in (0,1)$*, let* $X_\theta(D)$ *be given by*

$$X_\theta(D) = \{x \in [X_0, X_1]_\theta : \ Dx \in [Y_0, Y_1]_\theta\}$$

*equipped with the norm* $\|x\|_{X_\theta(D)} = \left(\|x\|^2_{[X_0,X_1]_\theta} + \|Dx\|^2_{[Y_0,Y_1]_\theta}\right)^{1/2}$. *Moreover, let* $K : Z_j \to X_j$ *and* $R : Z_j \to Y_j$ *be continuous linear operators with the property* $D \circ K = I + R$ *on the spaces* $Z_j$ *for* $j = 0, 1$. *Then*

$$[X_0(D), X_1(D)]_\theta = X_\theta(D)$$

*for all* $\theta \in (0, 1)$ *with equivalent norms.*

Here the operator $I_Z$ (or $I$, if $Z$ is clear from the context) denotes the identity map for a Hilbert space $Z$.

Note, that the operators $\mathcal{A}_0 : X_0 \to X_0^\star$ and $\mathcal{A}_1 : X_1 \to X_1^\star$ introduced in Corollaries 3.2 and 3.5, respectively, are different. However, we have:

**Lemma 3.15.**

$$\mathcal{A}_0 x = \mathcal{A}_1 x \quad \textit{for all} \;\; x \in X_0 \cap X_1 \tag{3.18}$$

*and*

$$\mathcal{A}_0^{-1} f = \mathcal{A}_1^{-1} f \quad \textit{for all} \;\; f \in X_0^\star \cap X_1^\star. \tag{3.19}$$

*Proof.* Let $x = (\boldsymbol{\sigma}, w) \in X_0 \cap X_1$, with $X_0 = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \times H_0^2(\Omega)$, $X_1 = \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}} \times L^2(\Omega)$ and $X_0 \cap X_1 = \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}} \times H_0^2(\Omega)$. We have $H_0^2(\Omega)$ is dense in $L^2(\Omega)$, and further we obtain from the density of $\mathbf{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$ in $\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ and in $\boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ (see appendix, Theorem 9.3) the density of $\boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ in $\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$. So $X_0 \cap X_1$ is dense in $X_0$ and $X_1$ and therefore, we obtain from the Duality Theorem 3.11 that $X_0^\star + X_1^\star = (X_0 \cap X_1)^\star$. Hence $\mathcal{A}_0 x - \mathcal{A}_1 x \in (X_0 \cap X_1)^\star$ and

$$\begin{aligned}
\left\langle \mathcal{A}_0 \begin{bmatrix} \boldsymbol{\sigma} \\ w \end{bmatrix}, \begin{bmatrix} \boldsymbol{\tau} \\ v \end{bmatrix} \right\rangle &= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \; dx - \int_\Omega \boldsymbol{\tau} : \nabla^2 w \; dx - \int_\Omega \boldsymbol{\sigma} : \nabla^2 v \; dx \\
&= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \; dx - \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, w \; dx - \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\sigma})\, v \; dx \\
&= \left\langle \mathcal{A}_1 \begin{bmatrix} \boldsymbol{\sigma} \\ w \end{bmatrix}, \begin{bmatrix} \boldsymbol{\tau} \\ v \end{bmatrix} \right\rangle
\end{aligned} \tag{3.20}$$

for all $(\boldsymbol{\tau}, v) \in X_0 \cap X_1$, where we obtain the second equality from Lemma 3.3. This completes the proof of (3.18).

Let $f \in X_0^\star \cap X_1^\star$ and let $x_0 = (\boldsymbol{\sigma}_0, w_0) \in X_0$, $x_1 = (\boldsymbol{\sigma}_1, w_1) \in X_1$ such that $x_0 = \mathcal{A}_0^{-1} f$ and $x_1 = \mathcal{A}_1^{-1} f$. We have $\mathcal{A}_0 x_0 - \mathcal{A}_1 x_1 = 0$ and hence

$$\langle \mathcal{A}_0 x_0, y \rangle = \langle \mathcal{A}_1 x_1, y \rangle \quad \text{for all } y \in X_0 \cap X_1,$$

or equivalently

$$\begin{gathered}
\int_\Omega \boldsymbol{\sigma}_0 : \boldsymbol{\tau} \; dx - \int_\Omega \boldsymbol{\tau} : \nabla^2 w_0 \; dx - \int_\Omega \boldsymbol{\sigma}_0 : \nabla^2 v \; dx = \\
\int_\Omega \boldsymbol{\sigma}_1 : \boldsymbol{\tau} \; dx - \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, w_1 \; dx - \int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}_1)\, v \; dx
\end{gathered} \tag{3.21}$$

for all $(\boldsymbol{\tau}, v) \in X_0 \cap X_1$. Using Lemma 3.3 for the second term on the left-hand side and the third term on the right-hand side in (3.21), we obtain

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}_0 : \boldsymbol{\tau}\, dx - \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, w_0\, dx - \int_\Omega \boldsymbol{\sigma}_0 : \nabla^2 v\, dx = \\ &\quad \int_\Omega \boldsymbol{\sigma}_1 : \boldsymbol{\tau}\, dx - \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, w_1\, dx - \int_\Omega \boldsymbol{\sigma}_1 : \nabla^2 v\, dx \end{aligned} \tag{3.22}$$

for all $(\boldsymbol{\tau}, v) \in X_0 \cap X_1$. From (3.22) it follows for $v = 0$,

$$\int_\Omega (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1) : \boldsymbol{\tau}\, dx - \int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\,(w_0 - w_1)\, dx = 0 \tag{3.23}$$

for all $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ and, for $\boldsymbol{\tau} = 0$,

$$\int_\Omega (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1) : \nabla^2 \boldsymbol{\tau}\, dx = 0 \tag{3.24}$$

for all $v \in H_0^2(\Omega)$. (3.24) implies

$$\operatorname{div} \mathbf{div}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1) = 0$$

and thus $\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1 \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. For the choice $\boldsymbol{\tau} = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1$ in (3.23), we obtain

$$\begin{aligned} &\int_\Omega (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1) : (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1)\, dx - \int_\Omega \operatorname{div} \mathbf{div}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1)\,(w_0 - w_1)\, dx \\ &= \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1\|_0^2 = 0 \end{aligned} \tag{3.25}$$

and hence $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_1 \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. Therefore, (3.23) reduces to

$$\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\,(w_0 - w_1)\, dx = 0 \tag{3.26}$$

for all $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. From (3.26) and (3.13) we obtain:

$$0 = \sup_{0 \neq \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}} \frac{\int_\Omega (\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\,(w_0 - w_1)\, dx}{\|\boldsymbol{\tau}\|_{\operatorname{div} \mathbf{div}, 0}} \geq \frac{1}{\sqrt{1 + C_F^4}} \|w_0 - w_1\|_0.$$

Therefore $w_0 = w_1 \in H_0^2(\Omega)$, which completes the proof of (3.19). $\square$

Because of property (3.18) the linear operator $\mathcal{A} : X_0 + X_1 \to X_0^\star + X_1^\star$ given by

$$\mathcal{A}x = \mathcal{A}_0 x_0 + \mathcal{A}_1 x_1 \tag{3.27}$$

for all $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$ is well-defined and $\mathcal{A}$ is an extension of $\mathcal{A}_0$ and $\mathcal{A}_1$. From property (3.19) it follows that $\mathcal{A}$ is bijective, where $\mathcal{A}^{-1} : X_0^\star + X_1^\star \to X_0 + X_1$ is given by

$$\mathcal{A}^{-1} f = \mathcal{A}_0^{-1} f_0 + \mathcal{A}_1^{-1} f_1$$

for all $f = f_0 + f_1$ with $f_0 \in X_0^\star$ and $f_1 \in X_1^\star$.

We have already shown that $\mathcal{A} : X_i \to X_i^\star$ is an isomorphism for $i = 1, 2$. Therefore the first part of the following theorem follows immediately from the Interpolation Theorem 3.13 applied to $\mathcal{A}$ and $\mathcal{A}^{-1}$, and the Duality Theorem 3.11.

**Theorem 3.16.** *The operator $\mathcal{A}$ is an isomorphism from $[X_0, X_1]_{1/2}$ to $[X_0, X_1]_{1/2}^\star$. Furthermore, we have*

$$[X_0, X_1]_{1/2} = X$$

*with equivalent norms, where*

$$X = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym} \times H_0^1(\Omega),$$

*with*

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{sym} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$

*equipped with the norm*

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2\right)^{1/2}. \tag{3.28}$$

*Proof.* We have

$$[X_0, X_1]_{1/2} = [\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2} \times [L^2(\Omega), H_0^2(\Omega)]_{1/2}.$$

For the representation of $[\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2}$, we apply Theorem 3.14: We have

$$\begin{aligned} \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} &= \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in H^{-2}(\Omega)\} \\ &= \{\boldsymbol{\tau} \in X_0 : D\boldsymbol{\tau} \in Y_0\} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}} &= \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} : \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in L^2(\Omega)\} \\ &= \{\boldsymbol{\tau} \in X_1 : D\boldsymbol{\tau} \in Y_1\}, \end{aligned}$$

with $D = \operatorname{div}\mathbf{div}$, $X_0 = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$, $Y_0 = H^{-2}(\Omega)$, $X_1 = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ and $Y_1 = L^2(\Omega)$. In addition, we set $Z_0 = Z_1 = H^{-2}(\Omega)$. $X_j, Y_j, Z_j$ for $j = 0, 1$ are Hilbert spaces, $Y_j$ is dense in $Z_j$, $X_0 \cap X_1 = X_0 = X_1$ and $D : X_0 = X_1 = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \to Z_0 = Z_1 = H^{-2}(\Omega)$ is bounded, since

$$|\langle D\boldsymbol{\tau}, v\rangle| = |\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle| = \left|\int_\Omega \boldsymbol{\tau} : \nabla^2 v \, dx\right| \le \|\boldsymbol{\tau}\|_0 \|v\|_2$$

for all $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$, $v \in H_0^2(\Omega)$. Then we follow the idea of the proof of Theorem 7.2, in [59]: We introduce the dual operator $D^\star : H_0^2(\Omega) \to [\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}]^\star = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ by

$$\langle D^\star v, \boldsymbol{\tau}\rangle = \langle D\boldsymbol{\tau}, v\rangle$$

for all $v \in H_0^2(\Omega)$, $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$, and consider further the operator $DD^\star + I : H_0^2(\Omega) \to H^{-2}(\Omega)$. Since

$$\langle (DD^\star + I)v, q\rangle = \int_\Omega \nabla^2 v : \nabla^2 q \, dx + \int_\Omega v\, q\, dx \quad \text{for all } v, q \in H_0^2(\Omega),$$

it follows immediately from the Lax-Milgram Theorem that $DD^\star + I$ is an isomorphism from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$. For the particular choice

$$K = D^\star \left(DD^\star + I\right)^{-1} \quad \text{and} \quad R = -\left(DD^\star + I\right)^{-1}$$

it is easy to see that $D \circ K = I + R$. To apply Theorem 3.14, it remains to prove the boundedness of $K$ and $R$:

For all $f \in H^{-2}(\Omega)$ we have:

$$\|Kf\|_0 = \|\nabla^2 v\|_0 = |v|_2 \leq \|\left(DD^\star + I\right)^{-1}\|_{\mathcal{L}(H^{-2}(\Omega)), H_0^2(\Omega))} \|f\|_{-2}$$

with $v = \left(DD^\star + I\right)^{-1} f \in H_0^2(\Omega)$. Therefore $K : Z_0 = Z_1 = H^{-2}(\Omega) \to X_0 = X_1 = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ is bounded.

Further for all $f \in H^{-2}(\Omega)$ we have

$$\|Rf\|_0 \leq C_F^2 |Rf|_2 \leq \|\left(DD^\star + I\right)^{-1}\|_{\mathcal{L}(H^{-2}(\Omega)), H_0^2(\Omega))} \|f\|_{-2}.$$

Therefore $R : Z_1 = H^{-2}(\Omega) \to Y_1 = L^2(\Omega)$ is bounded, which immediately implies that $R : Z_0 = H^{-2}(\Omega) \to Y_0 = H^{-2}(\Omega)$ is bounded.

So all assumptions of Theorem 3.14 are satisfied and we obtain

$$\begin{aligned}
&[\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2} \\
&\quad = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \colon \operatorname{div} \mathbf{div}\, \boldsymbol{\tau} \in [H^{-2}(\Omega), L^2(\Omega)]_{1/2}\}
\end{aligned}$$

with equivalent norms. Further we obtain from Example 3.12,

$$\begin{aligned}
&\{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \colon \operatorname{div} \mathbf{div}\, \boldsymbol{\tau} \in [H^{-2}(\Omega), L^2(\Omega)]_{1/2}\} \\
&\quad = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \colon \operatorname{div} \mathbf{div}\, \boldsymbol{\tau} \in H^{-1}(\Omega)\} \\
&\quad = \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}},
\end{aligned}$$

with equivalent norms, and hence

$$[\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2} = \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}, \tag{3.29}$$

with equivalent norms. Finally, from (3.29) and Example 3.8, we obtain

$$\begin{aligned}
[X_0, X_1]_{1/2} &= [\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2} \times [L^2(\Omega), H_0^2(\Omega)]_{1/2} \\
&= \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}} \times H_0^1(\Omega),
\end{aligned} \tag{3.30}$$

with equivalent norms. This completes the proof of the second statement. □

**Remark 3.17.** The space $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ was already introduced in [85, 73] in the context of linear elasticity problems.

In the next theorem we show that $\mathcal{A}$ has a representation of the form (2.3) on $X$.

**Theorem 3.18.** *The operator $\mathcal{A}$ from $X$ to $X^\star$ is given by*

$$\left\langle \mathcal{A}\begin{bmatrix}\boldsymbol{\sigma}\\ w\end{bmatrix}, \begin{bmatrix}\boldsymbol{\tau}\\ v\end{bmatrix}\right\rangle = a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) + b(\boldsymbol{\sigma}, v)$$

*with bilinear forms*

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx \quad \text{and} \quad b(\boldsymbol{\tau}, v) = -\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle. \tag{3.31}$$

*Proof.* From the density of $\mathbf{C}^\infty(\overline{\Omega})_{\text{sym}}$ in $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ (see appendix, Theorem 9.3) and the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ we obtain the density of $\mathbf{C}^\infty(\overline{\Omega})_{\text{sym}} \times C_0^\infty(\Omega)$ in $X$. Let

$$x = \begin{bmatrix} w\\ \boldsymbol{\sigma}\end{bmatrix} \in X \quad \text{and} \quad z = \begin{bmatrix} v\\ \boldsymbol{\tau}\end{bmatrix} \in \mathbf{C}^\infty(\overline{\Omega})_{\text{sym}} \times C_0^\infty(\Omega).$$

We have

$$\begin{aligned}\langle \mathcal{A}x, z\rangle &= \left\langle \mathcal{A}_0\begin{bmatrix}\boldsymbol{\sigma}\\ 0\end{bmatrix}, z\right\rangle + \left\langle \mathcal{A}_1\begin{bmatrix}0\\ w\end{bmatrix}, z\right\rangle\\ &= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx - \int_\Omega \boldsymbol{\sigma} : \nabla^2 v\, dx - \int_\Omega \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\, w\, dx \\ &= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle,\end{aligned} \tag{3.32}$$

where we used Lemma 3.3 for the last equality. Since all expressions in (3.32) are continuous for $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ in $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and for $w$ and $v$ in $H_0^1(\Omega)$, (3.32) is still satisfied for the closure of $\mathbf{C}^\infty(\overline{\Omega})_{\text{sym}} \times C^\infty(\overline{\Omega})$ in $X$. This completes the proof. □

So the new mixed variational formulation reads as follows: For given $f \in H^{-1}(\Omega)$, find $\boldsymbol{\sigma} \in \mathbf{V}$ and $w \in Q$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx \qquad - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle &&= 0 &&\text{for all } \boldsymbol{\tau} \in \mathbf{V},\\ &- \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle &&= -\langle f, v\rangle\ dx &&\text{for all } v \in Q,\end{aligned} \tag{3.33}$$

with

$$\mathbf{V} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} \quad \text{and} \quad Q = H_0^1(\Omega).$$

Since $\mathcal{A}$ is an isomorphism, we already know the existence of Brezzi's constants. Their concrete form is given by the following theorem. For the proof as well as for later use, we first introduce the following simple but useful notation for a function $v \in H_0^1(\Omega)$:

$$\boldsymbol{\pi}(v) = v\,\boldsymbol{I}_2. \tag{3.34}$$

Here $\boldsymbol{I}_k$ denotes the identity matrix in $\mathbb{R}^k$.

**Theorem 3.19.** *The bilinear forms $a$ and $b$, defined in (3.31), satisfy Brezzi's conditions on $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$ and $H_0^1(\Omega)$, equipped with the norms $\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}$ and $|v|_1$, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1+2C_F^2)^{-1/2}.$$

*Proof.* 1. Let $\boldsymbol{\sigma},\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$. Then

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0\|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{-1,\operatorname{div}\mathbf{div}}\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ and $v \in H_0^1(\Omega)$. Then

$$|b(\boldsymbol{\tau},v)| = |\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle| \le \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}\,|v|_1 \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}\,|v|_1.$$

3. Observe that $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} : \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} = 0\}$. Therefore,

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B.$$

4. Here we follow the proofs in [28, 14]. For $v \in H_0^1(\Omega)$ it is easy to see that

$$b(\boldsymbol{\pi}(v),v) = |v|_1^2 \quad \text{and} \quad \|\boldsymbol{\pi}(v)\|_{-1,\operatorname{div}\mathbf{div}}^2 = 2\|v\|_0^2 + |v|_1^2 \le (1+2C_F^2)\,|v|_1^2.$$

Therefore

$$\begin{aligned}\sup_{0\neq\boldsymbol{\tau}\in V} \frac{b(\boldsymbol{\tau},v)}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}} &\ge \frac{|b(\boldsymbol{\pi}(v),v)|}{\|\boldsymbol{\pi}(v)\|_{-1,\operatorname{div}\mathbf{div}}} = \frac{|v|_1^2}{(2\|v\|_0^2 + |v|_1^2)^{1/2}} \\ &\ge \frac{1}{(1+2C_F^2)^{1/2}}|v|_1.\end{aligned}$$

$\square$

Now from Theorem 2.2 it follows

$$c\|x\|_X \le \|\mathcal{A}x\|_{X^*} \le C\|x\|_X \quad \text{for all } x \in X,$$

with

$$c \ge \frac{\alpha}{1+\|a\|^2/\beta^2} = \frac{1}{2+2C_F^2}$$

and

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1+\sqrt{5}}{2} \approx 1.6180.$$

We have the following correspondence between the solutions of the mixed variational problem (3.33) and the primal variational formulation (3.2).

**Corollary 3.20.** *For $f \in H^{-1}(\Omega)$ the problems (3.2) and (3.33) are equivalent, i.e., if $w \in H_0^2(\Omega)$ solves (3.2), then $\boldsymbol{\sigma} = \nabla^2 w \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$ and $(\boldsymbol{\sigma}, w)$ solves (3.33). And, vice versa, if $(\boldsymbol{\sigma}, w) \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym} \times H_0^1(\Omega)$ solves (3.33), then $w \in H_0^2(\Omega)$ and $w$ solves (3.2).*

*Proof.* Both problems are uniquely solvable. Therefore, it suffices to show that $(w, \boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = \nabla^2 w$ solves (3.33), if $w$ solves (3.2). So, assume that $w \in H_0^2(\Omega)$ is a solution of (3.2). Then, obviously, $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ and

$$\int_\Omega \boldsymbol{\sigma} : \nabla^2 v \; dx = \langle f, v\rangle \quad \text{for all } v \in H_0^2(\Omega),$$

which implies that $\operatorname{div}\mathbf{div}\,\boldsymbol{\sigma} = f \in H^{-1}(\Omega)$. Therefore, $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ and the second row in (3.33) immediately follows.

By the definition of $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ in the distributional sense we have

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, it follows for $v = w$ that

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 w \; dx = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma} \; dx,$$

which shows the first row in (3.33). □

To summarize, in order to determine the space $X = V \times Q$ such that $\mathcal{A}$ is an isomorphism from $X$ to $X^\star$ , we shifted the smoothness between the spaces $V$ and $Q$ and interpolated the resulting spaces. Note, this technique can be applied to any operator $\mathcal{A}$ which represents a saddle point problem.

For the special case that $\mathcal{A}$ corresponds to a mixed variational formulation of an elliptic problem, as it is the case for the considered biharmonic problem, we are able to determine the space $X$ in a more direct way without interpolation. This technique is called Lagrangian multiplier technique and will be presented in the following section.

## 3.2 Lagrangian multiplier technique

The starting point is the formulation of the primal variational problem (3.2) as an unconstrained optimization problem: Find $w \in H_0^2(\Omega)$ that minimizes the objective functional

$$J(w) = \frac{1}{2}\int_\Omega \nabla^2 w : \nabla^2 w \; dx - \langle f, w\rangle. \tag{3.35}$$

It is well-known that this minimization problem is equivalent to (3.2). Actually, (3.2) can be seen as the optimality system characterizing the solution of (3.35). By introducing the auxiliary variable

$$\boldsymbol{\sigma} = \nabla^2 w \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \tag{3.36}$$

the objective functional becomes a functional depending on the original and the auxiliary variable:

$$J(w,\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\sigma}\ dx - \langle f, w\rangle. \tag{3.37}$$

The weak formulation of (3.36) leads to the constraint

$$c((w,\boldsymbol{\sigma}),\boldsymbol{\mu}) = 0 \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{M}, \tag{3.38}$$

where

$$c((v,\boldsymbol{\tau}),\boldsymbol{\mu}) = -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\mu}\ dx - \int_\Omega \nabla v \cdot \mathbf{div}\,\boldsymbol{\mu}\ dx,$$

and $\boldsymbol{M}$ is a (not yet specified) space of sufficiently smooth matrix-valued test functions. By this the unconstrained optimization problem from above is transformed to the following constrained optimization problem: Find $(w,\boldsymbol{\sigma}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$ that minimizes the objective functional (3.37) subject to the constraint (3.38). The Lagrangian functional associated with this constrained optimization problem is given by

$$\mathscr{L}((v,\boldsymbol{\tau}),\boldsymbol{\mu}) = J(v,\boldsymbol{\tau}) + c((v,\boldsymbol{\tau}),\boldsymbol{\mu}).$$

The first-order necessary optimality conditions, which are also sufficient for the problem considered here, are $\nabla\mathscr{L}(w,\boldsymbol{\sigma},\boldsymbol{\lambda}) = 0$, and read in detail

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\ dx + c((v,\boldsymbol{\tau}),\boldsymbol{\lambda}) &= \langle f, v\rangle \quad && \text{for all } (v,\boldsymbol{\tau}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}, \\ c((w,\boldsymbol{\sigma}),\boldsymbol{\mu}) &= 0 && \text{for all } \boldsymbol{\mu} \in \boldsymbol{M}. \end{aligned} \tag{3.39}$$

Here $\boldsymbol{\lambda} \in \boldsymbol{M}$ denotes the Lagrangian multiplier associated with the constraint (3.38). The optimality system is a saddle point problem on the space $X = H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$, equipped with the standard norm

$$\|(v,\boldsymbol{\tau})\|_X = \left(|v|_1^2 + \|\boldsymbol{\tau}\|_0^2\right)^{1/2}$$

for the primal variable $(v,\boldsymbol{\tau})$ and the (not yet specified) Hilbert space $\boldsymbol{M}$, equipped with a norm $\|\boldsymbol{\mu}\|_{\boldsymbol{M}}$ for the dual variable $\boldsymbol{\mu}$. An essential condition for the analysis of (3.39) is the inf-sup condition for the bilinear form $c$, which reads: There is a constant $\beta > 0$ such that

$$\sup_{0\neq(v,\boldsymbol{\tau})\in X} \frac{c((v,\boldsymbol{\tau}),\boldsymbol{\mu})}{\|(v,\boldsymbol{\tau})\|_X} \geq \beta\,\|\boldsymbol{\mu}\|_{\boldsymbol{M}}.$$

We have

$$\begin{aligned} &\sup_{0\neq(v,\boldsymbol{\tau})\in X} \frac{c((v,\boldsymbol{\tau}),\boldsymbol{\mu})}{\|(v,\boldsymbol{\tau})\|_X} \\ &\qquad = \left( \sup_{0\neq v\in H_0^1(\Omega)} \frac{\left(\int_\Omega \nabla v \cdot \mathbf{div}\,\boldsymbol{\mu}\ dx\right)^2}{|v|_1^2} + \sup_{0\neq\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega)_{\text{sym}}} \frac{\left(\int_\Omega \boldsymbol{\tau} : \boldsymbol{\mu}\ dx\right)^2}{\|\boldsymbol{\tau}\|_0^2} \right)^{1/2} \\ &\qquad = \left(\|\boldsymbol{\mu}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\mu}\|_{-1}^2\right)^{1/2} \end{aligned} \tag{3.40}$$

for sufficiently smooth functions $\boldsymbol{\mu}$, where, for the first equality, we use the following lemma, see [92, Lemma 2.1],

**Lemma 3.21.** *Let $X_1$ and $X_2$ be Hilbert spaces and $f_1 \in X_1^\star$ and $f_2 \in X_2^\star$. Then*

$$\|f\|_{X^\star}^2 = \|f_1\|_{X_1^\star}^2 + \|f_2\|_{X_2^\star}^2$$

*for $f \in X^\star$ with $X = X_1 \times X_2$, given by $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$.*

If the norm in $\boldsymbol{M}$ is chosen according to (3.40), i.e.

$$\|\boldsymbol{\mu}\|_{\boldsymbol{M}} = \left(\|\boldsymbol{\mu}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\mu}\|_{-1}^2\right)^{1/2},$$

then the inf-sup condition is trivially satisfied with constant $\beta = 1$. This motivates to set $\boldsymbol{M} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$. In order to have a well-defined bilinear form $c$ on $X \times \boldsymbol{M}$, the original definition has to be replaced by

$$c((v, \boldsymbol{\tau}), \boldsymbol{\mu}) = -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\mu}\, dx + \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\mu}, v\rangle,$$

which coincides with the original definition, if $\boldsymbol{\mu}$ is sufficiently smooth, see Lemma 3.3. In the next theorem we show that the remaining conditions of Brezzi's Theorem 2.1 are satisfied for (3.39) with $\mathbf{M} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$.

**Theorem 3.22.** *The bilinear forms*

$$a((w, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \quad \text{and} \quad b((v, \boldsymbol{\tau}), \boldsymbol{\mu}) = c((v, \boldsymbol{\tau}), \boldsymbol{\mu}) \tag{3.41}$$

*satisfy Brezzi's conditions on $H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{sym}$ and $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym}$, equipped with the standard product norm in $H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{sym}$ and $\|\cdot\|_{-1,\operatorname{div}\mathbf{div}}$, respectively, with the constants*

$$\|a\| = \|b\| = \beta = 1 \quad \text{and} \quad \alpha = \frac{1}{1 + 2C_F^2}.$$

*Proof.* 1. Let $(w, \boldsymbol{\sigma}), (v, \boldsymbol{\tau}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$. Then

$$a((w, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) \le \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 \le (\|\boldsymbol{\sigma}\|_0^2 + |w|_1^2)^{1/2} (\|\boldsymbol{\tau}\|_0^2 + |v|_1^2)^{1/2}. \tag{3.42}$$

2. Let $(v, \boldsymbol{\tau}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$ and $\boldsymbol{\mu} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$. Then

$$\begin{aligned} b((v, \boldsymbol{\tau}), \boldsymbol{\mu}) &= -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\mu}\, dx + \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\mu}, v\rangle \\ &\le \|\boldsymbol{\tau}\|_0 \|\boldsymbol{\mu}\|_0 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\mu}\|_{-1} |v|_1 \\ &\le (\|\boldsymbol{\tau}\|_0^2 + |w|_1^2)^{1/2} \|\boldsymbol{\mu}\|_{-1,\operatorname{div}\mathbf{div}}. \end{aligned}$$

3. Let $(w,\boldsymbol{\sigma}) \in \ker B = \{(v,\boldsymbol{\tau}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}} : c((v,\boldsymbol{\tau}),\boldsymbol{\mu}) = 0 \text{ for all } \boldsymbol{\mu} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}\}$. Then for the particular choice $\boldsymbol{\mu} = \boldsymbol{\pi}(w)$ we have

$$c((w,\boldsymbol{\sigma}),\boldsymbol{\mu}) = -\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\pi}(w)\,dx - \int_\Omega \nabla w \cdot \nabla w\,dx = 0.$$

So

$$|w|_1^2 = \left|\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\pi}(w)\,dx\right| \le \sqrt{2}\|w\|_0\|\boldsymbol{\sigma}\|_0 \le C_F\sqrt{2}|w|_1\|\boldsymbol{\sigma}\|_0,$$

which implies $|w|_1 \le C_F\sqrt{2}\|\boldsymbol{\sigma}\|_0$, and hence

$$|w|_1^2 + \|\boldsymbol{\sigma}\|_0^2 \le (2C_F^2+1)\|\boldsymbol{\sigma}\|_0^2 = (2C_F^2+1)a((w,\boldsymbol{\sigma}),(w,\boldsymbol{\sigma})).$$

□

Let us consider the operator $\tilde{\mathcal{A}} : \tilde{X} \to \tilde{X}^\star$ with $\tilde{X} = H_0^1(\Omega)\times\boldsymbol{L}^2(\Omega)_{\text{sym}}\times\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ given by

$$\left\langle \tilde{\mathcal{A}}\begin{bmatrix} w \\ \boldsymbol{\sigma} \\ \boldsymbol{\lambda}\end{bmatrix}, \begin{bmatrix} v \\ \boldsymbol{\tau} \\ \boldsymbol{\mu}\end{bmatrix}\right\rangle = a\left((w,\boldsymbol{\sigma}),(v,\boldsymbol{\tau})\right) + b\left((v,\boldsymbol{\tau}),\boldsymbol{\lambda}\right) + b\left((w,\boldsymbol{\sigma}),\boldsymbol{\mu}\right)$$

with bilinear forms $a$ and $b$ defined in (3.41), then the mixed variational problem (3.39) can be rewritten as the linear operator equation

$$\tilde{\mathcal{A}}\begin{bmatrix} w \\ \boldsymbol{\sigma} \\ \boldsymbol{\lambda}\end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0\end{bmatrix}. \tag{3.43}$$

From the Theorem 3.22, Brezzi's Theorem 2.1 and Theorem 2.2, it follows immediately that $\tilde{\mathcal{A}}$ is an isomorphism and further

$$c\|\tilde{x}\|_{\tilde{X}} \le \|\tilde{\mathcal{A}}\tilde{x}\|_{\tilde{X}^\star} \le C\|\tilde{x}\|_{\tilde{X}} \quad \text{for all } \tilde{x} \in \tilde{X} \tag{3.44}$$

with

$$c \ge \frac{\alpha}{1+\|a\|^2/\beta^2} = \frac{1}{2(1+2C_F^2)} \tag{3.45}$$

and

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1+\sqrt{5}}{2}. \tag{3.46}$$

For $v = 0$, we obtain from the first row of (3.39),

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\;dx - \int_\Omega \boldsymbol{\lambda} : \boldsymbol{\tau}\;dx = 0 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$$

and thus $\boldsymbol{\sigma} = \boldsymbol{\lambda}$. Therefore, the Lagrangian multiplier $\boldsymbol{\lambda}$ can be eliminated in (3.43). The operator equation (3.43) is equivalent to

$$\mathcal{A}\begin{bmatrix} w \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{3.47}$$

with

$$\mathcal{A}(v, \boldsymbol{\tau}) = \tilde{\mathcal{A}}(v, \boldsymbol{\tau}, \boldsymbol{\tau}) \quad \text{for all } v \in H_0^1(\Omega),\ \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}},$$

and we obtain from (3.44),

$$c\|x\|_X \leq c\|(v, \boldsymbol{\tau}, \boldsymbol{\tau})\|_{\tilde{X}} \leq \|\mathcal{A}x\|_{X^\star} \leq C\|(v, \boldsymbol{\tau}, \boldsymbol{\tau})\|_{\tilde{X}} \leq \sqrt{2}C\|x\|_X \tag{3.48}$$

for all $x = (v, \boldsymbol{\tau})$ in $X = H_0^1(\Omega) \times \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ with $c$ and $C$ from (3.45) and (3.46), respectively. Therefore $\mathcal{A} : X \to X^\star$ is an isomorphism. Finally we rewrite (3.47) in variational form: For $f \in H^{-1}(\Omega)$, find $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ and $w \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \qquad - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle &= 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}, \\ - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle \qquad\qquad &= -\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{3.49}$$

Note, that the variational formulations (3.49) and (3.33) coincide.

In the previous two subsections we presented two different techniques for a biharmonic model problem for the construction of the space $X$, such that the operator $\mathcal{A} : X \to X^\star$, representing the mixed variational problem is an isomorphism. Moreover, we have shown that the solution of the primal variational problem (3.2) coincides with the solution of the mixed variational problem (3.49).

In the next section we apply the presented interpolation technique and Lagrangian multiplier technique to another mixed method for biharmonic problems.

## 3.3 The Ciarlet-Raviart method for biharmonic problems

As in the previous two sections we discuss the first biharmonic problem introduced in (3.1).

As an alternative to (3.2) we consider the following standard primal variational formulation of (3.1): Find $w \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta w\, \Delta v\, dx = \langle f, v\rangle \quad \text{for all } v \in H_0^2(\Omega). \tag{3.50}$$

We focus here on the well-known mixed method by Ciarlet and Raviart, see [32], for which an auxiliary variable

$$\sigma = -\Delta w$$

is introduced. With this auxiliary variable the biharmonic problem (3.1) can be rewritten as a boundary value problem of a system of two second-order equations

$$-\Delta w = \sigma, \quad -\Delta \sigma = f \quad \text{in } \Omega, \quad w = \partial_n w = 0 \quad \text{on } \Gamma. \tag{3.51}$$

A choice for the Hilbert spaces which leads to a well-posed variational formulation for (3.51) was already given in [93]. We show here that theses spaces can be derived from each of the two techniques presented in the previous chapter.

### 3.3.1 Interpolation technique

Analogous to Section 3.1 we derive two variational formulations for (3.51), where we reduce the smoothness either for $\sigma$ or for $w$ by the use of integration by parts.

**A first variational formulation.** For given $f \in H^{-2}(\Omega)$, find $\sigma \in V$ and $w \in Q$ such that

$$\begin{aligned} \int_\Omega \sigma\,\tau\; dx \quad + \int_\Omega \Delta w\,\tau\; dx &= 0 && \text{for all } \tau \in V = L^2(\Omega), \\ \int_\Omega \sigma\,\Delta v\; dx &= -\langle f, v\rangle && \text{for all } v \in Q = H_0^2(\Omega), \end{aligned} \tag{3.52}$$

with $V = L^2(\Omega)$ and $Q = H_0^2(\Omega)$.

**A second variational formulation.** For given $f \in L^2(\Omega)$, find $\sigma \in V$ and $w \in Q$ such that

$$\begin{aligned} \int_\Omega \sigma\,\tau\; dx \quad + \int_\Omega w\,\Delta\tau\; dx &= 0 && \text{for all } \tau \in V = H(\Delta,\Omega), \\ \int_\Omega \Delta\sigma\,v\; dx &= -\langle f, v\rangle && \text{for all } v \in Q = L^2(\Omega) \end{aligned} \tag{3.53}$$

with $V = H(\Delta,\Omega)$ and $Q = L^2(\Omega)$, where

$$H(\Delta,\Omega) = \{\tau \in L^2(\Omega) \colon \Delta\tau \in L^2(\Omega)\},$$

equipped with the norm

$$\|\tau\|_{0,\Delta} = \left(\|\tau\|_0^2 + \|\Delta\tau\|_0^2\right)^{1/2}.$$

Here $\Delta$ denotes the Laplace operator in the distributional sense, i.e. for $\tau \in L^2(\Omega)$ we have

$$\langle \Delta\tau, v\rangle = \int_\Omega \tau\,\Delta v\, dx \quad \text{for all } v \in C_0^\infty(\Omega).$$

The next lemma gives several representations of $\Delta$ under additional smoothness assumptions.

**Lemma 3.23** (Green's formulas)**.** *We have:*

- *For all* $\tau \in L^2(\Omega)$ *and* $v \in H_0^2(\Omega)$*:*

$$\langle \Delta\tau, v\rangle = \int_\Omega \tau\, \Delta v\, dx.$$

- *For all* $\tau \in H^1(\Omega)$ *and* $v \in H_0^1(\Omega)$*:*

$$\langle \Delta\tau, v\rangle = -\int_\Omega \nabla\tau \cdot \nabla v\, dx.$$

- *For all* $\tau \in H(\Delta, \Omega)$ *and* $v \in L^2(\Omega)$*:*

$$\langle \Delta\tau, v\rangle = \int_\Omega \Delta\tau\, v.$$

*Proof.* Let $\tau \in C^\infty(\overline{\Omega})$, $v \in C_0^\infty(\Omega)$. Using integration by parts twice we obtain

$$\langle \Delta\tau, v\rangle = \int_\Omega \tau\, \Delta v\, dx = -\int_\Omega \nabla\tau \cdot \nabla v\, dx = \int_\Omega \Delta\tau\, v\, dx. \tag{3.54}$$

The formulas follow from the continuity of the second, third, and fourth term in (3.54), and the density of $C^\infty(\overline{\Omega})$ and $C_0^\infty(\Omega)$ in the corresponding spaces (see appendix, Theorem 9.4).

□

For both variational problems, (3.52) and (3.53), we have the well-posedness:

**Theorem 3.24.** *The bilinear forms*

$$a(\sigma,\tau) = \int_\Omega \sigma\,\tau\; dx \quad \text{and} \quad b(\tau, v) = \int_\Omega \tau\, \Delta v\; dx \tag{3.55}$$

*satisfy Brezzi's conditions on* $V = L^2(\Omega)$ *and* $Q = H_0^2(\Omega)$*, equipped with the norms* $\|\tau\|_0$ *and* $|v|_2$*, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = \beta = 1.$$

**Theorem 3.25.** *The bilinear forms*

$$a(\sigma,\tau) = \int_\Omega \sigma\,\tau\; dx \quad \text{and} \quad b(\tau, v) = \int_\Omega \Delta\tau\, v\; dx \tag{3.56}$$

*satisfy Brezzi's conditions on* $V = H_0^2(\Omega)$ *and* $Q = L^2(\Omega)$*, equipped with the norms* $|\tau|_2$ *and* $\|v\|_0$*, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \text{and} \quad \beta = \frac{1}{\sqrt{1 + C_F^4}}.$$

The proofs of Theorem 3.24 and Theorem 3.25 are completely analogous to the proofs of Theorem 3.1 and Theorem 3.4, respectively, and are, therefore, omitted.

So the operator $\mathcal{A}_0 : X_0 \to X_0^\star$ with $X_0 = L^2(\Omega) \times H_0^2(\Omega)$ given by

$$\left\langle \mathcal{A}_0 \begin{bmatrix} \sigma \\ w \end{bmatrix}, \begin{bmatrix} \tau \\ v \end{bmatrix} \right\rangle = a\,(\sigma, \tau) + b\,(\tau, w) + b\,(\sigma, v)$$

with bilinear forms $a$ and $b$ defined in (3.55), is an isomorphism. And, the operator $\mathcal{A}_1 : X_1 \to X_1^\star$ with $X_1 = H(\Delta, \Omega) \times L^2(\Omega)$ given by

$$\left\langle \mathcal{A}_1 \begin{bmatrix} \sigma \\ w \end{bmatrix}, \begin{bmatrix} \tau \\ v \end{bmatrix} \right\rangle = a\,(\sigma, \tau) + b\,(\tau, w) + b\,(\sigma, v)$$

with bilinear forms $a$ and $b$ defined in (3.56), is an isomorphism.

**A new variational formulation by interpolation.** Note, that the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are different. However, we have:

**Lemma 3.26.**

$$\mathcal{A}_0 x = \mathcal{A}_1 x \quad \textit{for all} \;\; x \in X_0 \cap X_1 \tag{3.57}$$

*and*

$$\mathcal{A}_0^{-1} f = \mathcal{A}_1^{-1} f \quad \textit{for all} \;\; f \in X_0^\star \cap X_1^\star. \tag{3.58}$$

*Proof.* Let $x = (\sigma, w) \in X_0 \cap X_1$, with $X_0 = L^2(\Omega) \times H_0^2(\Omega)$, $X_1 = H_0^2(\Omega) \times L^2(\Omega)$ and $X_0 \cap X_1 = H(\Delta, \Omega) \times H_0^2(\Omega)$. We have $H_0^2(\Omega)$ is dense in $L^2(\Omega)$, and further we obtain from the density of $C^\infty(\overline{\Omega})$ in $L^2(\Omega)$ and $H(\Delta, \Omega)$ (see appendix, Theorem 9.4) the density of $H(\Delta, \Omega)$ in $L^2(\Omega)$. So $X_0 \cap X_1$ is dense in $X_0$ and $X_1$ and therefore, we obtain from the Duality Theorem 3.11 that $X_0^\star + X_1^\star = (X_0 \cap X_1)^\star$. Hence $\mathcal{A}_0 x - \mathcal{A}_1 x \in (X_0 \cap X_1)^\star$and

$$\begin{aligned} \left\langle \mathcal{A}_0 \begin{bmatrix} \sigma \\ w \end{bmatrix}, \begin{bmatrix} \tau \\ v \end{bmatrix} \right\rangle &= \int_\Omega \sigma\,\tau\;dx - \int_\Omega \tau\,\Delta w\;dx - \int_\Omega \sigma\,\Delta v\;dx \\ &= \int_\Omega \sigma\,\tau\;dx - \int_\Omega \Delta\tau\,w\;dx - \int_\Omega \Delta\sigma\,v\;dx \\ &= \left\langle \mathcal{A}_1 \begin{bmatrix} \sigma \\ w \end{bmatrix}, \begin{bmatrix} \tau \\ v \end{bmatrix} \right\rangle \end{aligned}$$

for all $(\tau, v) \in X_0 \cap X_1$ where we obtain the second equality, from Lemma 3.23. This completes the proof of (3.57).

Let $f \in X_0^\star \cap X_1^\star$ and let $x_0 = (\sigma_0, w_0) \in X_0, x_1 = (\sigma_1, w_1) \in X_1$ such that $x_0 = \mathcal{A}_0^{-1} f$ and $x_1 = \mathcal{A}_1^{-1} f$. We have

$$\langle \mathcal{A}_0 x_0, y \rangle = \langle \mathcal{A}_1 x_1, y \rangle \quad \text{for all } y \in X_0 \cap X_1,$$

or equivalently

$$\begin{aligned}\int_\Omega \sigma_0\,\tau\;dx - \int_\Omega \tau\,\Delta w_0\;dx - \int_\Omega \sigma_0\,\Delta v\;dx \\ = \int_\Omega \sigma_1\,\tau\;dx - \int_\Omega \Delta\tau\,w_1\;dx - \int_\Omega \Delta\sigma_1\,v\;dx\end{aligned} \tag{3.59}$$

for all $(\tau, v) \in X_0 \cap X_1$. Using Lemma 3.23 for the second term on the left-hand side and for the third term on the right-hand side in (3.59), we obtain

$$\begin{aligned}\int_\Omega \sigma_0\,\tau\;dx - \int_\Omega \Delta\tau\,w_0\;dx - \int_\Omega \sigma_0\,\Delta v\;dx \\ = \int_\Omega \sigma_1\,\tau\;dx - \int_\Omega \Delta\tau\,w_1\;dx - \int_\Omega \sigma_1\,\Delta v\;dx\end{aligned} \tag{3.60}$$

for all $(\tau, v) \in X_0 \cap X_1$. For the choice $\tau = 0$ it follows from (3.60)

$$\int_\Omega (\sigma_0 - \sigma_1)\,\Delta v\;dx = 0 \tag{3.61}$$

for all $v \in H_0^2(\Omega)$, i.e. $\Delta(\sigma_0 - \sigma_1) = 0$ and thus $\sigma_0 - \sigma_1 \in H(\Delta, \Omega)$. Further we obtain from (3.60) for the choice $\tau = 0$

$$\int_\Omega (\sigma_0 - \sigma_1)\,\tau\;dx - \int_\Omega \Delta\tau\,(w_0 - w_1)\;dx = 0 \tag{3.62}$$

for all $\tau \in H(\Delta, \Omega)$. In particular for the choice $\tau = \sigma_0 - \sigma_1$ in (3.62) we obtain

$$\int_\Omega (\sigma_0 - \sigma_1) : (\sigma_0 - \sigma_1)\;dx = 0$$

and hence $\sigma_0 = \sigma_1 \in H(\Delta, \Omega)$. Therefore, (3.62) reduces to

$$\int_\Omega \Delta\tau\,(w_0 - w_1)\;dx = 0 \tag{3.63}$$

for all $\tau \in H(\Delta, \Omega)$. Further from Theorem 3.25 we have

$$\sup_{0 \neq \tau \in H(\Delta,\Omega)} \frac{\int_\Omega \Delta\tau\,v\;dx}{\|\tau\|_{\Delta,0}} \geq \frac{1}{\sqrt{1 + C_F^4}}\|v\|_0 \tag{3.64}$$

for all $v \in L^2(\Omega)$. Now from (3.63) and (3.64) we obtain

$$0 = \sup_{0 \neq \tau \in H(\Delta,\Omega)} \frac{\int_\Omega \Delta\tau\,(w_0 - w_1)\;dx}{\|\tau\|_{\Delta,0}} \geq \frac{1}{\sqrt{1 + C_F^4}}\|w_0 - w_1\|_0$$

and thus $w_0 = w_1 \in H_0^2(\Omega)$, which completes the proof of (3.58). □

Because of Lemma 3.26 there is a linear and bijective operator $\mathcal{A} : X_0 + X_1 \to X_0^\star + X_1^\star$, given by (3.27), such that $\mathcal{A}$ is an extension of $\mathcal{A}_0$ and $\mathcal{A}_1$.

We have already shown that $\mathcal{A} : X_i \to X_i^\star$ is an isomorphism for $i = 1, 2$. Therefore the first part of the following theorem follows immediately from the Interpolation Theorem 3.13 applied to $\mathcal{A}$ and $\mathcal{A}^{-1}$ and the Duality Theorem 3.11.

**Theorem 3.27.** *The operator $\mathcal{A}$ is an isomorphism from $[X_0, X_1]_{1/2}$ to $[X_0, X_1]_{1/2}^\star$. Moreover, we have*

$$[X_0, X_1]_{1/2} = X$$

*with equivalent norms, where*

$$X = H^{-1}(\Delta, \Omega) \times H_0^1(\Omega),$$

*with*

$$H^{-1}(\Delta, \Omega) = \{\tau \in L^2(\Omega) \colon \Delta\tau \in H^{-1}(\Omega)\},$$

*equipped with the norm*

$$\|\tau\|_{-1,\Delta} = \left(\|\tau\|_0^2 + \|\Delta\tau\|_{-1}^2\right)^{1/2}.$$

The proof of the second part is completely analogous to the proof of Theorem 3.16 and is, therefore, omitted.

For the operator $\mathcal{A}$ we have the following representation on $X$.

**Theorem 3.28.** *The operator $\mathcal{A} : X \to X^\star$ is given by*

$$\left\langle \mathcal{A} \begin{bmatrix} \sigma \\ w \end{bmatrix}, \begin{bmatrix} \tau \\ v \end{bmatrix} \right\rangle = a\,(\sigma, \tau) + b\,(\tau, w) + b\,(\sigma, v)$$

*with bilinear forms*

$$a(\sigma, \tau) = \int_\Omega \sigma\, \tau \; dx \quad \textit{and} \quad b(\tau, v) = \langle \Delta\tau, v\rangle. \tag{3.65}$$

The proof is completely analogous to the proof of Theorem 3.18, and is, therefore, omitted.

The new mixed variational formulation reads as follows: For $f \in H^{-1}(\Omega)$, find $\sigma \in H^{-1}(\Delta, \Omega)$ and $w \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \sigma\, \tau \; dx + \langle \Delta\tau, w\rangle &= 0 && \text{for all } \tau \in H^{-1}(\Delta, \Omega), \\ \langle \Delta\sigma, v\rangle &= -\,\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{3.66}$$

In the following theorem we give estimates for Brezzi's constants for (3.66).

**Theorem 3.29.** *The bilinear forms defined in (3.65) satisfy Brezzi's conditions on $H^{-1}(\Delta,\Omega)$ and $H^1_0(\Omega)$, equipped with the norms $\|\tau\|_{-1,\Delta}$ and $|v|_1$, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + 2C_F^2)^{-1/2}.$$

The proof is completely analogous to the proof of Theorem 3.19, and is, therefore, omitted.

From Theorem 2.2 it follows

$$c\|x\|_X \le \|\mathcal{A}x\|_{X^*} \le C\|x\|_X \quad \text{for all } x \in X,$$

with

$$c \ge \frac{\alpha}{1 + \|a\|^2/\beta^2} = \frac{1}{2 + 2C_F^2}$$

and

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1+\sqrt{5}}{2} \approx 1.6180.$$

The space $H^{-1}(\Delta,\Omega)$ coincides with the space used in [93].

### 3.3.2 Lagrangian multiplier technique.

Starting point is the reformulation of the primal variational formulation (3.50) as unconstrained optimization problem: Find $w \in H^2_0(\Omega)$ that minimizes the objective functional

$$J(w) = \frac{1}{2}\int_\Omega |\Delta w|^2 \, dx - \langle f, w\rangle.$$

By introducing the auxiliary variable

$$\sigma = -\Delta w \in L^2(\Omega) \tag{3.67}$$

the objective functional becomes a functional depending on the original and the auxiliary variable:

$$J(w,\sigma) = \frac{1}{2}\int_\Omega \sigma^2 \, dx - \langle f, w\rangle. \tag{3.68}$$

The weak formulation of (3.67) leads to the constraint

$$c((w,\sigma),\mu) = 0 \quad \text{for all } \mu \in M, \tag{3.69}$$

where

$$c((v,\tau),\mu) = \int_\Omega \tau\mu \, dx - \int_\Omega \nabla v \cdot \nabla \mu \, dx,$$

and $M$ is a (not yet specified) space of sufficiently smooth scalar-valued test functions. By this the unconstrained optimization problem from above is transformed to the following constrained optimization problem: Find $(w, \sigma) \in H_0^1(\Omega) \times L^2(\Omega)$ that minimizes the objective functional (3.68) subject to the constraint (3.69). The Lagrangian functional associated with this constrained optimization problem is given by

$$\mathscr{L}((v,\tau),\mu) = J(v,\tau) + c((v,\tau),\mu),$$

where $\mu \in M$ denotes the Lagrangian multiplier associated with the constraint (3.69). The first-order optimality conditions, which are also sufficient for the problem considered here, are $\nabla \mathscr{L}(w,\sigma,\lambda) = 0$, and read in detail

$$\begin{aligned} \int_\Omega \sigma\tau \, dx + c((v,\tau),\lambda) &= -\langle f, v\rangle && \text{for all } (v,\tau) \in H_0^1(\Omega) \times L^2(\Omega), \\ c((w,\sigma),\mu) &= 0 && \text{for all } \mu \in M. \end{aligned} \tag{3.70}$$

The optimality system is a saddle point problem on the space $X = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the standard norm

$$\|(v,\sigma)\|_X = \left(|v|_1^2 + \|\tau\|_0^2\right)^{1/2}$$

for the primal variable $(v,\tau)$ and the (not yet specified) Hilbert space $M$, equipped with a norm $\|\mu\|_M$ for the dual variable $\mu$.

Motivated by

$$\begin{aligned} &\sup_{0\neq(v,\tau)\in X} \frac{c((v,\tau),\mu)}{\|(v,\tau)\|_X} \\ &= \left( \sup_{0\neq v\in H_0^1(\Omega)} \frac{\left(\int_\Omega \nabla v\cdot\nabla\mu \, dx\right)^2}{|v|_1^2} + \sup_{0\neq\tau\in L^2(\Omega)} \frac{\left(\int_\Omega \tau\,\mu \, dx\right)^2}{\|\tau\|_0^2} \right)^{1/2} \\ &= \left(\|\mu\|_0^2 + \|\Delta\mu\|_{-1}^2\right)^{1/2}. \end{aligned} \tag{3.71}$$

where for the first equality, we use Lemma 3.21, we set $M = H^{-1}(\Delta,\Omega)$. In order to have a well-defined bilinear form $c$, the original definition has to be replaced by

$$c((v,\tau),\mu) = \int_\Omega \tau\mu \, dx + \langle \Delta\mu, v\rangle,$$

which coincides with the original definition, if $\mu$ is sufficiently smooth. For our choice of $M$ all Brezzi's conditions for (3.70), are satisfied.

**Theorem 3.30.** *The bilinear forms*

$$a((w,\sigma),(v,\tau)) = \int_\Omega \sigma\,\tau \, dx \quad \text{and} \quad b((v,\tau),\mu) = c((v,\tau),\mu),$$

*satisfy Brezzi's conditions on $H_0^1(\Omega) \times L^2(\Omega)$ and $H^{-1}(\Delta,\Omega)$ equipped with the standard product norm in $H_0^1(\Omega) \times L^2(\Omega)$ and $\|\cdot\|_{-1,\Delta}$, respectively, with the constants*

$$\|a\| = \|b\| = \beta = 1 \quad \text{and} \quad \alpha = (1 + C_F^2)^{-1}.$$

The proof is completely analogous to the proof of Theorem 3.22, and is, therefore, omitted.

For $v = 0$, it follows from the first row in (3.70) that $\lambda = \sigma$. So the Lagrangian multiplier $\lambda$ can be eliminated. Finally we obtain after reordering the reduced optimality system:

$$\begin{aligned} \int_\Omega \sigma\tau \, dx + \langle \Delta\tau, w\rangle &= 0 && \text{for all } \tau \in H^{-1}(\Delta, \Omega), \\ \langle \Delta\sigma, v\rangle &= -\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{3.72}$$

By an analogous discussion as we had in the end of Section 3.2, we obtain that Brezzi's conditions are still satisfied for the reduced optimality system (3.72). Note, that the variational formulations (3.66) and (3.72) coincide.

In the next chapter we apply the presented interpolation technique and Lagrangian multiplier technique to two model problems from optimal control.

# Chapter 4

# Distributed optimal control problems with time-periodic state equations

In this chapter we apply the presented interpolation technique to two model problems from optimal control, distributed time-periodic Stokes control and distributed time-periodic parabolic control.

## 4.1 Distributed optimal control with the time-periodic Stokes equations

Let $\Omega$ be an open and bounded domain in $\mathbb{R}^d$ for $d \in \{2, 3\}$ with a polygonal/polyhedral Lipschitz-continuous boundary $\Gamma$. For $T > 0$, we introduce the space-time cylinder $Q_T = \Omega \times (0, T)$ and its lateral surface $\Sigma_T = \Gamma \times (0, T)$.

We consider the following model problem: Find the velocity $\mathbf{u}(x, t)$, the pressure $p(x, t)$, and the force $\mathbf{f}(x, t)$ that minimize the cost functional

$$J(\mathbf{u}, \mathbf{f}) = \frac{1}{2} \int_0^T \int_\Omega \|\mathbf{u}(x, t) - \mathbf{u}_d(x, t)\|^2 \, dx \, dt + \frac{\nu}{2} \int_0^T \int_\Omega \|\mathbf{f}(x, t)\|^2 \, dx \, dt$$

subject to the time-periodic Stokes problem

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}(x, t) - \Delta\mathbf{u}(x, t) + \nabla p(x, t) &= \mathbf{f}(x, t) && \text{in } Q_T, \\
\operatorname{div} \mathbf{u}(x, t) &= 0 && \text{in } Q_T, \\
\mathbf{u}(x, t) &= 0 && \text{on } \Sigma_T, \\
\mathbf{u}(x, 0) &= \mathbf{u}(x, T) && \text{on } \Omega, \\
p(x, 0) &= p(x, T) && \text{on } \Omega, \\
\mathbf{f}(x, 0) &= \mathbf{f}(x, T) && \text{on } \Omega.
\end{aligned}$$

Here $\Delta$ denotes the vector Laplacian, $\mathbf{u}_d(x, t)$ is a given target velocity, $\nu > 0$ is a cost or regularization parameter, and $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$. We assume that

$\mathbf{u}_d(x,t)$ is time-harmonic, i.e.:

$$\mathbf{u}_d(x,t) = \mathbf{u}_d(x)\, e^{i\omega t} \quad \text{with} \quad \omega = \frac{2\pi k}{T} \quad \text{for some } k \in \mathbb{Z}.$$

Then there is a time-periodic solution to the original control problem of the form

$$\mathbf{u}(x,t) = \mathbf{u}(x)\, e^{i\omega t}, \quad p(x,t) = p(x)\, e^{i\omega t}, \quad \mathbf{f}(x,t) = \mathbf{f}(x)\, e^{i\omega t},$$

where $\mathbf{u}(x)$, $p(x)$, and $\mathbf{f}(x)$ solve the following time-independent optimal control problem: Minimize

$$J(\mathbf{u},\mathbf{f}) = \frac{1}{2}\int_\Omega \|\mathbf{u}(x) - \mathbf{u}_d(x)\|^2\, dx + \frac{\nu}{2}\int_\Omega \|\mathbf{f}(x)\|^2\, dx$$

subject to

$$\begin{aligned} i\omega\, \mathbf{u}(x) - \Delta\mathbf{u}(x) + \nabla p(x) &= \mathbf{f}(x) && \text{in } \Omega, \\ \operatorname{div} \mathbf{u}(x) &= 0 && \text{in } \Omega, \\ \mathbf{u}(x) &= 0 && \text{on } \Gamma. \end{aligned} \tag{4.1}$$

To obtain the uniqueness for $p$, we assume that $p$ has zero average, i.e.

$$\int_\Omega p(x)\, dx = 0.$$

The Lagrangian functional associated with this constrained optimization problem is given by

$$\begin{aligned} \mathscr{L}(\mathbf{u}, p, \mathbf{f}, \mathbf{w}, r) = J(\mathbf{u},\mathbf{f}) &+ \int_\Omega \mathbf{w}(x)^\star \cdot (i\omega\, \mathbf{u}(x) - \Delta\mathbf{u}(x) + \nabla p(x) - \mathbf{f}(x))\, dx \\ &+ \int_\Omega r(x)^\star \operatorname{div} \mathbf{u}(x)\, dx, \end{aligned}$$

where $\mathbf{w}$ and $r$ denote the Lagrangian multipliers associated with the constraints. Here the symbol $\star$ denotes the conjugate transpose of a vector. The first-order necessary optimality conditions, which are also sufficient for the problem considered here, are $\nabla\mathcal{L}(\mathbf{u}, p, \mathbf{f}, \mathbf{w}, r) =$

0, and read in detail

$$
\begin{aligned}
-i\omega\,\mathbf{w}(x) - \Delta\mathbf{w}(x) + \nabla r(x) &= \mathbf{u}_d(x) - \mathbf{u}(x) && \text{in } \Omega,\\
\operatorname{div}\mathbf{w}(x) &= 0 && \text{in } \Omega,\\
\mathbf{w}(x) &= 0 && \text{on } \Gamma,\\
\int_\Omega r(x)\,dx &= 0, &&\\
&&&\\
\nu\,\mathbf{f}(x) - \mathbf{w}(x) &= 0 && \text{in } \Omega, \qquad (4.2)\\
&&&\\
i\omega\,\mathbf{u}(x) - \Delta\mathbf{u}(x) + \nabla p(x) &= \mathbf{f}(x) && \text{in } \Omega,\\
\operatorname{div}\mathbf{u}(x) &= 0 && \text{in } \Omega,\\
\mathbf{u}(x) &= 0 && \text{on } \Gamma,\\
\int_\Omega p(x)\,dx &= 0. &&
\end{aligned}
$$

From the fifth equation it follows that $\mathbf{f} = \nu^{-1}\mathbf{w}$. So the control $\mathbf{f}$ can be eliminated, and one obtains the reduced optimality system

$$
\begin{aligned}
-i\omega\,\mathbf{w}(x) - \Delta\mathbf{w}(x) + \nabla r(x) &= \mathbf{u}_d(x) - \mathbf{u}(x) && \text{in } \Omega,\\
\operatorname{div}\mathbf{w}(x) &= 0 && \text{in } \Omega,\\
\mathbf{w}(x) &= 0 && \text{on } \Gamma,\\
\int_\Omega r(x)\,dx &= 0, &&\\
i\omega\,\mathbf{u}(x) - \Delta\mathbf{u}(x) + \nabla p(x) &= \nu^{-1}\mathbf{w} && \text{in } \Omega,\\
\operatorname{div}\mathbf{u}(x) &= 0 && \text{in } \Omega,\\
\mathbf{u}(x) &= 0 && \text{on } \Gamma,\\
\int_\Omega p(x)\,dx &= 0 &&
\end{aligned}
\tag{4.3}
$$

which reads in operator notation

$$
\begin{bmatrix}
\boldsymbol{I} & 0 & -\Delta - i\omega\boldsymbol{I} & \nabla\\
0 & 0 & -\operatorname{div} & 0\\
-\Delta + i\omega\boldsymbol{I} & \nabla & -\nu^{-1}\boldsymbol{I} & 0\\
-\operatorname{div} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}\mathbf{u}\\ p\\ \mathbf{w}\\ r\end{bmatrix}
=
\begin{bmatrix}\mathbf{u}_d\\ 0\\ 0\\ 0\end{bmatrix}
$$

with $\mathbf{w}(x) = \mathbf{u}(x) = 0$ on $\Gamma$ and zero average for $p$ and $r$. Here the operator $\boldsymbol{I}$ is given by $(\boldsymbol{I}\mathbf{v})_i = I\mathbf{v}_i$ for $i = 1, ., d$. A first essential observation is that, by swapping the second and the third rows and columns, we obtain a system in saddle point form with a vanishing

2-by-2 block in the right lower part:

$$\begin{bmatrix} \boldsymbol{I} & -\Delta - i\omega\boldsymbol{I} & 0 & \nabla \\ -\Delta + i\omega\boldsymbol{I} & -\nu^{-1}\boldsymbol{I} & \nabla & 0 \\ 0 & -\operatorname{div} & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \end{bmatrix} x = b \tag{4.4}$$

with

$$x = \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ p \\ r \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\mathbf{w}(x) = \mathbf{u}(x) = 0$ on $\Gamma$ and zero average for $p$ and $r$.

### 4.1.1 Transformation to a system with real operators

Elementary calculations show that:

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{I} & -\Delta - i\omega\boldsymbol{I} & 0 & \nabla \\ -\Delta + i\omega\boldsymbol{I} & -\nu^{-1}\boldsymbol{I} & \nabla & 0 \\ 0 & -\operatorname{div} & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \end{bmatrix} \\ &= \mathbf{T}^* \begin{bmatrix} (1+\nu\omega^2)^{1/2}\boldsymbol{I} & -\Delta & 0 & \nabla \\ -\Delta & -\nu^{-1}(1+\nu\omega^2)^{1/2}\boldsymbol{I} & \nabla & 0 \\ 0 & -\operatorname{div} & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \end{bmatrix} \mathbf{T}, \end{aligned}$$

where

$$\mathbf{T} = (1+\nu\omega^2)^{-1/4} \begin{bmatrix} \boldsymbol{I} & -i\omega\boldsymbol{I} & 0 & 0 \\ 0 & (1+\nu\omega^2)^{1/2}\boldsymbol{I} & 0 & 0 \\ 0 & 0 & I & -i\omega I \\ 0 & 0 & 0 & (1+\nu\omega^2)^{1/2} I \end{bmatrix}.$$

So, the original system (4.4) is equivalent to the system

$$\begin{bmatrix} \gamma^2\boldsymbol{I} & -\Delta & 0 & \nabla \\ -\Delta & -\nu^{-1}\gamma^2\boldsymbol{I} & \nabla & 0 \\ 0 & -\operatorname{div} & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} \gamma\mathbf{u}_d \\ -i\omega\gamma^{-1}\mathbf{u}_d \\ 0 \\ 0 \end{bmatrix}, \tag{4.5}$$

with $y = \mathbf{T}x$. Here and in the following we use

$$\gamma = (1+\nu\omega^2)^{1/4}.$$

So, instead of $\omega$ and $\nu$, we consider in the following a problem depending on the parameters $\gamma$ and $\nu$.

In order to solve Problem (4.5) we have to solve for the real and the imaginary parts of $y$ two real problems:

$$\begin{bmatrix} \gamma^2\boldsymbol{I} & -\Delta & 0 & \nabla \\ -\Delta & -\nu^{-1}\gamma^2\boldsymbol{I} & \nabla & 0 \\ 0 & -\operatorname{div} & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ p \\ r \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \\ 0 \\ 0 \end{bmatrix}, \tag{4.6}$$

where $\mathbf{u}(x) = \mathbf{w}(x) = 0$ on $\Gamma$ and $p$ and $r$ have zero average, for

$$\mathbf{g} = \gamma \operatorname{Re}(\mathbf{u}_d),\ \mathbf{h} = \omega\gamma^{-1} \operatorname{Im}(\mathbf{u}_d) \quad \text{and} \quad \mathbf{g} = \gamma \operatorname{Im}(\mathbf{u}_d),\ \mathbf{h} = -\omega\gamma^{-1} \operatorname{Re}(\mathbf{u}_d),$$

where $\operatorname{Re}(\mathbf{v})$ and $\operatorname{Im}(\mathbf{v})$ denote the real and imaginary part for a function $\mathbf{v}$, respectively. Here we use with a slight abuse of notation the same variables for the new unknown as for the original unknown $x$.

So instead of solving a complex system, we have to solve a real system for two different right-hand sides.

We proceed now as follows:

In a first step, we consider the operator given by the left upper 2-by-2 block in (4.6), i.e., we consider for general right hand-sides $\mathbf{g}$ and $\mathbf{h}$ the operator equation

$$\begin{bmatrix} \gamma^2\boldsymbol{I} & -\Delta \\ -\Delta & -\nu^{-1}\gamma^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} \tag{4.7}$$

with $\mathbf{u}(x) = \mathbf{w}(x) = 0$ on $\Gamma$. We derive a variational formulation for (4.7), which will be well-posed with bounds independent of $\nu$ and $\gamma$. The corresponding Hilbert space for the primal variable $(\mathbf{u}, \mathbf{w})$ will be denoted by $V$.

In a second step we derive the variational formulation for the entire problem (4.6), for which we choose the space $Q$ for the dual variable $(p, r)$ in such a way that the variational problem is well posed with bounds independent of $\nu$ and $\gamma$.

### 4.1.2 The space for the primal variable

For deriving the variational formulation for (4.7) we start in the usual way. We multiply the first and second equation in (4.7) by arbitrary test functions $\mathbf{v}$ and $\mathbf{z}$, respectively, and integrate over $\Omega$:

$$\begin{aligned} \gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx \quad - \int_\Omega \Delta\mathbf{w}\cdot\mathbf{v}\,dx \qquad &= \langle \mathbf{g}, \mathbf{v}\rangle, \\ -\int_\Omega \Delta\mathbf{u}\cdot\mathbf{z}\,dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w}\cdot\mathbf{z}\,dx &= \langle \mathbf{h}, \mathbf{z}\rangle. \end{aligned} \tag{4.8}$$

In a next step we derive two different variational formulations by reducing the smoothness assumptions either for $\mathbf{u}$ or for $\mathbf{w}$ by integration by parts.

**A first variational formulation.** To reduce the smoothness assumptions for $\mathbf{u}$ as much as possible, we apply integration by parts twice to the first term of the left-hand side of the second equation in (4.8):

$$\int_\Omega \Delta \mathbf{u} \cdot \mathbf{z}\, dx = \int_\Gamma (\nabla \mathbf{u} \cdot n) \cdot \mathbf{z}\, ds - \int_\Gamma \mathbf{u} \cdot (\nabla \mathbf{z} \cdot n)\, ds + \int_\Omega \mathbf{u} \cdot \Delta \mathbf{z}\, dx. \tag{4.9}$$

Assuming $\mathbf{z} = 0$ on $\Gamma$ for the test functions $\mathbf{v}$ the first boundary integral in (4.9) vanishes. Further the second boundary integral in (4.9) vanishes since $\mathbf{u} = 0$ on $\Gamma$. Together with the first unchanged equation from (4.8) we obtain a first mixed variational formulation: Find $\mathbf{u} \in L^2(\Omega)^d$ and $\mathbf{w} \in H^2_D(\Omega)^d$ such that

$$\begin{aligned} \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}\; dx \quad - \int_\Omega \Delta \mathbf{w} \cdot \mathbf{v}\, dx \qquad &= \langle \mathbf{g}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in L^2(\Omega)^d, \\ - \int_\Omega \mathbf{u} \cdot \Delta \mathbf{z}\, dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z}\; dx &= \langle \mathbf{h}, \mathbf{z} \rangle \quad \text{for all } \ \mathbf{z} \in H^2_D(\Omega)^d, \end{aligned} \tag{4.10}$$

where

$$H^2_D(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$$

equipped with $|\cdot|_2$. We have that $\|\cdot\|_2$ is equivalent to $|\cdot|_2$ on $H^2_D(\Omega)$ and that the identity $|v|_2 = \|\Delta v\|_0$ holds for all $v \in H^2_D(\Omega)$, see, e.g., [40, Theorem 2.2.3].

Problem (4.10) has saddle point structure (2.1) with

$$a(\mathbf{u}, \mathbf{v}) = \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}\; dx, \quad b(\mathbf{v}, \mathbf{z}) = - \int_\Omega \mathbf{v} \cdot \Delta \mathbf{z}\, dx \tag{4.11}$$

and

$$c(\mathbf{w}, \mathbf{z}) = \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z}\; dx. \tag{4.12}$$

Recall, necessary and sufficient assumptions on the bilinear forms $a$, $b$ and $c$ for the well-posed of (4.10) are given by Theorem 2.3. To obtain well-posedness for (4.10) with bounds independent of $\nu$ and $\gamma$, we equip in the following the spaces $L^2(\Omega)^d$ and $H^2_D(\Omega)^d$ in (4.10) with parameter dependent norms such that all assumptions of Theorem 2.3 are satisfied independently of $\nu$ and $\gamma$. Thereby we use the symbols $\cap$ and $+$ for the intersection and the sum of Hilbert spaces, introduced in (3.15), and the following notation:

**Notation.** *Let $\eta$ be a positive number and $H$ be a Hilbert space. Then $\eta H$ denotes the Hilbert space of all functions from $H$ equipped with the norm*

$$\|x\|_{\eta H} = \eta \|x\|_H$$

*for all $x \in H$.*

Motivated by the equality

$$a(\mathbf{v},\mathbf{v}) = \gamma^2\|\mathbf{v}\|_0^2 = \|\mathbf{v}\|^2_{\gamma L^2(\Omega)^d},$$

we equip the space $L^2(\Omega)^d$ in (4.10) with the norm $\|\cdot\|^2_{\gamma L^2(\Omega)^d}$, i.e. we replace the space $L^2(\Omega)$ in (4.10) by $\gamma L^2(\Omega)$.

Motivated by the condition (2.10) we equip the space $H^2_D(\Omega)^d$ with the norm

$$\Big(c(\mathbf{w},\mathbf{w}) + \|b(\cdot,\mathbf{w})\|^2_{(\gamma L^2(\Omega))^\star}\Big)^{1/2}.$$

Then obviously (2.10) is satisfied with $c_{II} = C_{II} = 1$. We have

$$\begin{aligned} c(\mathbf{w},\mathbf{w}) + \|b(\cdot,\mathbf{w})\|^2_{V^\star} &= c(\mathbf{w},\mathbf{w}) + \left(\sup_{\mathbf{v}\in\gamma L^2(\Omega)^d} \frac{\int_\Omega \mathbf{v}\cdot\Delta\mathbf{w}\,dx}{\gamma\|\mathbf{v}\|_0}\right)^2 \\ &= c(\mathbf{w},\mathbf{w}) + \big(\gamma^{-1}\|\Delta\mathbf{w}\|_0\big)^2 \\ &= c(\mathbf{w},\mathbf{w}) + \gamma^{-2}|\mathbf{w}|_2^2 \\ &= \nu^{-1}\gamma^2\|\mathbf{w}\|_0^2 + \gamma^{-2}|\mathbf{w}|_2^2 \\ &= \|\mathbf{w}\|^2_{\nu^{-1/2}\gamma L^2(\Omega)^d\cap\gamma^{-1}\left(H^2(\Omega)\cap H^1_0(\Omega)\right)^d}, \end{aligned}$$

i.e. we replace the space $H^2_D(\Omega)^d$ in (4.10) by $\nu^{-1/2}\gamma L^2(\Omega)^d \cap \gamma^{-1}H^2_D(\Omega)^d$.

In the following theorem we show that for

$$\mathbf{u},\mathbf{v} \in \gamma L^2(\Omega)^d \quad \text{and} \quad \mathbf{w},\mathbf{z} \in \nu^{-1/2}\gamma L^2(\Omega)^d \cap \gamma^{-1}H^2_D(\Omega)^d$$

the variational formulation (4.10) is well-posed with bounds independent of $\gamma$ and $\nu$.

**Theorem 4.1.** *The linear operator $A_0$ introduced by*

$$\left\langle A_0 \begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}, \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\rangle = a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},\mathbf{w}) + b(\mathbf{u},\mathbf{z}) - c(\mathbf{w},\mathbf{z})$$

*with bilinear forms $a$, $b$, and $c$ defined in (4.11) and (4.12) is an isomorphism from $V_0$ to $V_0^\star$ for*

$$V_0 = \gamma L^2(\Omega)^d \times \left(\nu^{-1/2}\gamma L^2(\Omega)^d \cap \gamma^{-1}H^2_D(\Omega)^d\right),$$

*whose natural norm is given by*

$$\|(\mathbf{v},\mathbf{z})\|_{V_0} = \left(\gamma^2\|\mathbf{v}\|_0^2 + \nu^{-1}\gamma^2\|\mathbf{z}\|_0^2 + \gamma^{-2}|\mathbf{z}|_2^2\right)^{1/2}$$

*for $(\mathbf{v},\mathbf{z}) \in V_0$. Furthermore, we have*

$$c\|\mathbf{v}\|_{V_0} \le \|A_0\mathbf{v}\|_{V_0^\star} \le C\|\mathbf{v}\|_{V_0} \quad \textit{for all } \mathbf{v}\in V_0,$$

*with*

$$c = \frac{3-\sqrt{5}}{32} \approx 0.0239 \quad \textit{and} \quad C = 2\sqrt{2} \approx 2.8284. \tag{4.13}$$

*Proof.* We apply Theorem 2.3 for $V = \gamma L^2(\Omega)^d$ and $Q = \nu^{-1/2}\gamma L^2(\Omega)^d \cap \gamma^{-1} H_D^2(\Omega)^d$. Obviously condition (2.10) is satisfied with $c_{II} = C_{II} = 1$.

It remains to check the condition (2.9): Let $\mathbf{u} \in V$. We have:

$$\|\mathbf{u}\|_V^2 = \gamma^2 \|\mathbf{u}\|_0^2 = a(\mathbf{u},\mathbf{u}) \le a(\mathbf{u},\mathbf{u}) + \|b(\mathbf{u},\cdot)\|_{Q^\star}^2,$$

i.e. $c_I = 1$ and

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) + \|b(\mathbf{u},\cdot)\|_{Q^\star}^2 &= a(\mathbf{u},\mathbf{u}) + \|b(\mathbf{u},\cdot)\|^2_{\left(\nu^{-1/2}\gamma L^2(\Omega)^d \cap \gamma^{-1} H_D^2(\Omega)^d\right)^\star} \\ &\le a(\mathbf{u},\mathbf{u}) + \left( \sup_{\mathbf{z} \in \gamma^{-1} H_D^2(\Omega)^d} \frac{\int_\Omega \mathbf{u} \cdot \Delta \mathbf{z} \, dx}{\gamma^{-1} |\mathbf{z}|_2} \right)^2 \\ &\le a(\mathbf{u},\mathbf{u}) + \gamma^2 \|\mathbf{v}\|_0^2 \\ &= 2\|\mathbf{u}\|_V^2, \end{aligned}$$

i.e. $C_I = 2$. Therefore, all assumptions of Theorem 2.3 are satisfied.

Finally we obtain the values for the constants $c$ and $C$ in (4.13) from (2.11) and (2.12). □

**A second variational formulation.** Now we reduce the smoothness assumptions for $\mathbf{w}$ as much as possible. Therefore we apply integration by parts twice to the second term of the left-hand side of the first equation in (4.8):

$$\int_\Omega \Delta \mathbf{w} \cdot \mathbf{v} \, dx = \int_\Gamma (\nabla \mathbf{w} \cdot n) \cdot \mathbf{v} \, ds - \int_\Gamma \mathbf{w} \cdot (\nabla \mathbf{v} \cdot n) \, ds + \int_\Omega \mathbf{w} \cdot \Delta \mathbf{v} \, dx \tag{4.14}$$

Assuming $\mathbf{v} = 0$ on $\Gamma$ for the test functions $\mathbf{v}$ the first boundary integral in (4.14) vanishes. Further the second boundary integral in (4.14) vanishes since $\mathbf{w} = 0$ on $\Gamma$. Together with the unchanged second equation from (4.8) this leads to a second mixed variational formulation: Find $\mathbf{u} \in H_D^2(\Omega)^d$ and $\mathbf{w} \in L^2(\Omega)^d$ such that

$$\begin{aligned} \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v} \; dx \quad - \int_\Omega \mathbf{w} \cdot \Delta \mathbf{v} \, dx \qquad &= \langle \mathbf{g}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in H_D^2(\Omega)^d, \\ - \int_\Omega \Delta \mathbf{u} \cdot \mathbf{z} \, dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z} \; dx &= \langle \mathbf{h}, \mathbf{z} \rangle \quad \text{for all } \mathbf{z} \in L^2(\Omega)^d. \end{aligned} \tag{4.15}$$

Problem (4.15) has saddle point structure (2.1) with

$$a(\mathbf{u},\mathbf{v}) = \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v} \; dx, \quad b(\mathbf{v},\mathbf{z}) = - \int_\Omega \Delta \mathbf{v} \cdot \mathbf{z} \, dx \tag{4.16}$$

and

$$c(\mathbf{w},\mathbf{z}) = \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z} \; dx. \tag{4.17}$$

As for the first variational formulation, we equip in the following the spaces $\left(H^2(\Omega) \cap H_0^1(\Omega)\right)^d$ and $L^2(\Omega)^d$ in (4.15) with parameter dependent norms such that all assumptions of Theorem 2.2 are satisfied independently of $\nu$ and $\gamma$.

Motivated by the equality

$$c(\mathbf{w}, \mathbf{w}) = \gamma^2 \nu^{-1} \|\mathbf{w}\|_0^2 = \|\mathbf{w}\|^2_{\nu^{-1/2}\gamma L^2(\Omega)^d},$$

we equip the space $L^2(\Omega)^d$ with the norm $\|\cdot\|_{\nu^{-1/2}\gamma L^2(\Omega)^d}$, i.e. we replace the space $L^2(\Omega)^d$ in (4.15) by $\nu^{-1/2}\gamma L^2(\Omega)^d$.

Motivated by the condition (2.9) we equip the space $H_D^2(\Omega)^d$ with the norm

$$\left(a(\mathbf{v}, \mathbf{v}) + \|b(\mathbf{v}, \cdot)\|^2_{Q^\star}\right)^{1/2}.$$

Then obviously (2.9) is satisfied with $c_I = C_I = 1$. We have

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) + \|b(\mathbf{v}, \cdot)\|^2_{Q^\star} &= a(\mathbf{v}, \mathbf{v}) + \left( \sup_{\mathbf{z} \in \nu^{1/2}\gamma L^2(\Omega)^d} \frac{\int_\Omega \mathbf{z} \cdot \Delta \mathbf{v}\, dx}{\nu^{-1/2}\gamma \|\mathbf{z}\|_0} \right)^2 \\ &= a(\mathbf{v}, \mathbf{v}) + \left(\nu^{1/2}\gamma^{-1} \|\Delta \mathbf{v}\|_0\right)^2 \\ &= a(\mathbf{v}, \mathbf{v}) + \nu\gamma^{-2} |\mathbf{v}|_2^2 \\ &= \gamma^2 \|\mathbf{v}\|_0^2 + \nu\gamma^{-2} |\mathbf{v}|_2^2 \\ &= \|\mathbf{v}\|^2_{\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d}, \end{aligned}$$

i.e. we replace the space $H_D^2(\Omega)^d$ in (4.15) by $\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d$.

In the following theorem we show that for

$$\mathbf{u}, \mathbf{v} \in \gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d \quad \text{and} \quad \mathbf{w}, \mathbf{z} \in \nu^{-1/2}\gamma L^2(\Omega)^d$$

the variational formulation (4.15) is well-posed with bounds independent of $\gamma$ and $\nu$.

**Theorem 4.2.** *The linear operator $A_1$ introduced by*

$$\left\langle A_1 \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \mathbf{w}) + b(\mathbf{u}, \mathbf{z}) - c(\mathbf{w}, \mathbf{z}),$$

*with bilinear forms $a$, $b$ and $c$ defined in (4.16) and (4.17) is an isomorphism from $V_1$ to $V_1^\star$ for*

$$V_1 = \left(\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right) \times \nu^{-1/2}\gamma L^2(\Omega)^d,$$

*whose natural norm is given by*

$$\|(\mathbf{v}, \mathbf{z})\|_{V_1} = \left(\gamma^2 \|\mathbf{v}\|_0^2 + \nu\gamma^{-2} |\mathbf{v}|_2^2 + \nu^{-1}\gamma^2 \|\mathbf{z}\|_0^2\right)^{1/2}$$

*for $(\mathbf{v}, \mathbf{z}) \in V_1$. Furthermore, we have*

$$c\|\mathbf{v}\|_{V_1} \le \|A_1 \mathbf{v}\|_{V_1^\star} \le C \|\mathbf{v}\|_{V_1} \quad \textit{for all } \mathbf{v} \in V_1, \tag{4.18}$$

*with the same positive constants $c$ and $C$ as in (4.13).*

*Proof.* We apply Theorem 2.3 for $V = \gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H^2_D(\Omega)^d$ and $Q = \nu^{-1/2}\gamma L^2(\Omega)^d$. Obviously condition (2.9) is satisfied with $c_I = C_I = 1$.

It remains to check the condition (2.10): Let $\mathbf{w} \in Q$. We have:

$$\|\mathbf{w}\|_Q^2 = \nu^{-1}\gamma^2 \|\mathbf{w}\|_0^2 = c(\mathbf{w},\mathbf{w}) \le c(\mathbf{w},\mathbf{w}) + \|b(\cdot,\mathbf{w})\|^2_{V^\star},$$

i.e. $c_{II} = 1$ and

$$\begin{aligned} c(\mathbf{w},\mathbf{w}) + \|b(\cdot,\mathbf{w})\|^2_{V^\star} &= c(\mathbf{w},\mathbf{w}) + \|b(\cdot,\mathbf{w})\|^2_{\left(\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H^2_D(\Omega)^d\right)^\star} \\ &\le c(\mathbf{u},\mathbf{u}) + \left( \sup_{\mathbf{v} \in \nu^{1/2}\gamma^{-1} H^2_D(\Omega)^d} \frac{\int_\Omega \Delta \mathbf{v} \cdot \mathbf{w}\, dx}{\nu^{1/2}\gamma^{-1} |\mathbf{v}|_2} \right)^2 \\ &\le c(\mathbf{w},\mathbf{w}) + \nu^{-1}\gamma^2 \|\mathbf{w}\|_0^2 \\ &= 2\|\mathbf{w}\|_Q^2, \end{aligned}$$

i.e. $C_{II} = 2$. Therefore all assumptions of Theorem 2.3 are satisfied.

Finally we obtain the same values as in (4.13) for the constants $c$ and $C$ in (4.18) from (2.11) and (2.12). □

**A new variational formulation by interpolation.** Note, that the operators $A_0 : V_0 \to V_0^\star$ and $A_1 : V_1 \to V_1^\star$ introduced in Theorems 4.1 and 4.2 are different. However, we have:

**Lemma 4.3.** *We have:*

$$A_0 \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = A_1 \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \quad \textit{for all } (\mathbf{u},\mathbf{w}) \in V_0 \cap V_1. \tag{4.19}$$

*Proof.* Let $(\mathbf{u},\mathbf{w}) \in V_0 \cap V_1$ with $V_0$ and $V_1$ given in Theorem 4.1 and Theorem 4.2 and

$$\begin{aligned} V_0 \cap V_1 = &\left(2\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H^2_D(\Omega)^d\right) \\ &\times \left(2\nu^{-1/2}\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H^2_D(\Omega)^d\right). \end{aligned}$$

It is easy to see that $V_0 \cap V_1$ is dense in $V_0$ and $V_1$. Therefore, we obtain from the Duality Theorem 3.11 that $V_0^\star + V_1^\star = (V_0 \cap V_1)^\star$. Hence

$$A_0 \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} - A_1 \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \in (V_0 \cap V_1)^\star$$

and

$$\begin{aligned}\left\langle A_0 \begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}, \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\rangle &= \gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \mathbf{u}\cdot\Delta\mathbf{z}\,dx + \int_\Omega \mathbf{v}\cdot\Delta\mathbf{w}\,dx \\ &\quad - \gamma^2\nu^{-1}\int_\Omega \mathbf{w}\cdot\mathbf{z}\,dx \\ &= \gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \Delta\mathbf{u}\cdot\mathbf{z}\,dx + \int_\Omega \Delta\mathbf{v}\cdot\mathbf{w}\,dx \\ &\quad - \gamma^2\nu^{-1}\int_\Omega \mathbf{w}\cdot\mathbf{z}\,dx \\ &= \left\langle A_1 \begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}, \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\rangle\end{aligned}$$

for all $(\mathbf{v},\mathbf{z}) \in V_0 \cap V_1$, where we obtain the second equality by applying integration by parts twice to the terms on the left-hand side involving the $\Delta$ operator. This proves (4.19). □

Because of Lemma 4.3 there is a linear operator $A : V_0 + V_1 \to V_0^\star + V_1^\star$ given by

$$A\mathbf{v} = A_0\mathbf{v}_0 + A_1\mathbf{v}_1,$$

for all $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $\mathbf{v}_0 \in V_0$ and $\mathbf{v}_1 \in V_1$, such that $A$ is an extension of $A_0$ and $A_1$.

**Remark 4.4.** *Contrary to the corresponding operators of the previous discussed problems, we were not able here to prove*

$$A_0^{-1}\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix} = A_1^{-1}\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix} \quad \textit{for all } (\mathbf{g},\mathbf{h}) \in V_0^\star \cap V_1^\star,$$

*without additional assumptions on the domain $\Omega$. This means we do not even know if $A^{-1}$ from $V_0^\star + V_1^\star$ to $V_0 + V_1$ exists. Anyway, as we will see in the following, we have that $A$ is an isomorphism from $[V_0, V_1]_{1/2}$ to $[V_0, V_1]_{1/2}^\star$.*

We have already shown that $A : V_i \to V_i^\star$ is an isomorphism for $i = 1, 2$. Therefore the following theorem follows immediately from the Interpolation Theorem 3.13 applied to $A$ and the Duality Theorem 3.11.

**Theorem 4.5.** *There exists a positive constant $C$ independent of $\gamma$ and $\nu$ such that:*

$$\left\| A\begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\|_{[V_0,V_1]_{1/2}^\star} \le C\|(\mathbf{v},\mathbf{z})\|_{[V_0,V_1]_{1/2}} \quad \textit{for all } (\mathbf{v},\mathbf{z}) \in [V_0,V_1]_{1/2}.$$

Next we give a representation result for $[V_0, V_1]_{1/2}$, where the following two results are essential:

For the interpolation of parameter dependent Hilbert spaces we have the following property.

**Lemma 4.6.** *Let $X_0, X_1$ and $X$ be Hilbert Spaces, with $X_0$ and $X_1$. Then it holds for all positive real numbers $\alpha$ and $\beta$:*

$$[\alpha X_0, \beta X_1]_\theta = \alpha^{1-\theta}\beta^\theta [X_0, X_1]_\theta.$$

*for all $\theta \in (0, 1)$, with equal norms.*

*Proof.* Let $x \in [\alpha X_0, \beta X_1]_\theta$. We have:

$$\begin{aligned}
\|x\|_{[\alpha X_0,\beta X_1]_\theta} &= \left(\int_0^\infty t^{-2\theta} K(t, x, \alpha X_0, \beta X_2)^2 \, dt/t\right)^{1/2} \\
&= \alpha \left(\int_0^\infty t^{-2\theta} K(t, x, X_0, \beta/\alpha X_2)^2 \, dt/t\right)^{1/2} \\
&= \alpha \left(\int_0^\infty t^{-2\theta} K(\beta\, t/\alpha, x, X_0, X_2)^2 \, dt/t\right)^{1/2} \\
&= \alpha \left(\int_0^\infty (\alpha\, t/\beta)^{-2\theta}) K(t, x, X_0, X_2)^2 \, dt/t\right)^{1/2} \\
&= \alpha(\alpha/\beta)^{-\theta} \left(\int_0^\infty t^{-2\theta}) K(t, x, X_0, X_2)^2 \, dt/t\right)^{1/2} \\
&= \alpha^{1-\theta}\beta^\theta \|x\|_{[X_0,X_1]_\theta}.
\end{aligned}$$

□

Further we have the following interpolation result. For a proof see Theorem 9.1 in the appendix.

**Theorem 4.7.**

$$[L^2(\Omega), H^2_D(\Omega)]_{1/2} = H^1_0(\Omega)$$

*with equivalent norms.*

Now we show a representation result for $[V_0, V_1]_{1/2}$.

**Theorem 4.8.**

$$[V_0, V_1]_{1/2} = V \tag{4.20}$$

*with equivalent norms independent of $\gamma$ and $\nu$, where*

$$V = \left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H^1_0(\Omega)^d\right) \times \nu^{-1/2} \left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H^1_0(\Omega)^d\right)$$

*whose natural norm is given by*

$$\|(\mathbf{v}, \mathbf{z})\|_V = \left(\gamma^2\|\mathbf{v}\|_0^2 + \nu^{1/2}|\mathbf{v}|_1^2 + \nu^{-1}\gamma^2\|\mathbf{z}\|_0^2 + \nu^{-1/2}|\mathbf{z}|_1^2\right)^{1/2}$$

*for $(\mathbf{v}, \mathbf{z}) \in V$.*

*Proof.* From Lemma 4.6 we obtain the identity

$$\begin{aligned}[V_0, V_1]_{1/2} = &\left[\gamma L^2(\Omega)^d, \left(\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right)\right]_{1/2} \\ &\times \nu^{-1/2} \left[\gamma L^2(\Omega)^d, \left(\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right)\right]_{1/2}\end{aligned} \tag{4.21}$$

with equal norms. Next we obtain from Lemma 3.10

$$\begin{aligned}&\left[\gamma L^2(\Omega)^d, \left(\gamma L^2(\Omega)^d \cap \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right)\right]_{1/2} \\ &= \gamma L^2(\Omega)^d \cap \left[\gamma L^2(\Omega)^d, \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right]_{1/2}\end{aligned} \tag{4.22}$$

with equivalent norms independent of $\gamma$ and $\nu$, and from Lemma 4.6 and Theorem 4.7

$$\left[\gamma L^2(\Omega)^d, \nu^{1/2}\gamma^{-1} H_D^2(\Omega)^d\right]_{1/2} = \nu^{1/4} \left[L^2(\Omega)^d, H_D^2(\Omega)^d\right]_{1/2} = \nu^{1/4} H_0^1(\Omega)^d \tag{4.23}$$

with equivalent norms independent of $\gamma$ and $\nu$.

Now, from (4.21), (4.22) and (4.23) we obtain

$$[V_0, V_1]_{1/2} = \left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H_0^1(\Omega)^d\right) \times \nu^{-1/2} \left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H_0^1(\Omega)^d\right)$$

with equivalent norms independent of $\gamma$ and $\nu$, which completes the proof. □

Moreover, we have the following representation for $A$ on $V$.

**Theorem 4.9.** *The operator $A : V \to V^\star$ is given*

$$\left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle = a(\mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w}) + b(\mathbf{u}, \mathbf{z}) - c(\mathbf{w}, \mathbf{z}),$$

*with bilinear forms*

$$a(\mathbf{u}, \mathbf{v}) = \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v} \; dx, \quad b(\mathbf{v}, \mathbf{z}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \; dx,$$

*and*

$$c(\mathbf{w}, \mathbf{z}) = \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z} \; dx.$$

*Proof.* It is easy to see that for all $\nu > 0$ and $\omega > 0$ there is a positive constant $C$ such that $\|\mathbf{v}\|_V \leq C\|\mathbf{v}\|_1$ for all $\mathbf{v} \in C_0^\infty(\Omega)^d \times C_0^\infty(\Omega)^d$. Now from the density of $C_0^\infty(\Omega)^d$ in $H_0^1(\Omega)^d$ it follows that $C_0^\infty(\Omega)^d \times C_0^\infty(\Omega)^d$ is dense in $V$.

Let $(\mathbf{u},\mathbf{w}) \in V$ and $(\mathbf{v},\mathbf{z}) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. We have:

$$\begin{aligned}
\left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle &= \left\langle A_0 \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle + \left\langle A_1 \begin{bmatrix} 0 \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle \\
&= \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx - \int_\Omega \mathbf{u} \cdot \Delta \mathbf{z}\, dx - \int_\Omega \Delta \mathbf{v} \cdot \mathbf{w}\, dx \\
&\quad + \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z}\, dx \qquad (4.24) \\
&= \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx + \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{z}\, dx + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{w}\, dx \\
&\quad + \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z}\, dx,
\end{aligned}$$

where we use integration by parts for the last equality. Since all expressions in (4.24) are continuous for $(\mathbf{u},\mathbf{w})$ in $V$, (4.24) is still satisfied for the closure of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ in $V$-norm. This completes the proof. $\square$

So the new variational formulation reads as follows: Find $\mathbf{u} \in V$ and $\mathbf{w} \in Q$ such that

$$\begin{aligned}
\gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx \quad \int_\Omega \nabla \mathbf{w} \cdot \nabla \mathbf{v}\, dx &= \langle \mathbf{g}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V, \\
\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{z}\, dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z}\, dx &= \langle \mathbf{h}, \mathbf{z} \rangle \quad \text{for all } \mathbf{z} \in Q
\end{aligned} \qquad (4.25)$$

with

$$V = \left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H_0^1(\Omega)^d\right) \quad \text{and} \quad Q = \nu^{-1/2}\left(\gamma L^2(\Omega)^d \cap \nu^{1/4} H_0^1(\Omega)^d\right).$$

In the next theorem we give an estimate for the upper bound of $A$ and further we show that $A$ satisfies an inf-sup condition for each $H$-invariant subspace $\hat{V} \subseteq V$, where the linear operator $H : V \to V$ is given by

$$H \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{v} + \nu^{-1/2}\mathbf{z} \\ \nu^{1/2}\mathbf{v} - \mathbf{z} \end{bmatrix}. \qquad (4.26)$$

**Theorem 4.10.** *1. For all $(\mathbf{v},\mathbf{z}) \in V$ :*

$$\left\| A \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\|_{V^\star} \le \|(\mathbf{v},\mathbf{z})\|_V.$$

*2. For each subspace $\hat{V} \subseteq V$ with $H(\hat{V}) \subseteq \hat{V}$ we have*

$$\sup_{(\mathbf{v},\mathbf{z}) \in \hat{V}} \frac{\left| \left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle \right|}{\|(\mathbf{v},\mathbf{z})\|_V} \ge \frac{1}{\sqrt{2}} \|(\mathbf{u},\mathbf{w})\|_V \qquad (4.27)$$

*for all $(\mathbf{u},\mathbf{w}) \in \hat{V}$.*

*Proof.* Let $(\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z}) \in V$. We have:

$$
\begin{aligned}
\left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle &= \left| \gamma^2 \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx + \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{z} \, dx + \int_\Omega \nabla \mathbf{w} \cdot \nabla \mathbf{v} \, dx \right. \\
&\quad \left. + \nu^{-1} \gamma^2 \int_\Omega \mathbf{w} \cdot \mathbf{z} \, dx \right| \\
&\leq \gamma^2 \|\mathbf{u}\|_0 \|\mathbf{v}\|_0 + |\mathbf{u}|_1 |\mathbf{z}|_1 + |\mathbf{w}|_1 |\mathbf{v}|_1 + \nu^{-1} \gamma^2 \|\mathbf{w}\|_0 \|\mathbf{z}\|_0 \\
&= \gamma^2 \|\mathbf{u}\|_0 \|\mathbf{v}\|_0 + \nu^{1/4} |\mathbf{u}|_1 \nu^{-1/4} |\mathbf{z}|_1 + \nu^{-1/4} |\mathbf{w}|_1 \nu^{1/4} |\mathbf{v}|_1 + \nu^{-1} \gamma^2 \|\mathbf{w}\|_0 \|\mathbf{z}\|_0 \\
&\leq \left( \gamma^2 \|\mathbf{u}\|_0^2 + \nu^{1/2} |\mathbf{u}|_1^2 + \nu^{-1/2} |\mathbf{w}|_1^2 + \nu^{-1} \gamma^2 \|\mathbf{w}\|_0^2 \right)^{1/2} \\
&\quad \left( \gamma^2 \|\mathbf{v}\|_0^2 + \nu^{1/2} |\mathbf{v}|_1^2 + \nu^{-1/2} |\mathbf{z}|_1^2 + \nu^{-1} \gamma^2 \|\mathbf{z}\|_0^2 \right)^{1/2} \\
&= \|(\mathbf{u}, \mathbf{w})\|_V \|(\mathbf{v}, \mathbf{z})\|_V .
\end{aligned}
$$

This proves the first part.

Further, we have:

$$
\begin{aligned}
\left| \left\langle A \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix}, H \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle \right| &= \gamma^2 \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx + \gamma^2 \nu^{-1/2} \int_\Omega \mathbf{v} \cdot \mathbf{z} \, dx \\
&\quad + \nu^{1/2} \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, dx - \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{z} \, dx \\
&\quad + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{z} \, dx + \nu^{-1/2} \int_\Omega \nabla \mathbf{z} \cdot \nabla \mathbf{z} \, dx \\
&\quad + \gamma^2 \int_\Omega \mathbf{v} \cdot \mathbf{z} \, dx - \nu^{-1/2} \gamma^2 \int_\Omega \mathbf{z} \cdot \mathbf{z} \, dx \\
&= \gamma^2 \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx + \nu^{1/2} \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, dx \\
&\quad + \nu^{-1/2} \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{z} \, dx + \gamma^2 \int_\Omega \mathbf{v} \cdot \mathbf{z} \, dx \\
&= \|(\mathbf{v}, \mathbf{z})\|_V^2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| H \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\|_V^2 &= \gamma^2 \|\mathbf{v} + \nu^{-1/2}\mathbf{z}\|_0^2 + \nu^{1/2}\|\mathbf{v} + \nu^{-1/2}\mathbf{z}\|_1^2 + \nu^{-1}\gamma^2\|\nu^{1/2}\mathbf{v} - \mathbf{z}\|_0^2 \\
&\quad + \nu^{-1/2}\|\nu^{1/2}\mathbf{v} - \mathbf{z}\|_1^2 \\
&= \gamma^2\|\mathbf{v}\|_0^2 + 2\gamma^2\nu^{-1/2}(\mathbf{v},\mathbf{z})_0 + \nu^{-1}\gamma^2\|\mathbf{z}\|_0^2 \\
&\quad + \nu^{1/2}\|\mathbf{v}\|_1^2 + 2(\mathbf{v},\mathbf{z})_1 + \nu^{-1/2}\|\mathbf{z}\|_1^2 \\
&\quad + \gamma^2\|\mathbf{v}\|_0^2 - 2\gamma^2\nu^{-1/2}(\mathbf{v},\mathbf{z})_0 + \nu^{-1}\gamma^2\|\mathbf{z}\|_0^2 \\
&\quad + \nu^{1/2}\|\mathbf{v}\|_1^2 - 2(\mathbf{v},\mathbf{z})_1 + \nu^{-1/2}\|\mathbf{z}\|_1^2 \\
&= 2\left(\gamma^2\|\mathbf{v}\|_0^2 + \nu^{-1}\gamma^2\|\mathbf{z}\|_0^2 + \nu^{1/2}\|\mathbf{v}\|_1^2 + \nu^{-1/2}\|\mathbf{z}\|_1^2\right) \\
&= 2\left(\gamma^2\|\mathbf{v}\|_0^2 + \nu^{1/2}\|\mathbf{v}\|_1^2 + \nu^{-1}\left(\gamma^2\|\mathbf{z}\|_0^2 + \nu^{1/2}\|\mathbf{z}\|_1^2\right)\right) \\
&= 2\|(\mathbf{v},\mathbf{z})\|_V^2.
\end{aligned}
$$

Therefore,

$$
\sup_{0 \neq (\mathbf{v},\mathbf{z}) \in \hat{V}} \frac{\left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle}{\|(\mathbf{v},\mathbf{z})\|_V} \geq \frac{\left| \left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, H \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \right\rangle \right|}{\left\| H \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \right\|_V} = \frac{1}{\sqrt{2}} \|(\mathbf{u},\mathbf{w})\|_V,
$$

which completes the proof of the second part. □

Now from Theorem 4.10 it follows immediately:

**Corollary 4.11.** *The operator $A$ is an isomorphism from $V$ to $V^\star$ with bounds independent of $\gamma$ and $\nu$.*

This completes the consideration of the left upper 2-by-2 block of the operator in (4.5). In the next subsection we consider the entire operator in (4.5).

### 4.1.3 The space for the dual variable

For deriving the variational formulation for the entire problem (4.6) we start in the usual way: We multiply the four equations by arbitrary test functions $\mathbf{v}$, $\mathbf{z}$, $q$ and $s$, and integrate over $\Omega$. For the 2-by-2 upper left block in (4.6) we use the variational formulation derived in (4.25). Finally we apply integration by parts to both terms of the 2-by-2 upper right block, where the appearing boundary integrals vanish under the assumption that $\mathbf{v} = \mathbf{z} = 0$ on $\Gamma$. This leads to the following mixed variational formulation of (4.6): Find $(\mathbf{u}, \mathbf{w}) \in V$

and $(p,\mathbf{z}) \in Q$ such that

$$
\begin{aligned}
\gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\; dx \quad &+ \int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{v}\, dx && + \int_\Omega (\operatorname{div}\mathbf{v})\, r\; dx = \langle \mathbf{g},\mathbf{v}\rangle, \\
\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{z}\; dx \quad &- \nu^{-1}\gamma^2 \int_\Omega \mathbf{w}\cdot\mathbf{z}\, dx + \int_\Omega (\operatorname{div}\mathbf{z})\, p\; dx && = \langle \mathbf{h},\mathbf{z}\rangle, \\
&- \int_\Omega (\operatorname{div}\mathbf{w})\, q\; dx && = 0, \\
- \int_\Omega (\operatorname{div}\mathbf{u})\, s\; dx & && = 0,
\end{aligned}
\tag{4.28}
$$

for all $(\mathbf{v},\mathbf{z}) \in V$ and $(q,s) \in Q = L^2_0(\Omega)^2$, where

$$
L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q(x)\, dx = 0 \right\}.
$$

Problem (4.28) is in saddle point form (2.1), with bilinear forms

$$
\begin{aligned}
a\left((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})\right) =& \gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\; dx + \int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{v}\; dx \\
&+ \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{z}\; dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w}\cdot\mathbf{z}\, dx
\end{aligned}
\tag{4.29}
$$

and

$$
b\left((\mathbf{v},\mathbf{z}),(q,s)\right) = -\int_\Omega (\operatorname{div}\mathbf{z})\, q\; dx - \int_\Omega (\operatorname{div}\mathbf{v})\, s\; dx \tag{4.30}
$$

and $c \equiv 0$ for all $(\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z}) \in V$ and $(p,r),(q,s) \in Q$.

For the next theorem the following result is essential, see, e.g., [38].

**Theorem 4.12.** *There exists a positive constant $c_D$ such that*

$$
\sup_{0\neq\mathbf{v}\in H^1_0(\Omega)} \frac{\int_\Omega (\operatorname{div}\mathbf{v})\, q\, dx}{|\mathbf{v}|_1} \geq c_D \|q\|_0
$$

*for all $q \in L^2_0(\Omega)$.*

Now we can show the following result for (4.28).

**Theorem 4.13.** *Let $Q = L^2_0(\Omega)^2$ be equipped with the norm*

$$
\|(q,s)\|_Q = \sup_{0\neq(\mathbf{v},\mathbf{z})\in V} \frac{b\left((\mathbf{v},\mathbf{z}),(q,s)\right)}{\|(\mathbf{v},\mathbf{z})\|_V} \tag{4.31}
$$

*for all* $(q,s) \in Q$. *Then the operator* $\mathcal{A}$ *introduced by*

$$\left\langle \mathcal{A} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ p \\ r \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \\ q \\ s \end{bmatrix} \right\rangle = a\left((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})\right) + b\left((\mathbf{v},\mathbf{z}),(p,r)\right) + b\left((\mathbf{u},\mathbf{w}),(p,r)\right) \tag{4.32}$$

*with bilinear forms* $a$ *and* $b$, *given in (4.29) and (4.30), respectively, is an isomorphism from* $X$ *to* $X^\star$, *for*

$$X = V \times Q.$$

*Moreover, we have*

$$c\|x\|_X \le \|\mathcal{A}x\|_{X^\star} \le C\|x\|_X \quad \textit{for all } x \in X,$$

*with*

$$c = \frac{1}{2}\left(-1+\sqrt{3}\right) \approx 0.366 \quad \textit{and} \quad C = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618. \tag{4.33}$$

*Proof.* In a first step we check that (4.31) defines a norm on $Q$. It is easy to see that $\|\alpha(p,r)\|_Q = |\alpha|\|(p,r)\|_Q$ and $\|(p,r)+(q,s)\|_Q \le \|(p,r)\|_Q + \|(q,s)\|_Q$ for all $\alpha \in \mathbb{R}$ and $(p,r),(q,s) \in Q$. It remains to show that for $(p,r) \in Q$: If $\|(p,r)\|_Q = 0$ then $p = r = 0$. Let $(p,r) \in Q$ with $\|(p,r)\|_Q = 0$, then it follows from Lemma 3.21 and Theorem 4.12:

$$\begin{aligned}
0 &= \|(p,r)\|_Q^2 \\
&= \left( \sup_{0 \ne (\mathbf{v},\mathbf{z}) \in V} \frac{\int_\Omega (\operatorname{div} \mathbf{z})\, p\; dx + \int_\Omega (\operatorname{div} \mathbf{v})\, r\; dx}{\left(\gamma^2\|\mathbf{v}\|_0^2 + \nu^{1/2}|\mathbf{v}|_1^2 + \nu^{-1}\left(\gamma^2\|\mathbf{z}\|_0^2 + \nu^{1/2}|\mathbf{z}|_1^2\right)\right)^{1/2}} \right)^2 \\
&\ge \left( \sup_{0 \ne \mathbf{v},\mathbf{z} \in H_0^1(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \mathbf{z})\, p\; dx + \int_\Omega (\operatorname{div} \mathbf{v})\, r\; dx}{\left((\gamma^2 C_F^2 + \nu^{1/2})|\mathbf{v}|_1^2 + \nu^{-1}(\gamma^2 C_F^2 + \nu^{1/2})|\mathbf{z}|_1^2\right)^{1/2}} \right)^2 \\
&= \left( \sup_{0 \ne \mathbf{v} \in H_0^1(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \mathbf{z})\, p\; dx}{(\gamma^2 C_F^2 + \nu^{1/2})|\mathbf{v}|_1^2} \right)^2 + \left( \sup_{0 \ne \mathbf{v} \in H_0^1(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \mathbf{v})\, r\; dx}{\nu^{-1}(\gamma^2 C_F^2 + \nu^{1/2})|\mathbf{z}|_1} \right)^2 \\
&\ge c_D(\gamma^2 C_F^2 + \nu^{1/2})^{-2}\left(\|p\|_0^2 + \nu^2\|r\|_0^2\right),
\end{aligned}$$

for a constant $c > 0$, and thus $p = r = 0$.

Now we apply Brezzi's Theorem 2.1:

As consequence of the identity

$$a((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})) = \left\langle A \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} \right\rangle \quad \text{for all } (\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z}) \in V,$$

with operator $A$ given in Theorem 4.9, and Theorem 4.10 it follows immediately

$$a((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})) \le \|a\|\|(\mathbf{u},\mathbf{w})\|_V\|(\mathbf{v},\mathbf{z})\|_V \quad \text{for all } (\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z}) \in V$$

with $\|a\| = 1$.

Further we have

$$\sup_{0\neq(\mathbf{v},\mathbf{z})\in V} \frac{b\left((\mathbf{v},\mathbf{z}),(q,r)\right)}{\|(\mathbf{v},\mathbf{z})\|_V} = \|b\left(\cdot,(q,r)\right)\|_{V^\star} = \|(q,r)\|_Q \quad \text{for all } (q,r)\in Q.$$

Thus the inf-sup condition is trivially satisfied with constant $\beta = 1$ and $b$ is bounded with $\|b\| = 1$.

To apply Brezzi's Theorem, it remains to check the inf-sup condition for the bilinear forms $a$:

It is easy to see that $\ker B \subset V$ and $\ker B$ is an $H$-invariant subspace of $V$, i.e. $H(\ker B) \subseteq \ker B$ with the operator $H$ introduced in (4.26). Thus we obtain from Theorem 4.10:

$$\inf_{0\neq(\mathbf{u},\mathbf{w})\in\ker B}\ \sup_{0\neq\hat{\mathbf{v}}\in\ker B} \frac{a\left((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})\right)}{\|(\mathbf{u},\mathbf{w})\|_V\,\|(\mathbf{v},\mathbf{z})\|_V} \geq \inf_{0\neq\mathbf{v}\in\ker B} \frac{\left\langle A\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}, H\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}\right\rangle}{\|(\mathbf{u},\mathbf{w})\|_V^2} \geq \alpha$$

with $\alpha = 1/2$.

So all conditions of Brezzi's Theorem are satisfied with $\|a\| = \|b\| = \beta = 1$ and $\alpha = 1/2$ and thus we have: $\mathcal{A}$ is an isomorphism from $X$ to $X^\star$.

Finally we obtain from Theorem 2.2:

$$c\|x\|_X \leq \|\mathcal{A}x\|_{X^\star} \leq C\|x\|_X \quad \text{for all } x\in X,$$

with

$$C = \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4\|b\|^2}\right) = \frac{1}{2}\left(1+\sqrt{5}\right) \quad \text{and} \quad c = \frac{1}{2}\left(-1+\sqrt{3}\right),$$

where $c$ is the smallest positive root $\eta$ for the cubic equation

$$\eta^3 - \frac{3}{2}\eta + \frac{1}{2} = 0.$$

$\square$

For the next result the following theorem is essential, see, e.g. [60].

**Theorem 4.14.** *There exist positive constants $\hat{c}$ and $\hat{C}$ such that*

$$\hat{c}\|q\|_{H^1(\Omega)+\varepsilon L^2(\Omega)} \leq \sup_{0\neq\mathbf{v}\in H_0^1(\Omega)^d} \frac{\int_\Omega (\operatorname{div}\mathbf{v})\, q\, dx}{\|\mathbf{v}\|_{L^2(\Omega)^d\cap\varepsilon^{-1}H_0^1(\Omega)^d}} \leq \hat{C}\|q\|_{H^1(\Omega)+\varepsilon L^2(\Omega)},$$

*holds for all $\varepsilon > 0$ and $q \in L_0^2(\Omega)$.*

**Lemma 4.15.** *Let*

$$\|(q,s)\|_{\hat{Q}} = \left(\|q\|^2_{\nu^{1/2}\left(\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)^d\right)} + \|s\|^2_{\left(\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)^d\right)},\right)^{1/2}$$

*for all* $(q,s) \in Q$*. Then*

$$\hat{c}\|(q,s)\|_{\hat{Q}} \leq \|(q,s)\|_Q \leq \hat{C}\|(q,s)\|_{\hat{Q}} \quad \textit{for all } (q,s) \in Q$$

*with constants* $\hat{c}$ *and* $\hat{C}$ *from Theorem 4.14.*

*Proof.* From Lemma 3.21 and the Duality Theorem 3.11 we obtain

$$\begin{aligned}
\|(q,s)\|^2_Q &= \|b\left(\cdot,(q,s)\right)\|^2_{V^\star} \\
&= \left(\sup_{0\neq(\mathbf{v},\mathbf{z})\in V} \frac{b\left((\mathbf{v},\mathbf{z}),(q,s)\right)}{\|(\mathbf{v},\mathbf{z})\|_V}\right)^2 \\
&= \left\|\int_\Omega (\operatorname{div}\cdot)\, s\, dx\right\|^2_{\left(\gamma L^2(\Omega)^d\cap\nu^{1/4}H^1_0(\Omega)^d\right)^\star} + \left\|\int_\Omega (\operatorname{div}\cdot)\, q\, dx\right\|^2_{\nu^{1/2}\left(\gamma L^2(\Omega)^d\cap\nu^{1/4}H^1_0(\Omega)^d\right)^\star} \\
&= \left\|\int_\Omega (\operatorname{div}\cdot)\, s\, dx\right\|^2_{\gamma^{-1}L^2(\Omega)^d+\nu^{-1/4}H^{-1}(\Omega)^d} + \left\|\int_\Omega (\operatorname{div}\cdot)\, q\, dx\right\|^2_{\nu^{1/2}\left(\gamma^{-1}L^2(\Omega)^d+\nu^{-1/4}H^{-1}(\Omega)^d\right)}
\end{aligned}$$

for all $(q,s) \in Q$. The rest follows from Theorem 4.14. □

Now from the previous lemma and Theorem 4.13 it follows:

**Corollary 4.16.** *Let*

$$\|\,(\mathbf{v},\mathbf{z},q,s)\,\|_{\hat{X}} = \left(\|(\mathbf{v},\mathbf{z})\|^2_V + \|(q,s)\|^2_{\hat{Q}}\right)^{1/2}$$

*for all* $(\mathbf{v},\mathbf{z},q,s) \in X$*. Then we have*

$$c\|x\|_{\hat{X}} \leq \|\mathcal{A}x\|_{\hat{X}^\star} \leq C\|x\|_{\hat{X}} \quad \textit{for all } x \in X,$$

*where*

$$\|\mathcal{A}x\|_{\hat{X}^\star} = \sup_{0\neq w\in X} \frac{\langle \mathcal{A}x, w\rangle}{\|w\|_{\hat{X}}}$$

*and*

$$c = \frac{1}{2}\left(-1+\sqrt{3}\right)\min(1,\hat{c})^2 \quad \textit{and} \quad C = \frac{1}{2}\left(1+\sqrt{5}\right)\max(1,\hat{C})^2$$

*with constants* $\hat{c}$ *and* $\hat{C}$ *from Theorem 4.14.*

*Proof.* From the previous lemma it follows that

$$\min(1, \hat{c})\|x\|_{\hat{X}} \leq \|x\|_X \leq \max(1, \hat{C})\|x\|_{\hat{X}} \quad \text{for all } x \in X.$$

This implies together with Theorem 4.13:

$$\begin{aligned}
\frac{1}{\max(1,\hat{C})} \sup_{0\neq w\in X} \frac{\langle \mathcal{A}x, w\rangle}{\|w\|_{\hat{X}}} \leq \sup_{0\neq w\in X} \frac{\langle \mathcal{A}x, w\rangle}{\|w\|_X} &\leq \frac{1}{2}\left(1+\sqrt{5}\right)\|x\|_X \\
&\leq \frac{1}{2}\left(1+\sqrt{5}\right)\max(1,\hat{C})\|x\|_{\hat{X}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\min(1,\hat{c})} \sup_{0\neq w\in X} \frac{\langle \mathcal{A}x, w\rangle}{\|w\|_{\hat{X}}} \geq \sup_{0\neq w\in X} \frac{\langle \mathcal{A}x, w\rangle}{\|w\|_X} &\geq \frac{1}{2}\left(-1+\sqrt{3}\right)\|x\|_X \\
&\geq \frac{1}{2}\left(-1+\sqrt{3}\right)\min(1,\hat{c})\|x\|_{\hat{X}}
\end{aligned}$$

for all $x \in X$, which completes the proof.

□

## 4.2 Distributed optimal control with the time-periodic parabolic equations

We consider the following model problem: Find the state $y(x,t)$ and the control $u(x,t)$ that minimizes the cost functional

$$J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y(x,t) - y_d(x,t)|^2 \, dx \, dt + \frac{\nu}{2}\int_0^T \int_\Omega u(x,t)^2 \, dx \, dt$$

subject to the time-periodic parabolic problem

$$\begin{aligned}
\frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) &= u(x,t) && \text{in } Q_T, \\
y(x,t) &= 0 && \text{on } \Sigma_T, \\
y(x,0) &= y(x,T) && \text{on } \Omega, \\
u(x,0) &= u(x,T) && \text{on } \Omega.
\end{aligned}$$

Here $y_d(x,t)$ is a given target (or desired) state and $\nu > 0$ is a cost or regularization parameter. We assume that $y_d(x,t)$ is time-harmonic, i.e.:

$$y_d(x,t) = y_d(x)\, e^{i\omega t} \quad \text{with} \quad \omega = \frac{2\pi k}{T} \quad \text{for some } k \in \mathbb{Z}.$$

Then there is a time-periodic solution to the original control problem of the form

$$y(x,t) = y(x)\, e^{i\omega t}, \quad u(x,t) = u(x)\, e^{i\omega t},$$

where $y(x)$ and $u(x)$ solve the following time-independent optimal control problem: Minimize

$$\frac{1}{2}\int_\Omega |y(x) - y_d(x)|^2 \, dx + \frac{\nu}{2}\int_\Omega |u(x)|^2 \, dx$$

subject to

$$\begin{aligned} i\omega\, y(x) - \Delta y(x) &= u(x) && \text{in } \Omega, \\ y(x) &= 0 && \text{on } \Gamma. \end{aligned}$$

The Lagrangian functional for this constrained optimization problem is given by

$$\mathscr{L}(y,u,p) = J(y,u) + \int_\Omega p^\star(x)\left(i\omega\, y(x) - \Delta y(x) - u(x)\right) dx,$$

where $p$ denotes the Lagrangian multiplier associated with the constraint. The first-order necessary optimality conditions, which are also sufficient for the problem considered here, are $\nabla\mathscr{L}(y,u,p) = 0$, and read in detail:

$$\begin{aligned} -i\omega\, p(x) - \Delta p(x) &= y_d(x) - y(x) && \text{in } \Omega, \\ p(x) &= 0 && \text{on } \Gamma, \\ \\ \nu\, u - p &= 0 && \text{in } \Omega, \\ \\ i\omega\, y(x) - \Delta y(x) &= u(x) && \text{in } \Omega, \\ y(x) &= 0 && \text{on } \Gamma. \end{aligned}$$

From the third equation it follows that $u = \nu^{-1}p$. So the control $u$ can be eliminated, and one obtains the reduced optimality system

$$\begin{aligned} -i\omega\, p(x) - \Delta p(x) &= y_d(x) - y(x) && \text{in } \Omega, \\ p(x) &= 0 && \text{on } \Gamma, \\ i\omega\, y(x) - \Delta y(x) &= \nu^{-1}p(x) && \text{in } \Omega, \\ y(x) &= 0 && \text{on } \Gamma, \end{aligned}$$

which reads in operator notation

$$\begin{bmatrix} I & -\Delta - i\omega I \\ -\Delta + i\omega I & -\nu^{-1}I \end{bmatrix} x = b \tag{4.34}$$

with

$$x = \begin{bmatrix} y \\ p \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} y_d \\ 0 \end{bmatrix},$$

where $y(x) = p(x) = 0$ on $\Gamma$.

A choice for the Hilbert spaces which leads to a well-posed variational formulation for (3.51) was already given in [54]. We show here that theses spaces can be derived with the interpolation technique presented in the previous chapter.

### 4.2.1 Transformation to a system with real operators

Observe that the structures of (4.34) and the 2-by-2 block of (4.4) are similar. As a consequence we obtain analogously

$$\begin{bmatrix} I & -\Delta - i\omega I \\ -\Delta + i\omega I & -\nu^{-1} I \end{bmatrix} = T^* \begin{bmatrix} (1+\nu\omega^2)^{1/2} I & -\Delta \\ -\Delta & -\nu^{-1}(1+\nu\omega^2)^{1/2} I \end{bmatrix} T,$$

with

$$T = (1+\nu\omega^2)^{-1/4} \begin{bmatrix} (1+\nu\omega^2)^{1/2} I & -iI \\ 0 & I \end{bmatrix}.$$

So the original system (4.34) is equivalent to the system

$$\begin{bmatrix} \gamma^2 I & -\Delta \\ -\Delta & -\nu^{-1}\gamma^2 I \end{bmatrix} z = \begin{bmatrix} \gamma y_d \\ -i\omega\gamma^{-1} y_d \end{bmatrix} \tag{4.35}$$

with $z = Tx$. Here and in the following we use

$$\gamma = (1+\nu\omega^2)^{1/4}.$$

So, instead of $\omega$ and $\nu$, we consider in the following a problem depending on the parameters $\gamma$ and $\nu$.

We have the same situation as in the Stokes case: In order to solve Problem (4.35) two real problems for the real and the imaginary parts of $z$ must be solved:

$$\begin{bmatrix} \gamma^2 I & -\Delta \\ -\Delta & -\nu^{-1}\gamma^2 I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} \tag{4.36}$$

where $y(x) = p(x) = 0$ on $\Gamma$, for

$$g = \gamma \operatorname{Re}(y_d),\ h = \omega\gamma^{-1} \operatorname{Im}(y_d) \quad \text{and} \quad g = \gamma \operatorname{Im}(y_d),\ h = -\omega\gamma^{-1} \operatorname{Re}(y_d).$$

Here we use with a slight abuse of notation the same variables for the new unknown as for the original unknown $x$.

So instead of solving a complex system, we have to solve a real system for two different right-hand sides.

Similar to Subsection 4.1.2 (cf., (4.25)), we can derive the following variational formulation for (4.36): Find $y \in V$ and $p \in Q$ such that

$$\begin{aligned} \gamma^2 \int_\Omega y\, v\, dx \quad + \int_\Omega \nabla p \cdot \nabla v\, dx &= \langle g, v\rangle, \\ \int_\Omega \nabla y \cdot \nabla q\, dx - \nu^{-2}\gamma^2 \int_\Omega p\, q\, dx &= \langle h, q\rangle, \end{aligned} \tag{4.37}$$

for all $v \in V$ and $q \in Q$, with

$$V = \left(\gamma L^2(\Omega) \cap \nu^{1/4} H_0^1(\Omega)\right) \quad \text{and} \quad Q = \nu^{-1/2}\left(\gamma L^2(\Omega) \cap \nu^{1/4} H_0^1(\Omega)\right).$$

Similar as for the variational problem (4.25) we can prove for (4.37) the following result:

**Theorem 4.17.** *The operator $\mathcal{A}$ introduced by*

$$\left\langle \mathcal{A} \begin{bmatrix} y \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle = a(y,v) + b(v,p) + b(y,q) - c(p,q),$$

*with bilinear forms*

$$a(y,v) = \gamma^2 \int_\Omega y\, v \ dx, \quad b(v,q) = \int_\Omega \nabla v \cdot \nabla q \ dx$$

*and*

$$c(p,q) = -\nu^{-2}\gamma^2 \int_\Omega p\, q\, dx,$$

*is an isomorphism from $X$ to $X^\star$, for*

$$X = V \times Q = \left(\gamma L^2(\Omega) \cap \nu^{1/4} H_0^1(\Omega)\right) \times \nu^{-1/2} \left(\gamma L^2(\Omega) \cap \nu^{1/4} H_0^1(\Omega)\right),$$

*equipped with the standard product norm, i.e.*

$$\|(v,q)\|_X = \left(\gamma^2 \|v\|_0^2 + \nu^{1/2}\|v\|_1^2 + \nu^{-1}\gamma^2\|q\|_0^2 + \nu^{-1/2}\|q\|_1^2\right)^{1/2}$$

*for all $(v,q) \in X$. Moreover, we have*

$$\frac{1}{\sqrt{2}}\|x\|_X \le \|\mathcal{A}x\|_{X^\star} \le \|x\|_X \quad \textit{for all } x \in X.$$

The proof is analogous to the proofs of Theorem 4.5, Theorem 4.9 and Theorem 4.10, and is therefore, omitted.

# Chapter 5

# Properties of $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathbf{sym}}$

According to [93] an essential tool for the further analysis of the Ciarlet-Raviart method is the following Helmholtz-like decomposition of $H^{-1}(\Delta,\Omega)$,

$$H^{-1}(\Delta,\Omega) = H_0^1(\Omega) \oplus \mathscr{H}(\Omega)$$

with

$$\mathscr{H}(\Omega) = \{\tau \in L^2(\Omega) \colon \Delta\tau = 0\},$$

where $\oplus$ denotes the direct sum of Hilbert spaces.

This is also the case for the variational problem (3.33) and therefore we derive in the following a Helmholtz-like decomposition for the space $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$.

## 5.1 A Helmholtz-like decomposition

Recall, for the mixed method (3.3) for the biharmonic problem, we end up with the following mixed variational formulation, see (3.33): For $f \in H^{-1}(\Omega)$, find $w \in H_0^1(\Omega)$ and $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \qquad - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle &= 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}, \\ -\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle \qquad &= -\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega) \end{aligned} \tag{5.1}$$

with

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2\right)^{1/2}.$$

We have the following first decomposition result for $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$.

**Theorem 5.1.** *For each* $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$*, there is a unique decomposition*

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1, \tag{5.2}$$

*where* $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ *with* $\boldsymbol{\pi}(p) = p\boldsymbol{I}_2$*, cf. (3.34), for some* $p \in H_0^1(\Omega)$ *and* $\boldsymbol{\tau}_1 \in \boldsymbol{L}^2(\Omega)_{sym}$ *with* $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$*. Moreover,*

$$c\,\left(|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2\right) \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}^2 \le C\,\left(|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2\right)$$

*for all* $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{sym}$*, with positive constants* $c$ *and* $C$ *which depend only on the constant* $C_F$ *of Friedrichs' inequality.*

*Proof.* For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$, let $p \in H_0^1(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla p \cdot \nabla v\;dx = -\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle \quad \text{for all } v \in H_0^1(\Omega) \tag{5.3}$$

and set $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$. Since

$$-\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0, v\rangle = \int_\Omega \nabla p \cdot \nabla v\;dx,$$

it follows that $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0 = \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$, and, therefore, $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$ for $\boldsymbol{\tau}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_0$ in the distributional sense. On the other hand, if $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1$ with $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ and $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = 0$, then $-\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_0 = -\operatorname{div}\mathbf{div}\,\boldsymbol{\tau} + \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}_1 = -\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$, which implies (5.3). This shows the uniqueness.

Furthermore, (5.3) implies $|\boldsymbol{\tau}_0|_1^2 = 2\,|p|_1^2 = 2\,\|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2$. Hence

$$\begin{aligned}\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}^2 &= \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2 = \|\boldsymbol{\tau}_0 + \boldsymbol{\tau}_1\|_0^2 + \frac{1}{2}\,|\boldsymbol{\tau}_0|_1^2 \\ &\le 2\,\|\boldsymbol{\tau}_0\|_0^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2 + \frac{1}{2}\,|\boldsymbol{\tau}_0|_1^2 \le \left(\frac{1}{2} + 2C_F^2\right)\,|\boldsymbol{\tau}_0|_1^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2\end{aligned}$$

and

$$\begin{aligned}|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2 &= |\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\|_0^2 \le |\boldsymbol{\tau}_0|_1^2 + 2\,\|\boldsymbol{\tau}\|_0^2 + 2\,\|\boldsymbol{\tau}_0\|_0^2 \\ &\le 2\,\|\boldsymbol{\tau}\|_0^2 + (1 + 2C_F^2)\,|\boldsymbol{\tau}_0|_1^2 = 2\,\|\boldsymbol{\tau}\|_0^2 + 2(1 + 2C_F^2)\,\|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2.\end{aligned}$$

Then the estimates immediately follow with the constants $1/c = 2(1 + 2C_F^2)$ and $C = \max(2, 1/2 + 2C_F^2)$. $\square$

In short, we have algebraically as well as topologically

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}} = \boldsymbol{\pi}(H_0^1(\Omega)) \oplus \mathscr{H}(\operatorname{div}\mathbf{div},\Omega) \tag{5.4}$$

with

$$\mathscr{H}(\operatorname{div}\mathbf{div},\Omega) = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}\colon\ \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} = 0\}.$$

**Remark 5.2.** The Helmholtz decomposition of $\boldsymbol{L}^2(\Omega)_{\text{sym}}$ in [52], based on previous results in [11], has the same second component. The first component in [11, 52] is different and requires the solution of a biharmonic problem in contrast to Theorem 5.1, where the first component requires to solve only a Poisson problem.

Next we give an explicit characterization of $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$.

**Theorem 5.3.** *Let $\Omega$ be additionally simply-connected. For each $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$, there is a function $\phi \in \left(H^1(\Omega)\right)^2$ such that*

$$\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \quad \text{with} \quad \boldsymbol{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon}(\phi)_{ij} = \frac{1}{2}\left(\partial_j\phi_i + \partial_i\phi_j\right). \tag{5.5}$$

*And vice versa, each function of the form $\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}$ with $\phi \in \left(H^1(\Omega)\right)^2$ lies in $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$.*

*The function $\phi$ is unique up to an element from*

$$RM = \left\{\boldsymbol{\tau}(x) = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \colon a \in \mathbb{R},\ b \in \mathbb{R}^2\right\},$$

*and there is a constant $c_K$ such that*

$$c_K\,\|\phi\|_1 \leq \|\boldsymbol{\tau}\|_0 = \|\boldsymbol{\varepsilon}(\phi)\|_0 \leq \|\phi\|_1 \quad \text{for all } \phi \in \left(H^1(\Omega)\right)^2/RM. \tag{5.6}$$

*Proof.* In [52] it was shown that $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$ can be written in the following way:

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix} + \mathbf{Curl}\,\psi \quad \text{with} \quad \rho = \frac{1}{2}\operatorname{div}\psi, \quad \mathbf{Curl}\,\psi = \begin{bmatrix} -\partial_2\psi_1 & \partial_1\psi_1 \\ -\partial_2\psi_2 & \partial_1\psi_2 \end{bmatrix} \tag{5.7}$$

for some $\psi \in H^1(\Omega)^2$. Setting $\phi = (-\psi_2, \psi_1)^T$ yields the representation in (5.5).

Let $\boldsymbol{\tau} = 0$ and $\psi \in H^1(\Omega)^2$ such that (5.7) holds. Then it is easy to see that $\psi$ has the following representation:

$$\psi_1 = -bx_2 + a_1 \quad \text{and} \quad \psi_2 = bx_1 + a_2, \tag{5.8}$$

with $a_1, a_2, b \in \mathbb{R}$. On the other hand for $\psi \in H^1(\Omega)^2$ with representation (5.8) we have

$$\begin{bmatrix} 0 & -\frac{1}{2}\operatorname{div}\psi \\ \frac{1}{2}\operatorname{div}\psi & 0 \end{bmatrix} + \mathbf{Curl}\,\psi = 0.$$

Therefore the function $\phi = (-\psi_2, \psi_1)$ is unique up to an element from RM.

Finally the estimates in (5.6) follow from Korn's inequality. □

Therefore, we have the following representation of the solution $\boldsymbol{\sigma}$ to (5.1):

$$\boldsymbol{\sigma} = \boldsymbol{\pi}(p) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}.$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\psi)\boldsymbol{H}$ leads to the following equivalent formulation of (5.1). Find $p \in H_0^1(\Omega)$, $\phi \in (H^1(\Omega))^2/\mathrm{RM}$, $w \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\pi}(q)\;dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi)\;dx + \int_\Omega \nabla w \cdot \nabla q\;dx &&= 0,\\
&\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\varepsilon}(\psi)\;dx + \int_\Omega \boldsymbol{\varepsilon}(\phi) : \boldsymbol{\varepsilon}(\psi)\;dx &&= 0,\\
&\int_\Omega \nabla p \cdot \nabla v\;dx &&= -\langle f, v\rangle,
\end{aligned} \tag{5.9}$$

for all $q \in H_0^1(\Omega)$, $\psi \in (H^1(\Omega))^2/\mathrm{RM}$, $v \in H_0^1(\Omega)$.

Observe that $\boldsymbol{\pi}(p) : \boldsymbol{\pi}(q) = 2\,p\,q$ and $\boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\psi) = q\;\mathrm{div}\,\psi$, which allows to simplify parts of (5.9).

In summary, the biharmonic problem is equivalent to three (consecutively to solve) elliptic second-order problems. The first problem is a Poisson problem with Dirichlet boundary conditions for $p$, which reads in strong form

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma.$$

The second problem is a pure traction problem in linear elasticity with Poisson ratio 0 for $\phi$, which reads in strong form

$$-\,\mathbf{div}\,\boldsymbol{\varepsilon}(\phi) = \nabla p \quad \text{in } \Omega, \quad \phi\,n = \boldsymbol{\varepsilon}(\phi)\,n = 0 \quad \text{on } \Gamma.$$

And, finally, the third problem is a Poisson problem with Dirichlet boundary conditions for the original variable $w$, which reads in strong form

$$\Delta w = 2\,p + \mathrm{div}\,\phi \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma.$$

**Remark 5.4.** *In the case $\Omega \subset \mathbb{R}^3$ we have for*

$$\boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega)_{sym} = \{\boldsymbol{\tau} \in \boldsymbol{L}(\Omega)^{3\times 3}_{sym} :\; \mathrm{div}\,\mathbf{div}\,\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$

*the following similar decomposition result*

$$\boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega)_{sym} = \boldsymbol{\pi}(H_0^1(\Omega)) \oplus \mathscr{H}(\mathrm{div}\,\mathbf{div},\Omega)$$

*with*

$$\mathscr{H}(\mathrm{div}\,\mathbf{div},\Omega) = \{\boldsymbol{\tau} \in \boldsymbol{L}(\Omega)^{3\times 3}_{sym} \colon\; \mathrm{div}\,\mathbf{div}\,\boldsymbol{\tau} = 0\}.$$

*Here we use $\boldsymbol{\pi}(v) = v\boldsymbol{I}_3$ for $v \in H_0^1(\Omega)$ and*

$$\boldsymbol{L}(\Omega)^{3\times 3}_{sym} = \{\boldsymbol{\tau} \colon \boldsymbol{\tau}_{ji} = \boldsymbol{\tau}_{ij} \in L^2(\Omega),\; i,j = 1,2,3\},$$

*equipped with the standard $L^2$-norm $\|\boldsymbol{\tau}\|_0$ for matrix-valued functions $\boldsymbol{\tau}$.*

## 5.2 A trace operator

There is a natural trace operator associated with $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$, which was discussed in [85, 73]. We shortly summarize here the basic properties.

Let the boundary $\Gamma$ of the polygonal domain $\Omega$ be written in the form

$$\Gamma = \bigcup_{k=1}^{K} \overline{\Gamma}_k, \tag{5.10}$$

where $\Gamma_k$, $k = 1, 2, \dots, K$, are the edges of $\Gamma$, considered as open line segments. $\overline{\Gamma}_k$ denotes the corresponding closed line segment. For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ which is additionally twice continuously differentiable and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ we obtain the following identity by integration by parts.

$$\int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, v\; dx = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\; dx - \int_\Gamma \boldsymbol{\tau}_{nn}\,\partial_n v\; ds \tag{5.11}$$

with

$$\boldsymbol{\tau}_{nn} = n^T \boldsymbol{\tau} n.$$

Following standard procedures this identity allows to extend the trace $\boldsymbol{\tau}_{nn}$ to all functions $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ as an element of the dual of the image of the Neumann traces of functions from $H^2(\Omega) \cap H_0^1(\Omega)$, i.e.

$$\boldsymbol{\tau}_{nn} \in H_{pw}^{-1/2}(\Gamma) = \Pi_{k=1}^{K} H^{-1/2}(\Gamma_k),$$

where $H^{-1/2}(\Gamma_k)$ is the dual of $\widetilde{H}^{1/2}(\Gamma_k)$, see [41] for details. Another widely used notation for $\widetilde{H}^{1/2}(\Gamma_k)$ is $H_{00}^{1/2}(\Gamma_k)$, see [59].

From (5.11) we obtain the corresponding Green's formula for $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\mathrm{sym}}$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v\; dx - \langle \boldsymbol{\tau}_{nn}, \partial_n v\rangle_\Gamma . \tag{5.12}$$

Here $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product in a Hilbert space of functions on $\Gamma$.

# Chapter 6

# Discretization

In this chapter we discuss the discretization of the *mixed method of the first biharmonic boundary value problem (3.33)* and the *distributed optimal control problem with the time-periodic Stokes equations*, quite in the spirit of the corresponding continuous problems in Chapter 3 and Section 4.1, respectively.

## 6.1 A mixed finite element method for the first biharmonic boundary value problem

In the following two subsections we study the well known Hellan-Herrmann-Johnson (HHJ) method and a modified conforming variant, for the mixed variational formulation derived in Chapter 3, see (3.33):

For $f \in H^{-1}(\Omega)$, find $\boldsymbol{\sigma} \in \mathbf{V}$ and $w \in Q$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx && - \langle \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}, w \rangle = 0 && \text{for all } \boldsymbol{\tau} \in \mathbf{V}, \\ &- \langle \operatorname{div} \mathbf{div}\, \boldsymbol{\sigma}, v \rangle && = -\langle f, v \rangle && \text{for all } v \in Q, \end{aligned} \tag{6.1}$$

where

$$\mathbf{V} = \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}} \quad \text{and} \quad Q = H_0^1(\Omega).$$

### 6.1.1 The Hellan-Herrmann-Johnson (HHJ) method

Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal domain $\Omega$. For $k \in \mathbb{N}$ the standard finite element spaces $\mathcal{S}_h$ and $\mathcal{S}_{h,0}$ are given by

$$\mathcal{S}_h = \{v \in C(\overline{\Omega}) \colon v|_T \in P_k \text{ for all } T \in \mathcal{T}_h\} \quad \text{and} \quad \mathcal{S}_{h,0} = \mathcal{S}_h \cap H_0^1(\Omega),$$

where $P_k$ denotes the set of bivariate polynomials of total degree less than or equal to $k$.

The Hellan-Herrmann-Johnson method uses for the approximation of the auxiliary variable $\boldsymbol{\sigma}$ and the original variable $w$ in (6.1), the finite element spaces

$$\begin{aligned}\mathbf{V}_h = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}|_T \in P_{k-1} \text{ for all } T \in \mathcal{T}_h, \text{ and} \\ \boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\},\end{aligned}$$

and the standard finite element space

$$Q_h = \mathcal{S}_{h,0},$$

respectively.

For later use we show the following dimension result of $\mathbf{V}_h$.

**Lemma 6.1.** *We have*

$$\dim \mathbf{V}_h = \dim \mathcal{S}_{h,0}(\Omega) + 2 \dim \mathcal{S}_h(\Omega) - 3.$$

*Proof.* Let $N_v$, $N_e$, and $N_t$ denote the number of vertices, the number of edges, and the number of triangles of the mesh $\mathcal{T}_h$, respectively. Then it is well-known that

$$N_v = N_i + N_b, \quad N_e = 3N_i + 2N_b - 3, \quad N_t = 2N_i + N_b - 2, \tag{6.2}$$

where $N_i$ and $N_b$ denote the number of interior vertices and the number of boundary vertices, respectively.

The degrees of freedom (dofs) of the standard finite element space $\mathcal{S}_h(\Omega)$ consist of one dof at each vertex, $k-1$ dofs on each edge, and additional $(k-2)(k-1)/2$ dofs for each triangle of the triangulation $\mathcal{T}_h$, hence

$$\dim \mathcal{S}_h(\Omega) = N_v + (k-1)\, N_e + \frac{(k-2)(k-1)}{2}\, N_t$$

and

$$\dim \mathcal{S}_{h,0}(\Omega) = \dim \mathcal{S}_h(\Omega) - k\, N_b. \tag{6.3}$$

For the Hellan-Herrmann-Johnson method the dofs of the auxiliary variable $\boldsymbol{\tau}$ consist of $k$ dofs for $\boldsymbol{\tau}_{nn}$ on each edge, and, for each of the three independent entries of $\boldsymbol{\tau}$, $(k-1)k/$ dofs for each triangle. Therefore,

$$\begin{aligned}\dim \mathbf{V}_h &= k\, N_e + 3 \cdot \frac{(k-1)k}{2}\, N_t \\ &= 3 \dim \mathcal{S}_h(\Omega) - 3\, N_v - (2k-3)\, N_e + 3(k-1)\, N_t.\end{aligned}$$

From (6.3) we obtain

$$\dim \mathbf{V}_h = 2 \dim \mathcal{S}_h(\Omega) + \dim \mathcal{S}_{h,0}(\Omega) + k\, N_b - 3\, N_v - (2k-3)\, N_e + 3(k-1)\, N_t.$$

Finally it follows from (6.2):

$$k\, N_b - 3\, N_v - (2k-3)\, N_e + 3(k-1)\, N_t = -3,$$

which completes the proof. □

The HHJ method reads as follows: Find $\boldsymbol{\sigma}_h \in \mathbf{V}_h$ and $w_h \in Q_h$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau} \, dx \quad & - \langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, w_h \rangle = 0 && \text{for all } \boldsymbol{\tau} \in \mathbf{V}_h, \\ - \langle \operatorname{div} \mathbf{div}_h \boldsymbol{\sigma}_h, v \rangle \quad & = - \langle f, v \rangle && \text{for all } v \in Q_h, \end{aligned} \tag{6.4}$$

with

$$\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v \rangle = \sum_T \left\{ \int_T \boldsymbol{\tau} : \nabla^2 v \, dx - \int_{\partial T} \boldsymbol{\tau}_{nn} \, \partial_n v \, ds \right\} \quad \text{for } \boldsymbol{\tau} \in \mathbf{V}_h, \ v \in Q_h. \tag{6.5}$$

It was shown is [73, Theorem 3.36], that $\mathbf{V}_h \not\subset \mathbf{V}$ and thus the HHJ method is a non-conforming method for (3.33).

Comparing (6.5) with (5.12), the definition of $\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v \rangle$ for $\boldsymbol{\tau} \in \mathbf{V}_h$ and $v \in Q_h$ in the HHJ method is just an element-wise assembled version of corresponding expression on the continuous level, a standard technique in non-conforming methods.

**Remark 6.2.** Using integration by parts we obtain

$$\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v \rangle = - \sum_{T \in \mathcal{T}_h} \left\{ \int_T \mathbf{div} \, \boldsymbol{\tau} \cdot \nabla v \, dx - \int_{\partial T} \boldsymbol{\tau}_{ns} \, \partial_s v \, ds \right\}$$

with the vector $s = (-n_2, n_1)^T$, which is tangent to $\Gamma$, the tangential derivative $\partial_s v$, and

$$\boldsymbol{\tau}_{ns} = s^T \boldsymbol{\tau} n.$$

The HHJ method is often formulated with this representation, which allows an extension for all functions $\boldsymbol{\tau}$ from the (mesh-dependent) infinite dimensional function space

$$\begin{aligned} \widetilde{\mathbf{V}} = \{ \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}|_T \in \boldsymbol{H}^1(T)_{\text{sym}} \text{ for all } T \in \mathcal{T}_h, \text{ and} \\ \boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries} \}. \end{aligned}$$

This space was used for the analysis of the method in [28, 5, 37], and others. Existence and uniqueness of a solution for the corresponding variational problem on the continuous level could be shown under additional smoothness assumptions. For the approach taken in this thesis, this is not required.

Similar to the continuous case, the well-posedness of the discrete problem can be shown. For the proof of the discrete inf-sup condition the discrete analogue to $\boldsymbol{\pi}(v)$, see (3.34), is needed. For $v_h \in \mathcal{S}_{h,0}$, we define

$$\boldsymbol{\pi}_h(v_h) = \boldsymbol{\Pi}_h \boldsymbol{\pi}(v_h)$$

with the linear operator $\boldsymbol{\Pi}_h$, introduced in [28] by the conditions

$$\int_e ((\boldsymbol{\tau}_h)_{nn} - \boldsymbol{\tau}_{nn}) \, q \, ds = 0, \quad \text{for all } q \in P_{k-1}, \text{ for all edges } e \text{ of } T, \ T \in \mathcal{T}_h,$$

and

$$\int_T (\boldsymbol{\tau}_h - \boldsymbol{\tau})\, q \; dx = 0, \quad \text{for all } q \in P_{k-2},\ T \in \mathcal{T}_h,$$

for $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h \boldsymbol{\tau} \in \mathbf{V}_h$ and $\boldsymbol{\tau} \in \boldsymbol{\pi}(Q_h)$. Observe that $\boldsymbol{\Pi}_h$ was originally introduced in [28] as a linear operator on the infinite dimensional space $\widetilde{\mathbf{V}}$ from above.

From the corresponding properties of $\boldsymbol{\Pi}_h$ in [28], Lemma 4, the next result directly follows.

**Lemma 6.3.** *Assume that $\mathcal{T}_h$ is a regular family of triangulation. Then there exists a constant $c_B > 0$ which is independent of $h$ such that*

$$\|\boldsymbol{\pi}_h(v)\|_0 \le c_B\, |v|_1 \quad \textit{for all } v \in \mathcal{S}_{h,0}.$$

Moreover, we need the following simple identity.

**Lemma 6.4.** *For all $p, v \in \mathcal{S}_{h,0}$, we have*

$$-\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\pi}_h(p), v \rangle = \int_\Omega \nabla p \cdot \nabla v \, dx.$$

Now the well-posedness of the discrete problem can be shown.

**Theorem 6.5.** *The bilinear forms*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \; dx, \quad b_h(\boldsymbol{\tau}, v) = -\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, v \rangle$$

*satisfy Brezzi's conditions on $\mathbf{V}_h$ and $Q_h$, equipped with the norms $\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}$ and $|v|_1$, respectively, where*

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} = \left( \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}\|_{-1,h}^2 \right)^{1/2} \tag{6.6}$$

*and*

$$\|\ell\|_{-1,h} = \sup_{v_h \in \mathcal{S}_{h,0}} \frac{\langle \ell, v_h \rangle}{|v_h|_1} \quad \textit{for } \ell \in (\mathcal{S}_{h,0})^*,$$

*with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + c_B^2)^{-1/2},$$

*where $c_B$ denotes the constant in Lemma 6.3.*

*Proof.* 1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbf{V}_h$. Then

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0\, \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{-1,\operatorname{div}\mathbf{div},h}\, \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}.$$

2. Let $\boldsymbol{\tau} \in \mathbf{V}_h$ and $v \in Q_h$. Then

$$|b(\boldsymbol{\tau}, v)| = |\langle \operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}, v \rangle \le \|\operatorname{div} \mathbf{div}_h\, \boldsymbol{\tau}\|_{-1,h} |v|_1 \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} \|v\|_1.$$

3. Observe that $\ker B_h = \{\boldsymbol{\tau} \in \mathbf{V}_h : \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}_h = 0\}$. Therefore,
$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|^2_{-1,\operatorname{div}\mathbf{div},h} \quad \text{for } \boldsymbol{\tau} \in \ker B_h.$$

4. From Lemma 6.3 and Lemma 6.4 we obtain for $v \in Q_h$
$$b_h(\boldsymbol{\pi}_h(v), v) = |v|_1^2$$
and
$$\|\boldsymbol{\pi}_h(v)\|^2_{-1,\operatorname{div}\mathbf{div},h} = \|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2 \le (1 + c_B^2)\, |v|_1^2.$$
Therefore,
$$\begin{aligned} \sup_{0\neq\boldsymbol{\tau}\in\mathbf{V}_h} \frac{b_h(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}} &\ge \frac{|b_h(\boldsymbol{\pi}_h(v), v)|}{\|\boldsymbol{\pi}_h(v)\|_{-1,\operatorname{div}\mathbf{div},h}} = \frac{|v|_1^2}{(\|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2)^{1/2}} \\ &\ge \frac{1}{(1 + c_B^2)^{1/2}} |v|_1. \end{aligned}$$

□

Observe that the norms introduced for the space $\mathbf{V} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ in (3.28) and its discrete counterpart $\mathbf{V}_h$ in (6.6) are similar but different. For the discrete problem the norm is mesh-dependent.

### A discrete Helmholtz decomposition

We have the following discrete version of Theorem 5.1.

**Theorem 6.6.** *For each $\boldsymbol{\tau} \in \mathbf{V}_h$, there is a unique decomposition*
$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}_0 + \hat{\boldsymbol{\tau}}_1,$$
*where $\hat{\boldsymbol{\tau}}_0 = \boldsymbol{\pi}_h(\hat{p})$ for some $\hat{p} \in Q_h$ and $\hat{\boldsymbol{\tau}}_1 \in \mathbf{V}_h$ with $\operatorname{div}\mathbf{div}_h\, \hat{\boldsymbol{\tau}}_1 = 0$. Moreover,*
$$c\, \left(|\hat{\boldsymbol{\tau}}_0|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2\right) \le \|\boldsymbol{\tau}\|^2_{-1,\operatorname{div}\mathbf{div},h} \le C\, \left(|\hat{\boldsymbol{\tau}}_0|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2\right)$$
*for all $\boldsymbol{\tau} \in \mathbf{V}_h$, with positive constants $c$ and $C$, which depend only on the constant $c_B$ of the inequality in Lemma 6.3.*

The proof is completely analogous to the proof for the continuous case and is, therefore, omitted. The only difference is the use of the estimate from Lemma 6.3 instead of Friedrichs' inequality.

So, in short,
$$\mathbf{V}_h = \boldsymbol{\pi}_h(\mathcal{S}_{h,0}) \oplus \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$$
with
$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \{\boldsymbol{\tau}_h \in \mathbf{V}_h \colon \langle \operatorname{div}\mathbf{div}_h\, \boldsymbol{\tau}_h, v_h\rangle = 0 \text{ for all } v_h \in Q_h\}.$$
For describing the space $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$ more explicitly, we consider the subspace of all functions in $\mathscr{H}(\operatorname{div}\mathbf{div}, \Omega)$ which can be represented by a finite element function $\phi \in (\mathcal{S}_h)^2$, for which we show the following result.

**Theorem 6.7.** $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h,\Omega) = \{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\}$.

*Proof.* Let $\phi \in (\mathcal{S}_h)^2$. Then $\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \in P_{k-1}$ for all triangles $T \in \mathcal{T}_h$. Furthermore, let $e$ be an edge of a triangle $T$ with outer unit normal vector $n = (n_1, n_2)^T$ and unit tangent vector $s = (-n_2, n_1)^T$. By elementary computations we obtain

$$\boldsymbol{\tau}_{nn} = n^T\boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}n = s \cdot \frac{\partial \phi}{\partial s}.$$

So, $\boldsymbol{\tau}_{nn}$ depends only on values of $\phi$ on the edge $e$, which immediately implies that $\boldsymbol{\tau}_{nn}$ is continuous on inter-element boundaries. This shows that $\boldsymbol{\tau}$ lies in $\mathbf{V}_h$, and, therefore, the inclusion $\{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} \subset \mathscr{H}_h(\operatorname{div}\mathbf{div}_h,\Omega)$ follows.

The equality follows by comparing the dimensions. We have

$$\dim\{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} = 2\dim\mathcal{S}_h - \dim \mathrm{RM} = 2\dim\mathcal{S}_h - 3.$$

On the other hand, by Theorem 6.6 and Lemma 6.1, it follows that

$$\begin{aligned}\dim\mathscr{H}_h(\operatorname{div}\mathbf{div}_h,\Omega) =& \dim\mathbf{V}_h - \dim\mathcal{S}_{h,0},\\ =&2\,\dim\mathcal{S}_h - 3.\end{aligned}$$

Therefore, $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h,\Omega) = \{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\}$, which completes the proof. □

**Remark 6.8.** A consequence of the last theorem is the important inclusion

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h,\Omega) \subset \mathscr{H}(\operatorname{div}\mathbf{div},\Omega),$$

which resembles the corresponding result of Lemma 5 in [28].

Therefore, we have the following representation of the approximate solution $\boldsymbol{\sigma}_h \in \mathbf{V}_h$ of (6.4):

$$\boldsymbol{\sigma}_h = \boldsymbol{\pi}_h(p_h) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi_h)\boldsymbol{H}.$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}_h(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\psi)\boldsymbol{H}$ leads to the following equivalent formulation of (6.4). Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q_h)dx + \int_\Omega \hat{\boldsymbol{\pi}}_h(q_h) : \boldsymbol{\varepsilon}(\phi_h)\ dx + \int_\Omega \nabla w_h \cdot \nabla q_h\ dx &&= 0,\\
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx &&= 0,\\
&\int_\Omega \nabla p_h \cdot \nabla v_h\ dx &&= -\langle f, v_h\rangle,
\end{aligned} \tag{6.7}$$

for all $q_h \in \mathcal{S}_{h,0}$, $\psi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $v_h \in \mathcal{S}_{h,0}$, and with

$$\hat{\boldsymbol{\pi}}_h(q) = \boldsymbol{H}\boldsymbol{\pi}_h(q)\boldsymbol{H}^T.$$

### 6.1.2 A conforming variant of the HHJ methdod

Comparing (6.7) with the continuous problem (5.9) it is natural to replace the operator $\hat{\boldsymbol{\pi}}_h$ in (6.7) by the operator $\boldsymbol{\pi}$. This leads to the problem: Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q_h) dx \ + \int_\Omega \boldsymbol{\pi}(q_h) : \boldsymbol{\varepsilon}(\phi_h)\ dx + \int_\Omega \nabla w_h \cdot \nabla q_h\ dx &= 0, \\ \int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx &= 0, \\ \int_\Omega \nabla p_h \cdot \nabla v_h\ dx &= -\langle f, v_h \rangle, \end{aligned} \tag{6.8}$$

for all $q_h \in \mathcal{S}_{h,0}$, $\psi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $v_h \in \mathcal{S}_{h,0}$. Here, contrary to the HHJ method the space for the approximate solution

$$\boldsymbol{\sigma}_h = \boldsymbol{\pi}(p_h) + \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi_h) \boldsymbol{H}.$$

is given by

$$\mathbf{V}_h = \boldsymbol{\pi}\left(\mathcal{S}_{h,0}\right) \oplus \boldsymbol{H}^T \phi \left( (\mathcal{S}_h)^2 / RM \right) \boldsymbol{H}.$$

From (5.4) we have $\mathbf{V}_h \subset \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$, e.g. (6.8) is a confirming method for (6.1).

Observe that the only difference between the conforming variant (6.8) and the HHJ method (6.7) is that the operator $\mathbf{\Pi}_h$ does not occur in (6.8). As a consequence the conforming variant is slightly less costly than the HHJ method.

### 6.1.3 Computational aspects

The obvious procedure for solving (6.7) consists of three consecutive steps.

step 1. For given $f \in H^{-1}(\Omega)$, solve

$$\int_\Omega \nabla p_h \cdot \nabla v_h\ dx = -\langle f, v_h \rangle$$

by the preconditioned conjugate gradient (PCG) method with a standard multigrid preconditioner for a Poisson problem.

step 2. For $p_h$, computed in step 1, solve

$$\int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi_h)\ dx \tag{6.9}$$

by the PCG method with a standard multigrid preconditioner for a pure traction problem.

step 3. For $p_h$ and $\phi_h$, computed in step 1 and 2, respectively, solve

$$\int_\Omega \nabla w_h \cdot \nabla q_h \; dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q_h) \; dx + \int_\Omega \hat{\boldsymbol{\pi}}_h(q_h) : \boldsymbol{\varepsilon}(\phi_h) \; dx \tag{6.10}$$

by the PCG method with a standard multigrid preconditioner for a Poisson problem.

In particular, we choose for each of the three multigrid preconditioners, one V-cycle of a geometric multigrid with one forward and one backward Gauss-Seidel sweep for pre- and post-smoothing, respectively. This leads in each step to a condition number for the preconditioned system that is independent of the mesh paramter $h$, see, e.g., [44].

For the conforming variant (6.8), the right-hand sides in (6.9) and (6.10) have to re-placed by the simpler expressions

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi) \; dx = -\int_\Omega p_h \operatorname{div} \psi \; dx$$

and

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q) dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi_h) \; dx = -2 \int_\Omega p_h \, q \; dx + \int_\Omega q \operatorname{div} \phi_h \; dx,$$

respectively.

## 6.2 A finite element method for the distributed optimal control problem with the time-periodic Stoke equations

Recall, the solution of the distributed optimal control problem with the time-periodic equations, introduced at the beginning of Section 4.1, requires to solve for

$$\mathbf{g} = \gamma \operatorname{Re}(\mathbf{u}_d), \; \mathbf{h} = \omega\gamma^{-1} \operatorname{Im}(\mathbf{u}_d) \quad \text{and} \quad \mathbf{g} = \gamma \operatorname{Im}(\mathbf{u}_d), \; \mathbf{h} = -\omega\gamma^{-1} \operatorname{Re}(\mathbf{u}_d),$$

the mixed variational problem: Find $(\mathbf{u}, \mathbf{w}) \in V$ and $(p, r) \in Q$ such that

$$\left\langle \mathcal{A} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ p \\ r \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \\ q \\ s \end{bmatrix} \right\rangle = \langle \mathbf{g}, \mathbf{v} \rangle + \langle \mathbf{h}, \mathbf{z} \rangle \quad \text{for all } (\mathbf{v}, \mathbf{z}) \in V, \; (q, s) \in Q, \tag{6.11}$$

with spaces $V$ and $Q$, introduced in Theorem 4.8 and Theorem 4.13, respectively, and the operator $\mathcal{A} : X \to X^\star$ for $X = V \times Q$ introduced in (4.32).

An approximate solution of the mixed problem (6.11) is obtained by chosen appropriate finite-dimensional subspaces

$$V_h \subset V \quad \text{and} \quad Q_h \subset Q.$$

Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal/polyhedral domain $\Omega$. We choose:

$$V_h = V_{\mathrm{TH}}^2 \quad \text{and} \quad Q_h = Q_{\mathrm{TH}}^2,$$

with

$$V_{\mathrm{TH}} = \{\mathbf{v} \in C_0(\overline{\Omega})^d : \ \mathbf{v}|_T \in P_2 \text{ for all } T \in \mathcal{T}_h\}$$

and

$$Q_{\mathrm{TH}} = \{q \in C(\overline{\Omega}) \cap L^2(\Omega)_0 : \ q|_T \in P_1 \text{ for all } T \in \mathcal{T}_h\}.$$

The pair of finite element spaces $(V_{\mathrm{TH}}, Q_{\mathrm{TH}})$ is well-known as the Taylor-Hood element.

By Galerkin's principle the approximate solutions are given by the discrete variational problem: Find $(\mathbf{u}_h, \mathbf{w}_h) \in V_h$ and $(p_h, r_h) \in Q_h$ such that

$$\left\langle \mathcal{A} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{w}_h \\ p_h \\ r_h \end{bmatrix}, \begin{bmatrix} \mathbf{v}_h \\ \mathbf{z}_h \\ q_h \\ s_h \end{bmatrix} \right\rangle = \langle \mathbf{g}, \mathbf{v}_h \rangle + \langle \mathbf{h}, \mathbf{z}_h \rangle \quad \text{for all } (\mathbf{v}_h, \mathbf{z}_h) \in V_h, \ (q_h, s_h) \in Q_h. \tag{6.12}$$

Our goal is to find a norm for $X_h$ such that:

$$c\|x_h\|_{X_h} \leq \sup_{0 \neq w_h \in X_h} \frac{\langle \mathcal{A} x_h, w_h \rangle}{\|w_h\|_{X_h}} \leq C\|x\|_{X_h}$$

for all $x_h \in X_h$, is satisfied for positive constants $c$ and $C$ that are independent of $\gamma$, $\nu$ and $h$.

Now, for the construction of this norm, we proceed similar as in the continuous case:

In a first step, we consider for general right hand-sides $\mathbf{g}$ and $\mathbf{h}$ the variational problem

$$\left\langle A \begin{bmatrix} \mathbf{u}_h \\ \mathbf{w}_h \end{bmatrix}, \begin{bmatrix} \mathbf{v}_h \\ \mathbf{z}_h \end{bmatrix} \right\rangle = \langle \mathbf{g}, \mathbf{v}_h \rangle + \langle \mathbf{h}, \mathbf{z}_h \rangle \quad \text{for all } (\mathbf{v}_h, \mathbf{z}_h) \in V_h, \tag{6.13}$$

with operator $A : V \to V^\star$ introduced in Theorem 4.9. Recall, we have

$$\left\langle A \begin{bmatrix} \mathbf{u}_h \\ \mathbf{w}_h \end{bmatrix}, \begin{bmatrix} \mathbf{v}_h \\ \mathbf{z}_h \end{bmatrix} \right\rangle = \left\langle \mathcal{A} \begin{bmatrix} \mathbf{u}_h \\ \mathbf{w}_h \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_h \\ \mathbf{z}_h \\ 0 \\ 0 \end{bmatrix} \right\rangle \quad \text{for all } (\mathbf{v}_h, \mathbf{z}_h) \in V_h.$$

We will show that (6.13) is well posed with bounds independent of $\gamma$, $\nu$ and $h$, for the choice $\|\cdot\|_{V_h} = \|\cdot\|_V$.

In a second step we choose the norm $\|\cdot\|_{Q_h}$ in such a way that the entire variational problem (6.12) is well posed with bounds independent of $\gamma$, $\nu$ and $h$.

### 6.2.1 The norm for the primal variable

We consider for general right hand-sides $\mathbf{g}$ and $\mathbf{h}$ the variational problem

$$\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}_h\\ \mathbf{z}_h\end{bmatrix}\right\rangle = \langle \mathbf{g}, \mathbf{v}_h\rangle + \langle \mathbf{h}, \mathbf{z}_h\rangle \quad \text{for all } (\mathbf{v}_h, \mathbf{z}_h) \in V_h,$$

with operator $A : V \to V^\star$ introduced in Theorem 4.9.

Our goal is to find $\|\cdot\|_{V_h}$ such that:

$$c\|(\mathbf{u}_h, \mathbf{w}_h)\|_{V_h} \le \sup_{0\neq(\mathbf{v}_h,\mathbf{z}_h)\in V_h} \frac{\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}_h\\ \mathbf{z}_h\end{bmatrix}\right\rangle}{\|(\mathbf{v}_h, \mathbf{z}_h)\|_{V_h}} \le C\|(\mathbf{u}_h, \mathbf{w}_h)\|_{V_h} \tag{6.14}$$

for all $(\mathbf{u}_h, \mathbf{w}_h) \in V_h$, is satisfied for positive constants $c$ and $C$ that are independent of $\gamma$, $\nu$ and $h$. Recall, we have that $A$ is an isomorphism from $V$ to $V^\star$ satisfying

$$\frac{1}{\sqrt{2}}\|(\mathbf{u}, \mathbf{w})\|_V \le \sup_{0\neq(\mathbf{v},\mathbf{z})\in V} \frac{\left\langle A\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix}, \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\rangle}{\|(\mathbf{v}, \mathbf{z})\|_V} \le \|(\mathbf{u}, \mathbf{w})\|_V \tag{6.15}$$

for all $(\mathbf{u}, \mathbf{w}) \in V$, see Theorems 4.9 and 4.10. Therefore, a natural choice for the norm in $V_h$ is $\|\cdot\|_{V_h} = \|\cdot\|_V$. For this choice, the upper estimate in (6.14) follows immediately from (6.15),

$$\sup_{0\neq(\mathbf{v}_h,\mathbf{z}_h)\in V_h} \frac{\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}_h\\ \mathbf{z}_h\end{bmatrix}\right\rangle}{\|(\mathbf{v}_h, \mathbf{z}_h)\|_V} \le \sup_{0\neq(\mathbf{v},\mathbf{z})\in V} \frac{\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}\right\rangle}{\|(\mathbf{v}, \mathbf{z})\|_V} \le \|(\mathbf{u}_h, \mathbf{w}_h)\|_V$$

for all $(\mathbf{u}_h, \mathbf{w}_h) \in V_h$.

Next, we show that also the lower bound in (6.14) holds for $\|\cdot\|_{V_h} = \|\cdot\|_V$:

**Theorem 6.9.**

$$\sup_{0\neq(\mathbf{v}_h,\mathbf{z}_h)\in V_h} \frac{\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}_h\\ \mathbf{z}_h\end{bmatrix}\right\rangle}{\|(\mathbf{v}_h, \mathbf{z}_h)\|_V} \ge \frac{1}{\sqrt{2}}\|(\mathbf{u}_h, \mathbf{w}_h)\|_V$$

*for all* $(\mathbf{u}_h, \mathbf{w}_h) \in V_h$.

*Proof.* We apply Theorem 4.10 for $\hat{V} = V_h$. We have $\hat{V} \subset V$ and further it is easy to see that $H(\hat{V}) \subset \hat{V}$. Thus,

$$\sup_{0\neq(\mathbf{v}_h,\mathbf{z}_h)\in \hat{V}} \frac{\left\langle A\begin{bmatrix}\mathbf{u}_h\\ \mathbf{w}_h\end{bmatrix}, \begin{bmatrix}\mathbf{v}_h\\ \mathbf{z}_h\end{bmatrix}\right\rangle}{\|(\mathbf{v}_h, \mathbf{z}_h)\|_V} \ge \frac{1}{\sqrt{2}}\|(\mathbf{u}_h, \mathbf{w}_h)\|_V$$

for all $(\mathbf{u}_h, \mathbf{w}_h) \in \hat{V}$, which completes the proof. $\square$

### 6.2.2 The space for the dual variable

Now we consider the entire variational problem (6.12). Recall, the operator $\mathcal{A} : X \to X^\star$ for $X = V \times Q$ is of the form (2.3), with bilinear forms

$$\begin{aligned} a\left((\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z})\right) =& \gamma^2 \int_\Omega \mathbf{u}\cdot\mathbf{v}\; dx + \int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{v}\; dx \\ &+ \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{z}\; dx - \nu^{-1}\gamma^2 \int_\Omega \mathbf{w}\cdot\mathbf{z}\, dx \end{aligned}$$

and

$$b\left((\mathbf{v},\mathbf{z}),(q,s)\right) = -\int_\Omega (\operatorname{div}\mathbf{z})\, q\; dx - \int_\Omega (\operatorname{div}\mathbf{v})\, s\; dx$$

and $c \equiv 0$, for all $(\mathbf{u},\mathbf{w}),(\mathbf{v},\mathbf{z}) \in V$ and $(p,r),(q,s) \in Q$.

For the further consideration the following assumption on $\mathcal{T}_h$ is essential:

**Assumption 1.** Let each element $T \in \mathcal{T}_h(\Omega)$ has at least two internal edges (in the case $d = 2$) or at least three internal faces (in the case $d = 3$).

Under Assumption 1 one can proof for the Taylor Hood element the following discrete analogue of Theorem 4.12, see, e.g., [27], which is essential for the proof of the next theorem.

**Theorem 6.10.** *Let Assumption 1 be satisfied. Then there exists a positive constant $c$ independent of $h$ such that*

$$\sup_{0\neq\mathbf{v}_h\in V_{TH}} \frac{\int_\Omega (\operatorname{div}\mathbf{v}_h)\, q_h\, dx}{|\mathbf{v}_h|_1} \geq c\|q_h\|_0$$

*for all $q_h \in Q_{TH}$.*

We have the following mapping properties for the operator $\mathcal{A}$.

**Theorem 6.11.** *Let Assumption 1 be satisfied. Let $Q_h$ be equipped with the norm*

$$\|(q_h,s_h)\|_{Q_h} = \sup_{0\neq(\mathbf{v}_h,\mathbf{z}_h)\in V_h} \frac{b\left((\mathbf{v}_h,\mathbf{z}_h),(q_h,s_h)\right)}{\|(\mathbf{v}_h,\mathbf{z}_h)\|_V} \tag{6.16}$$

*for all $(q_h,s_h) \in Q_h$. Then the operator $\mathcal{A}$ is an isomorphism from $X_h$ to $X_h^\star$, for*

$$X_h = V_h \times Q_h$$

*equipped with the norm*

$$\|(\mathbf{v}_h,\mathbf{z}_h,q_h,s_h)\|_{X_h} = \left(\|(\mathbf{v}_h,\mathbf{z}_h)\|_V^2 + \|(q_h,s_h)\|_{Q_h}^2\right)^{1/2}$$

*for all $(\mathbf{v}_h,\mathbf{z}_h,q_h,s_h) \in X_h$. Moreover, we have*

$$c\|x_h\|_{X_h} \leq \|\mathcal{A}x_h\|_{X_h^\star} \leq C\|x_h\|_{X_h} \quad \text{for all } x_h \in X_h, \tag{6.17}$$

*with constants $c$ and $C$ of Theorem 4.13.*

The proof is completely analogous to the proof of Theorem 4.13 and is, therefore, omitted.

To summarize, we have shown that the discrete mixed variational problem (6.12) is well posed in $X_h$ equipped with the norm $\|\cdot\|_{X_h}$ with bounds independent of $\gamma$, $\nu$ and $h$.

### 6.2.3 The theoretical preconditioner $\mathcal{P}$

Let $\{\boldsymbol{\phi}_i\}_{i=1,\dots,n_h}$ be a basis for $V_{\mathrm{TH}}$ and $\{\psi_k\}_{k=1,\dots,m_h}$ be a basis for $Q_{\mathrm{TH}}$. Then each function $\mathbf{v}_h \in V_{\mathrm{TH}}$ and $q_h \in Q_{\mathrm{TH}}$ can be uniquely represented in the following form:

$$\mathbf{v}_h(x) = \sum_{i=1}^{n_h} v_i \boldsymbol{\phi}_i(x) \quad \text{and} \quad q_h(x) = \sum_{j=1}^{m_h} q_j \psi_j(x)$$

and is therefore uniquely determined by the coefficient vector $\underline{\mathbf{v}} = (v_i) \in \mathbb{R}^{n_h}$ and $\underline{q} = (q_i) \in \mathbb{R}^{m_h}$, respectively.

For the discrete variational problem (6.12) a linear system of equations is obtained: Find $\underline{\mathbf{u}}, \underline{\mathbf{w}} \in \mathbb{R}^{n_h}$ and $\underline{p}, \underline{r} \in \mathbb{R}^{m_h}$ such that:

$$\mathcal{A}_h \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mathbf{w}} \\ \underline{p} \\ \underline{r} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{g}} \\ \underline{\mathbf{h}} \\ 0 \\ 0 \end{bmatrix}. \tag{6.18}$$

with

$$\mathcal{A}_h = \begin{bmatrix} \gamma^2 \mathbf{M} & \mathbf{K} & 0 & -\mathbf{D}^T \\ \mathbf{K} & -\nu^{-1}\gamma^2 \mathbf{M} & -\mathbf{D}^T & 0 \\ 0 & -\mathbf{D} & 0 & 0 \\ -\mathbf{D} & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2(n_h+m_h)\times 2(n_h+m_h)},$$

where

$$\mathbf{M} = \left( \int_\Omega \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j \, dx \right) \in \mathbb{R}^{n_h \times n_h}, \quad \mathbf{K} = \left( \int_\Omega \nabla \boldsymbol{\phi}_i \cdot \nabla \boldsymbol{\phi}_j \, dx \right) \in \mathbb{R}^{n_h \times n_h},$$
$$\mathbf{D} = \left( \int_\Omega (\operatorname{div} \boldsymbol{\phi}_i) \; \psi_l \, dx \right) \in \mathbb{R}^{m_h \times n_h}$$

$$M_p = \left( \int_\Omega \psi_k \cdot \psi_l \, dx \right) \in \mathbb{R}^{m_h \times m_h}, \quad K_p = \left( \int_\Omega \nabla \psi_k \cdot \nabla \psi_l \, dx \right) \in \mathbb{R}^{m_h \times m_h}$$

and

$$\underline{\mathbf{g}} = (\langle \mathbf{g}, \boldsymbol{\phi}_j \rangle) \in \mathbb{R}^{n_h}, \quad \underline{\mathbf{h}} = (\langle \mathbf{h}, \boldsymbol{\phi}_j \rangle) \in \mathbb{R}^{n_h}.$$

**Lemma 6.12.** *We have:*

$$\|x_h\|_{X_h} = \langle \mathcal{P}\underline{x}, \underline{x} \rangle^{1/2} \quad \textit{for all } x_h \in X_h \tag{6.19}$$

*with*

$$\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \tag{6.20}$$

*where*

$$P = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \nu^{-1}\mathbf{P} \end{bmatrix} \quad \textit{with} \quad \mathbf{P} = \gamma^2 \mathbf{M} + \nu^{1/2}\mathbf{K}$$

*and*

$$R = \begin{bmatrix} \nu S & 0 \\ 0 & S \end{bmatrix} \quad \textit{with} \quad S = \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T.$$

*Proof.* Let $x_h = (\mathbf{v}_h, \mathbf{z}_h, q_h, s_h) \in X_h$. We have:

$$\begin{aligned}
\|x_h\|_{X_h}^2 &= \|(\mathbf{v}_h, \mathbf{z}_h)\|_V^2 + \|(q_h, s_h)\|_{Q_h}^2 \\
&= \gamma^2 \|\mathbf{v}_h\|_0^2 + \nu^{1/2} |\mathbf{v}_h|_1^2 + \nu^{-1/2} |\mathbf{z}_h|_1^2 + \nu^{-1}\gamma^2 \|\mathbf{z}_h\|_0^2 \\
&\quad + \left( \sup_{0 \neq (\mathbf{u}_h, \mathbf{w}_h) \in V_h} \frac{b\left((\mathbf{u}_h, \mathbf{w}_h), (q_h, s_h)\right)}{\|(\mathbf{u}_h, \mathbf{w}_h)\|_V} \right)^2 \\
&= \gamma^2 (\mathbf{M}\underline{\mathbf{v}}, \underline{\mathbf{v}}) + \nu^{1/2} (\mathbf{K}\underline{\mathbf{v}}, \underline{\mathbf{v}}) + \nu^{-1/2} (\mathbf{K}\underline{\mathbf{z}}, \underline{\mathbf{z}}) + \nu^{-1}\gamma^2 (\mathbf{K}\underline{\mathbf{z}}, \underline{\mathbf{z}}) \\
&\quad + \left( \sup_{0 \neq (\mathbf{u}_h, \mathbf{w}_h) \in V_h} \frac{b\left((\mathbf{u}_h, \mathbf{w}_h), (q_h, s_h)\right)}{\|(\mathbf{u}_h, \mathbf{w}_h)\|_V} \right)^2 \\
&= \langle \mathbf{P}\underline{\mathbf{v}}, \underline{\mathbf{v}} \rangle + \nu^{-1} \langle \mathbf{P}\underline{\mathbf{z}}, \underline{\mathbf{z}} \rangle \\
&\quad + \left( \sup_{0 \neq (\mathbf{u}_h, \mathbf{w}_h) \in V_h} \frac{-\langle \mathbf{D}\underline{\mathbf{w}}, \underline{q} \rangle - \langle \mathbf{D}\underline{\mathbf{u}}, \underline{s} \rangle}{\left( \langle \mathbf{P}\underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle + \nu^{-1} \langle \mathbf{P}\underline{\mathbf{w}}, \underline{\mathbf{w}} \rangle \right)^{1/2}} \right)^2 .
\end{aligned}$$

Using Lemma 3.21, we obtain further:

$$\begin{aligned}
&\left( \sup_{0 \neq (\mathbf{u}_h, \mathbf{w}_h) \in V_h} \frac{-\langle \mathbf{D}\underline{\mathbf{w}}, \underline{q} \rangle - \langle \mathbf{D}\underline{\mathbf{u}}, \underline{s} \rangle}{\left( \langle \mathbf{P}\underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle + \nu^{-1} \langle \mathbf{P}\underline{\mathbf{w}}, \underline{\mathbf{w}} \rangle \right)^{1/2}} \right)^2 \\
&\quad = \left( \sup_{0 \neq \mathbf{u}_h \in V_h} \frac{-\langle \mathbf{D}\underline{\mathbf{u}}, \underline{s} \rangle}{\langle \mathbf{P}\underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle^{1/2}} \right)^2 + \left( \sup_{0 \neq \mathbf{w}_h \in V_h} \frac{-\langle \mathbf{D}\underline{\mathbf{w}}, \underline{q} \rangle}{\nu^{1/2} \langle \mathbf{P}\underline{\mathbf{w}}, \underline{\mathbf{w}} \rangle^{1/2}} \right)^2 \\
&\quad = \nu \langle \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T \underline{q}, \underline{q} \rangle + \langle \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T \underline{s}, \underline{s} \rangle
\end{aligned}$$

and thus

$$\begin{aligned}
\|x_h\|_{X_h}^2 &= \langle \mathbf{P}\underline{\mathbf{v}}, \underline{\mathbf{v}} \rangle + \nu^{-1} \langle \mathbf{P}\underline{\mathbf{z}}, \underline{\mathbf{z}} \rangle + \nu \langle \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T \underline{q}, \underline{q} \rangle + \langle \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T \underline{s}, \underline{s} \rangle \\
&= \langle \mathcal{P}\underline{x}, \underline{x} \rangle ,
\end{aligned}$$

which completes the proof.

□

Using the identity

$$\langle \mathcal{A} x_h, w_h \rangle = \langle \mathcal{A}_h \underline{x}, \underline{w} \rangle \quad \text{for all } x_h, w_h \in X_h$$

and the representation (6.19) the estimates in (6.17) of Theorem 6.11 reads as:

$$c\|\underline{x}\|_{\mathcal{P}} \leq \sup_{0\neq\underline{w}\in\mathbb{R}^{2(n_h+m_h)}} \frac{\langle \mathcal{A}_h\underline{x}, \underline{w}\rangle}{\|\underline{w}\|_{\mathcal{P}}} \leq C\|\underline{x}\|_{\mathcal{P}} \quad \text{for all } \underline{x} \in \mathbb{R}^{2(n_h+m_h)} \tag{6.21}$$

with

$$c = \frac{\sqrt{3}-1}{2} \quad \text{and} \quad C = \frac{1+\sqrt{5}}{2}. \tag{6.22}$$

Here the following notation is used:

**Notation.** *For a positive definite matrix $M \in \mathbb{R}^{n\times n}$ the associated inner product is given by $\langle x, y\rangle_M = \langle Mx, y\rangle$. Both the vector norm and the matrix norm associated with the inner product $\langle\cdot,\cdot\rangle_M$ are denoted by $\|\cdot\|_M$.*

Moreover, we have the following representation:

$$\begin{aligned}
\sup_{0\neq\underline{w}\in\mathbb{R}^{2(n_h+m_h)}} \frac{\langle \mathcal{A}_h\underline{x}, \underline{w}\rangle}{\|\underline{w}\|_{\mathcal{P}}} &= \sup_{0\neq\underline{w}\in\mathbb{R}^{2(n_h+m_h)}} \frac{\langle \mathcal{P}^{-1/2}\mathcal{A}_h\underline{x}, \underline{w}\rangle}{\|\mathcal{P}^{-1/2}\underline{w}\|_{\mathcal{P}}} \\
&= \sup_{0\neq\underline{w}\in\mathbb{R}^{2(n_h+m_h)}} \frac{\langle \mathcal{P}^{-1/2}\mathcal{A}_h\underline{x}, \underline{w}\rangle}{\|\underline{w}\|} \\
&= \|\mathcal{P}^{-1/2}\mathcal{A}_h\| = \|\mathcal{P}^{-1}\mathcal{A}_h\|_P
\end{aligned} \tag{6.23}$$

for all $\underline{x} \in \mathbb{R}^{2(n_h+m_h)}$, where for a symmetric and positive matrix $M \in \mathbb{R}^{n\times n}$, $M^{1/2} \in \mathbb{R}^{n\times n}$ denotes the unique symmetric and positive matrix, satisfying $M = M^{1/2}M^{1/2}$.

Now from(6.21) and (6.23) we obtain

$$c\|\underline{x}\|_{\mathcal{P}} \leq \|\mathcal{P}^{-1}\mathcal{A}_h\underline{x}\|_{\mathcal{P}} \leq C\|\underline{x}\|_{\mathcal{P}} \quad \text{for all } \underline{x} \in \mathbb{R}^{2(n_h+m_h)} \tag{6.24}$$

with constants $c$ and $C$ from (6.22). In a next step, we consider the condition number

$$\kappa(\mathcal{P}^{-1}\mathcal{A}_h) = \|\mathcal{P}^{-1}\mathcal{A}_h\|_{\mathcal{P}}\|(\mathcal{P}^{-1}\mathcal{A}_h)^{-1}\|_{\mathcal{P}}.$$

**Lemma 6.13.** *Let $M \in \mathbb{R}^{n\times n}$ be symmetric and non-singular, and let $P \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Then*

$$\|P^{-1}M\|_P = \lambda_{\max}(P^{-1}M) \quad \textit{and} \quad \|(P^{-1}M)^{-1}\|_P = \frac{1}{\lambda_{\min}(P^{-1}M)},$$

*where for $N \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(N)$ and $\lambda_{\max}(N)$ denote the eigenvalues of $N$ with minimal and maximal modulus.*

*Proof.* First, we have:

$$\|P^{-1}M\|_P = \sup_{0\neq\underline{x}\in\mathbb{R}^n} \frac{\|P^{-1}M\underline{x}\|_P}{\|\underline{x}\|_P} = \sup_{0\neq\underline{x}\in\mathbb{R}^n} \left(\frac{\langle P^{-1}M\underline{x}, P^{-1}M\underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P}\right)^{1/2} \tag{6.25}$$

and

$$
\begin{aligned}
\left\|(P^{-1}M)^{-1}\right\|_P &= \sup_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\|(P^{-1}M)^{-1}\underline{x}\|_P}{\|\underline{x}\|_P} \\
&= \left( \inf_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\|P^{-1}M\underline{x}\|_P}{\|\underline{x}\|_P} \right)^{-1} \\
&= \inf_{0\neq \underline{x}\in\mathbb{R}^n} \left( \frac{\langle P^{-1}M\underline{x}, P^{-1}M\underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P} \right)^{-1/2}.
\end{aligned}
\tag{6.26}
$$

Now let us consider the matrix $(P^{-1}M)^2$. It is easy to see, that if $\lambda$ is an eigenvalue of $(P^{-1}M)^2$ with corresponding eigenvector $\underline{x}\in\mathbb{R}^n$, then $\lambda$ is an eigenvalue of the symmetric and positive definite matrix $P^{1/2}\,(P^{-1}M)^2\,P^{-1/2} = (P^{-1/2}MP^{-1/2})^2$ with corresponding eigenvector $P^{1/2}\underline{x}$, i.e., the have the same positive eigenvalues. Further, we have that $(P^{-1}M)^2$ is self adjoint w.r.t. to $\langle\cdot,\cdot\rangle_P$ and thus we obtain by Raleigh's quotient the following representation of $\lambda_{\min}\left((P^{-1}M)^2\right)$ and $\lambda_{\max}\left((P^{-1}M)^2\right)$:

$$
\begin{aligned}
\lambda_{\min}\left((P^{-1}M)^2\right) &= \inf_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\langle (P^{-1}M)^2\underline{x}, \underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P} \\
&= \inf_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\langle P^{-1}M\underline{x}, P^{-1}M\underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P}
\end{aligned}
\tag{6.27}
$$

and

$$
\begin{aligned}
\lambda_{\max}\left((P^{-1}M)^2\right) &= \sup_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\langle (P^{-1}M)^2\underline{x}, \underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P} \\
&= \sup_{0\neq \underline{x}\in\mathbb{R}^n} \frac{\langle P^{-1}M\underline{x}, P^{-1}M\underline{x}\rangle_P}{\langle \underline{x}, \underline{x}\rangle_P}.
\end{aligned}
\tag{6.28}
$$

Finally, we obtain from (6.25), (6.26), (6.27) and (6.28):

$$
\|P^{-1}M\|_P = \sqrt{\lambda_{\max}\left((P^{-1}M)^2\right)} = \lambda_{\max}(P^{-1}M)
$$

and

$$
\|(P^{-1}M)^{-1}\|_P = \frac{1}{\sqrt{\lambda_{\min}\left((P^{-1}M)^2\right)}} = \frac{1}{\lambda_{\min}(P^{-1}M)}.
$$

□

Now, Lemma 6.13 and (6.24) imply the following estimate for the condition number of $\mathcal{P}^{-1}\mathcal{A}_h$:

**Theorem 6.14.** *We have:*

$$
\lambda_{\min}(\mathcal{P}^{-1}\mathcal{A}_h) \leq \frac{2}{\sqrt{3}-1} \quad \textit{and} \quad \lambda_{\max}(\mathcal{P}^{-1}\mathcal{A}_h) \leq \frac{(1+\sqrt{5})}{2}
$$

*and moreover,*

$$
\kappa(\mathcal{P}^{-1}\mathcal{A}_h) = \frac{\lambda_{\max}(\mathcal{P}^{-1}\mathcal{A}_h)}{\lambda_{\min}(\mathcal{P}^{-1}\mathcal{A}_h)} \leq \frac{1+\sqrt{5}}{\sqrt{3}-1} \approx 4.4205.
\tag{6.29}
$$

### 6.2.4 Computational aspects

For solving the linear system in (6.18) we use the preconditioned MINRES (minimal residual) method. For the preconditioned MINRES method with preconditioner $P$ we have the following estimate

$$\|r_{2k}\|_{P^{-1}} \leq \frac{2q^k}{1+q^{2l}} \|r_0\|_{P^{-1}} \quad \text{with} \quad q = \frac{\kappa\,(P^{-1}\mathcal{A}_h) - 1}{\kappa\,(P^{-1}\mathcal{A}_h) + 1}, \tag{6.30}$$

for the residual $r_k$ of the k-th iterate, see, e.g., [39].

For the choice $P = \mathcal{P}$, it follows from (6.29) and (6.30) that the number of iterations, which is needed to decrease the relative error of the $k$-th residual measured in the $\|\cdot\|_{\mathcal{P}^{-1}}$-norm by a factor $\varepsilon > 0$, is independent of $\gamma$, $\nu$ and $h$.

#### 6.2.4.1 The practical preconditioner $\tilde{\mathcal{P}}$

The usage of $\mathcal{P}$ as preconditioner for $\mathcal{A}_h$ requires the evaluation of $\mathbf{P}^{-1}\underline{\mathbf{g}}$ and $S^{-1}\underline{e}$ for some given vectors $\underline{\mathbf{g}}$ and $\underline{e}$ in every step of the MINRES method. These, especially the evaluation of $S^{-1}\underline{e}$, are nontrivial tasks, due to the potentially high number of involved unknowns. To decrease the computational costs we want to replace $\mathbf{P}$ and $S$ by efficient approximations $\tilde{\mathbf{P}}$ and $\tilde{S}$, respectively.

This leads to a preconditioner of the form:

$$\tilde{\mathcal{P}} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{R} \end{bmatrix} \quad \text{with} \quad \tilde{P} = \begin{bmatrix} \nu^{-1}\tilde{\mathbf{P}} & 0 \\ 0 & \tilde{\mathbf{P}} \end{bmatrix} \quad \text{and} \quad \tilde{R} = \begin{bmatrix} \nu\tilde{S} & 0 \\ 0 & \tilde{S} \end{bmatrix}. \tag{6.31}$$

**Notation.** *For symmetric and positive definite matrices $M, N \in \mathbb{R}^{n\times n}$, we write $M \sim N$, if there exists positive constants $c$ and $C$ independent of $\gamma, \nu$ and $h$, such that $c\langle M\underline{x}, \underline{x}\rangle \leq \langle N\underline{x}, \underline{x}\rangle \leq C\langle M\underline{x}, \underline{x}\rangle$ for all $\underline{x} \in \mathbb{R}^n$.*

Obviously we have that $\kappa\,(\mathcal{P}^{-1}\mathcal{A}) \leq C$ for a positive constant $C$ independent of $\gamma$, $\nu$ and $h$, if $\tilde{\mathbf{P}} \sim \mathbf{P}$ and $\tilde{S} \sim S$. We will now present a possible choice for $\tilde{\mathbf{P}}$ and $\tilde{S}$ with $\tilde{\mathbf{P}} \sim \mathbf{P}$ and $\tilde{S} \sim S$, which guarantee $\mathcal{P} \sim \tilde{\mathcal{P}}$, and further an efficient evaluation of $\tilde{\mathcal{P}}^{-1}\underline{x}$ for a vector $\underline{x}$:

**Choice for $\tilde{\mathbf{P}}$:** Due to the analysis in [70] we replace the application of the inverse of the matrix $\mathbf{P}$, which corresponds to the discretization of the second order operator

$$\gamma^2\, I - \nu^{1/2}\Delta,$$

by one V-cycle of a geometric multigrid method with one forward and with one backward Gauss-Seidel sweep for pre- and post-smoothing, shortly denoted by $\tilde{\mathbf{P}}^{-1}$. In [70] it was shown that $\tilde{\mathbf{P}} \sim \mathbf{P}$.

**Choice for $\tilde{\mathbf{S}}$:**

We have the following discrete analogue of Theorem 4.14 is essential, see, e.g., [30], [61], [62] and [69].

**Theorem 6.15.** *Let Assumption 1 be satisfied. Then there exist positive constants $c$ and $C$ such that*

$$c\|q_h\|_{Q_{TH},L^2(\Omega)+\varepsilon H^1(\Omega)} \leq \sup_{0\neq \mathbf{v}_h\in V_{TH}} \frac{\int_\Omega (\operatorname{div}\mathbf{v}_h)\, q_h\, dx}{\|\mathbf{v}_h\|_{L^2(\Omega)^d\cap\varepsilon^{-1}H^1_0(\Omega)^d}} \leq C\|q_h\|_{Q_{TH},L^2(\Omega)+\varepsilon H^1(\Omega)}$$

*holds for all $\varepsilon > 0$ and $q_h \in Q_{TH}$.*

Here the following notation is used:

**Definition 6.16.** *Let $X$ and $Y$ be Hilbert spaces containing $Q_{TH}$. We define:*

$$\|p_h\|_{Q_{TH},X+Y} = \inf_{q_h,r_h\in Q_{TH}\colon p_h=q_h+r_h} \left(\|q_h\|_X^2 + \|r_h\|_Y^2\right)^{1/2}$$

*for all $p_h \in Q_{TH}$.*

Now, as a consequence of Theorem 6.15 we obtain the following result:

**Theorem 6.17.** *Let Assumption 1 be satisfied. We have $S \sim S_{CH}$ with*

$$S_{CH} = \left(\sqrt{\nu}\, M_p^{-1} + \gamma^2 K_p^{-1}\right)^{-1}.$$

*Proof.* We proceed in three steps:

- Firstly, it follows immediately from Theorem 6.15 that there exist positive constants $c$ and $C$ independent of $\gamma$, $\nu$ and $h$ such that

  $$\begin{aligned} c\|q_h\|_{V_{\mathrm{TH}},\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)} &\leq \sup_{0\neq \mathbf{v}_h\in V_{\mathrm{TH}}} \frac{\int_\Omega (\operatorname{div}\mathbf{v}_h)\, q_h\, dx}{\|\mathbf{v}_h\|_{\gamma L^2(\Omega)^d+\nu^{1/4}H^1(\Omega)^d}} \\ &\leq C\|q_h\|_{Q_{\mathrm{TH}},\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)} \end{aligned}$$

  for all $q_h \in Q_{\mathrm{TH}}$.

- Secondly, we show:

  $$\sup_{0\neq \mathbf{v}_h\in V_{\mathrm{TH}}} \frac{\int_\Omega (\operatorname{div}\mathbf{v}_h)\, q_h\, dx}{\|\mathbf{v}_h\|_{\gamma L^2(\Omega)^d+\nu^{1/4}H^1(\Omega)^d}} = \langle S\underline{q},\underline{q}\rangle^{1/2}$$

  for all $q_h \in Q_{\mathrm{TH}}$.

  Let $q_h \in Q_{\mathrm{TH}}$. We have:

  $$\begin{aligned} \sup_{0\neq \mathbf{v}_h\in V_{\mathrm{TH}}} \frac{\int_\Omega (\operatorname{div}\mathbf{v}_h)\, q_h\, dx}{\|\mathbf{v}_h\|_{\gamma L^2(\Omega)^d+\nu^{1/4}H^1(\Omega)^d}} &= \sup_{0\neq \underline{\mathbf{v}}\in\mathbb{R}^{n_h}} \frac{\langle \mathbf{D}\underline{\mathbf{v}},\underline{q}\rangle}{\langle \mathbf{P}\underline{\mathbf{v}},\underline{\mathbf{v}}\rangle^{1/2}} = \sup_{0\neq \underline{\mathbf{v}}\in\mathbb{R}^{n_h}} \frac{\langle \underline{\mathbf{v}},\mathbf{P}^{-1/2}\mathbf{D}^T\underline{q}\rangle}{\langle \underline{\mathbf{v}},\underline{\mathbf{v}}\rangle^{1/2}} \\ &= \langle \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T q, q\rangle^{1/2} = \langle Sq,q\rangle^{1/2}. \end{aligned}$$

- Finally, we show that:

$$\|s_h\|_{Q_{\mathrm{TH}},\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)^d} = \langle S_{CH}\underline{s},\underline{s}\rangle^{1/2} \tag{6.32}$$

for all $s_h \in Q_{\mathrm{TH}}$, with

$$S_{CH} = \left(\sqrt{\nu}\, M_p^{-1} + \gamma^2 K_p^{-1}\right)^{-1}.$$

Let $s_h \in Q_{\mathrm{TH}}$. We have:

$$\begin{aligned}\|s_h\|_{Q_{\mathrm{TH}},\gamma^{-1}H^1(\Omega)+\nu^{-1/4}L^2(\Omega)^d} &= \inf_{q_h,r_h\in Q_{\mathrm{TH}}:\, s_h=q_h+r_h} \left(\gamma^{-2}\|q_h\|_1^2 + \nu^{-1/2}\|r_h\|_0^2\right)\\ &= \inf_{\underline{q},\underline{r}\in\mathbb{R}^{m_h}:\, \underline{s}=\underline{q}+\underline{r}} \left(\gamma^{-2}\langle K_p\underline{q},\underline{q}\rangle + \nu^{-1/2}\langle M_p\underline{r},\underline{r}\rangle\right)\\ &= \inf_{\underline{q}\in\mathbb{R}^{m_h}} J_s(\underline{q}),\end{aligned}$$

with

$$J_s(\underline{q}) = \gamma^{-2}\langle K_p\underline{q},\underline{q}\rangle + \nu^{-1/2}\langle M_p(\underline{s}-\underline{q}),(\underline{s}-\underline{q})\rangle.$$

The functional $J_s$ is convex on $\mathbb{R}^{m_h}$ and thus the solution $\underline{\hat{q}}$ of $\inf_{\underline{q}\in\mathbb{R}^{m_h}} J_s(\underline{q})$ is given by the solution of the linear equation $\nabla J_s(\underline{\hat{q}}) = 0$, which can be easily solved. We have $\underline{\hat{q}} = \nu^{-1/2}(\gamma^{-2}K_p + \nu^{-1/2}M_p)^{-1}M_p$ and hence

$$J_s(\underline{\hat{q}}) = \left\langle \left(\nu^{1/2}\, M_p^{-1} + \gamma^2 K_p^{-1}\right)^{-1} \underline{s},\underline{s}\right\rangle,$$

which, shows the representation (6.32).

□

Therefore, we replace in a first step of approximation $S$ by $S_{CH}$.

In a second step of approximation we replace the evaluation of $M_p^{-1}\underline{e}$ by one step of a symmetric Gauß-Seidel iteration applied to $M_p\underline{q} = \underline{e}$ and the evaluation of $K_p^{-1}\underline{e}$ by one $V$-cycle of a geometric multigrid method with one forward and with one backward Gauss-Seidel sweep for pre- and post-smoothing to $K_p\underline{q} = \underline{e}$, shortly denoted by $\tilde{M}_p^{-1}\underline{e}$ and $\tilde{K}_p^{-1}\underline{e}$, respectively. Again from [70] we have $M_p \sim \tilde{M}_p$ and $K_p \sim \tilde{K}_p$ . As result of this replacements we replace $S$ by

$$\tilde{S}_{CH} = (\sqrt{\nu}\tilde{M}_p^{-1} + \gamma^2\tilde{K}_p^{-1})^{-1}.$$

We have $\tilde{S}_{CH} \sim S$ and the inverse of $\tilde{S}_{CH}$ can be applied efficiently.

In addition we consider also the multiple application of $\tilde{S}_{CH}$:

We replace the evaluation $S^{-1}\underline{e}$ by applying $r$-steps (typically $r = 1, 2, 3$) of the preconditioned Richardson method to the equation $S\underline{q} = \underline{e}$, with scaling parameters $\tau_i > 0$, the preconditioner $\tilde{S}_{CH}$ and the initial vector $\underline{q}_0 = 0$. The corresponding preconditioner is given by

$$\tilde{S}_{CH}^{(r)} = S \left( I_{2m} - \prod_{i=1}^{r} (I_{2m} - \tau_i \, \tilde{S}_{CH}^{-1} S)^i \right)^{-1}, \tag{6.33}$$

In order to guarantee that $\tilde{S}$ is positive definite, it is easy to see that the condition

$$1 - \prod_{i=1}^{r} (1 - \tau_i \, \lambda)^i > 0 \quad \forall \lambda \in (0, 1].$$

suffices. In particular if we choose $\tau_1 > 0$ fixed and $\tau_i = 1$ for $i \geq 2$, then it follows that $\tilde{S}_{CH}^{(r)}$ is symmetric, positive definite, and $\tilde{S}_{CH}^{(r)} \sim S$.

In summary we obtain:

**Theorem 6.18.** *$\tilde{\mathcal{P}}$ defined in (6.31) with the previous presented $\tilde{\mathbf{P}}$ and $\tilde{S} = \tilde{S}_{CH}^{(r)}$ from (6.33), is symmetric, positive definite and*

$$\kappa(\tilde{\mathcal{P}}^{-1} \mathcal{A}_h) \leq C$$

*for a constant $C$ independent of $\gamma$, $\nu$ and $h$.*

#### 6.2.4.2 Alternative stopping criteria

Based on the convergence result (6.30) an intuitive choice for the stopping criterion of the preconditioned MINRES method is

$$\|r_k\|_{\tilde{\mathcal{P}}^{-1}} \leq \varepsilon \, \|r_0\|_{\tilde{\mathcal{P}}^{-1}}. \tag{6.34}$$

Another natural measure for the error is $\|x - x_k\|_{\tilde{\mathcal{P}}}$. This quantity is not directly computable but can be estimated by using the relation:

$$c\|x - x_k\|_{\tilde{\mathcal{P}}} \leq \|r_k\|_{\tilde{\mathcal{P}}^{-1}} \leq C\|x - x_k\|_{\tilde{\mathcal{P}}} \tag{6.35}$$

with $c = \lambda_{\min}(\tilde{\mathcal{P}}^{-1} \mathcal{A}_h)$ and $C = \lambda_{\max}(\tilde{\mathcal{P}}^{-1} \mathcal{A}_h)$. Approximations $\tilde{c}$ and $\tilde{C}$ for $c$ and $C$, respectively, can be computed by using the so-called harmonic Ritz values, see [83]. Therefore, the stopping criterion

$$\|x - x_k\|_{\mathcal{P}} \leq \varepsilon \, \|x - x_0\|_{\mathcal{P}} \tag{6.36}$$

is asymptotically satisfied, if we prescribe (6.34) with $\varepsilon$ replaced by $\varepsilon_* = \tilde{c}/\tilde{C} \, \varepsilon$.

**Standard norm for stopping criterion:** Finally we present an analytic convergence result, for the standard norm

$$\|(u, p, w, r)\|_{\mathcal{N}}^2 := \|u\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2 + \|r\|_{L^2(\Omega)}^2.$$

For $\nu \leq 1$, it is easy to see that:

$$\|x - x_k\|_{\mathcal{N}} / \|x - x_0\|_{\mathcal{N}} \leq 2 \left( \max(2, \omega)/\nu \right)^2 \|x - x_k\|_{\mathcal{P}} / \|x - x_0\|_{\mathcal{P}}.$$

This allows to use this standard norm for the stopping criterion in an efficient manner via (6.36). Using this estimate in combination with (6.35) and (6.30), we obtain:

$$\|r_{2k}\|_{\mathcal{N}} \leq \frac{2C\,(\max(2,\omega)/\nu)^2}{c} \frac{2q^k}{1+q^{2k}} \|r_0\|_{\mathcal{N}} \quad \text{with} \quad q = \frac{\kappa\,(\mathcal{P}^{-1}\mathcal{A}_h) - 1}{\kappa\,(\mathcal{P}^{-1}\mathcal{A}_h) + 1} \tag{6.37}$$

for the residual $r_k$ of the k-th iterate.

From (6.37) we obtain that the number of iterations $k^\star$ which is needed to decrease the initial error by a factor $\varepsilon > 0$, depends only mildly on the parameters $\omega$ and $\nu$, namely, logarithmically on $(\max(2,\omega)/\nu)^2$.

# Chapter 7

# Numerical results

In this chapter we perform some numerical experiments for the *mixed method of the first biharmonic boundary value problem (3.3)* and the *distributed optimal control problem with the time-periodic Stokes equations*, see Section 4.1.

All computations are carried out on a OPENSuse Linux machine with Intel(R) Xeon(R) CPU W3680 @ 3.33GHz.

## 7.1 The first biharmonic boundary value problem

For illustrating the theoretical results for the Hellan-Herrmann-Johnson finite element method (6.7) and its conforming variant (6.8) we consider the following simple biharmonic test problem:

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma$$

on two domains, the square $\Omega = \Omega_S = (-1,1)^2$ and the $L$-shaped domain $\Omega = \Omega_L$ depicted in figures 7.1 and 7.2, where also the initial mesh (level $\ell = 0$) is shown. The right-hand side

$$f(x) = 16\pi^4 \Big[\cos(2\pi x_1)\big[25\cos(4\pi x_2) - 1\big] - 16\cos(4\pi x_2)\Big]$$

is chosen such that

$$w(x) = \big[1 - \cos(2\pi x_1)\big]\,\big[1 - \cos(4\pi x_2)\big]$$

is the exact solution to the problem. The initial meshes are uniformly refined until the final level $\ell = L$. In all experiments the polynomial degree $k$ as introduced in the beginning of Subsection 6.1.1 is chosen equal to 1, which represents the lowest order HHJ method.

In each step of the solution procedure as described in Subsection 6.1.3 a reduction of the Euclidean norm of the initial residual by a factor of $10^{-8}$ was used as stopping criterion for the PCG methods with initial guess equal to 0.

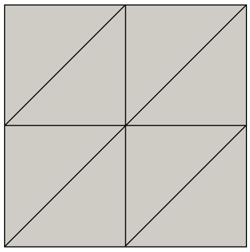

Figure 7.1: $\Omega = \Omega_S$.

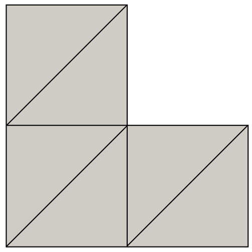

Figure 7.2: $\Omega = \Omega_L$.

Table 7.1 shows the observed number of iterations for the solution procedure for $\Omega = \Omega_S$. The first column contains the size $h = 2^{-\ell}$. The next three pairs of columns show the total number $N_i$ of degrees of freedom and the number of iterations $\mathrm{iter}_i$ of the PCG method for the linear system in step $i = 1, 2, 3$.

Table 7.1: Number of iterations, $\Omega = \Omega_S$ (square).

| $h$ | $N_1$ | $\mathrm{iter}_1$ | $N_2$ | $\mathrm{iter}_2$ | $N_3$ | $\mathrm{iter}_3$ |
|---|---|---|---|---|---|---|
| $2^{-7}$ | 64 001 | 10 | 132 098 | 14 | 64 001 | 10 |
| $2^{-8}$ | 261 221 | 10 | 526 338 | 15 | 261 221 | 10 |
| $2^{-9}$ | 1 046 530 | 11 | 2 101 250 | 15 | 1 046 530 | 11 |
| $2^{-10}$ | 4 190 210 | 11 | 8 396 802 | 15 | 4 190 210 | 11 |

Table 7.2 shows the corresponding results for the $L$-shaped domain $\Omega = \Omega_L$ representing a non-convex case.

Table 7.2: Number of iterations, $\Omega = \Omega_L$ ($L$-shaped domain).

| $h$ | $N_1$ | $\mathrm{iter}_1$ | $N_2$ | $\mathrm{iter}_2$ | $N_3$ | $\mathrm{iter}_3$ |
|---|---|---|---|---|---|---|
| $2^{-7}$ | 48 665 | 11 | 99 330 | 16 | 48 665 | 11 |
| $2^{-8}$ | 195 585 | 11 | 395 266 | 16 | 195 585 | 11 |
| $2^{-9}$ | 784 385 | 11 | 1 576 962 | 16 | 784 385 | 11 |
| $2^{-10}$ | 3 141 630 | 12 | 6 299 650 | 17 | 3 141 630 | 12 |

In accordance with well-established convergence results for multigrid methods, see, e.g., [44], the number of iterations is bounded uniformly with respect to the mesh size.

Finally, in Table 7.3 the discretization error of the non-conforming method (6.7) and its conforming variant (6.8) for $\Omega = \Omega_L$ are compared. For both methods the $H^1$-error of the original variable $w$ has almost same size and decreases with the order $h$. This is for the original HHJ method in accordance with known estimates, see [5, 37].

Table 7.3: Discretization error $\|w - w_h\|_1$.

| $h$ | (6.7) | (6.8) |
|---|---|---|
| $2^{-6}$ | 1.2710 | 1.2612 |
| $2^{-7}$ | 0.6322 | 0.6310 |
| $2^{-8}$ | 0.3157 | 0.3156 |
| $2^{-9}$ | 0.1578 | 0.1578 |
| $2^{-10}$ | 0.0789 | 0.0789 |

## 7.2 Distributed optimal control problem with the time-periodic Stoke equations

In this section we illustrate the theoretical results for the discretization method presented in Section 6.2 by some numerical examples on the unit square domain $\Omega = (0,1) \times (0,1)$.

Following Example 1 in [42] we choose the target velocity

$$\mathbf{u}_d(x,y) = [(U(x,y), V(x,y)]^T,$$

given by

$$U(x,y) = 10\,\varphi(x)\varphi'(y) \quad \text{and} \quad V(x,y) = -10\,\varphi'(x)\varphi(y)$$

with

$$\varphi(z) = \big[1 - \cos(0.8\pi z)\big](1-z)^2.$$

This target velocity $\mathbf{u}_d(x,y)$ is divergence free.

The initial mesh contains four triangles obtained by connecting the two diagonals. The final mesh was constructed by applying $\ell$ uniform refinement steps to the initial mesh, leading to a mesh size $h = 2^{-\ell}$, see figure 7.3. Obviously, each triangle of the mesh has at least two internal edges and therefore, Assumption 1 is satisfied.

All presented numerical experiments refer to the first of the two systems from (4.6). The results for the second system are completely identical. Therefore, they are omitted. For each system, the total number of unknowns on the finest level $\ell = 7$ is 1 184 780.

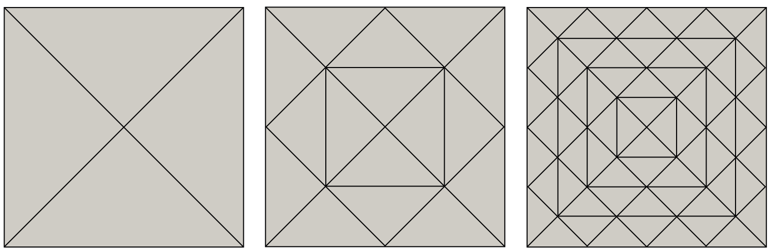

Figure 7.3: $\mathcal{T}_h(\Omega)$ with $h = 2^{-\ell}$ for $\ell = 0, 1, 2$.

Table 7.4: $\omega = 10^4$

| | $\nu$ | | | | |
|---|---|---|---|---|---|
| $h$ | $10^{-8}$ | $10^{-4}$ | $1$ | $10^4$ | $10^8$ |
| $2^{-4}$ | 44 | 46 | 46 | 46 | 46 |
| $2^{-5}$ | 48 | 50 | 50 | 50 | 48 |
| $2^{-6}$ | 50 | 52 | 52 | 52 | 52 |
| $2^{-7}$ | 54 | 56 | 56 | 56 | 56 |

Table 7.5: $\nu = 10^{-4}$

| | $\omega$ | | | | |
|---|---|---|---|---|---|
| $h$ | $10^{-8}$ | $10^{-4}$ | $1$ | $10^4$ | $10^8$ |
| $2^{-4}$ | 87 | 87 | 87 | 46 | 38 |
| $2^{-5}$ | 99 | 99 | 99 | 52 | 34 |
| $2^{-6}$ | 101 | 101 | 101 | 51 | 30 |
| $2^{-7}$ | 105 | 105 | 105 | 56 | 34 |

Tables 7.4 and 7.5 contain the numerical results produced by the preconditioned MINRES method with the preconditioner $\tilde{\mathcal{P}}$ as described in (6.31), where we choose $r = 1$ with $\tau_1 = 1$ (i.e. $\tilde{S} = \tilde{S}_{CH}$). The considered values for the frequency $\omega$, the regularization parameter $\nu$, and the mesh size $h$, are specified in the table captions, the first rows and first columns. The other entries of the tables contain the numbers of MINRES iterations that are required for reducing the initial error of the residual in the $\tilde{\mathcal{P}}^{-1}$-norm by a factor of $\varepsilon = 10^{-8}$ with initial vector $x_0 = 0$, respectively.

As expected from the results of Theorem 6.18, the condition numbers are bounded away from $\infty$ independently of $h$, $\nu$ and $\omega$ , leading to a uniform bound for the number of iterations.

Next, we compare the performance of the practical preconditioner $\tilde{\mathcal{P}}$ with the original (typically better but impractical) preconditioner $\mathcal{P}$ from (6.20) for the particular parameter choice $\omega = 1$, $\nu = 1$ and $h = 2^{-7}$. In this case the number of iterations for $\tilde{\mathcal{P}}$ is 118, which is roughly four times higher than the expected number of iterations for $\mathcal{P}$, see Table 1 in [54]. Since the difference is relatively high, it is worthwhile to consider other options for the inner iteration in order to reduce this gap. Table 7.6 shows the numbers of iterations $\tilde{k}$ and the computational costs, measured in the CPU-time, for $\tilde{S} = \tilde{S}_{CH}^{(r)}$ with $r \in \{1, 2, 3\}$, for different values of $\tau_1 \in \{1, 4\}$ and $\tau_i = 1$ for $i \geq 2$. These and similar further numerical experiments show that a significant improvement of the numbers of iterations can be achieved by a proper choice for $\tau_1$ and not so much by a higher number $r$ of inner iterations. It turned out that $\tau_1 = 4$ is a very good choice, for all cases considered.

Table 7.6: $\omega = 1$, $\nu = 1$, $h = 2^{-7}$

| $r$ | scaling parameters | $\tilde{k}$ | CPU-time $[sec]$ |
|---|---|---|---|
| 1 | $\tau_1 = 1$ (i.e. $\tilde{S} = \tilde{S}_{CH}$) | 118 | 577.65 |
| 1 | $\tau_1 = 4$ (i.e. $\tilde{S} = 4\,S_{CH}$) | 69 | 341.69 |
| 2 | $\tau_1 = 4$, $\tau_2 = 1$ | 46 | 333.7 |
| 3 | $\tau_1 = 4$, $\tau_2 = \tau_3 = 1$ | 41 | 402.73 |

### Alternative stopping criteria

Finally we test the use of the stopping criterion in (6.36) with a numerical example. For the parameter choice $\omega = 1, \nu = 1$ and $h = 2^{-7}$, we computed the numbers of iterations $\tilde{k}$ produced by the preconditioned MINRES method, for the two different stopping criteria (6.34) and (6.36). Thereby we choose $\tilde{S} = \tilde{S}_{\mathrm{CH}}$. As result we obtain $\tilde{k} = 118$ and $\tilde{k} = 130$, using (6.34) and (6.36), respectively. The computed approximations for the constants $c$ and $C$ in (6.35) are $\tilde{c} = 0.152077$ and $\tilde{C} = 1.60562$.

# Chapter 8

# Conclusions

In this thesis we demonstrated the power of the Lagrange multiplier technique as well as the interpolation technique, since we were able to construct an appropriate Sobolev space $X$ for all considered problems.

In every studied case we were able to carry over the results from the continuous level to the discrete level, using the discrete analogue of the continuous norms. This emphasises the importance of understanding the continuous problem.

Nevertheless, as we have seen, handling the obtained (nonstandard) Sobolev spaces can be a major challenge.

# Chapter 9

# Appendix

## Interpolation between $H^2_D(\Omega)$ and $L^2(\Omega)$

In the following theorem we give a representation result for the interpolation space $[H^2_D(\Omega), L^2(\Omega)]_{1/2}$.

**Theorem 9.1.** *We have:*

$$[H^2_D(\Omega), L^2(\Omega)]_{1/2} = H^1_0(\Omega)$$

*with equivalent norms.*

*Proof.* We proceed in three steps:

- Firstly, we show:

$$H^1_0(\Omega) \subseteq [H^2_D(\Omega), L^2(\Omega)]_{1/2} \subseteq H^1(\Omega). \tag{9.1}$$

From $H^2_0(\Omega) \subset H^2_D(\Omega) \subset L^2(\Omega)$ it follows immediately by the definition of the interpolation norm and Example 3.8:

$$\begin{aligned} H^1_0(\Omega) = [L^2(\Omega), H^2_0(\Omega)]_{1/2} &\subseteq [L^2(\Omega), H^2_D(\Omega)]_{1/2} \\ &\subseteq [L^2(\Omega), H^2(\Omega)]_{1/2} = H^1(\Omega) \end{aligned}$$

and thus (9.1) is satisfied.

- Secondly, we show that there exists a positive constant $C$ such that

$$\|Dv\|_{[L^2(\Omega)^d, H(\mathrm{div},\Omega)]^\star_{1/2}} \leq C\|v\|_{[L^2(\Omega), H^2_D(\Omega)]_{1/2}} \tag{9.2}$$

for all $v \in [L^2(\Omega), H^2_D(\Omega)]_{1/2}$, where the operator $D : L^2(\Omega) \to [H(\mathrm{div}, \Omega)]^\star$ is given by

$$\langle Dv, \phi\rangle = -\int_\Omega v \,\mathrm{div}\, \phi \,dx$$

for all $v \in L^2(\Omega)$, $\phi \in H(\operatorname{div}, \Omega)$ and

$$H(\operatorname{div}, \Omega) = \{\phi \in L^2(\Omega)^d \colon \operatorname{div} \phi \in L^2(\Omega)\}$$

equipped with

$$\|\phi\|_{\operatorname{div}} = \left(\|\phi\|_0^2 + \|\operatorname{div} \phi\|_0^2\right)^{1/2}$$

for all $\phi \in H(\operatorname{div}, \Omega)$.

We have:

- Let $v \in L^2(\Omega)$. Then:

$$|\langle Dv, \phi\rangle| = \left| - \int_\Omega v \operatorname{div} \phi \, dx \right| \leq \|v\|_0 \|\phi\|_{\operatorname{div}}$$

  for all $\phi \in H(\operatorname{div}, \Omega)$, i.e., $\|Dv\|_{H(\operatorname{div},\Omega)^\star} \leq \|v\|_0$.
- Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Using the Green formula we obtain

$$|\langle Dv, \phi\rangle| = \left| - \int_\Omega v \operatorname{div} \phi \, dx \right| = \left| - \int_\Omega \nabla v \cdot \phi \, dx \right| \leq |v|_1 \|\phi\|_0 \tag{9.3}$$

  for all $\phi \in L^2(\Omega)^d$. Further we have the estimate

$$|v|_1^2 = \int_\Omega \nabla v \cdot \nabla v \, dx = - \int_\Omega v \, \Delta v \, dx \leq \|v\|_0 \|\Delta v\|_0 \leq C_F |v|_1 \|\Delta v\|_0 \tag{9.4}$$

  for all $v \in H_D^2(\Omega)$.

  From (9.3) and (9.4) we obtain

$$|\langle Dv, \phi\rangle| \leq C_F \|\Delta v\|_0 \|\phi\|_0 = C_F |v|_2 \|\phi\|_0,$$

  for all $\phi \in L^2(\Omega)^d$, i.e., $\|Dv\|_{L^2(\Omega)^d} \leq C_F |v|_2$.

Now we obtain from the Interpolation Theorem 3.13

$$\|Dv\|_{[L^2(\Omega)^d, H(\operatorname{div},\Omega)^\star]_{1/2}} \leq C \|v\|_{[L^2(\Omega), H_D^2(\Omega)]_{1/2}} \tag{9.5}$$

for all $v \in [L^2(\Omega), H_D^2(\Omega)]_{1/2}$. Further, it follows from the Duality Theorem 3.11

$$[L^2(\Omega)^d, H(\operatorname{div}, \Omega)^\star]_{1/2} = [L^2(\Omega)^d, H(\operatorname{div}, \Omega)]_{1/2}^\star$$

with equivalent norms. This together with (9.5) implies the estimate (9.2).

- Thirdly, we show that there exists a positive constant $C$ such that

$$\|Dv\|_{\left(H^{1/2}(\Omega)^d\right)^\star} \le C\|v\|_{[L^2(\Omega),H^2_D(\Omega)]_{1/2}} \tag{9.6}$$

for all $v \in [L^2(\Omega), H^2_D(\Omega)]_{1/2}$.

From $H^1(\Omega)^d \subset H(\mathrm{div},\Omega) \subset L^2(\Omega)^d$ and the definition of the interpolation norm we obtain

$$[L^2(\Omega)^d, H^1(\Omega)^d]_{1/2} \subseteq [L^2(\Omega)^d, H(\mathrm{div},\Omega)]_{1/2}.$$

Since $H^{1/2}(\Omega)^d = [L^2(\Omega)^d, H^1(\Omega)^d]_{1/2}$, see, e.g., [1], we have

$$H^{1/2}(\Omega)^d \subseteq [L^2(\Omega)^d, H(\mathrm{div},\Omega)]_{1/2},$$

which together with (9.2) implies (9.6).

- Finally, we show:

$$[H^2_D(\Omega), L^2(\Omega)]_{1/2} \subseteq H^1_0(\Omega).$$

Let $v \in [L^2(\Omega), H^2_D(\Omega)]_{1/2}$. Using the the Green formula we have:

$$\begin{aligned} |\langle \phi\cdot n, v\rangle_\Gamma| &= \left|\int_\Omega \nabla v\cdot\phi\,dx - \int_\Omega v\,(\mathrm{div}\,\phi)\,dx\right| \\ &= \left|\langle Dv,\phi\rangle - \int_\Omega \nabla v\cdot\phi\,dx\right| \\ &\le \left(\|Dv\|_{\left(H^{1/2}(\Omega)^d\right)^\star}\|\phi\|_{H^{1/2}(\Omega)^d} + \|v\|_1\|\phi\|_0\right) \\ &= C\|\phi\|_{H^{1/2}(\Omega)^d} \end{aligned} \tag{9.7}$$

for all $\phi \in C^\infty(\overline{\Omega})^d$ with $C = \|Dv\|_{\left(H^{1/2}(\Omega)^d\right)^\star} + \|v\|_1$.

Let $\phi \in C^\infty(\overline{\Omega})^d$. Since $C_0^\infty(\Omega)^d$ is dense in $H^{1/2}(\Omega)^d$, see, e.g., [41, Theorem 1.4.2.4], there exists a sequence $(\phi_m)_{m\in\mathbb{N}}$ which converges to $\phi$ in $H^{1/2}(\Omega)^d$. Now from (9.7) it follows:

$$\begin{aligned} \langle v,\phi\cdot n\rangle_\Gamma &= \lim_{m\to\infty}\langle v,\phi_m\cdot n\rangle_\Gamma \\ &= \lim_{m\to\infty}\left|\int_\Omega \nabla v\cdot\phi_m\,dx - \int_\Omega v\ \mathrm{div}\,\phi_m\,dx\right| \\ &= \lim_{m\to\infty} 0 = 0, \end{aligned}$$

i.e.,

$$\langle v,\phi\cdot n\rangle_\Gamma = 0 \quad \text{for all } \phi \in C^\infty(\overline{\Omega})^d. \tag{9.8}$$

We have $v \in H^1(\Omega)$. Therefore, $v \in H^{1/2}(\Gamma)$, see, e.g., [41, Theorem 1.4.2.4], and thus $v \in L^2(\Gamma)$. Further we have $C_0^\infty(\Gamma_k)$ is dense in $L^2(\Gamma_k)$ for $k = 1, \ldots, K$, and thus $\Pi_{k=1}^K C_0^\infty(\Gamma_k)$ is dense in $L^2(\Gamma) = \Pi_{k=1}^K L^2(\Gamma_k)$.

Next we show that for each $\mu \in \Pi_{k=1}^K C_0^\infty(\Gamma_k)$ there exists a $\phi \in C^\infty(\overline{\Omega})^d$ such that $\phi \cdot n = \mu$: Let $\mu = (\mu_k)_{k=1,\ldots,K} \in \Pi_{k=1}^K C_0^\infty(\Gamma_k)$, $k \in \{1, \ldots, K\}$ and $i \in \{1, .., d\}$ such that $n_i \neq 0$ on $\Gamma_k$. It is easy to see, that there exists $\phi_k = (\phi_{k,j})_{j=1,..d} \in C^\infty(\overline{\Omega})^d$ such that $\phi_k = 0$ on $\Gamma/\Gamma_k$, $\phi_{k,j} = \mu_k/n_j$ for $j = i$ and $\phi_{k,j} = 0$ for $j \neq i$. For $\phi = \sum_{k=1}^K \phi_k$ we have $\phi \in C^\infty(\overline{\Omega})^d$ and $\phi \cdot n = \mu$ on $\Gamma$.

Finally it follows from the density of $\Pi_{k=1}^K C_0^\infty(\Gamma_k)$ in $L^2(\Gamma)$ and (9.8), that $v = 0$ on $\Gamma$, i.e., $v \in H_0^1(\Omega)$. This completes the proof.

□

# Some density results

Here we show density results for the spaces $\boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$, $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ and $H(\Delta, \Omega)$. The proofs are based upon the following density criterion for Hilbert spaces, see, e.g., [4, Theorem 1]:

**Lemma 9.2.** *Let $X$ and $Y$ be Hilbert spaces with $X \subset Y$. Then $X$ is dense in $Y$ iff every element of $Y^\star$ that vanishes on $X$ also vanishes on $Y$.*

**Theorem 9.3.** *We have:*

1. $\mathbf{C}^\infty(\overline{\Omega})_{sym}$ *is dense in* $\boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{sym}$.
2. $\boldsymbol{C}^\infty(\overline{\Omega})_{sym}$ *is dense in* $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym}$.

*Proof.* We follow the idea of the proof of Theorem 2.4 in [38].

Let $g \in \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)^\star_{\mathrm{sym}}$. By the Riesz representation Theorem there exists a unique $\boldsymbol{\sigma} \in \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ such that:

$$\langle g, \boldsymbol{\tau} \rangle = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx + \int_\Omega w\, (\operatorname{div}\mathbf{div}\, \boldsymbol{\tau})\, dx \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}, \tag{9.9}$$

where

$$w = \operatorname{div}\mathbf{div}\, \boldsymbol{\sigma} \quad \text{in } \Omega.$$

Now assume that $g$ vanishes on $\mathbf{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$, and let $\tilde{\boldsymbol{\sigma}}$ and $\tilde{w}$ denote the extensions of $\boldsymbol{\sigma}$ and $w$ by zero outside of $\Omega$, respectively. Then it follows from (9.9) that

$$\int_{\mathbb{R}^2} \tilde{\boldsymbol{\sigma}} : \boldsymbol{\phi}\, dx + \int_{\mathbb{R}^2} \tilde{w}\, (\operatorname{div}\mathbf{div}\, \boldsymbol{\phi})\, dx = 0 \quad \text{for all } \boldsymbol{\phi} \in \boldsymbol{C}^\infty\left(\mathbb{R}^2\right)_{\mathrm{sym}}.$$

This equality implies that in sense of distributions on $\mathbb{R}^2$

$$\tilde{\boldsymbol{\sigma}} = -\nabla^2 \tilde{w}.$$

Thus all second derivatives of $\tilde{w}$ exist in $L^2(\mathbb{R}^2)$.

From [29, Lemma 6], we obtain $\tilde{w} \in H^2(\mathbb{R}^2)$, and further we obtain from [38, Theorem 1.2] that $w \in H_0^2(\Omega)$. Then from (9.9) it follows immediately that $\boldsymbol{\sigma} = -\nabla^2 w$. Therefore, we have the following representation for $g$:

$$\langle g, \boldsymbol{\tau} \rangle = -\int_\Omega \nabla^2 w : \boldsymbol{\tau}\, dx + \int_\Omega w \,(\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, dx \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}.$$

Now let $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ that tends to $w$ in $H_0^2(\Omega)$. From Green's identity we obtain

$$-\int_\Omega \nabla^2 w_n : \boldsymbol{\tau}\, dx + \int_\Omega w_n \,(\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, dx = 0$$

for all $n \in \mathbb{N}$, and, subsequently, for the limit

$$\langle g, \boldsymbol{\tau} \rangle = \lim_{n\to\infty} \left( -\int_\Omega \nabla^2 w_n : \boldsymbol{\tau}\, dx + \int_\Omega w_n \,(\operatorname{div} \mathbf{div}\, \boldsymbol{\tau})\, dx \right) = 0.$$

Therefore, $g$ also vanishes on $\boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$, which proves the first statement.

From Lemma 3.9 we have that $\boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \cap \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}} = \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ is dense in $[\boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}]_{1/2} = \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. So we have $\boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ is dense in $\boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ and $\boldsymbol{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$ is dense in $\boldsymbol{H}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$ and therefore, $\boldsymbol{C}^\infty(\overline{\Omega})_{\mathrm{sym}}$ is dense in $\boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\mathrm{sym}}$. □

We have the following density result for the space $H(\Delta, \Omega)$.

**Theorem 9.4.** *The space $C^\infty(\overline{\Omega})$ is dense in $H(\Delta, \Omega)$.*

*Proof.* We have $C^\infty(\overline{\Omega})$ is dense in $H^2(\Omega)$. Moreover, we have $H^2(\Omega)$ is dense in $H(\Delta, \Omega)$, see, e.g., [40], and thus the density of $C^\infty(\overline{\Omega})$ in $H(\Delta, \Omega)$. □

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# Eidesstattliche Erklärung

Ich, Wolfgang Krendl, erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch.

Linz, Mai 2015

Wolfgang Krendl

# Curriculum Vitae

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W. Krendl, V. Simoncini, and W. Zulehner. *Stability estimates and structural spectral properties of saddle point problems*, Numerische Mathematik, 124(1), pp. 183-213, 2013, see [54].

W. Krendl, V. Simoncini, and W. Zulehner. *Efficient preconditioning for an optimal control problem with the time-periodic Stokes equations*, Numerical Mathematics and Advanced Applications, 2014, (to appear), see [55].

W. Krendl, and W. Zulehner. *The Hellan-Herrmann-Johnson method for Biharmonic problems: Mapping properties and preconditioning*, 2014, (submitted), see [56].