JOHANNES KEPLER
UNIVERSITÄT LINZ

Eingereicht von
Matteo Gallet

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Johannes Radon Institute for Computational and Applied Mathematics Doktoratskolleg Computational Mathematics

Erstbeurteiler
Josef Schicho

Zweitbeurteiler
Herwig Hauser

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JOHANNES KEPLER
UNIVERSITÄT LINZ
Altenbergerstraße 69
4040 Linz, Österreich
www.jku.at
DVR 0093696

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ZUSAMMENFASSUNG

Kinematik, verstanden als die Untersuchung der Mobilität von mechanischen Vorrichtungen, oder *Gestängen*, hat Mathematiker immer schon interessiert, insbesondere algebraische Geometer. Seit dem 19. Jahrhundert werden algebraische und geometrische Techniken entwickelt, um solche Mechanismen zu klassifizieren und die Existenz von neuen Familien zu untersuchen. Wichtige Ergebnisse wurden bereits erreicht, wie zum Beispiel die Klassifizierung der beweglichen geschlossenen Ketten bestehend aus höchstens fünf Stangen und verbunden durch Drehgelenke, oder verschiedene Algorithmen, die derzeit in der Robotik eingesetzt werden.

In dieser Arbeit beschreiben wir zwei Methoden, nämlich *Bond-Theorie* und *Möbius-Photogrammetrie*, für das Studium einer bestimmten Klasse von Gestängen, nämlich *Poden*, welche aus zwei starren Körpern gebildet weden, die sogenannte *Basis* und die *Plattform*, verbunden durch mehrere Stangen, die sogenannten *Beine*, die an der Basis und an der Plattform mit sphärische Gelenken befestigt werden. Wir zeigen, wie Bond-Theorie für die Mobilität von Gestängen in Bezug auf die Geometrie von deren Basis und Plattform notwendigen Voraussetzungen liefert, und wie Möbius-Photogrammetrie diese Bedingungen im Fall von Pentapoden, das heißt Gestängen mit fünf Beinen, mit unerwarteter Mobilität verfeinern kann. Die kombinierte Verwendung dieser beiden Methoden, zusammen mit einigen elementaren Fakten aus *Liaison-Theorie* — die die Eigenschaften von zwei algebraische Varietäten, deren Vereinigung ein kompletter Durchschnitt ist, beschreibt — ergibt eine Konstruktion für eine neue Familie von mobilen Hexapoden, das heißt Gestängen mit sechs Beinen. Durch den Einsatz einiger aktueller Ergebnisse über *spectrahedra* — Objekte, die im Zusammenhang mit der semidefiniten Programmierung entstehen — zeigen wir, dass es möglich ist, eine konkrete Instanz eines mobilen Icosapods, nämlich eines Gestänges mit 20 Beinen, zu erhalten.

Der Ansatz, den wir verwenden, ist in erster Linie geometrisch: Ausgangspunkt für die Bond-Theorie ist, jedem Pod eine Teilvarietät einer festen projektiven Varietät — die alle möglichen Konfigurationen des Pods kodiert — zuzuordnen und einige ihrer Punkte, die "Grenzwerte" von Konfigurationen sind, zu studieren; in Möbius-Photogrammetrie, fügen wir zu jeden Tupel von Punkten im reellen Raum eine komplexe Kurve dazu, die das Verhalten der orthogonalen Projektionen dieser Punkte unter Möbius-Transformationen widerspiegelt. In beiden Situationen ist der Schüsselfaktor, das Vorhandensein von komplexen Strukturen in Problemen, die in reellen Situationen — nämlich für die die Eingangsdaten durch reelle Zahlen codiert werden — auftreten, auszunutzen.

ABSTRACT

Kinematics, understood as the study of the mobility of mechanical devices, or *linkages*, has always interested mathematicians, and in particular algebraic geometers. Since the 19th century, algebraic and geometric techniques have been developed to classify such mechanisms and to investigate the existence of new families. Important results have been achieved so far, as for example the classification of mobile closed chains composed of at most five rods and connected by revolute joints, or several algorithms that are currently used in robotics.

In this work we describe two techniques, called *bond theory* and *Möbius photogrammetry*, for the study of a particular class of linkages, namely *pods*, devices constituted of two rigid bodies, called the *base* and the *platform*, connected by several rods, called *legs*, that are attached to the base and the platform via spherical joints. We show how bond theory provides necessary conditions for the mobility of pods in terms of the geometry of its base and platform, and how Möbius photogrammetry can refine these conditions in the case of pentapods, i.e. pods with five legs, with unexpected mobility. The combined use of these two methods, together with some elementary facts from *liaison theory* — which describes the properties of two algebraic varieties whose union is a complete intersection — yields a construction for a new family of mobile hexapods, i.e. pods with six legs. By employing some recent results on *spectrahedra* — objects that arise in the context of *semidefinite programming* — we show that it is possible to obtain a concrete instance of a mobile icosapod, namely a pod with 20 legs.

The approach we use is mainly geometric: the starting point for bond theory is to associate to every pod a subvariety of a fixed projective variety — that encodes all the possible configurations of the pod — and to study some of its points that are "limits" of configurations; in Möbius photogrammetry, we attach to every tuple of points in real space a complex curve reflecting how the orthogonal projections of such points behave under Möbius transformations. In both situations, the driving principle is to exploit the presence of complex structures in problems that arise from real situations — namely for which the input data can be encoded via real numbers.

A Lorenzo,
la persona più coraggiosa che io conosca,
ed alla sua fiducia nella ragione.

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Moving to Linz was not very easy for me, especially at the beginning: it was the first time I lived alone, not so close to my family and my friends in Italy anymore. Luckily for me, I met great friends here in Linz, who helped me to be more comfortable with the new environment, and with whom I shared many joyful moments: Hamid, Jakob, Manuela, Max and Zaf have been great office-mates at RISC, and I would like to thank them for all our discussions about mathematics and politics, for the beers on the boat and in other places in Linz, for the walks through the streets of Istanbul. Later I met Elisa, with whom I enjoyed many coffees and several nice expat discussions about living and working in Austria. Rika joined then the Linz math community, and so I found a new friend: I would like to thank her for being always helpful, for lunches and coffees we had together in these years, and for the discussions we had passing by each other's office.

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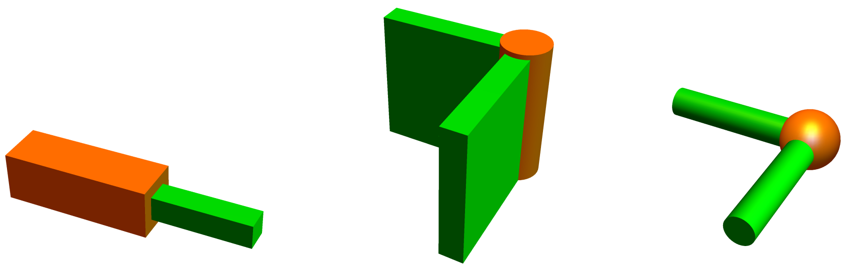
CONTENTS

INTRODUCTION

Kinematics, meant as the study of mechanical devices — also called *linkages* — has been interesting mathematicians and in particular algebraic geometers for centuries. By a mechanical device we mean an object formed by rigid bodies, called *links*, connected by *joints*. Joints constrain the relative motion between two links, and they are in general classified according to the kind of relative motion they allow: we speak of *prismatic* joint if a translation is allowed, *rotational* joints when a rotation around a line is permitted, and *spherical joints* when an arbitrary rotation around a point is allowed (see Figure 1).



(a) A prismatic joint. (b) A rotational joint. (c) A spherical joint.

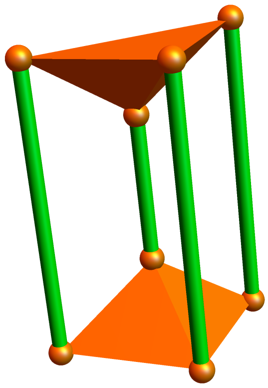
Figure 1: Three different kinds of joints.

Algebraic and geometric techniques have been developed to tackle the many difficult problems that arise in this area, and some of them form the ground on which currently used algorithms are based. For an introductory discussion one may refer to the survey [HS10] or to [WS11].

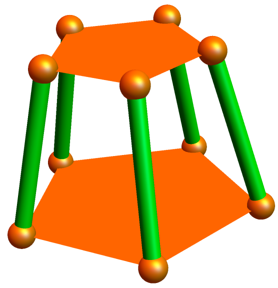
In this work we will focus on a particular type of mechanical devices, called *pods*: a pod is a linkage constituted of two rigid bodies, called the *basis* and the *platform*, that are connected by a number of other rigid bodies, called *legs*; legs are anchored to base and platform via spherical joints. One can distinguish between *planar* and *non-planar* pods, depending on whether or not both the basis and the platform lie each on a plane; examples these kinds of pods are provided in Figure 2.

Given a pod, one can ask whether it is *mobile* or not. By this we mean whether the following condition is satisfied: suppose the basis is fixed, then it is possible to move the platform respecting all constraints imposed by the legs. It is possible to show that pods with n legs are always mobile provided that $n \leqslant 5$, and so the first case for which it is interesting to investigate whether a pod is mobile or not is the one of hexapods. A slightly more refined analysis allows to speak about *degrees of freedom* of a pod, namely to codify "how much" a pod can move. Then the previous statement about mobility follows from the fact that a pod with n legs has at least $6 - n$ degrees of freedom, and equality is obtained in for a general[1] pod. This result determines an interesting class

1 Here by "general" we mean that the cases for which the statement is not true form a proper algebraic subvariety of the space of all possible pods.



(a) A non-planar tetrapod.



(b) A planar pentapod.

Figure 2: Two examples of pods.

of pods containing the one of mobile hexapods, namely the family of pods with unexpected degrees of freedom: pentapods with 2 degrees of freedom, tetrapods with 3 degrees of freedom and so on.

One notices that, when a pod is mobile, all points in the platform where the legs are anchored move on a sphere, whose center is the corresponding anchor point in the base, and whose radius is the length of the leg. From this perspective, investigating the mobility and the degrees of freedom of pods is equivalent to understanding the possible configurations of a rigid body, in which a fixed number of points move on spherical paths. This formulation of the problem was adopted by the French Academy of Science in 1904 to assign the Prix Vaillant: the problem was considered hard, and indeed only partial answers could be provided. Several trivial solutions were known at the time, and the focus was on the existence of non-trivial ones. Two papers, one by Borel and the other by Bricard, were awarded the prize (for more details about the original papers of Borel and Bricard see [Hus00]), but the problem is still object of ongoing research.

This work collects a few contributions whose common denominator is the use of techniques from algebraic geometry in the investigation of the mobility of pods. This idea is not new, and actually dates back to the works of Study ([Stu91]) and of Clifford ([Cli71] and [Cli82]), who introduced algebraic and geometric tools to study direct isometries of three-dimensional Euclidean space (for a nice survey about the work of Study, see [Gfr00]). Since then, many new results have been obtained using the formalism of algebraic geometry: consider for example the computation of the number of possible postures of a chain of six general rotational joints connected by rigid bars, which can be accomplished by a simple computation in the Chow ring of the Study quadric (see [Sel05, Section 11.5.1]); or the computation of the number of configurations of a general hexapod, accomplished in a very elegant way by Ronga and Vust using intersection theory methods (see [RV95]), and by Ranghavan with numerical continuation methods (see [Rag93]), and then investigated also by Wampler (see [Wam96]), Mourrain (see [Mou93]) and Husty (see [Hus96]);

or the classification of closed chains with five rotational joints, completed by Karger (see [Kar98]) and then re-proved by Hegedüs, Schicho and Schröcker within the framework of the so-called *bond theory* (see [HSS13]).

A GLIMPSE OF THE STATE OF THE ART

The sheer number of publications dealing with pods, and the huge variety of families of mobile ones produced in the last century makes difficult to provide an overview of all the achieved results. Quadripods with unexpected mobility are studied in [Naw14c]. Pentapods with unexpected mobility have been classified by Nawratil and Schicho (see [NS15a], [NS15b] and [NS16]) using results that will be presented in this work, and so we do not spend time on them and we refer the interested reader to the three cited papers, and the references therein. Instead, the classification of mobile hexapods seems, at the moment, very far from being complete, so we report some of the families that have been constructed so far. The credit for the creation of the following list should be entirely given to Georg Nawratil, and I would like to thank him for sharing this piece of information with me; the list is reported here for the lack — to the author's knowledge — of a reference and under his permission.

We start mentioning the two papers by Borel and Bricard ([Bor08] and [Bri06]), which contain families of pods that are not covered by the more recent works we refer to in the following, and may be source of inspiration for new research in this field. Moreover, one can refer to [HK02] and to [Naw12b] for general surveys on this topic.

To introduce the first family, we recall the notion of *architecturally singular* hexapods. Given a hexapod in a prescribed configuration, there are in general 6 degrees of freedom for the leg lengths: this means that, in general, when a hexapod is in a given configuration, there is a 6-dimensional variety of ways to perturb infinitesimally the leg lengths and obtain another hexapod. There are cases, however, when the number of degrees of freedom is strictly smaller than 6. Merlet in [Mer89] proved that this happens if and only if the lines supporting the legs of the hexapod belong to a *linear line complex*, namely to the intersection of Grassmannian of lines in three dimensional space with a hyperplane. There are many papers dealing with the case of architecturally singular pods, for example [HK00], [RM98] and [Naw08] for the planar case, and [Kar08a] and [Naw09] for the non-planar case. Duporcq stated in [Dup98] that — translated in the language of hexapods — once we are given five pairs of base and platform points, it is possible to obtain an architecturally singular hexapod whose set of configuration equals the set of configurations of the given pentapod, and is therefore mobile. However, the claim of Duporcq, in this very formulation, is unfortunately not correct as pointed out by Nawratil in [Naw14b]; there the author thoroughly inspects the original paper by Duporcq and provides a corrected version of the statement. This gives a first family of mobile hexapods.

A second family was described by Bricard in [Bri97], and is constituted of hexapods in which three pairs of base points and three pairs of platform points are identified, but there are still six legs, as in Figure 3. Since in this way we obtain a polyhedron with 8 faces, these special mobile hexapods are

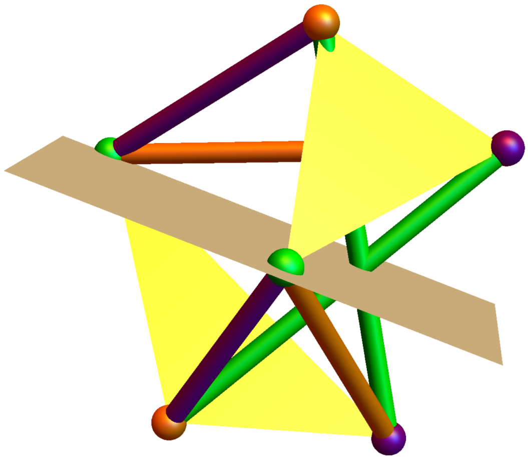


Figure 3: A Bricard flexible octahedron of Type II: two pairs of base and platform points (the orange and the purple ones) are symmetric with respect to a plane passing through the third pair (the green one).

called *flexible octahedra*. Bricard distinguished three types of flexible octahedra, depending on whether or not the base and platform points present some symmetries. For a short and nice account on these structures one can refer to [Gol78], and for more detailed information to [Naw11a] and to [Naw11b], and to the references therein.

A third family of mobile hexapods, already described by Bricard and by Borel in theis papers for the Prix Vaillant, admits a *Schoenflies motion*, namely the platform moves by a translation and a rotation along a fixed axis. Further details about this family are provided in [HZM94].

The class of planar mobile hexapods for which the base and the platform are related by a projectivity has been studied in [Naw12a] and in the subsequent papers [Naw13b] and [Naw13c], showing that the mobile ones that are not architecturally singular admit only translational motions.

Mobile hexapods for which the base and the platform are congruent, namely only differ by an isometry of $\mathbb{R}^3$, are studied in [Naw14a]. Mobile hexapods for which we allow that base and platform can differ, after an isometry, also by a scaling factor are studied in [Naw13d].

Using a completely new technique, Geiß and Schreyer constructed in [GS09] a new family of mobile hexapods by solving appropriate equations over finite fields, and then lifting the obtained results to a number field, for which it was then ensured the existence of a real embedding via an *ad hoc* computation.

Another family of mobile icosapods, already discussed by Borel and Bricard, is the one in which the motion of the platform is *line-symmetric*, namely obtained as the sequence of rotations of the platform by 180° along a ruling of a ruled surface: the non-planar case was considered by Husty and Zsombor-Murray in [HZM94] and by Hartman in [Har95], while the planar case was

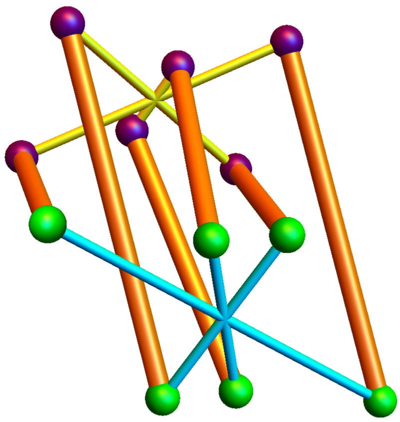


Figure 4: A point-symmetric hexapod: the lines (cyan) between three pairs of points in the base (green) meet in a point, and the same happens for the lines (yellow) between three pairs of points in the platform (purple).

investigated by Karger in [Kar08b] and in [Kar08c]. For further and more detailed references, see [GNSS16, Section 1.2].

Eventually, we mention one last family of mobile hexapods, called *point-symmetric hexapods*, obtained by Nawratil in [Naw14d] following ideas from Dietmaier in [Die96] and using also some results that will be presented in this work. A hexapod is point-symmetric when the base is constituted of pairs of points that are symmetric with respect to one fixed point, and the same happens for the platform, as shown in Figure 4.

STRUCTURE OF THE THESIS

In Chapter 1 we give a rigorous definition of the main concepts related to the mobility of pods. We start by introducing a projective compactification of the group of direct isometries of three-dimensional Euclidean space, called the *conformal embedding* (Definition 1.5). After considering how direct isometries act on this projective variety, we define the main object of our work, namely *bonds* (Definition 1.16). The rest of the chapter is devoted to describing the various possible kinds of bonds, and to associate to each of them a "geometric meaning" in terms of the geometry of base and platform of a pod (Propositions 1.21, 1.23, and 1.25). Eventually necessary conditions for the mobility of a pod are inferred starting from the analysis of bonds (Theorem 1.27 and Theorem 1.28).

Chapter 2 takes the cue from the findings of Chapter 1. In particular, we focus on one of the necessary conditions for the mobility of a pod listed in Theorem 1.27: the existence of a pair of orthogonal projections from $\mathbb{R}^3$ to $\mathbb{R}^2$, one for the base and one for the platform, such that the images of the base points and of the platform forms differ by an inversion, or a similarity. This suggests to consider the following problem: given a tuple of points in $\mathbb{R}^3$, understand the behavior of its orthogonal projections on $\mathbb{R}^2$, up to inversions and similarities. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, we notice that inversions and similarities are special cases of Möbius transformations, that in turn constitute the automorphisms of the projective line $\mathbb{P}^1_{\mathbb{C}}$ in which $\mathbb{C}$ can be compactified. From this perspective, once we fix an n-tuple in $\mathbb{R}^3$ we obtain a map associating to every oriented direction in $\mathbb{R}^3$ —- namely, to every point in the unit sphere S^2 — an element in the moduli space M_n of n points in $\mathbb{P}^1_{\mathbb{C}}$. Since the unit sphere S^2 can be identified with an algebraic curve over $\mathbb{C}$, we get a morphism from such a curve to M_n; we call it the *Möbius map*, and its image the *Möbius curve* associated to an n-tuple (Definition 2.5). With this terminology, the previous necessary condition for the mobility of a pod can be rephrased by saying that the Möbius curves of the base and the platform points intersect non-trivially. The study of some properties of Möbius curves allows to obtain refined results on pentapods with unexpected mobility (Theorem 2.22).

Chapter 3 exploits Möbius photogrammetry in the case of hexapods. This time, instead of looking for necessary conditions for the mobility of hexapods — which are in general very difficult to find — we produce a maximal family of mobile hexapods. By this we mean that such a family cannot be obtained as a particular case of another one depending on more parameters. The key observation for the construction of this family is the following: a general 6-tuple of points in $\mathbb{R}^3$ admits a Möbius curve of degree 6, to which it is possible to associate another curve of degree 6 via a procedure called *liaison* (Section 3.1). This second curve shares many properties satisfied by Möbius curves, and in concrete examples it is possible to show that is actually is the Möbius curve of a 6-tuple of points in $\mathbb{R}^3$. In this way, starting from 6 general base points we can construct 6 platform points; in concrete instances one observes that, once we scale the platform in a suitable way, there is a three dimensional set of leg lengths that determine a mobile hexapod. The maximality of such a family is proved by showing that some invariant, called the *conformal degree* (Definition 1.9) attains the maximum for the elements of the family (Theorem 3.17).

Chapter 4 deals with an extremal case of mobile pods, namely icosapods, objects constituted of 20 legs. It is known that 20 is the maximal finite number of legs for a mobile pod; by this we mean that if a pod is mobile and admits more than 20 legs, then it is possible to introduce an infinite number of legs without changing the mobility. Because of this fact, which we re-prove using the notions we introduced so far (Proposition 4.4), mobile icosapods are of particular interest among pods. Already in 1904 Borel proposed a method to obtain mobile icosapods, but he was not able to show that the system of equations he set up admits real solutions, and hence leads to pods whose base and platform points have real coordinates. We prove that, under mild assumptions, every mobile icosapod arises as an instance of Borel's construction (Theorem 4.14), and we exhibit a concrete example of such a pod exploiting the theory of the so-called *spectrahedra* (Example 4.21).

A PRIMER IN PROJECTIVE ALGEBRAIC GEOMETRY

We provide a very short summary of some of the main concepts in projective geometry, which we will freely use in the next chapters. An excellent reference for the material covered here is [Arr15]. Other standard introductory books include [Rei88] and [Sha13]; for more advanced readers the classical reference is [Har77], and an excellent new one is [Vak15]. More specific material will be introduced in the intermezzos between the chapters.

Definition (Affine sets). Let $f_1, \dots, f_r$ be polynomials in $\mathbb{C}[x_1, \dots, x_n]$. Their *zero set*, denoted $V(f_1, \dots, f_r)$, is the set

$$\{a \in \mathbb{C}^n : f_1(a) = \cdots = f_r(a) = 0\}.$$

Every subset of $\mathbb{C}^n$ of the form $V(f_1, \dots, f_r)$ is called an *affine set*.

Definition (Affine Zariski topology). Affine sets satisfy the axioms of closed sets for a topology on $\mathbb{C}^n$, called the *Zariski topology*.

Definition (Projective space). Consider on $\mathbb{C}^{n+1} \setminus \{0\}$ the equivalence relation $a \sim b$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda a = b$. We define the *projective space* $\mathbb{P}^n_{\mathbb{C}}$ as the set $\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\sim$ of equivalence classes of $\sim$.

Definition (Projective sets). Let $f_1, \dots, f_r$ be homogeneous polynomials in $\mathbb{C}[x_0, x_1, \dots, x_n]$. Their *zero set*, denoted $V(f_1, \dots, f_r)$, is the set

$$\{p \in \mathbb{P}^n_{\mathbb{C}} : f_1(p) = \cdots = f_r(p) = 0\}.$$

Every subset of $\mathbb{P}^n_{\mathbb{C}}$ of the form $V(f_1, \dots, f_r)$ is called a *projective set*.

Definition (Projective Zariski topology). Projective sets satisfy the axioms of closed sets for a topology on $\mathbb{P}^n_{\mathbb{C}}$, called the *Zariski topology*.

Definition (Irreducible spaces). A topological space X is called *irreducible* if X cannot be written as $X = X_1 \cup X_2$ where X_1 and X_2 are closed proper non-empty subsets of X.

Definition (Affine and projective varieties). An affine irreducible set is called an *affine variety*; *projective varieties* are defined in a similar way.

Definition (Irreducible components). Every affine (or projective) set can be decomposed as the union of irreducible subsets that are maximal with respect to inclusion. Such subsets are called *irreducible* components.

Definition (Ideals of affine and projective sets). Let $X \subseteq \mathbb{C}^n$ be an affine set. The set

$$I(X) = \{f \in \mathbb{C}[x_1, \dots, x_n] : f(a) = 0 \ \forall a \in X\}$$

is an ideal, called the *ideal of* X.

Similarly, one defines the *ideal* of a projective set; in this case the ideal is homogeneous.

Definition (Quasi-projective varieties). A *quasi-projective variety* is any (Zariski) open set of a projective variety.

Remark. An affine/projective set is irreducible if and only if its ideal is prime.

Proposition (Hilbert polynomial). *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a projective set, and for every $k \in \mathbb{N}$ define*

$$f(k) = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x_1, \dots, x_n]}{I(X)}\right)_k,$$

where with $(\cdot)_k$ we denoted the homogeneous component of degree k. Then there exists $\bar{k} \in \mathbb{N}$ and a polynomial $\mathrm{HP}_X \in \mathbb{Q}[t]$ such that $f(k) = \mathrm{HP}_X(k)$ for all $k \geqslant \bar{k}$. The polynomial HP_X is called the Hilbert polynomial *of* X.

Definition (Dimension and degree). Let X be a projective variety and write $HP_X(t) = a_r t^r + \cdots$. We define the *dimension* of X as the degree r of the Hilbert polynomial of X. We define the *degree* of X as the number $a_r\, r!$. If $X \subseteq \mathbb{P}^n_{\mathbb{C}}$, the *codimension* of X in $\mathbb{P}^n_{\mathbb{C}}$ is the number $n - \dim X$.

Definition (Linear varieties). A projective variety $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is called a *linear variety* if its ideal is defined by linear polynomials. For linear varieties there is an intuitive notion of dimension coming from linear algebra: if the number of linearly independent linear polynomials in the ideal is r, then X has dimension $n - r$. This notion of dimension coincides with the one previously defined via the Hilbert polynomial.

Proposition. *Let X be a projective variety. Then*

a. the dimension of X coincides with the longest length of chains $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_r = X$ of subvarieties of X; such number coincides also with the smallest number r such that a general linear subvariety of codimension $r+1$ does not intersect X;

b. the degree of X coincides with the (finite) number of intersections of X and a general linear subvariety of codimension $\dim X$.

Definition (Regular map). Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $Y \subseteq \mathbb{P}^m_{\mathbb{C}}$ be quasi-projective varieties and let $f\colon X \longrightarrow Y$ be a function. Then f is called a *regular map*, or a *morphism*, if there exist homogeneous polynomials $f_0, \dots, f_m$ of the same degree such that $f(x) = \big(f_0(x) : \cdots : f_m(x)\big)$ for all $x \in X$.

A bijective regular map whose inverse is also a regular map is called an *isomorphism*.

Definition (Rational map). Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $Y \subseteq \mathbb{P}^m_{\mathbb{C}}$ be quasi-projective varieties. A *rational map* $f\colon X \dashrightarrow Y$ is a regular map $f\colon U \longrightarrow Y$ where U is a (Zariski) open subset of X. The notation $\dashrightarrow$ is used to mean that f is not define on the whole X.

A rational map admitting an inverse[2] that is also a rational map is called a *birational map*.

Proposition. *Let $f\colon X \longrightarrow Y$ be a regular map between quasi-projective varieties. Then* $\dim(X) \geqslant \dim f(X)$. *Moreover, birational maps — hence, a fortiori, isomorphisms — preserve dimension.*

Definition (Rational varieties). A quasi-projective variety X is called *rational* if there exists a birational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.

Definition (Veronese embedding). Given $n, d \in \mathbb{N} \setminus \{0\}$, consider the map

$$\begin{array}{rccc} \nu_{n,d}\colon & \mathbb{P}^n_{\mathbb{C}} & \longrightarrow & \mathbb{P}^{\binom{n+d}{d}-1}_{\mathbb{C}} \\ & (x_0 : \cdots : x_n) & \mapsto & \left(\begin{array}{c} \text{monomials of degree } d \\ \text{in } x_0, \dots, x_n \end{array} \right) \end{array}$$

One can prove that $\nu_{n,d}$, called the *Veronese embedding*, is always an isomorphism on its image. The image of $\nu_{n,d}$ is called a *Veronese variety*.

2 Here by "inverse" we mean the following: if $f\colon X \dashrightarrow Y$ $g\colon Y \dashrightarrow X$ are rational functions, then f is defined on open subset $U \subseteq X$ and g is defined on an open subset $V \subseteq Y$; we say that g is an inverse for f is V intersects non-trivially the image of f, and the composition $g \circ f$ is the identity on $f^{-1}(V)$.

Definition (Segre embedding)**.** Let $n, m \in \mathbb{N}$ and consider the Cartesian product $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$. The function

$$\begin{array}{rccc} \sigma_{n,m}\colon & \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} & \longrightarrow & \mathbb{P}^{(n+1)(m+1)-1}_{\mathbb{C}} \\ & (x_0 : \cdots : x_n) \times (y_0 : \cdots : y_m) & \mapsto & (\cdots : x_i y_j : \cdots)_{\substack{0 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m}} \end{array}$$

is injective, and is called the *Segre embedding*. The image of the Segre embedding is a projective variety, called the *Segre variety*, and in this way can endow the product of two projective spaces with the structure of a projective variety.

Definition (Smoothness)**.** Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a quasi-projective variety of dimension r. We say that a point $P \in X$ is *smooth*, or *non-singular*[3], if the following condition is satisfied: let $I(X) = (f_1, \ldots, f_k)$ and form the Jacobian matrix $J_X = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \leqslant i \leqslant k \\ 0 \leqslant j \leqslant n}}$, where $x_0, \ldots, x_n$ are coordinates on $\mathbb{P}^n_{\mathbb{C}}$; then we ask that the matrix $J_X(P)$ has rank at least $n - r$.

A quasi-projective variety is called *smooth* if all its points are smooth. Otherwise it is called *singular*.

Definition (Curve)**.** A projective variety of dimension 1 is called a *projective curve*. If X is a curve, one defines its *arithmetic genus* $p_a(X)$ to be the number $1 - HP_X(0)$.

Definition (Surface)**.** A projective variety of dimension 2 is called a *projective surface*.

3 These two concepts have, in general, different meanings but for our purposes they can be taken as synonyms.

1 BOND THEORY FOR PODS

This chapter introduces a technique, called *bond theory*, that has already been used in the framework of the classification of linkages constituted of closed chains of bars connected by rotational links (see the foundational works [HSS13] and [HLSS15], and also [Li14], [LS15] and [ALS15]); the first appearance of this technique in the study of pods is in [Naw13a]. The idea is to provide a suitable compactification of the group of direct isometries of $\mathbb{R}^3$ and use it to describe the admissible configurations of a pod (Section 1.1); then one analyzes what points in the compactification may arise as limits of admissible configurations (Section 1.2), and from this one determines necessary conditions on the geometry of the base and the platform of mobile pods (Section 1.3).

1.1 THE CONFORMAL EMBEDDING

We start our investigations by introducing an object that will accompany us from now on: a compactification of the group SE_3 of direct isometries of $\mathbb{R}^3$ well-suited for analyzing mobility of pods. The content of this chapter is mainly taken from [GNS15a].

Why are we interested in direct isometries when studying pods? One can consider a pod as a pair of tuples of points in $\left(\mathbb{R}^3\right)^n$, representing the base and the platform. One way of codifying the set of configurations that can be achieved by this pod is to consider the base as fixed, and look for all possible tuples of platform points that satisfy the constrains imposed by the legs. One notices that these tuples can be obtained one from the other by applying a direct isometry: hence the idea to fix once for all one of such tuples, and consider the set of direct isometries that send such configuration to the other admissible ones. The understanding of the possible configurations of a pod is translated into the analysis of equations — determined by the presence of legs — on the algebraic group SE_3 of direct isometries of $\mathbb{R}^3$. Such group is not complete as a complex variety, and so one can try to find suitable projective varieties, in which to embed SE_3 as a dense subset, to unravel the structure of configuration sets of pods.

The compactification of SE_3 as the Study quadric in $\mathbb{P}^7_{\mathbb{R}}$ seems to be the most widely used when approaching kinematics' problems using tools from algebraic geometry (see for a reference [Sel05, Chapter 11]). By this we mean that the elements of SE_3 are in bijective correspondence with the real points of the quasi-projective variety

$$S = \left\{ (\vec{e}, \vec{f}) \in \mathbb{P}^7_{\mathbb{R}} \ : \ \begin{array}{l} e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 = 0 \\ e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0 \end{array} \right\},$$

where we take $(\vec{e}, \vec{f}) = (e_0, \ldots, e_3, f_0, \ldots, f_3)$ as coordinates in $\mathbb{P}^7_{\mathbb{R}}$. The Zariski closure of S is the quadric

$$e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 \ = \ 0,$$

which is called the Study quadric. The geometry of this variety is rich and well-understood, and the kinematic interpretation of its properties has led to many important results (many of the works cited so far rely on these methods). This, however, is not going to be the compactification of SE_3 we will use.

In order to understand one possible reason to use another compactification when studying pods, let us consider two points $p = (a, b, c)$ and $P = (A, B, C)$ in $\mathbb{R}^3$ and a real number d. The key condition in this setting is the constrain on isometries describing those elements $\sigma \in SE_3$ that send the point p to a point distant d from P; such constrain is called the *spherical condition* imposed on SE_3 by the points p and P and the distance d. In other words, we focus on the set

$$\big\{\sigma \in SE_3 \,:\, \|\sigma(p) - P\| = d\big\}.$$

If we express the spherical condition in terms of the Study coordinates $(\vec{e}, \vec{f})$ we obtain (see [Naw13a, Equation (2)])

$$\begin{aligned}
(a^2+b^2+c^2+A^2+B^2+C^2-d^2)\,(\mathbf{e}_0^2+\mathbf{e}_1^2+\mathbf{e}_2^2+\mathbf{e}_3^2)&\\
-2(a\,A+b\,B+c\,C)\,\mathbf{e}_0^2-2(a\,A-b\,B-c\,C)\,\mathbf{e}_1^2&\\
+2(a\,A-b\,B+c\,C)\,\mathbf{e}_2^2+2(a\,A+b\,B-c\,C)\,\mathbf{e}_3^2&\\
+4(c\,B-b\,C)\,\mathbf{e}_0\mathbf{e}_1-4(c\,A-a\,C)\,\mathbf{e}_0\mathbf{e}_2+4(b\,A-a\,B)\,\mathbf{e}_0\mathbf{e}_3&\\
-4(b\,A+a\,B)\,\mathbf{e}_1\mathbf{e}_2-4(c\,A+a\,C)\,\mathbf{e}_1\mathbf{e}_3-4(c\,B+b\,C)\,\mathbf{e}_2\mathbf{e}_3&\\
+4(a-A)(\mathbf{e}_0\mathbf{f}_1-\mathbf{e}_1\mathbf{f}_0)+4(b-B)(\mathbf{e}_0\mathbf{f}_2-\mathbf{e}_2\mathbf{f}_0)&\\
+4(c-C)(\mathbf{e}_0\mathbf{f}_3-\mathbf{e}_3\mathbf{f}_0)+4(a+A)(\mathbf{e}_3\mathbf{f}_2-\mathbf{e}_2\mathbf{f}_3)&\\
+4(b+B)(\mathbf{e}_1\mathbf{f}_3-\mathbf{e}_3\mathbf{f}_1)+4(c+C)(\mathbf{e}_2\mathbf{f}_1-\mathbf{e}_1\mathbf{f}_2)&\\
+4(\mathbf{f}_0^2+\mathbf{f}_1^2+\mathbf{f}_2^2+\mathbf{f}_3^2)=0,&
\end{aligned} \tag{1}$$

where we highlighted in bold the variables e_i and f_j to distinguish them from the parameters a, b, c and A, B, C. One immediately notices that this is a quadratic condition in the Study coordinates. Understanding the possible configurations of a pod becomes then equivalent to understanding the solution set of a system of quadratic equations on the Study quadric. The idea underlying the compactification we are going to introduce is to try to simplify the system of equations we need to deal with (maybe at the expense of the number of variables involved).

1.1.1 *A compactification tuned on pods*

A possibility is to impose, by a suitable change of coordinates, that all quadratic terms appearing in Equation (1) become linear. We consider therefore the following change of variables:

$$\begin{cases} n = f_0^2 + f_1^2 + f_2^2 + f_3^2, & \\ c_{ij} = e_i e_j & \text{for } 0 \leqslant i \leqslant j \leqslant 3, \\ b_{ij} = e_i f_j - e_j f_i & \text{for } 0 \leqslant i < j \leqslant 3. \end{cases} \tag{2}$$

Equation (2) defines a rational map

$$\Phi \colon \mathbb{P}^7_{\mathbb{R}} \dashrightarrow \mathbb{P}^{16}_{\mathbb{R}}, \tag{3}$$

where on $\mathbb{P}^{16}_{\mathbb{R}}$ we take coordinates n, b_{ij} and c_{ij}, which is a morphism on S.

Definition 1.1. We define $X_{\mathbb{R}}$ to be the Zariski closure in $\mathbb{P}^{16}_{\mathbb{R}}$ of the image of S via the rational map Φ defined by Equation (2).

In Proposition 1.2 we will prove that the map Φ embeds SE_3 into $\mathbb{P}^{16}_{\mathbb{R}}$, this implying that $X_{\mathbb{R}}$ is a compactification of SE_3. Before doing that, we introduce a piece of notation that will be helpful in dealing with the map Φ.

Notation. From now on, denote

$$\vec{b} = (b_{01} : \cdots : b_{23}) \quad \text{and} \quad \vec{c} = (c_{00} : \cdots : c_{33}).$$

The defining equations for $\{b_{ij}\}$ and $\{c_{ij}\}$ can be written as:

$$\vec{b} = \vec{e} \wedge \vec{f} \quad \text{and} \quad \vec{c} = \vec{e} \odot \vec{e},$$

where we denote by $\wedge$ the wedge product and by $\odot$ the symmetric tensor product.

From this compact description of Φ we can guess some equations for $X_{\mathbb{R}}$:

- if we take $\vec{e}$ to be the coordinates of $\mathbb{P}^3_{\mathbb{R}}$, then $\vec{c}$ satisfies the equations of the Veronese embedding of $\mathbb{P}^3_{\mathbb{R}}$ by quadrics;
- being by definition a rank one skew-symmetric tensor, $\vec{b}$ must satisfy the Plücker equation $\vec{b} \wedge \vec{b} = 0$ for the Grassmannian $G(1,3)$;
- from the definition we immediately see that $\vec{e} \wedge \vec{b} = 0$, hence by expressing $\vec{e}$ in terms of $\vec{c}$ we get equations for $X_{\mathbb{R}}$;
- $(e_0^2 + \cdots + e_3^2)(f_0^2 + \cdots + f_3^2) = \sum_{0 \leqslant i < j \leqslant 3} (e_i f_j - e_j f_i)^2$, so we obtain an equation for n.

A direct Gröbner basis computation shows that these are exactly the equations defining $X_{\mathbb{R}}$. Moreover such computation shows that $X_{\mathbb{R}}$ is a variety of dimension 6 and degree 40.

Proposition 1.2. *The map* $SE_3 \longrightarrow \mathbb{P}^{16}_{\mathbb{R}}$ *obtained as the composition of the immersion* $SE_3 \hookrightarrow S$ *followed by* Φ *from Equation* (3) *is injective. Hence* $X_{\mathbb{R}}$ *is a compactification of* SE_3*.*

Proof. We exhibit a morphism Ψ defined on $\Phi(S)$ that is the inverse of $\Phi|_S$. As remarked before, the map $\vec{e} \mapsto \vec{c} = \vec{e} \odot \vec{e}$ is exactly the second Veronese embedding of $\mathbb{P}^3_{\mathbb{R}}$, hence we can take its inverse for the first components of Ψ.

We are left to prove that, given a vector $(n, \vec{b}, \vec{c}) \in \Phi(Y)$, there exists a vector $\vec{f} \in \mathbb{C}^4$, unique up to scalar multiplication, that satisfies:

- $(\vec{e}, \vec{f}) \in S$, where $\vec{e}$ is the previously found vector via the Veronese inverse;
- $\Phi(\vec{e}, \vec{f}) = (n, \vec{b}, \vec{c})$.

Therefore we want an $\vec{f}$ such that $\vec{b} = \vec{e} \wedge \vec{f}$ and $n = f_0^2 + \cdots + f_3^2$. Hence we look at the set of solutions of the linear system $\vec{e} \wedge \vec{x} = \vec{b}$, which is of the form $\{\vec{f}_* + \lambda \vec{e} : \lambda \in \mathbb{C}\}$, where $\vec{f}_*$ is any particular solution. We find our desired $\vec{f}$ by determining the unique λ that ensures $(\vec{e}, \vec{f}_* + \lambda \vec{e}) \in S$. To do so, let us denote by Q the polarization of standard Euclidean quadratic form on $\mathbb{C}^4$:

$$Q(\vec{x}, \vec{y}) = x_0\, y_0 + x_1 y_1 + x_2\, y_2 + x_3\, y_3 \qquad \text{for } \vec{x}, \vec{y} \in \mathbb{C}^4.$$

Then the condition $e_0\, f_0 + \cdots + e_3\, f_3 = 0$, which ensures $(\vec{e}, \vec{f}) \in S$, can be rephrased as $Q(\vec{e}, \vec{f}) = 0$. So

$$Q(\vec{e}, \vec{f}_* + \lambda\, \vec{e}) = 0 \quad \Leftrightarrow \quad \lambda \;=\; -\frac{Q(\vec{e}, \vec{f}_*)}{Q(\vec{e}, \vec{e})}$$

and the last expression is always well-defined since $\vec{e} \neq 0$. For this choice of $\vec{f}$, we have

$$\begin{aligned} f_0^2 + \cdots + f_3^2 \;&=\; Q(\vec{f}, \vec{f}) \;=\; Q(\vec{f}_* + \lambda\, \vec{e}, \vec{f}_* + \lambda\, \vec{e}) \\ &=\; Q(\vec{f}_*, \vec{f}_*) - 2\lambda\, Q(\vec{e}, \vec{f}_*) + \lambda^2\, Q(\vec{e}, \vec{e}) \\ &=\; \frac{Q(\vec{f}_*, \vec{f}_*)\, Q(\vec{e}, \vec{e}) - Q(\vec{e}, \vec{f}_*)^2}{Q(\vec{e}, \vec{e})} \end{aligned}$$

Now, using the fact that by construction $\vec{e} \wedge \vec{f}_* = \vec{b}$ and by explicit computations, one finds that the numerator of the previous expression equals $\sum b_{ij}^2$. So

$$(e_0^2 + \cdots + e_3^2)(f_0^2 + \cdots + f_3^2) - \sum_{0 \leqslant i < j \leqslant 3} b_{ij}^2 \;=\; 0,$$

and this proves that $n = f_0^2 + \ldots + f_3^2$.

In this way we see that the vector $\vec{f}$ we were looking for is unique, so the statement follows. □

Remark 1.3. In the previous proof, we were able to reconstruct $\vec{e}$ and $\vec{f}$ only using $\vec{b}$ and $\vec{c}$, namely n did not play any role. This means that if we consider the projection on the last 16 coordinates, namely the projection π from the point $(1 : 0 : \cdots : 0)$

$$\pi\colon \mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{15}_{\mathbb{C}} \qquad (n : \vec{b} : \vec{c}) \;\mapsto\; (\vec{b} : \vec{c}),$$

then this induces a rational map on $X_{\mathbb{R}}$ (one can check that the center of the projection lies on $X_{\mathbb{R}}$), and we can define $X^0_{\mathbb{R}} = \overline{\mathrm{image}(\pi|_{X_{\mathbb{R}}})}$. By our considerations the following diagram commutes

$$\begin{tikzcd} SE_3 \arrow[r, "\Phi"] \arrow[dr, "\Phi_0"'] & X_{\mathbb{R}} \arrow[d, dashed, "\pi|_{X_{\mathbb{R}}}"] \\ & X^0_{\mathbb{R}} \end{tikzcd}$$

and Φ and Φ_0 are both immersions. A direct computation shows that the variety $X^0_{\mathbb{R}}$ has degree 20.

We could continue our analysis in this setting, but in our investigations we found that one can use a more direct approach (that already appeared in [Mou96, Section 5] and in [Sel13]) to obtain the same variety $X_{\mathbb{R}}$. This different perspective allows for easier computations, and makes some arguments more transparent. This is why from now on we will use this other formulation.

Any isometry of $\mathbb{R}^3$ can be written as a pair (M, y), where $M \in SO_3$ is an orthogonal matrix, and $y \in \mathbb{R}^3$ is the image of the origin under the isometry. We define $x := -M^t y = -M^{-1} y$ and $r := \langle x, x \rangle = \langle y, y \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. The isometry (M, y) is then associated to a point in $\mathbb{P}^{16}_{\mathbb{R}}$ with coordinates

- $m_{11}, \ldots, m_{33}$ (the entries of the matrix),
- x_1, x_2, x_3 (the coordinates of x),
- y_1, y_2, y_3 (the coordinates of y),
- r (the square of the norm of x or y),
- h (a homogenization coordinate).

The elements of the group SE_3 are then defined by the inequality $h \neq 0$ and by the equations

$$\begin{aligned} MM^t = M^t M = h^2 \cdot \mathrm{id}_{\mathbb{R}^3}, &\quad \det(M) = h^3, \\ M^t y + hx = 0, &\quad Mx + hy = 0, \\ \langle x, x \rangle = \langle y, y \rangle = rh. & \end{aligned} \tag{4}$$

Notice that not all equations are needed: for instance, $M^t y + hx = 0$ is a consequence of the other equations and the inequality; we list them only for symmetry reasons.

The following proposition links this construction to the initial one, and shows that they are essentially the same object:

Proposition 1.4. *Let $\widetilde{X}_{\mathbb{R}} \subseteq \mathbb{P}^{16}_{\mathbb{R}}$ be the projective variety defined by Equations (4). There is a projective automorphism of $\mathbb{P}^{16}_{\mathbb{R}}$ into itself identifying $\widetilde{X}_{\mathbb{R}}$ with $X_{\mathbb{R}}$.*

Proof. We start relating the coordinates h and m_{ij} to the coordinates $e_0, \ldots, e_3$ in $\mathbb{P}^7_{\mathbb{C}}$ where the Study quadric lives. We consider the bijection between points $(e_0 : e_1 : e_2 : e_3) \in \mathbb{P}^3_{\mathbb{R}}$ and orthogonal matrices (see [GPS01, Section 4.5]) given by

$$\begin{gathered} (e_0 : e_1 : e_2 : e_3) \\ \updownarrow \\ \frac{1}{e_0^2+e_1^2+e_2^2+e_3^2} \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2e_1e_2-2e_0e_3 & 2e_0e_2+2e_1e_3 \\ 2e_1e_2+2e_0e_3 & e_0^2-e_1^2+e_2^2-e_3^2 & -2e_0e_1+2e_2e_3 \\ -2e_0e_2+2e_1e_3 & 2e_0e_1+2e_2e_3 & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix} . \end{gathered} \tag{5}$$

Equation (5) shows then that one can perform a linear change of coordinates $\big(h : (m_{ij})_{ij}\big) = A \cdot (c_{ij})_{ij}$, where A is the 10×10 matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

The coordinates $(b_{ij})_{ij}$ and y are connected by the equations

$$\begin{cases} y_1 = b_{01} - b_{23}, \\ y_2 = b_{02} + b_{13}, \\ y_3 = b_{03} - b_{12}, \end{cases} \tag{7}$$

and similar relations hold for $(b_{ij})_{ij}$ and x. Eventually, we have that the coordinates n and r in the two $\mathbb{P}^{16}_{\mathbb{R}}$ play essentially the same role in the two compactifications, and this concludes the proof. □

In this new setting, the spherical condition $\|\sigma(p) - P\| = d$ can be expressed by:

$$\begin{aligned} d^2 &= \langle Mp + y - P, Mp + y - P\rangle \\ &= \langle Mp, Mp\rangle + 2\langle Mp, y\rangle + r + \langle P, P\rangle - 2\langle Mp, P\rangle - 2\langle y, P\rangle \\ &= \langle p, p\rangle + \langle P, P\rangle + r + 2\langle p, M^t y\rangle - 2\langle Mp, P\rangle - 2\langle y, P\rangle \\ &= \langle p, p\rangle + \langle P, P\rangle + r - 2\langle p, x\rangle - 2\langle y, P\rangle - 2\langle Mp, P\rangle. \end{aligned} \tag{8}$$

We remark again that, by using this compactification of SE_3, we turn the problem of understanding the constrains imposed by the presence of legs on isometries that determine admissible configurations of a pod into the problem of understanding linear equations on $X_{\mathbb{R}}$. To investigate this we will make use of algebraic geometry, and to do so it turns out to be useful to extend our ground field to the complex numbers.

Definition 1.5. We define the variety $X \subseteq \mathbb{P}^{16}_{\mathbb{C}}$ to be the complexification — that is, the base change under the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ — of $\widetilde{X}_{\mathbb{R}}$. This means that X is defined in $\mathbb{P}^{16}_{\mathbb{C}}$ by the same equations that define $X_{\mathbb{R}}$ in $\mathbb{P}^{16}_{\mathbb{R}}$. Hence X is a projective variety in $\mathbb{P}^{16}_{\mathbb{C}}$ of complex dimension 6 and degree 40. The embedding $SE_3 \hookrightarrow X$ is called the *conformal embedding*.

Remark 1.6. From the definition of the variety X we see that there is a natural projection from $\mathbb{P}^{16}_{\mathbb{C}}$ to $\mathbb{P}^{9}_{\mathbb{C}}$ sending

$$(h : M : x : y : r) \mapsto (h : M).$$

At the level of isometries, this corresponds to mapping an isometry to its linear part, namely a rotation.

From the proof of Proposition 1.4, it follows that the image X_m of X under this map is a variety isomorphic to the Veronese variety obtained as the image of the embedding $\mathbb{P}^3_{\mathbb{C}} \hookrightarrow \mathbb{P}^9_{\mathbb{C}}$ determined by the quadrics in $\mathbb{P}^3_{\mathbb{C}}$.

We are eventually in the position to describe the set of configurations of a pod as a subvariety of X. Here and in the following chapters, we will think of an n-pod as a triple

$$\Pi = \Big((p_1, \ldots, p_n), (P_1, \ldots, P_n), (d_1, \ldots, d_n)\Big),$$

where $\vec{p} = (p_1, \ldots, p_n)$ are the platform points, $\vec{P} = (P_1, \ldots, P_n)$ are the base points, and $\vec{d} = (d_1, \ldots, d_n)$ are the leg lengths. The description of admissible configurations we are going to provide allows independent choices for two coordinate systems, one for the base and one for the platform: this is why also the leg lengths have to be included in the definition of a pod.

Definition 1.7. Let $\Pi = (\vec{p}, \vec{P}, \vec{d})$ be a pod. For every leg (p_i, P_i, d_i), the spherical condition given by Equation (8) determines a linear form l_i on $\mathbb{P}^{16}_{\mathbb{C}}$. The intersection of the variety X with the zero set of the linear forms $l_1, \ldots, l_n$ is called the *complex configuration set* of Π and is denoted by K_Π; the real points of this intersection are called the *real configuration set* of Π. The dimension of $K_\Pi \cap \{h \neq 0\}$ as a complex algebraic variety is called the *mobility* of Π. As one notices from the definition, in the notion of mobility only the subvariety of X whose real points correspond to direct isometries is involved. If the mobility of Π is greater than or equal to 1, then Π is said to be *mobile*.

Notice that by construction the set of real points in K_Π that correspond to direct isometries is in bijection with the set of direct isometries σ such that $\|\sigma(p_i) - P_i\|^2 = d_i^2$ for all $i \in \{1, \ldots, n\}$.

Notice moreover that in Definition 1.7 we do not require that the identity isometry belongs to the complex configuration set of a pod, namely we do not fix any initial position of the pod.

Remark 1.8. From the fact that the degree of X is 40 and its dimension is 6 we see that a general hexapod admits at most 40 configurations. In fact a general hexapod determines six general linear forms l_i, and so their zero set is a general linear space of codimension 6. By the characterization of the degree of a projective variety, hence, the complex configuration set of a general hexapods is constituted of 40 points.

If Π is a pod of mobility one, then its configuration set is a curve in X, and therefore we can consider its degree: this invariant will play a major role in Chapter 3.

Definition 1.9. Let Π be a pod of mobility one. The degree of K_Π is called the *conformal degree* of Π. This number is at most 40, since the degree of X is 40.

1.1.2 *The action of direct isometries*

In this subsection we extend the natural actions of SE_3 on itself, given by composition on the left and on the right, to actions of SE_3 on $\mathbb{P}^{16}_{\mathbb{C}}$ that determine projective automorphisms of X. This will be useful in Section 1.2, since it allows us to exploit the symmetries of X.

We investigate how the composition of two direct isometries looks like in the coordinates of $\mathbb{P}^{16}_{\mathbb{C}}$. Let $\sigma_1 \colon v \mapsto M_1 v + y_1$ and $\sigma_2 \colon v \mapsto M_2 v + y_2$ be two direct isometries. The composition $\sigma_{12} = \sigma_1 \circ \sigma_2$ sends

$$v \;\mapsto\; (M_1 M_2) v + (M_1 y_2 + y_1).$$

We set $M_{12} := M_1 M_2$ and $y_{12} := M_1 y_2 + y_1$. The remaining affine coordinates of σ_{12} are

$$\begin{aligned} x_{12} &= -M_{12}^t y_{12} = -M_2^t M_1^t M_1 y_2 - M_2^t M_1^t y_1 \\ &= -M_2^t y_2 - M_2^t M_1^t y_1 = x_2 + M_2^t x_1, \\ r_{12} &= \langle y_{12}, y_{12} \rangle = \langle y_1, y_1 \rangle + \langle M_1 y_2, M_1 y_2 \rangle + 2 \langle M_1 y_2, y_1 \rangle \\ &= r_1 + r_2 - 2 \langle x_1, y_2 \rangle. \end{aligned}$$

The product operation on SE_3 determines therefore a bilinear form on $\mathbb{P}^{16}_{\mathbb{C}}$. The coordinates of σ_{12} as a point in X are

$$\begin{aligned}\big(\overbrace{h_1h_2}^{h_{12}} : \overbrace{M_1M_2}^{M_{12}} : \overbrace{M_2^t x_1 + h_1 x_2}^{x_{12}} : \\ \underbrace{h_2y_1 + M_1y_2}_{y_{12}} : \underbrace{h_2r_1 + h_1r_2 - 2\langle x_1, y_2\rangle}_{r_{12}}\big).\end{aligned} \tag{9}$$

Remark 1.10. Notice that the product defined by Equation (9) is well-defined as long as the coordinates of one of the two factors satisfy $h \neq 0$. Therefore for every element in SE_3 we obtain a projective automorphism of X into itself.

We specialize Equation (9) to the cases of left and right multiplication by translations or rotations around the origin. We fix an element $\sigma \in X$ with coordinates $\sigma = (h : M : x : y : r)$.

a. Given a vector $t \in \mathbb{R}^3$, the translation τ by t has the following coordinates in $\mathbb{P}^{16}_{\mathbb{R}}$:

$$\tau = \big(1 : \mathrm{id} : -t : t : \langle t, t\rangle\big).$$

Left multiplication by τ gives

$$\tau\,\sigma = \big(h : M : -M^t t + x : ht + y : h\,\langle t, t\rangle + r + 2\langle t, y\rangle\big), \tag{10}$$

while right multiplication by τ gives

$$\sigma\,\tau = \big(h : M : x - ht : y + Mt : r + h\,\langle t, t\rangle - 2\langle x, t\rangle\big). \tag{11}$$

b. Given an orthogonal matrix $O \in SO_3$, the rotation ρ around the origin by O has the following coordinates in $\mathbb{P}^{16}_{\mathbb{C}}$:

$$\rho = (1 : O : 0 : 0 : 0).$$

Left multiplication by ρ gives

$$\rho\,\sigma = \big(h : OM : x : Oy : r\big), \tag{12}$$

while right multiplication by ρ gives

$$\sigma\,\rho = \big(h : MO : Ox : y : r\big). \tag{13}$$

1.2 BOUNDARY POINTS

In this section we study the boundary of X, namely the subvariety of X constituted of points that do not come from isometries. These results are preparatory to Section 1.3, which constitutes the cornerstone of this chapter.

Definition 1.11. The *boundary* of X is defined to be $B = X \setminus H$, where H is the hyperplane in $\mathbb{P}^{16}_{\mathbb{C}}$ defined by the equation $h = 0$. A direct computation shows that X and H do not intersect transversally, and that actually $\deg(X \cap H) = 2\deg(X \cap H)_{\mathrm{red}}$ — where with $(\cdot)_{\mathrm{red}}$ we denote the reduce structure. In this and in the next chapter we will be concerned only with the set-theoretic version of B; its non-reduced structure will play an important role in Chapter 3.

Points in the boundary have a peculiar structure, presented in Lemma 1.12.

Lemma 1.12. *All points* $\beta = (h : M : x : y : r) \in B$ *are of the form*

$$\beta = (0 : vw^t : x : y : r)$$

for two vectors $v, w \in \mathbb{C}^3$*, and two vectors* $x, y \in \mathbb{C}^3$ *that satisfy the condition* $\langle x, x\rangle = \langle y, y\rangle = 0$*. Moreover, if* $M \neq 0$*, then there exist complex numbers* $\lambda, \mu \in \mathbb{C}$ *such that* $x = \lambda w$ *and* $y = \mu v$*.*

Proof. A direct inspection of Equation (4) shows that for a boundary point we have $\mathrm{rk}(M) \leqslant 1$, and the first part of the statement follows. If $M \neq 0$, we have that $\langle v, v\rangle = \langle w, w\rangle = 0$. Recalling the equations

$$M^t y = Mx = 0,$$

it follows $\langle x, w\rangle = \langle y, v\rangle = 0$. This implies that x and w are linearly independent, and the same holds for y and v. From this the statement follows. □

We describe now a partition of the boundary in five subsets; for each of them, we show how the automorphisms of X of the form described in Section 1.1.2 can be used to find "normal forms" for the points. The nomenclature we use will become meaningful after Section 1.3.

Vertex

Any real point in B satisfies $v = w = x = y = 0$; this follows from the equations

$$\langle v, v\rangle = \langle w, w\rangle = 0,$$
$$\langle x, x\rangle = \langle y, y\rangle = 0.$$

Therefore there exists a unique real point in B, whose coordinates are $(0 : \ldots, : 0 : 1)$, called the *vertex*.

Inversion points.

A boundary point β is called an *inversion point* if it satisfies $M \neq 0$ and $N \neq 0$, where N is the matrix $rM + 2yx^t$. Hence by construction inversion points form an open subset of the boundary. Lemma 1.12 in this case implies $x = \lambda w$ and $y = \mu v$ for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Thus an inversion point is of the form $(0 : vw^t : \lambda w : \mu v : r)$.

We look for a normal form for inversion points up to the action of SE_3 on X. To do so, we first consider the action by rotations. In order to compute normal forms, we first apply rotations. Multiplication from the right by a rotation of matrix M' gives — see Equation (13):

$$(0 : vw^t M' : \lambda\, M'w : \mu\, v : r),$$

so it leaves v fixed. Being M' orthogonal, it is in particular unitary, so it preserves both the scalar product and the Hermitian norm of w, and the action is transitive on vectors with $\langle w, w\rangle = 0$ and of the same Hermitian norm. Hence w can be taken to a vector of the form $\delta\,(1, i, 0)^t$, where $\delta \in \mathbb{C}^*$ since $M \neq 0$. Multiplication from the left acts analogously on v.

Thus using rotations we obtain $v = \gamma\,(1,i,0)^t$ and $w = \delta\,(1,i,0)^t$ with both γ and δ different from zero. Then by multiplying all coordinates by an appropriate non-zero number we can suppose that $M = \left(\begin{smallmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

Applying from the left a translation by a vector $s \in \mathbb{R}^3$ gives — see Equation (10):

$$\big(0 : vw^t : (-\langle v, s\rangle + \lambda)\, w : \mu\, v : r + 2\mu\,\langle v, s\rangle\big),$$

and similarly if we act from the right via a translation by a vector $t \in \mathbb{R}^3$ — see Equation (11):

$$\big(0 : vw^t : \lambda\, w : (\langle w, t\rangle + \mu)\, v : r - 2\lambda\,\langle w, t\rangle\big).$$

This shows that we can achieve $\lambda = \mu = 0$ by multiplication by translations from both sides (for example, since we reduced to the situation $v = (1,i,0)^t$, one can take $s = (s_1, s_2, s_3)^t$ where $s_1 = \mathrm{Re}\lambda$, $s_2 = \mathrm{Im}\lambda$ and any s_3, and similarly for t). It also shows that the matrix N is invariant under translations. So by translations from both sides, we obtain $x = y = 0$. The value of r cannot be changed by any rotation that fixes $x = y = 0$, but we still can apply a rotation of the form $\left(\begin{smallmatrix} c & d & 0 \\ -d & c & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, with $c^2 + d^2 = 1$, from the left. The effect on M is multiplication by $(c + id)$, and we have no effect on r. Projectively, this is the same as leaving M untouched and multiplying r by $(c + id)^{-1}$. Hence we can reach the situation where $r \in \mathbb{R}_{>0}$. We notice that r cannot be zero, otherwise we would have $N = 0$. So inversion points have the following normal forms:

$$\beta \;=\; (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : r),$$

with $r \in \mathbb{R}_{>0}$. A direct computation shows that inversion points are smooth points of the boundary.

Butterfly points.

A boundary point β with $M \neq 0$ and $N = 0$ is called a *butterfly point*. The complex dimension of the set of butterfly points is 4: as for inversion point, we can choose v and w satisfying $\langle v, v\rangle = \langle v, v\rangle = 0$ and $\lambda, \mu \in \mathbb{C}^*$. The normal form is constructed similarly as above. In this case, when we reach $x = y = 0$, the conditions $M \neq 0$ and $N = 0$ force r to be zero. We have then only a single normal form, namely

$$\beta \;=\; (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0).$$

Similarity points.

The points $\beta = (0 : M : x : y : r) \in B$ such that $M = 0$, $x \neq 0$ and $y \neq 0$ are called *similarity points*. Since x and y satisfy $\langle x, x\rangle = \langle y, y\rangle = 0$, the complex dimension of the set of similarity points is 4.

To compute normal forms of similarity points, we first apply rotations. As for inversion points, right multiplication fixes y and r and can transform x to $\gamma\,(1,i,0)^t$, and left multiplication fixes x and r and can transform y to $\delta\,(1,i,0)^t$, with both γ and δ in $\mathbb{C}^*$. Hence projectively we can always suppose that $\delta = 1$, so we can reduce any similarity point to one such that $x = \gamma\,(1, i, 0)^t$ and $y = (1, i, 0)^t$. Then translations act transitively on r, thus we may get to the situation with $r = 0$. Eventually we can perform rotations ensuring that γ is a real positive number. We get normal forms of the following kind

$$\beta \;=\; (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{\gamma : i\gamma : 0}_{x} : \underbrace{1:i:0}_{y} : 0),$$

with $\gamma \in \mathbb{R}_{>0}$.

Collinearity points.

For the last group of points β in B we have $M = 0$ and either $x = 0$, $y \neq 0$ or $x \neq 0$, $y = 0$. These points are called *collinearity points*. There are two subsets of collinearity points, one with $x = 0$ and one with $y = 0$. Both subsets have complex dimension 2 (since there is still a free value for r to choose).

By applying rotations, we can achieve either $x = (1 : i : 0)^t$ or $y = (1 : i : 0)^t$. Translations act transitively on r, so we get two normal forms, namely

$$\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{1:i:0}_{x} : \underbrace{0:0:0}_{y} : 0),$$
$$\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{0:0:0}_{x} : \underbrace{1:i:0}_{y} : 0).$$

We summarize the results obtained so far in Proposition 1.13.

Proposition 1.13. *Let $\beta \in B$ be a boundary point different form the vertex. Up to the left and right action of SE_3 on X, the point β is equivalent to one of the following normal forms:*

- *(inversion points)*
  $\beta = (0 : \underbrace{1:i:0:i:-1:0:0:0:0}_{M} : \underbrace{0:0:0}_{x} : \underbrace{0:0:0}_{y} : r)$;
- *(butterfly points)*
  $\beta = (0 : \underbrace{1:i:0:i:-1:0:0:0:0}_{M} : \underbrace{0:0:0}_{x} : \underbrace{0:0:0}_{y} : 0)$;
- *(similarity points)*
  $\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{\gamma : i\gamma : 0}_{x} : \underbrace{1:i:0}_{y} : 0)$;
- *(collinearity points)*
  $\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{1:i:0}_{x} : \underbrace{0:0:0}_{y} : 0)$ *or*
  $\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{0:0:0}_{x} : \underbrace{1:i:0}_{y} : 0)$.

Remark 1.14. Notice that the center of the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ defined in Remark 1.6 intersects X in the union of the sets of similarity points, collinearity points and the vertex.

We conclude this section by associating to each inversion, butterfly and similarity point a pair of oriented direction in $\mathbb{R}^3$, where the latter are directions together with the choice of one of the two corresponding vectors of norm 1. This means that an oriented direction can be identified with a point on the unit sphere S^2 in $\mathbb{R}^3$.

Recall that in the case of inversion and butterfly points the matrix M is of the form vw^t for two non-zero vectors whose coordinates satisfy

$$v_1^2 + v_2^2 + v_3^2 = 0 \quad \text{and} \quad w_1^2 + w_2^2 + w_3^2 = 0.$$

Consider now the conic

$$C = \left\{ (\alpha : \beta : \gamma) \, : \, \alpha^2 + \beta^2 + \gamma^2 = 0 \right\} \subseteq \mathbb{P}^2_{\mathbb{C}},$$

called the *absolute conic*. The vectors v and w determine hence a pair of points in C, and one notices that such pair of points does not depend on the choice of v and w as long as $M = vw^t$. If we are able to provide an identification between C and S^2, then we achieve our goal, at least as inversion and butterfly points are concerned. We illustrate the bijection between C and S^2 we are going to consider in two steps: first we show that C is isomorphic (as a curve over $\mathbb{C}$) to $\mathbb{P}^1_{\mathbb{C}}$, and then we identify S^2 and $\mathbb{P}^1_{\mathbb{C}}$ via the stereographic projection.

The isomorphism $C \cong \mathbb{P}^1_{\mathbb{C}}$ is given by the parametrization

$$\mathbb{P}^1_{\mathbb{C}} \ni (s, t) \mapsto \left((s^2 - t^2) : i(s^2 + t^2) : 2st \right) \in C \tag{14}$$

and its inverse

$$\begin{cases} (\alpha : \beta : \gamma) & \mapsto & (\alpha - i\beta : \gamma) & \text{if } (i\alpha + \beta, \gamma) \neq (0,0), \\ (\alpha : \beta : \gamma) & \mapsto & (\gamma : -\alpha - i\beta) & \text{otherwise.} \end{cases} \tag{15}$$

The identification between $\mathbb{P}^1_{\mathbb{C}}$ and S^2 by stereographic projection is provided by the following equations:

$$\begin{cases} (0,0,1) & \mapsto & (0:1) \in \mathbb{P}^1_{\mathbb{C}}, \\ (\lambda, \mu, \nu) & \mapsto & \left(1 : \frac{\lambda + i\mu}{1-\nu}\right) \in \mathbb{P}^1_{\mathbb{C}} \quad \text{for } (\lambda, \mu, \nu) \in S^2 \setminus \left\{(0,0,1)\right\}, \end{cases}$$
$$\begin{cases} (0:1) & \mapsto & (0,0,1) \in S^2, \\ (1 : a + ib) & \mapsto & \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1} \right) \quad \text{for } a, b \in \mathbb{R}. \end{cases} \tag{16}$$

We would like to do the same for similarity points. We cannot repeat the previous procedure, because by definition for similarity points all the m_{ij}-coordinates are zero. On the other hand, for all boundary points the two matrices M and xy^t are linear dependent, and for similarity points the matrix xy^t is non-zero. Moreover, x and y satisfy $\langle x, x \rangle = \langle y, y \rangle = 0$. So we can associate to a similarity point the pair of elements of S^2 coming from the vectors x and y.

Definition 1.15. Let $\beta \in B$ be an inversion, butterfly or similarity point. By the previously described procedure it is possible to associate to β a pair (L, R) of points in the unit sphere S^2. Such points are called the *left* and the *right vector* of β, respectively.

1.3 GEOMETRIC INTERPRETATION OF BOUNDARY POINTS

Lying on the boundary of X, boundary points do not correspond by construction to direct isometries of $\mathbb{R}^3$. However, and this is the main result of this section and of the whole chapter, it is possible to give them a geometric meaning, since their presence in the configuration set of a pod implies that the base and the platform must satisfy certain constrains.

Definition 1.16. Let Π be an n-pod, we define its set of *bonds* B_Π as the intersection of K_Π and the hyperplane $\{h = 0\}$. Namely, B_Π is the intersection of K_Π with the boundary B of X. A direct inspection at the spherical condition shows that bonds do not depend on the leg lengths.

Definition 1.17. Let $\beta = (0 : M : x : y : r)$ be a boundary point, and let p and P be points in $\mathbb{R}^3$. The linear form determined by the spherical condition from Equation (8), once instantiated at a boundary point, has the form:

$$r - 2\langle p, x\rangle - 2\langle y, P\rangle - 2\langle Mp, P\rangle \;=\; 0. \tag{17}$$

We call the condition imposed by Equation (17) the *pseudo spherical condition* for the points (p, P) at the boundary point β.

Remark 1.18. The vertex (see Section 1.2) can never be a bond for a pod, because by a direct inspection one sees that it can never satisfy Equation (17). Since the vertex is the only real point in the boundary, it follows that bonds are never real points. However, they are defined by real equations, and this implies by a direct check that in the case of inversion and similarity points they must appear in complex conjugated pairs, and we will use this fact in the next chapters.

Remark 1.19. If an n-pod Π is mobile, then by definition $\dim K_\Pi \geqslant 1$. Since B_Π is a hyperplane section of K_Π, it follows that B_Π is not empty. By the same argument, if the mobility is greater than or equal to 2, then Π admits infinitely many bonds.

Definition 1.20. Given a unit vector $\varepsilon \in S^2$, we say that a linear map $\pi_\varepsilon : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is an *orthogonal projection along* ε if $\ker(\pi_\varepsilon) = \langle \varepsilon \rangle$ and π_ε is an isometry on $\langle \varepsilon \rangle^\perp$. Moreover we ask that the preimages of the standard basis of $\mathbb{R}^2$ lying on $\langle \varepsilon \rangle^\perp$ form, together with ε, a positively oriented basis. Note that in this way π_ε is well-defined only up to rotations around the origin in $\mathbb{R}^2$.

We are ready to state and prove the main results of this section.

Proposition 1.21. *There is a one-to-one correspondence between inversion (respectively, similarity) points* β *with both left and right vectors* L *and* R *equal to the South pole* $(0, 0, -1) \in S^2$ *and inversions (respectively, similarities)* κ *of the plane such that for any pair of points* (p, P) *in* $\mathbb{R}^3$ *the pseudo spherical condition for* (p, P) *at* β *is equivalent to the fact that* $\kappa(q) = Q$*, where* $q = \pi_L(p)$ *and* $Q = \pi_R(P)$.

Proof. We start treating the case of inversion points. Suppose that $\beta_0 \in B$ is an inversion point with $L = R = (0, 0, -1)$. Suppose furthermore that β_0 is in the normal form:

$$\beta_0 \;=\; (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : r),$$

with $r \in \mathbb{R}_{>0}$. We get that π_L and π_R can be both taken to be the projection on the first two coordinates. Thus if $p = (a, b, c)$ and $P = (A, B, C)$, then $q = (a, b)$ and $Q = (A, B)$. If we instantiate the pseudo spherical condition from Equation (17) for (p, P) at β_0 we get the relations:

$$\begin{cases} aA - bB = r/2, \\ bA + aB = 0, \end{cases} \tag{18}$$

which define an inversion κ_0 mapping q to Q. Conversely, suppose we are given an inversion κ_0 described by Equation (18). Then going backwards in the previous argument we can see that we obtain an inversion point in normal form as in the thesis.

Suppose now that the $\beta \in B$ is an inversion point with $L = R = (0, 0, -1)$, but not necessarily in normal form. Then we know from Proposition 1.13 that we can find two isometries $\sigma_1, \sigma_2 \in SE_3$ that fix left and right vectors such that $\sigma_1 \beta \sigma_2 = \beta_0$ is in normal form. Moreover σ_1 and σ_2 induce isometries τ_1 and τ_2 of $\mathbb{R}^2$ such that the following two diagrams commute:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\sigma_1"] \arrow[d, "\pi_L"'] & \mathbb{R}^3 \arrow[d, "\pi_L"] \\ \mathbb{R}^2 \arrow[r, "\tau_1"] & \mathbb{R}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^3 \arrow[r, "\sigma_2"] \arrow[d, "\pi_R"'] & \mathbb{R}^3 \arrow[d, "\pi_R"] \\ \mathbb{R}^2 \arrow[r, "\tau_2"] & \mathbb{R}^2 \end{tikzcd}$$

If κ_0 is the inversion associated to β_0, then we define $\kappa = \tau_1 \kappa_0 \tau_2$, and one can check that the association $\beta \leftrightarrow \kappa$ satisfies the requirements of the thesis.

We consider now the case of similarity points. Suppose that $\beta_0 \in B$ is a similarity point with $L = R = (0, 0, -1)$. Suppose furthermore that β_0 is in the normal form:

$$\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{\gamma : i\gamma : 0}_{x} : \underbrace{1 : i : 0}_{y} : 0),$$

with $\gamma \in \mathbb{R}_{>0}$. Again for this kind of points π_L and π_R are both the projection on the first two coordinates. Performing analogous computations as before we get the relations:

$$\begin{cases} A = -\gamma\, a, \\ B = -\gamma\, b. \end{cases} \tag{19}$$

These define a similarity κ_0 mapping q to Q. Now we argue as for inversion points. $\square$

Corollary 1.22. *Assume that $\beta \in B_\Pi$ is an inversion (respectively, similarity) bond of Π. Let $L, R \in S^2$ be the left and right vector of β. For $i \in \{1, \dots, n\}$, set $q_i = \pi_L(p_i)$ and $Q_i = \pi_R(P_i)$. Then there is an inversion (respectively, similarity) of $\mathbb{R}^2$ mapping $q_1, \dots, q_n$ to $Q_1, \dots, Q_n$.*

Conversely, let $L, R \in S^2$ be two unit vectors such that the images of $(p_1, \dots, p_n)$ under π_L and of $(P_1, \dots, P_n)$ under π_R differ by an inversion (respectively, similarity). Then Π has an inversion (respectively, similarity) bond with left vector L and right vector R.

Proof. In both cases of inversion and similarity points we can apply suitable isometries in order to put β in normal form. Then it is enough to apply Proposition 1.21. $\square$

Proposition 1.23. *There is a one-to-one correspondence between butterfly points β and pairs (g_L, g_R) of oriented lines in $\mathbb{R}^3$ such that for any pair of points (p, P) in $\mathbb{R}^3$ the pseudo spherical condition for (p, P) at β is equivalent to the fact that $p \in g_L$ or $P \in g_R$.*

Proof. Suppose that $\beta_0 \in B$ is a butterfly point in the normal form

$$\beta_0 = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{M} : \underbrace{0 : 0 : 0}_{x} : \underbrace{0 : 0 : 0}_{y} : 0).$$

In this case we associate to β_0 the lines $g_L = g_R = \{z\text{—axis}\}$, both oriented to the South pole $(0, 0, -1) \in S^2$. If we instantiate the pseudo spherical condition for (p, P) at β_0 given by Equation (17) we get the relations:

$$\begin{cases} aA - bB = 0, \\ aB + bA = 0. \end{cases} \tag{20}$$

One checks that the only solutions to Equation (20) are either $(a, b) = (0, 0)$ or $(A, B) = (0, 0)$. Hence either p is of the form $(0, 0, c)$ (namely it lies on g_L) or P is of the form $(0, 0, C)$ (namely it lies on g_R).

If $\beta \in B$ is an arbitrary butterfly point, then there exist isometries $\sigma_1, \sigma_2 \in$ SE_3 such that $\sigma_1 \beta \sigma_2 = \beta_0$ is in normal form. Then we associate to β the pair of lines

$$(g_L, g_R) \;=\; \Big((\sigma_1)^{-1} \big(\{z\text{—axis}\}\big), (\sigma_2)^{-1} \big(\{z\text{—axis}\}\big) \Big)$$

with orientation given by the left and right vectors of β. One can check that the equivalence in the thesis holds. Conversely, given two oriented lines g_L and g_R we can find isometries $\sigma_1, \sigma_2 \in SE_3$ such that $g_L = \sigma_1\big(\{z\text{—axis}\}\big)$ and $g_R = \sigma_2\big(\{z\text{—axis}\}\big)$, both oriented to the South pole $(0, 0, -1) \in S^2$. Then we associate to (g_L, g_R) the butterfly point $\sigma_1 \beta \sigma_2$. $\square$

Corollary 1.24. *Assume that $\beta \in B_\Pi$ is a butterfly bond of Π. Let $L, R \in S^2$ be the left and right vector of β. Then, up to permutation of the indices $1, \ldots, n$, there exists $m \leqslant n$ such that $p_1, \ldots, p_m$ are collinear on a line parallel to L, and $P_{m+1}, \ldots, P_n$ are collinear on a line parallel to R.*

Conversely, let $L, R \in S^2$ be two unit vectors such that $p_1, \ldots, p_m$ are collinear on a line parallel to L, and $P_{m+1}, \ldots, P_n$ are collinear on a line parallel to R. Then Π has a butterfly bond with left vector L and right vector R.

Notation. Recall from Section 1.2 that the set of collinearity points is partitioned into two subsets: if the y-coordinate of a collinearity point is zero we call it a *left collinearity point*, while if the x-coordinate is zero we call it a *right collinearity point*.

Proposition 1.25. *There is a one-to-one correspondence between left (respectively right) collinearity points β and oriented lines g in $\mathbb{R}^3$ such that for any pair of points (p, P) in $\mathbb{R}^3$ the pseudo spherical condition for (p, P) at β is equivalent to the condition $p \in g$ (respectively $P \in g$).*

Proof. Suppose that $\beta_0 \in B$ is a left collinearity point and suppose that it is in normal form:

$$\beta = (0 : \underbrace{0:0:0:0:0:0:0:0:0}_{M} : \underbrace{1:i:0}_{x} : \underbrace{0:0:0}_{y} : 0).$$

We associate to β_0 the line $g = \{z\text{–axis}\}$, directed to the South pole $(0, 0, -1) \in S^2$. If we instantiate the pseudo spherical condition for (p, P) at β_0 given by Equation (17) we get the relations:

$$0 \; = \; -2(a + ib) \quad \Leftrightarrow \quad a = b = 0,$$

which is equivalent to $p \in g$.

If $\beta \in B$ is an arbitrary left collinearity point we proceed as in the proof of Proposition 1.23. Analogous arguments prove the statement about right collinearity points. □

Corollary 1.26. *Assume that $\beta \in B_\Pi$ is a collinearity bond of Π. Then either $p_1, \ldots, p_n$ are collinear or $P_1, \ldots, P_n$ are collinear (or both).*

Conversely, if $p_1, \ldots, p_n$ are collinear or $P_1, \ldots, P_n$ are collinear, then Π has a collinearity bond.

Proof. This follows from Proposition 1.25. □

Recalling Remark 1.19, one gets the following theorem.

Theorem 1.27. *If an n-pod is mobile, then one of the following conditions holds:*

(a) There exists at least one pair of orthogonal projections π_L and π_R such that the projections of the platform points $p_1, \ldots, p_n$ by π_L and of the base points $P_1, \ldots, P_n$ by π_R differ by an inversion or a similarity.

(b) There exists $m \leqslant n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_n$ are collinear, up to permutation of indices.

Proof. By hypothesis B_Π is not empty. Therefore there is at least one inversion/similarity/collinearity/butterfly bond, and then the result follows from Corollaries 1.22, 1.24 and 1.26. □

We conclude stating our last result, concerning constraints on base and platform points of n-pods with higher mobility.

Theorem 1.28. *Let $\Pi = (\vec{p}, \vec{P}, \vec{d})$ be an n-pod with mobility 2 or higher. Then one of the following holds:*

(a) there are infinitely many pair (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_n)$ and $\pi_R(P_1), \ldots, \pi_R(P_n)$ differ by an inversion or a similarity;

(b) there exists $m \leqslant n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1} = \ldots = P_n$, up to permutation of indices and interchange between base and platform;

(c) there exists $m \leqslant n$ with $1 < m < n - 1$ such that $p_1, \ldots, p_m$ lie on a line $g \subseteq \mathbb{R}^3$ and $p_{m+1}, \ldots, p_n$ lie on a line $g' \subseteq \mathbb{R}^3$ parallel to g, and $P_1, \ldots, P_m$ lie on a line $G \subseteq \mathbb{R}^3$ and $P_{m+1}, \ldots, P_n$ lie on a line $G' \subseteq \mathbb{R}^3$ parallel to G, up to permutation of indices.

Proof. Since Π has mobility at least 2, it has infinitely many bonds (see Remark 1.19). If Π admits one collinearity bond, then we have Case (b) with $m = n$ from Corollary 1.26. Assume that Π admits infinitely many butterfly points, then by Corollary 1.24 there exists $m \leqslant n$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_n$ lie on infinitely many different lines, and therefore we have Case (b) again. We are left with the case when Π admits infinitely many inversion or similarity bonds. If these bonds provide infinitely many different left and right vectors, we are in Case (a). Otherwise we have that there are infinitely many inversion or similarity points with the same left and right vectors (L, R). Consider the sets $\mathcal{U} = \{\pi_L(p_1), \ldots, \pi_L(p_n)\}$ and $\mathcal{V} = \{\pi_R(P_1), \ldots, \pi_R(P_n)\}$. By Corollary 1.22, each such inversion or similarity maps $\pi_L(p_i)$ to $\pi_R(P_i)$, so $\mathcal{U}$ and $\mathcal{V}$ have the same cardinality. On the other hand, any inversion or similarity is completely specified if we prescribe the image of three points, so if the cardinality of $\mathcal{U}$ were greater than 2 then we would have only one inversion or similarity. Moreover, when both $\mathcal{U}$ and $\mathcal{V}$ are given by one point we are in Case (b). So from now on we can suppose that $|\mathcal{U}| = |\mathcal{V}| = 2$. Hence $p_1, \ldots, p_n$ are arranged on two parallel lines, and the same holds for $P_1, \ldots, P_n$. From this and the fact that the inversions/similarities should map $\pi_L(p_i)$ to $\pi_R(P_i)$ it follows that the only possible configurations are the ones described in Case (c). $\square$

INTERMEZZO: MODULI SPACES OF POINTS IN THE LINE

Moduli and parameter spaces are ubiquitous in algebraic geometry. Roughly speaking, a moduli or parameter space is a geometric object whose points parametrize algebraic varieties with some prescribed properties: one can use the word *moduli space* while dealing with intrinsic properties, like the genus, and the word *parameter space* when considering extrinsic ones, like the degree. The algebraic structure on the moduli or parameter space is defined in such a way that points that are "close" in the moduli space determine varieties that are also "close"; making this statement precise is beyond the scope of this very short discussion. One of the areas where the use of moduli and parameter space has been really fruitful is *enumerative geometry*, namely the computation of the number of algebraic varieties satisfying certain geometric conditions (for a nice short account on the use of moduli spaces in curve enumeration, see for example [Pie01]; for a nice text covering these aspects with lots of examples, see [EH16]).

One of the techniques developed to construct moduli spaces is Geometric Invariant Theory; see [MFK94] for a comprehensive discussion. We mention a few of its main ingredients that will be useful for us, following [Dol94] and keeping the technical level as low as possible.

We start from the notion of *algebraic group* over $\mathbb{C}$ ([Dol94, Definition 1.1]): this is a complex algebraic variety carrying also the structure of a group such that the group multiplication and the map associating an element to its inverse are regular maps in the sense of algebraic geometry. For example, both the affine space $\mathbb{C}^n$ with its natural additive group structure and the complex torus $(\mathbb{C} \setminus \{0\})^n$ with its natural multiplicative group structure are algebraic groups. The example that will interest us the most is $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$, the automorphism group of the projective line $\mathbb{P}^1_{\mathbb{C}}$: it has a natural group structure and it can be endowed with an algebraic structure by interpreting it as the complement of the determinant hypersurface in the projectivization $\mathbb{P}(\mathbb{C}^{2\times 2})$ of the vector space of 2×2 matrices (and recalling that the complement of a hypersurface in projective space is an affine variety).

An *algebraic group action* of an algebraic group G on an algebraic variety X is a regular map $G \times X \longrightarrow X$ satisfying the usual axioms of group actions. For example, $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}$ by changes of coordinates, and such action is called *linear*.

One may be now tempted to consider the set of orbits of such a group action as the space whose points parametrize equivalence classes of points in X up to the action of G: unfortunately, such a set does almost never allows an algebraic structure that has the right categorical properties that a moduli space should have. A technique developed by Mumford — based on the concept of *stability* — allows, when the algebraic group G is *reducible*, to identify a suitable open set U of the variety X such that there exists an algebraic variety, called a *good geometric quotient*, that comprises both good categorical and geometric properties, and this is the kind of construction we are going to use.

In Chapters 2 and 3 we will be dealing with one very special class of moduli spaces, namely the compactifications $\overline{M}_{0,n}$ of the moduli spaces of n points in $\mathbb{P}^1_{\mathbb{C}}$, that we will simply denote by M_n: its elements are equivalence classes of tuples $(m_1, \ldots, n\}$ of points in $\mathbb{P}^1_{\mathbb{C}}$ considered up to the action of the group $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$ of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$. These are instances of the more general family of compactifications $\overline{M}_{g,n}$ of moduli spaces of smooth curves of genus g with n marked points (the interested reader can refer to [HM98] and to the monumental [ACG11] for more information about these spaces): since, up to isomorphisms, there exists only one curve of genus 0, namely the projective line, we recover immediately the notion we are interested in from this more general one.

The moduli space M_3 of three points on $\mathbb{P}^1_{\mathbb{C}}$ reduces just to a point because of the so-called Fundamental Theorem of projective geometry (see for example [PW01, Proposition 1.1.5]):

Theorem. *Let $\vec{P} = (P_0, \ldots, P_{n+1})$ and $\vec{Q} = (Q_0, \ldots, Q_{n+1})$ be two $n+2$-tuples of points in $\mathbb{P}^n_{\mathbb{C}}$, both in general linear position. Then there exists a unique projective automorphism of $\mathbb{P}^n_{\mathbb{C}}$ sending $\vec{P}$ to $\vec{Q}$.*

In fact, by virtue of the previous theorem, if we have three distinct points on $\mathbb{P}^1_{\mathbb{C}}$ we can always find a suitable change of coordinates mapping them to the points $(0:1)$, $(1:0)$ and $(1:1)$.

When we consider four points the situation is more interesting. In fact, if we are given four distinct points A,B,C,D in $\mathbb{P}^1_{\mathbb{C}}$, then by the Fundamental Theorem of projective geometry we can find a system of projective coordinates such that $A = (1:0)$, $B = (0:1)$, $C = (1,1)$ and $D = (1:\lambda)$. We see hence that the complex number λ is a projective invariant of the ordered 4-tuple (A, B, C, D), and we call it the *cross ratio* of (A, B, C, D). One can show that this is the only projective invariant, namely that two 4-tuples of distinct points are projectively equivalent if and only if their cross ratios are equal (for a different view on the cross ratio, see for example [Lab08]). From the fact that the cross ration can take any complex value, it follows that $M_4 \cong \mathbb{P}^1_{\mathbb{C}}$.

In order to deal with five or more points (our interest in the next chapters will be in particular for the cases $n = 5$ and $n = 6$), we can try to provide a general way to describe the moduli spaces M_n. We adopt the view proposed in [HMSV09] which, among many other things, provides an explicit way of constructing a quotient map $\left(\mathbb{P}^1_{\mathbb{C}}\right)^n \dashrightarrow \mathbb{P}^{\beta(n)}_{\mathbb{C}}$ whose image is M_n and such that, if we take a point x outside some subvariety of M_n, the preimage $\delta_n^{-1}(x)$ is exactly one orbit of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^n$ under the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$. Such rational map can be in fact obtained by the following "recipe":

1. Fix a natural number $k_n \geqslant 1$ (this number will determine the so-called *linerization* of the action of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$; different choices for the linearization provide different compactifications for the moduli space, for more information see [Dol94, Chapter 3]).

2. Consider the vertices of a regular n-gon in the plane, and label them by the indices $1, \ldots, n$.

3. Construct all possible topological realizations of graphs whose set of vertices is given by the points $\{1, \ldots, n\}$, whose edges are non-crossing

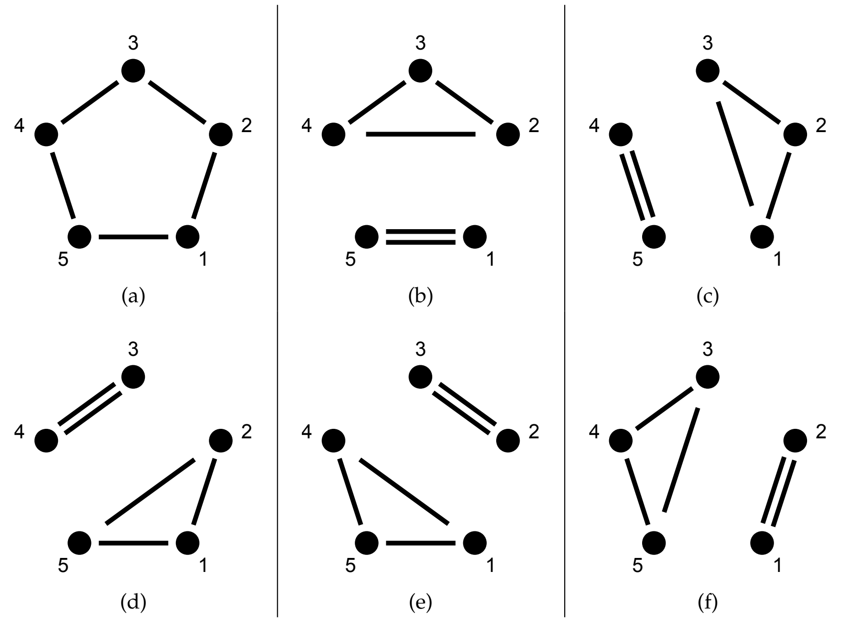


Figure 5: The six planar realizations of graphs with vertices on a regular pentagon, valency 2 and non-intersecting edges.

segments and such that each vertex has valency k_n. Recall that a vertex has valency k_n if there are exactly k_n edges having such vertex as an endpoint.

4. Consider coordinates $\big((a_i : b_i)\big)_{i=1}^{n}$ on $\big(\mathbb{P}^1_{\mathbb{C}}\big)^n$, and to every topological realization Γ from the previous step associate the polynomial

$$\delta_\Gamma \;=\; \prod_{\substack{(i,j)\in E\\ i<j}} \big(a_i\, b_j - a_j\, b_i\big).$$

Let $\beta(n)$ be the number of the polynomials δ_Γ.

5. Define the rational map $\delta_n \colon \big(\mathbb{P}^1_{\mathbb{C}}\big)^n \dashrightarrow \mathbb{P}^{\beta(n)}_{\mathbb{C}}$ to be the one whose components are given by the polynomials δ_Γ.

6. Consider the subset $\mathcal{U}_n \subseteq \big(\mathbb{P}^1_{\mathbb{C}}\big)^n$ defined by

$$\mathcal{U}_n \;=\; \big\{(m_1, \ldots, m_n) \;:\; \text{at most } {}^{n}\!/_{2} \cdot k_n \text{ of the points } m_i \text{ coincide}\big\}.$$

The image of $\mathcal{U}_n$ under δ_n is the moduli space M_n.

Let us apply this procedure in the case $n = 5$: the smallest value we can assign to k_5 is 2, since there are no realizations satisfying the conditions prescribed by the recipe for $k_5 = 1$. We obtain a rational map

$$\delta_5 \colon \big(\mathbb{P}^1_{\mathbb{C}}\big)^5 \dashrightarrow \mathbb{P}^5_{\mathbb{C}},$$

because there are exactly six possible realizations in this case, showed in Figure 5. The image of δ_5 is a non-degenerate surface of degree 5, which turns out to be a Del Pezzo surface. If we take coordinates $t, w_0, \ldots, w_5$ in $\mathbb{P}^5_{\mathbb{C}}$ — where t corresponds to the graph in Figure 5, Case (a) — then a direct computation shows that the ideal of M_5 is generated by

$$w_{i-2}\, w_{i+2} \;=\; t\, w_i + t^2 \qquad \forall i \in \{1, \ldots, 5\},$$

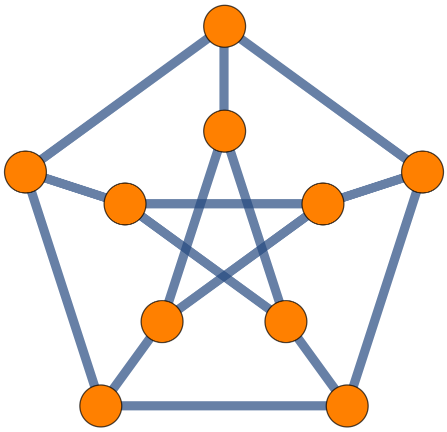


Figure 6: The Petersen graph, the intersection graph of lines in a smooth Del Pezzo surface of degree 5: each vertex corresponds to one of the 10 lines, and two vertices are connected by an edge if the corresponding lines intersect.

where the indices are taken modulo 5.

From the theory of Del Pezzo surfaces we know that there are exactly 10 lines on M_5, and we can give them a modular meaning noticing that the points in each line correspond to equivalence classes of 5-tuples $(m_1, \ldots, m_5)$ in which two points coincide. We denote by L_{ij} the line corresponding to tuples for which $m_i = m_j$. This modular interpretation clarifies also when two such lines intersect, once we take into account the condition defining the set $\mathcal{U}_5$ in the recipe, namely that at most three points can coincide: the intersection graph we obtain — where vertices correspond to lines, and there is an edge if two lines meet — is called the *Petersen graph* and is depicted in Figure 6; for more information about the configuration of lines on M_5, see [Dol12, Section 8.5.1].

The surface M_5 contains 5 families of conics, and every irreducible conic in M_5 belongs exactly to one of them. These families arise in the following way: fix an index $i \in \{1, \ldots, 5\}$ and consider the map $M_5 \longrightarrow M_4 \cong \mathbb{P}^1_{\mathbb{C}}$ sending the equivalence class of $(m_1, \ldots, m_5)$ to the equivalence class of $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_5)$, namely remove the i-th point; the fibers of this map give one family of conics. The i-th family of conics intersects only 4 lines, namely L_{ij} for $j \neq i$.

Consider now the situation $n = 6$: here we can take $k_6 = 1$ and we get a rational map

$$\delta_6 \colon \left(\mathbb{P}^1_{\mathbb{C}}\right)^6 \dashrightarrow \mathbb{P}^4_{\mathbb{C}},$$

since in this case we get exactly five graphs, as shown in Figure 7. The image of the map δ_6 is a three-dimensional variety, of equation

$$x_3 x_4 (x_0 + x_1 + x_2 + x_3 + x_4) - x_0 x_1 x_2 = 0.$$

where x_0, x_1 and x_1 correspond to the graphs in Figure 7, Case (a), (b) and (c) respectively, and x_3 and x_4 correspond to the graphs in Figure 7, Case (d) and (e) respectively. This variety is called the *Segre cubic primal* and admits only 10 points as singularities, all of which are nodes. In general, a cubic

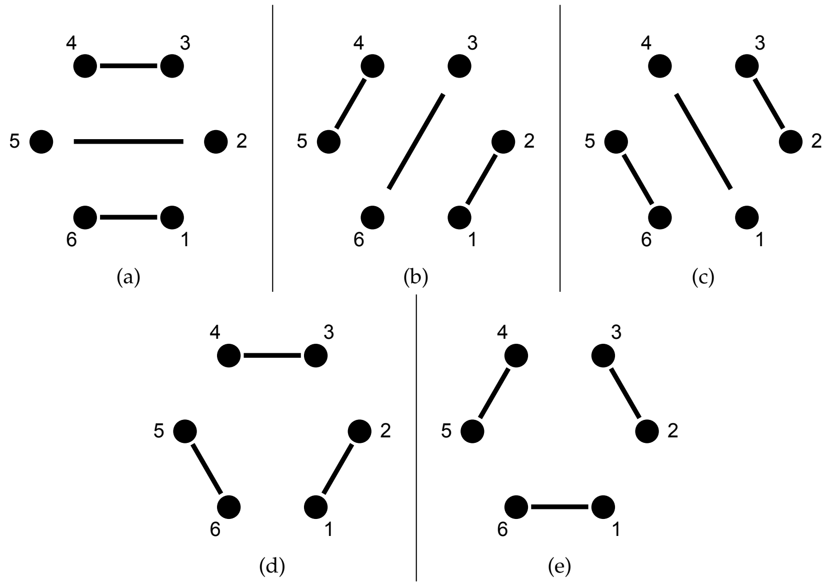


Figure 7: The five planar realizations of graphs with vertices on a regular hexagon, valency 1 and non-intersecting edges.

hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ has at most 10 nodes, and all hypersurfaces for which the maximal number of nodes is attained are projectively equivalent. The nodes of M_6 have a modular meaning, namely they correspond to classes of 6-tuples for which three points coincide. There are $\binom{6}{3} = 20$ different ways to have exactly three points coinciding out of 6, and the fiber of δ_6 over a node is not constituted by only one orbit, but rather two orbis: in fact, if we partition the set $\{1, \ldots, 6\}$ in two subsets $\{i, j, k\}$ and $\{u, v, w\}$, then the map δ_6 sends the tuples $(m_1, \ldots, m_6)$ for which $m_i = m_j = m_k$ to the same node of M_6 as the tuples for which $m_u = m_v = m_w$. Moreover, similarly as in the case $n = 5$, the variety M_6 contains linear spaces (in this case, planes) that parametrize 6-tuples for which two points coincide; there are $\binom{6}{2} = 15$ such planes. Here and in the following we denote by T_{ij} the plane in M_6 parametrizing 6-tuples $(m_1, \ldots, m_6)$ such that $m_i = m_j$. We have that each node is contained in 6 planes, and each plane contains 6 nodes (see also [Dol15]). Further properties of the threefold M_6 are explained in [Dol12, Subsection 9.4.4].

2 MÖBIUS PHOGRAMMETRY

The content of this chapter is based on the theory developed in [GNS15b], [GNS15c] and [GNS16].

Consider the outcome of Theorem 1.28: one of the possible necessary conditions for a pod Π to be mobile is the existence of two oriented directions in $\mathbb{R}^3$ such that, by orthogonally projecting the base and the platform points of Π along such directions, we obtain tuples of points in the plane that differ by an inversion or a similarity. Inversions and similarities are both particular instances of a more general family of transformations of the plane, called *Möbius transformations*. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then Möbius transformations are of the form

$$z \mapsto \frac{a\,z+b}{c\,z+d}, \qquad \text{where } ad-bc \neq 0 \text{ and } z \in \mathbb{C}.$$

These maps extend to morphisms of $\mathbb{P}^1_{\mathbb{C}}$ into itself given by

$$\begin{cases} (z:1) & \mapsto & \left(\frac{a\,z+b}{c\,z+d} : 1\right) & \text{if } z \neq -d/c \\ (-d/c:1) & \mapsto & (1:0) & \\ (1:0) & \mapsto & (a/c:1) & \end{cases}$$

with the convention that $(1:0) \mapsto (1:0)$ if $c=0$. If we are given two n-tuples $(m_1,\ldots,m_n)$ and $(\ell_1,\ldots,\ell_n)$ of points in plane $\mathbb{P}^1_{\mathbb{C}}$, we say that they are *Möbius equivalent* if there is a Möbius transformation sending m_i to ℓ_i for every $i \in \{1,\ldots,n\}$.

We can therefore read Case (a) of Theorem 1.28 as saying that one of the possible necessary conditions for an n-pod to be mobile is that "there exists at least one pair of orthogonal projections π_L and π_R such that the points $\pi_L(p_1),\ldots,\pi_L(p_n)$ and $\pi_R(P_1),\ldots,\pi_R(P_n)$ are Möbius equivalent".

It may hence be interesting to study the behavior of tuples of points in space under orthogonal projections, whose planar images are considered up to Möbius equivalence. In this chapter we will focus on 5-tuples and 6-tuples, and the problem we will deal with is the following: given a vector $\vec{A} = (A_1,\ldots,A_n)$ of n points in $\mathbb{R}^3$, where $n \in \{5,6\}$, we want to define a map $f_{\vec{A}}$, which we will call *Möbius map*, associating to each oriented direction ε the orthogonal projection of $\vec{A}$ along ε, considered up to Möbius equivalence (this is the content of Section 2.1). Then we want to be able to infer some properties of $\vec{A}$ from the knowledge of its Möbius map, and in particular we want to prove that two tuples of points whose Möbius maps have the same image are related by a similarity or an affine equivalence (this is the content of Section 2.2). Once we know the Möbius curves of the base and the platform of a hexapod we can bound the degree of its configuration curve, as we are going to prove in Section 2.3. Eventually, in Section 2.4 we will use the knowledge about Möbius maps to refine the results of Theorem 1.28.

2.1 DEFINITION OF THE MÖBIUS MAP

The domain of the Möbius map we want to define is, as we specified in the previous paragraph, the set of oriented directions in $\mathbb{R}^3$, which are parametrized by points in S^2. As we already saw in Chapter 1, the unit sphere S^2 can be put in bijection with an algebraic curve $C \subseteq \mathbb{P}^2_{\mathbb{C}}$: we are going to show that such a bijection not only allows to consider S^2 as an algebraic variety, but endows it also with the structure of a real variety, meant in the sense of the following definition.

Definition 2.1. A *real structure* on a complex variety is a pair (X, α), where X is a complex variety and α is an anti-holomorphic involution (see [Sil89, Proposition 1.3]).

Remark 2.2. An example of a real structure is given by the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ together with componentwise complex conjugation. One can prove that there exist exactly two real structures on $\mathbb{P}^1_{\mathbb{C}}$ (up to isomorphism), and they are given by the following two involutions:

$$(s, t) \mapsto (\bar{s}, \bar{t}) \quad \text{and} \quad (s : t) \mapsto (-\bar{t} : \bar{s}).$$

The fixed points of the first involution are exactly the points of $\mathbb{P}^1_{\mathbb{R}}$, while the second one does not have any fixed point.

Under the bijection between $\mathbb{P}^1_{\mathbb{C}}$ and the unit sphere S^2 provided by Equation (16), the second involution in Remark 2.2 becomes the antipodal map, namely the function that sends $S^2 \ni \varepsilon \mapsto -\varepsilon$. In this way S^2 becomes a real algebraic variety, whose anti-holomorphic involution is given by the antipodal map.

We read the identification between S^2 and the absolute conic in $\mathbb{P}^2_{\mathbb{C}}$ provided at the end of Section 1.2 in the light of the real structure of S^2.

Lemma 2.3. *The bijection* $\eta \colon S^2 \longrightarrow C = \{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ *determined by Equations* (14), (15) *and* (16) *makes the following diagram commutative:*

$$\begin{tikzcd} S^2 \arrow[r, "\text{antipodal map}"] \arrow[d, "\eta"', "\cong"] & S^2 \arrow[d, "\eta", "\cong"'] \\ C \arrow[r, "\text{componentwise}", "\text{conjugation}"'] & C \end{tikzcd} \tag{21}$$

Proof. By a direct computation, one sees that the map η is obtained in the following way. Let $\varepsilon \in S^2$, then pick $\varepsilon', \varepsilon''$ in the orthogonal space $\langle \varepsilon \rangle^{\perp}$ such that $\varepsilon', \varepsilon''$ form an orthonormal basis of $\langle \varepsilon \rangle^{\perp}$ and $\det\left(\,\varepsilon\ \varepsilon'\ \varepsilon''\,\right) > 0$. If $\varepsilon' = (\lambda', \mu', \nu')$ and $\varepsilon'' = (\lambda'', \mu'', \nu'')$, then η sends ε to

$$\varepsilon' + i\,\varepsilon'' = (\lambda' + i\lambda'', \mu' + i\,\mu'', \nu' + i\,\nu'') \in \mathbb{C}^3.$$

One can now check that η makes the diagram in Equation (21) commute. $\square$

Remark 2.4. We have the following triangle of isomorphisms of real varieties.

$$\begin{tikzcd} \left(S^2, \text{antipodal map}\right) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \left(\{x^2 + y^2 + z^2 = 0\}, \text{componentwise conj.}\right) \arrow[dl, leftrightarrow] \\ & \left(\mathbb{P}^1_{\mathbb{C}}, (s:t) \mapsto (-\bar{t} : \bar{s})\right) & \end{tikzcd}$$

Given $\varepsilon \in S^2$, one sees that the vectors ε' and ε'' determined in the proof of Lemma 2.3 satisfy $(\,\varepsilon\ \varepsilon'\ \varepsilon''\,) \in \mathrm{SO}(3, \mathbb{R})$. From this it follows that $(\,\varepsilon'\ \varepsilon''\,)^t$ gives the matrix of π_ε, an orthogonal projection along ε according to Definition 1.20. If we fix a point $A = (p, q, r)$ in $\mathbb{R}^3$ and we write $\eta(\varepsilon) = (x : y : z)$, then we can write $\pi_\varepsilon(A)$ as $p\,x + q\,y + r\,z$ once we identify the plane $\mathbb{R}^2$ with $\mathbb{C}$. If we change the representative $(x : y : z)$ of $\eta(\varepsilon)$, this modifies the image under the orthogonal projection π_ε by possibly a rotation and a dilation.

Hence we can realize the orthogonal projection π_ε as the dot product

$$\langle (x, y, z), \cdot \rangle : \mathbb{R}^3 \longrightarrow \mathbb{C},$$

where $(x : y : z)$ is any representative of $\eta(\varepsilon)$ with Hermitian norm equal to $\sqrt{2}$, namely such that $x\bar{x} + y\bar{y} + z\bar{z} = 2$. Thus we can view any orthogonal projection of n points $\vec{A} = (A_1, \ldots, A_n)$ as an n-tuple of points in $\mathbb{C}$, and so $\pi_\varepsilon(\vec{A})$ can be encoded as a single point in $\mathbb{C}^n$. We use the embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ sending z to $(z : 1)$ to identify $\pi_\varepsilon(\vec{A})$ with a point in $\left(\mathbb{P}^1_{\mathbb{C}}\right)^n$, and in this way we are in the position to consider the class of $\pi_\varepsilon(\vec{A})$ under the action of Möbius transformations, namely of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$.

We are ready to define the Möbius map for tuples of 5 or 6 points in $\mathbb{R}^3$.

Definition 2.5. Let $\vec{A}$ be a vector of n points in $\mathbb{R}^3$ where $n \in \{5, 6\}$. Let $\delta_n \colon \left(\mathbb{P}^1_{\mathbb{C}}\right)^n \dashrightarrow M_n$ be the quotient map defined in the Intermezzo. The *Möbius map* for $\vec{A}$ is the regular map

$$\begin{array}{rccl} f_{\vec{A}} : & C & \longrightarrow & M_n \\ & (x : y : z) & \mapsto & \delta_n\Big(\big(\pi_\varepsilon(A_1) : 1\big), \ldots, \big(\pi_\varepsilon(A_n) : 1\big) \Big) \\ & & & = \delta_n\Big(\big(\langle (x : y : z), A_1 \rangle : 1\big), \ldots, \big(\langle (x : y : z), A_n \rangle : 1\big) \Big) \end{array}$$

where C is the curve $\{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$, and $(x : y : z) \leftrightarrow \varepsilon$ under the bijection η established in Lemma 2.3. By pre-composing $f_{\vec{A}}$ by the parametrization of C described in Equation (14) we obtain a morphism $\mathbb{P}^1_{\mathbb{C}} \longrightarrow M_n$ that we denote by $\widetilde{f}_{\vec{A}}$.

Remark 2.6. Notice that the value of $f_{\vec{A}}$ on a point $(x : y : z) \in C$ is well-defined, namely it does not depend on the choice of the representative of $(x : y : z)$. This is in accordance to what we said so far, because changing the representative modifies the orthogonal projections $\langle (x : y : z), A_i \rangle$ by possibly a rotation and a dilation, and since both of them are Möbius transformations the image under the quotient map δ_n does not change.

Notice moreover that the point

$$\big(\langle (x : y : z), A_1 \rangle : 1\big), \ldots, \big(\langle (x : y : z), A_n \rangle : 1\big) \in \left(\mathbb{P}^1_{\mathbb{C}}\right)^n$$

does not necessarily lie in the domain of δ_n, and so a priori the maps $f_{\vec{A}}$ are only rational maps. However, since C is smooth, they extend to morphisms on the whole curve C.

Remark 2.7. The maps δ_5 and δ_6 are given by homogeneous polynomials of degree 5 and 3 respectively in the coordinates $\big((a_i : b_i)\big)$ of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^5$ and $\left(\mathbb{P}^1_{\mathbb{C}}\right)^3$. Hence the two corresponding maps $\widetilde{f}_{\vec{A}}$ are given by homogeneous polynomials of degree 10 and 6 respectively in the coordinates $(s : t)$ of $\mathbb{P}^1_{\mathbb{C}}$.

2.2 PROPERTIES OF THE MÖBIUS MAP

The following lemmata describe the behavior of the image of a Möbius map depending on the geometry of the vector $\vec{A}$.

Lemma 2.8. *Let* $\vec{A} = (A_1, \ldots, A_5)$ *be a* 5*-tuple of different coplanar points that are not collinear. Then the Möbius map* $f_{\vec{A}} \colon C \longrightarrow M_5$ *is* $2:1$ *to a rational curve of degree* 5, 4, 3, *or* 2 *in* M_5.

Proof. Since the points $\left(A_i\right)_{i=1}^5$ are planar, we can find a system of coordinates such that $A_i = (p_i, q_i, 0)$ for every i. Then the Möbius map $f_{\vec{A}}$ factors through the restriction to C of the projection $\tau_{x,y} \colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ sending $(x:y:z) \mapsto (x:y)$, which is a $2:1$ map. Hence we get

$$\begin{tikzcd} C \arrow[rr, "f_{\vec{A}}"] \arrow[dr, dashed, "\tau_{x,y}"'] & & M_5 \\ & \mathbb{P}^1_{\mathbb{C}} \arrow[ur, "g_{\vec{A}}"'] & \end{tikzcd}$$

If we show that $g_{\vec{A}}$ is birational, then $f_{\vec{A}}$ is $2:1$. The map $g_{\vec{A}}$ is given by 6 components, each of which is the product of five linear polynomials in x and y. Each of these polynomials is of the form $G_{ij} = x(p_i - p_j) + y(q_i - q_j)$. This in particular shows that if the lines $\overrightarrow{A_iA_j}$ and $\overrightarrow{A_hA_k}$ are parallel, then $G_{ij} = \lambda\, G_{hk}$ for some $\lambda \in \mathbb{C}$. The map $g_{\vec{A}}$ has the following structure:

$$\begin{array}{lcccccc} \left(g_{\vec{A}}\right)_0 & = & G_{12} & G_{23} & G_{34} & G_{45} & G_{15} \\ \left(g_{\vec{A}}\right)_1 & = & G_{12} & G_{25} & G_{15} & G_{34} & G_{34} \\ \left(g_{\vec{A}}\right)_2 & = & G_{12} & G_{23} & G_{13} & G_{45} & G_{45} \\ \left(g_{\vec{A}}\right)_3 & = & G_{23} & G_{34} & G_{24} & G_{15} & G_{15} \\ \left(g_{\vec{A}}\right)_4 & = & G_{34} & G_{45} & G_{35} & G_{12} & G_{12} \\ \left(g_{\vec{A}}\right)_5 & = & G_{14} & G_{45} & G_{15} & G_{23} & G_{23} \end{array} \qquad (22)$$

Since the 5 points are not collinear, only four configurations are allowed (after possibly relabeling the points), as shown in Figure 8. We deal with each of the four cases.

CASE (A) The components of $g_{\vec{A}}$ do not have factors in common, so

$$\deg\left(g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})\right) \cdot \deg\left(g_{\vec{A}}\right) = 5.$$

Hence either $g_{\vec{A}}$ is a birational map to a curve of degree 5, or it is a $5:1$ map to a line. However, if $g_{\vec{A}}\left(\mathbb{P}^1_{\mathbb{C}}\right)$ were a line, then it would coincide with one of the 10 lines in M_5. Recalling from the Intermezzo the modular meaning of lines in M_5, this would imply that every orthogonal projection maps the same two points of $\vec{A}$ to the same image, which is not possible. Hence $g_{\vec{A}}$ is birational.

CASE (B) Here G_{12}, G_{23} and G_{13} are equal up to scalar multiplication, so all the components have one factor in common, which can be removed. Hence

$$\deg\left(g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})\right) \cdot \deg\left(g_{\vec{A}}\right) = 4,$$

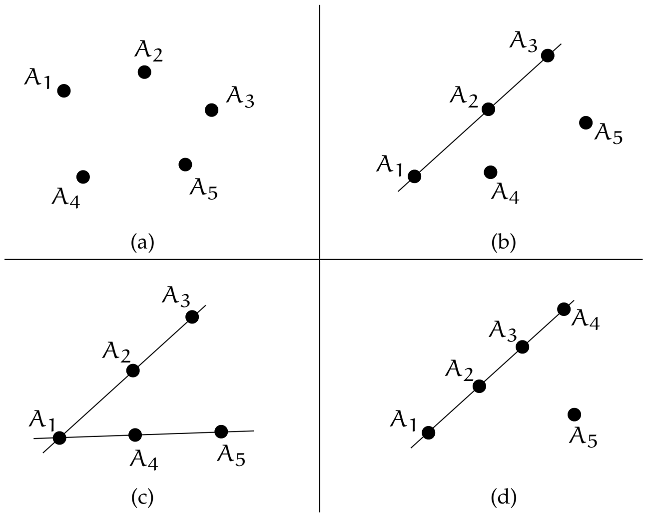


Figure 8: Possible configurations of 5 points in the plane: (a) no 3 points are aligned, (b) exactly 3 points are aligned, (c) $3+3$ points are collinear, (d) exactly 4 points are collinear.

this leading to three possibilities: $\deg(g_{\vec{A}}) = 1$, 2 or 4. The case when $\deg(g_{\vec{A}})$ is 4 can be discarded as in Case (a), hence it is enough to prove that it cannot happen that $\deg(g_{\vec{A}}) = 2$. In this case the image of $g_{\vec{A}}$ would be a conic, and by a direct check we see that it intersects the lines L_{14}, L_{15}, L_{24}, L_{25}, L_{34} and L_{35} (in general it will also intersect the line L_{45}, but this does not happen if $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_4A_5}$ are parallel). However, recall from the Intermezzo that conics in M_5 intersect only four lines, thus $g_{\vec{A}}(\mathbb{P}^1_{\mathbb{C}})$ cannot be one of them. Hence $g_{\vec{A}}$ can only be birational to a curve of degree 4.

CASE (C) Here G_{12}, G_{23} and G_{13} are equal up to scalar multiplication and the same holds for G_{14}, G_{45} and G_{15}. It follows that the components of $g_{\vec{A}}$ have two factors in common. Arguing as in Case (a) we can prove that $g_{\vec{A}}$ is birational to a curve of degree 3.

CASE (D) In this case G_{12}, G_{23}, G_{13}, G_{24}, G_{34} and G_{14} are equal up to scalar multiplication. One deduces that all components have three factors in common and so analogously as in Case (c) we have that $g_{\vec{A}}$ is birational to a curve of degree 2. □

Lemma 2.9. *Let $\vec{A} = (A_1, \ldots, A_6)$ be a 6-tuple of different points in $\mathbb{R}^3$. If the points $\vec{A}$ are coplanar, but not collinear, then the Möbius map $f_{\vec{A}} \colon C \longrightarrow M_6$ is $2:1$ to a rational curve of degree 3, 2 or 1 in M_6.*

Proof. As in the proof of Lemma 2.8, one sees that $f_{\vec{A}}$ factors through a $2:1$ map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ and a map $g_{\vec{A}} \colon \mathbb{P}^1_{\mathbb{C}} \longrightarrow M_6$. The following three cases are possible:

CASE (A) Suppose that no 4 points of $\vec{A}$ are collinear. Then the components of $f_{\vec{A}}$ do not have any factor in common, and $g_{\vec{A}}$ is birational to a cubic.

CASE (B) Suppose that 4 points of $\vec{A}$ are collinear, but no 5 points are so. Then the components of $f_{\vec{A}}$ have exactly one factor in common, and $g_{\vec{A}}$ is birational to a conic.

CASE (C) Suppose that 5 points of $\vec{A}$ are collinear. Then the components of $f_{\vec{A}}$ have exactly two factors in common, and $g_{\vec{A}}$ is birational to a line. □

Lemma 2.10. *Let $\vec{A} = (A_1, \ldots, A_5)$ be a 5-tuple of different points. If the points $\vec{A}$ are not coplanar, then the Möbius map $f_{\vec{A}} : C \longrightarrow M_5$ is birational to a rational curve of degree 10 or 8 in M_5.*

Proof. We argue as in the proof of Lemma 2.8: if we write H_{ij} for the linear polynomial $x(p_i - p_j) + y(q_i - q_j) + z(r_i - r_j)$, then the components of $f_{\vec{A}}$ have the same structure as described by Equation (22), where we replace G_{ij} by H_{ij}. Since the $\{A_i\}$ are not coplanar, we can have only three possibilities (after a possible relabeling of the points), showed in Figure 9.

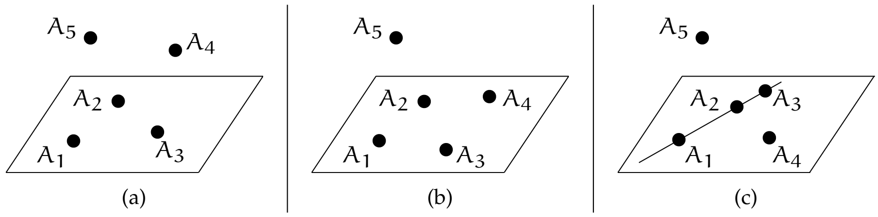


Figure 9: Possible configurations of 5 non-coplanar points in the space: (a) no 4 points are coplanar, (b) 4 coplanar points, no 3 of them aligned, (c) 3 aligned points.

CASE (A/B) Here the components of $f_{\vec{A}}$ do not have any common factor, hence either $f_{\vec{A}}$ is a birational map with image a degree 10 curve, or $f_{\vec{A}}$ is 2 : 1 to a curve of degree 5. We prove that the second case cannot happen, since it would imply that the points $\vec{A}$ are coplanar.

Claim. If $f_{\vec{A}}$ is 2 : 1, then there exists a regular map $r_{\vec{A}} : C \longrightarrow C$ respecting the real structure of C and such that

$$r_{\vec{A}}^2 = \mathrm{id} \quad \text{and} \quad f_{\vec{A}}\left(r_{\vec{A}}(\varepsilon)\right) = f_{\vec{A}}(\varepsilon).$$

Proof. Let $D = f_{\vec{A}}(C)$ and let $\widehat{D}$ be the normalization of D. Then $f_{\vec{A}}$ admits a lifting $\hat{f}_{\vec{A}} : C \longrightarrow \widehat{D}$. We define the involution $r_{\vec{A}}$ by sending a point $P \in C$ to the point $\hat{f}^*_{\vec{A}}\left(\hat{f}_{\vec{A}*}(P)\right) - P$, where $\hat{f}_{\vec{A}*} : \mathrm{Div}(C) \longrightarrow \mathrm{Div}(\widehat{D})$ and $\hat{f}^*_{\vec{A}} : \mathrm{Div}(\widehat{D}) \longrightarrow \mathrm{Div}(C)$ denote respectively the pushforward and the pullback induced by $\hat{f}_{\vec{A}}$ between the groups of divisors of the curves C and $\widehat{D}$ (for definitions and properties of these notions see, for example, [Har77, Appendix A]). The map $r_{\vec{A}}$ generically swaps the two elements in a fiber of $\hat{f}_{\vec{A}}$, and we passed to the normalization to have well-behaved properties of the pullback and pushforward. To see that $r_{\vec{A}}$ is a regular map one can employ a direct computation: locally the map $\hat{f}_{\vec{A}}$ can be written as the morphism corresponding to an inclusion

$$R \longrightarrow {}^{R[x]}\!\big/\!_{(x^2 + bx + c)},$$

and so $r_{\vec{A}}$ is locally given by the homomorphism $x \mapsto -b - x$ because it exchanges the two roots of the polynomial $x^2 + bx + c$, and therefore it is regular.

If we think of C as the unit sphere S^2, because of its properties $r_{\vec{A}}$ has to be a rotation of S^2 of 180° along an axis, which proves that $r_{\vec{A}}$ has two fixed points (the intersections of S^2 with the axis of rotation). By the modular interpretation of the lines L_{ij} provided in the Intermezzo we have that

$$f_{\vec{A}}^{-1}(L_{ij}) = \left\{ \frac{A_i - A_j}{\|A_i - A_j\|}, \frac{A_j - A_i}{\|A_i - A_j\|} \right\},$$

where we identify the domain C of $f_{\vec{A}}$ with the set of oriented directions in $\mathbb{R}^3$. On the other hand, if $\varepsilon \in f_{\vec{A}}^{-1}(L_{ij})$, then also $r_{\vec{A}}(\varepsilon) \in f_{\vec{A}}^{-1}(L_{ij})$, so there are only two options:

i. either $r_{\vec{A}}(\varepsilon)$ is the opposite direction of ε, and in this case we have that ε lies on a great circle of S^2 (the one orthogonal to the axis determined by $r_{\vec{A}}$) since $r_{\vec{A}}$ coincides with the antipodal map only on this great circle;

ii. or $r_{\vec{A}}(\varepsilon) = \varepsilon$, implying that ε is one of the two fixed points of $r_{\vec{A}}$.

If possibility i. happens for every L_{ij}, then the oriented directions of all lines $\overrightarrow{A_iA_j}$ lie on the same great circle of S^2, this implying that the points $\vec{A}$ are coplanar. If, instead, possibility ii. happens for some L_{ij}, then the assumption that no three points A_i are collinear implies that possibility ii. happens for exactly one line L_{ij}. Let us suppose that this line is L_{12}: this would imply that the points A_2, A_3, A_4 and A_5 are coplanar (in Case (a) here we would have already reached a contradiction) and the line $\overrightarrow{A_1A_2}$ is orthogonal to the plane on which the other points lie. On the other hand, the fact that all lines but L_{12} fall on possibility i. implies that also A_1, A_2, A_3 and A_4 are coplanar. Hence all points are coplanar, and we get a contradiction.

CASE (C) Here we have that H_{12}, H_{23} and H_{13} are equal up to a scalar factor, so the components of $f_{\vec{A}}$ have one factor in common, which can be removed. Thus four situations are possible: either $f_{\vec{A}}$ is birational to a curve of degree 8, or it is 2 : 1 to a curve of degree 4, or it is 4 : 1 to a conic, or it is 8 : 1 to a line. Arguments similar to the ones in the proof of Lemma 2.8, Case (a), rule out the last two situations. In order to prove that the 2 : 1 situation is not possible, we proceed as in Case (a): the curve $D = f_{\vec{A}}(C)$ does not meet all the lines L_{ij}, but the hypothesis on the configuration $\vec{A}$ ensures that D intersects L_{14}, L_{24}, L_{34}, L_{15}, L_{25}, L_{35} and L_{45}, which is enough to prove that the points are coplanar. □

Lemma 2.11. *Let $\vec{A} = (A_1, \ldots, A_6)$ be a 6-tuple of distinct points in $\mathbb{R}^3$. If the points $\{A_i\}$ are not coplanar, then the Möbius map $f_{\vec{A}} : C \longrightarrow M_6$ is birational to a rational curve of degree 6 or 4 in M_6.*

Proof. As we did in the proof of Lemma 2.10, we distinguish two cases that differ by the number of factors that the components of $f_{\vec{A}}$ have in common.

CASE (A) Here no 4 points are collinear. In this case the components of $f_{\vec{A}}$ do not have any factor in common, so

$$\deg\left(f_{\vec{A}}(C)\right) \cdot \deg\left(f_{\vec{A}}\right) = 6.$$

Thus there are only four possibilities: either $f_{\vec{A}}$ is 6 : 1 to a line, or it is 3 : 1 to a conic, or it is 2 : 1 to a cubic curve, or 1 : 1 to a sextic curve.

We prove that the first three situations can never happen. First of all, notice that there are exactly two directions in $\mathbb{R}^3$ for which the images of A_i and A_j under the projection coincide, namely the directions of the oriented lines $\overrightarrow{A_iA_j}$ and $\overrightarrow{A_jA_i}$. Denote by T_{ij} the plane in M_6 of classes of 6-tuples $(m_1, \ldots, m_6)$ where $m_i = m_j$. Then there are exactly 2 points in C that are mapped to the plane T_{ij} — and they are complex conjugate, since complex conjugation in C corresponds to the antipodal map in the unit sphere S^2.

If $f_{\vec{A}}$ is 3 : 1 or 6 : 1, then those two points have to be branching points of $f_{\vec{A}}$. Setting $A_i = (p_i, q_i, r_i)$, one can check that

$$f_{\vec{A}}^{-1}(T_{ij}) \;=\; \big\{(x : y : z) \in C \,:\, H_{ij}(x, y, z) = 0\big\},$$

where we recall that $H_{ij} = (p_i - p_j)\,x + (q_i - q_j)\,y + (r_i - r_j)\,z$. If the points in the preimage of T_{ij} were branching points, then the line $\{H_{ij} = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ would intersect C tangentially at those points. However, this is impossible, since both C and the line are real varieties, so if they are tangent their intersection point is real, but C has no real points. In this way we rule out the 3 : 1 and the 6 : 1 case.

Suppose now that $f_{\vec{A}}$ is 2 : 1. Then we argue exactly as in Lemma 2.10, obtaining a contradiction. Hence the birational case is the only possible one.

CASE (B) Here exactly 4 points are collinear. In this case the components of $f_{\vec{A}}$ have one factor in common, leading to

$$\deg\big(f_{\vec{A}}(C)\big) \cdot \deg\big(f_{\vec{A}}\big) \;=\; 4.$$

We have three possibilities: either $f_{\vec{A}}$ is 4 : 1 to a line, or it is 2 : 1 to a conic, or 1 : 1 to a quartic curve. Arguing as in Case (a) we prove the statement. $\square$

Remark 2.12. The properties of the Möbius map can be used to prove the following statement, proposed as a conjecture in [Lic07, Conjecture 2] and later proved in [Lic12, Proposition 3 and Theorem 4]: *there can only exist an infinite number of cylinders of revolution passing through five distinct points in $\mathbb{R}^3$ if the points are located on two parallel lines*. The key observation here is that when 5 points in $\mathbb{R}^2$ lie on a circle, then their corresponding class in M_5 is a real point. Therefore, if there are infinitely many cylinders passing through 5 points, then the Möbius curve of those points admits infinitely many real points. Since the conic C does not have any real point, it follows that the Möbius map cannot be birational. Hence the 5 points must be coplanar. From this it is well-known that the points actually have to lie on two parallel lines.

Now we state and prove the main result regarding 5-tuples of points.

Theorem 2.13. *Let $\vec{A}$ and $\vec{B}$ be two 5-tuples of points in $\mathbb{R}^3$ such that no 4 points in each tuple are collinear. Assume that $f_{\vec{A}}(C)$ and $f_{\vec{B}}(C)$ are equal as curves in M_5 and have degree $\geqslant 4$. If $\vec{A}$ is coplanar, then $\vec{B}$ is also coplanar and affine equivalent to $\vec{A}$. If $\vec{A}$ is not coplanar, then $\vec{B}$ is similar to $\vec{A}$.*

Proof. Suppose that $\vec{A}$ is not coplanar. Then by Lemma 2.10 we know that $f_{\vec{A}}$ is birational to a curve of degree 10 or 8. It follows that also $\vec{B}$ must be non-planar, because otherwise by Lemma 2.8 the degree of its Möbius curve would not be compatible with the assumptions. Thus $f_{\vec{B}}$ is birational, and by composing $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}^{-1}$ we get an isomorphism $\rho\colon \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} \mathbb{P}^1_{\mathbb{C}}$ that respects the real structure of $\mathbb{P}^1_{\mathbb{C}}$, since both $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}$ do so. Hence ρ is a rotation of S^2. Define $\vec{A}'$ to be the vector obtained applying ρ to $\vec{A}$. Then the diagram

$$\begin{tikzcd} & M_5 & \\ \mathbb{P}^1_{\mathbb{C}} \arrow[ur, "\widetilde{f}_{\vec{A}}"] \arrow[r, "\rho"'] & \mathbb{P}^1_{\mathbb{C}} \arrow[u, "\widetilde{f}_{\vec{A}'}"'] \arrow[r, "\mathrm{id}"'] & \mathbb{P}^1_{\mathbb{C}} \arrow[ul, "\widetilde{f}_{\vec{B}}"'] \end{tikzcd}$$

commutes, namely $\widetilde{f}_{\vec{A}}$ and $\widetilde{f}_{\vec{B}}$ coincide as maps. The goal now is to show that the directions $\overrightarrow{A_i'A_j'}$ and $\overrightarrow{B_iB_j}$ coincide for every i and j, this proving that $\vec{A}'$ and $\vec{B}$ are similar, and so the statement is proved. Suppose that $D = \widetilde{f}_{\vec{A}'}(\mathbb{P}^1_{\mathbb{C}})$ has degree 10. As in the proof of Lemma 2.10 we have

$$f_{\vec{A}'}^{-1}(L_{ij}) = \left\{ \frac{A_i' - A_j'}{\left\| A_i' - A_j' \right\|}, \frac{A_j' - A_i'}{\left\| A_i' - A_j' \right\|} \right\}$$

and similarly for $f_{\vec{B}}^{-1}(L_{ij})$. Since the two maps $\widetilde{f}_{\vec{A}'}$ and $\widetilde{f}_{\vec{B}}$ coincide, our claim is proved. In the situation when D has degree 8 the argument is the same, but in this case D does not intersect all the lines L_{ij}; however, knowing that $f_{\vec{A}'}^{-1}(D \cap L_{ij})$ and $f_{\vec{B}}^{-1}(D \cap L_{ij})$ are equal for $ij \in \{14, 24, 34, 15, 25, 35, 45\}$ (see Case (c) of Lemma 2.10) gives already enough information to prove that $\vec{A}'$ and $\vec{B}$ are similar.

Suppose that $\vec{A}$ is coplanar, then from Lemma 2.8 the map $f_{\vec{A}}$ is $2:1$ to a curve of degree 5 or 4 (we deal with the cubic and conic case in Remark 3.13). As in the previous case, also $\vec{B}$ must be coplanar for reasons concerning the degree of its Möbius curve. As in the proof of Lemma 2.8, we know that both $f_{\vec{A}}$ and $f_{\vec{B}}$ factor through a $2:1$ map from the conic C to $\mathbb{P}^1_{\mathbb{C}}$ followed by a birational map. By a change of coordinates we can suppose that this $2:1$ map is given by $(x : y : z) \mapsto (x : y)$. The situation is depicted in the following diagram:

$$\begin{tikzcd} & & & D & & & \\ C \arrow[urrr, "f_{\vec{A}}"] \arrow[r, "2:1"'] & \mathbb{P}^1_{\mathbb{C}} \arrow[urr, "\cong"'] & & & & \mathbb{P}^1_{\mathbb{C}} \arrow[ull, "\cong"] & C \arrow[ulll, "f_{\vec{B}}"'] \arrow[l, "2:1"] \end{tikzcd}$$

Thus we get an isomorphism $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} \mathbb{P}^1_{\mathbb{C}}$ that makes the previous diagram commute. If M is the invertible 2×2 matrix representing it, and we denote by $\vec{A}'$ the vector of points obtained by applying to $\vec{A}$ the affinity associated to M , then the following diagram commutes:

$$\begin{tikzcd}
& D & \\
C \arrow[ur, "f_{\vec{A}'}"] \arrow[r] & \mathbb{P}^1_{\mathbb{C}} \arrow[u] & C \arrow[ul, "f_{\vec{B}}"'] \arrow[l]
\end{tikzcd}$$

$$(x{:}y{:}z) \longmapsto (x{:}y) \longleftarrow\!\shortmid\ (x{:}y{:}z)$$

In this way we reached the point where $f_{\vec{A}'}$ and $f_{\vec{B}}$ are equal as maps, thus we can proceed as in the non-planar case, proving that $\vec{A}'$ and $\vec{B}$ are similar, so $\vec{A}$ and $\vec{B}$ are affine equivalent. □

We can describe an algorithm that takes as input the image of the Möbius map of a vector of points $\vec{A}$ satisfying the conditions of Theorem 2.13 and returns a vector of points $\vec{B}$ that is similar to $\vec{A}$. We describe the procedure in the case of non-planar points, when the degree of $f_{\vec{A}}(C)$ is 10. This is the easiest situation, because we have information about all the directions of the lines passing through the points of $\vec{A}$.

Algorithm Non-planar point reconstruction

Input: $D \subseteq M_5$, a degree 10 curve such that $f_{\vec{A}}(C) = D$.
Output: a vector $\vec{B}$ similar to $\vec{A}$.

1: **Parametrize** D via $\varphi\colon S^2 \longrightarrow D$ respecting the real structure of D.
2: **Compute** $\{\varepsilon_{ij}, -\varepsilon_{ij}\} = \varphi^{-1}(L_{ij})$ for all i, j.
3: **Set** $B_1 = (0, 0, 0)$.
4: **Pick** B_2 arbitrary on the line $\{B_1 + t\,\varepsilon_{12} : t \in \mathbb{R}\}$.
5: **Construct** B_3 as the intersection of the lines $\{B_1 + t\,\varepsilon_{13}\}$ and $\{B_2 + t\,\varepsilon_{23}\}$.
6: **Construct** B_4 using ε_{24} and ε_{34} as in Step 5.
7: **Construct** B_5 using ε_{35} and ε_{45} as in Step 5.
8: **Return** $\vec{B} = (B_1, \ldots, B_5)$.

Steps 5, 6 and 7 can be always executed, since the lines that are involved in their instructions always intersect, because we suppose we start from an existing configuration of points.

When the curve D has degree 8, 5, or 4 the algorithm is almost the same, we just have to take into account that D will not intersect all the lines L_{ij}: the ones that are disjoint from the image of $f_{\vec{A}}$ reveal which points in $\vec{B}$ are collinear, and the others can be used to identify the whole configuration.

Remark 2.14. The reconstruction algorithm cannot be performed when 4 points are collinear, i.e. when the degree of the image of the Möbius map is 2. In this case, in fact, it is not possible to reconstruct the direction $\overrightarrow{A_1A_4}$. In fact, $f_{\vec{A}}(C) \cap L_{14} = \emptyset$ since projecting along the direction $\overrightarrow{A_1A_4}$ gives a configuration where four points coincide, which is not allowed in M_5. In this case one can show that the images of two Möbius maps $f_{\vec{A}}(C)$ and $f_{\vec{B}}(C)$ are equal if and only if the cross ratios of the two 4-tuples of collinear points are equal. On the other hand, also when we have only three aligned points the image of the Möbius map does not intersect the line L_{13}, but in this case we can reconstruct the whole configuration regardless of the knowledge of $\overrightarrow{A_1A_3}$, since we can use $\overrightarrow{A_1A_4}$ and $\overrightarrow{A_1A_5}$ to determine A_1 starting from A_4 and A_5, and do the same for A_2 and A_3 — this procedure cannot be applied to the previous configuration. The two situations are described in Figure 10. We have to avoid also the situation when the curve

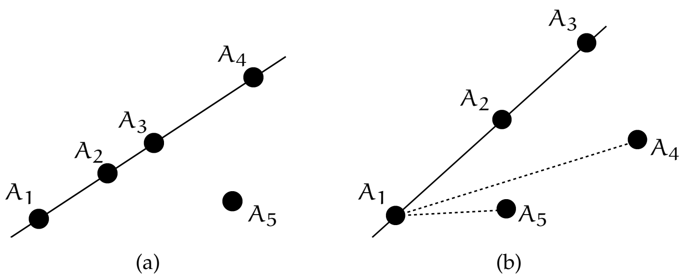


Figure 10: In the case of (a) four collinear points, the reconstruction algorithm does not work, since it is not possible to recover the direction of the line on which the four points lie. Instead, if we only allow (b) three collinear points, then the algorithm succeeds since we can reconstruct the aligned points using the other ones.

$f_{\vec{A}}(C) = f_{\vec{B}}(C)$ has degree 3, and this happens when, up to relabelling, the points A_1, A_2, A_3 are collinear, and also A_3, A_4, A_5 are collinear. In this case, in fact, the reconstruction of $\vec{B}$ from the curve $f_{\vec{A}}(C)$ is not unique up to similarity. Still, one knows that $\vec{B}$ lies on two lines and one can prove that B_3 is their intersection. In fact, denote $D = f_{\vec{A}}(C)$ and consider M_5 as the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at 4 points, so that its Picard group is generated by $L, E_1, \ldots, E_4$, where L is the strict transform of a line, and the E_i are the exceptional divisors. One can show that D is a component of a hyperplane section of M_5, and the other component is a conic. Since there are only five classes of conics in the Picard group of M_5, namely $L - E_i$ and $2L - E_1 - \cdots - E_4$, and since a hyperplane class is given by $H = 3L - E_1 - \cdots - E_4$, one concludes that it is possible to choose the blowup model of M_5 so that $[D] = L$, where $[\cdot]$ denotes the class in the Picard group. This means that D intersects the six lines $L - E_i - E_j$, while does not intersect the lines E_i. In our case, this means that D intersects L_{ij} with pairwise distinct $i, j \neq 3$, which implies that B_3 lies on both the lines carrying $\vec{B}$, namely it is their intersection. This leads to three possible cases, different up to similarity, which cannot be distinguished from the knowledge of $f_{\vec{A}}(C)$:

- B_1, B_2, B_3 and B_3, B_4, B_5 are collinear;
- B_1, B_4, B_3 and B_3, B_2, B_5 are collinear;
- B_1, B_5, B_3 and B_3, B_2, B_4 are collinear.

It is possible to extend the consequences of Theorem 2.13 to 6-tuples of points, and in general to n-tuples when $n > 5$. In order to do this, starting from an n-tuple $\vec{A}$ one can define a Möbius map $f_{\vec{A}} \colon C \longrightarrow M_n$, where M_n is the moduli space of n points in $\mathbb{P}^1_{\mathbb{C}}$, in the same way as we did in Definition 2.5. Then for every subtuple of 5 elements of $\vec{A}$, say $(A_1, \ldots, A_5)$, one has a commutative diagram:

$$\begin{tikzcd} C \arrow[r, "f_{\vec{A}}"] \arrow[dr, "f_{(A_1,\ldots,A_5)}"'] & M_n \arrow[d, dashed, "\xi"] \\ & M_5 \end{tikzcd} \tag{23}$$

where ξ is the rational map associating the equivalence class of the n-tuple $(m_1, \ldots, m_n)$ to the equivalence class of the 5-tuple $(m_1, \ldots, m_5)$.

Corollary 2.15. *Theorem 2.13 is true also in the case of two n-tuples $\vec{A}$ and $\vec{B}$, where $n \geqslant 5$, for which no $n-1$ points are collinear, and no 5 points lie on two lines.*

Proof. We prove the statement by reducing to the $n = 5$ case and applying Theorem 2.13.

Suppose that $\vec{A}$ is not coplanar; we want to prove that $\vec{A}$ and $\vec{B}$ are similar. After possibly relabeling the points, we can suppose that $A_1, \ldots, A_4$ are not coplanar. By hypothesis we have that $f_{\vec{A}}(C) = f_{\vec{B}}(C)$, so using suitable "forgetful" projections $M_n \dashrightarrow M_5$ as in Diagram (23) we get that for every $k \geqslant 5$ it holds

$$f_{(A_1,\ldots,A_4,A_k)}(C) = f_{(B_1,\ldots,B_4,B_k)}(C).$$

From Theorem 2.13 it follows that for all $k \geqslant 5$ the two 5-tuples $(A_1, \ldots, A_4, A_k)$ and $(B_1, \ldots, B_4, B_k)$ are not coplanar, and they are similar. Since there exists a unique similarity sending $(A_1, \ldots, A_4)$ to $(B_1, \ldots, B_4)$, the same similarity must send A_k to B_k for all $k \geqslant 5$. Hence $\vec{A}$ and $\vec{B}$ are similar.

If $\vec{A}$ is coplanar, then from the commutativity of Diagram (23) and by Theorem 2.13 we obtain that also $\vec{B}$ is coplanar. Now we can proceed as before to prove the statement, but here in order to be able to use Theorem 2.13 we have to make sure that we can choose $A_1, \ldots, A_4$ so that for every $k \geqslant 5$ there are no 4 collinear points among $A_1, \ldots, A_4, A_k$. This is ensured by the hypothesis that no $n-1$ among the points $\vec{A}$ are collinear, since the latter is the only case when this choice cannot be made. Hence we can conclude as before, since an affinity is completely determined by the image of 3 non-collinear points. □

We conclude the section with two results that will be of help in Chapter 3.

Lemma 2.16. *Let $\vec{A}$ be a 6-tuple of points in $\mathbb{R}^3$ and let $f_{\vec{A}}\colon C \longrightarrow M_6$ be its Möbius map, where the resulting Möbius curve is of degree 6. Then $f_{\vec{A}}$ extends to a morphism $F_{\vec{A}}\colon \mathbb{P}^2_{\mathbb{C}} \longrightarrow M_6$.*

Proof. We need to prove that the map $F_{\vec{A}}$ does not have base points in $\mathbb{P}^2_{\mathbb{C}}$. By a direct inspection of the structure of the map $f_{\vec{A}}$ we notice that a base point has to vanish on at least one polynomial H_{ij} (defined as in Lemma 2.10) for each component of $F_{\vec{A}}$. This would imply that at least 4 points of $\vec{A}$ are collinear, but this is impossible because otherwise the curve $f_{\vec{A}}(C)$ would not have degree 6, contradicting the hypothesis. □

Proposition 2.17. *Let D be a smooth Möbius curve of degree 6. Then*

- *D can be defined by real polynomials, but has no real points;*

· D *is contained in a linear projection, defined by real polynomials, of the Veronese embedding of* $\mathbb{P}^2_{\mathbb{C}}$ *given by cubics.*

Proof. By construction D is a real variety, since $f_{\vec{A}}$ is a real map and C is a real variety; since D is smooth and using Lemma 2.11 we have that $f_{\vec{A}}$ is an isomorphism, hence D has no real points, because this holds for C. From Lemma 2.16 we get that D is contained in a linear projection, defined by real polynomials, of the third Veronese embedding of $\mathbb{P}^2_{\mathbb{C}}$, which is real by construction. One notices that such projections is the complete intersection of M_6 with another cubic hypersurface. □

2.3 BOUNDING THE CONFORMAL DEGREE

At the beginning of the chapter, the study of Möbius curves was suggested by results coming from the study of bonds of pods; in particular Möbius curves are defined in such a way that the the presence of an inversion or a similarity bond implies that the Möbius curves of the base and the platform intersect. In this section we refine this connection showing that, if we exclude some degenerate situations, in the case of pods with one degree of freedom the number of intersections of the Möbius curves of base and platform gives a bound for the degree of the configuration curve of the pod. We focus in particular on the case of hexapods, because this will be useful in Chapter 3.

Definition 2.18. A pod $\Pi = \left(\vec{p}, \vec{P}, \vec{d}\right)$ is called *equiform* if there exists a similarity of $\mathbb{R}^3$ sending the base $\vec{P}$ to the platform $\vec{p}$.

From Corollary 2.15 we see that for a non-planar and non-equiform pod the Möbius curves of the base and the platform are different, and this justifies why we are going to suppose that the hexapods we will deal with satisfy these two properties.

Let Π be a hexapod of mobility one, and let K_Π be its configuration curve. In order to obtain a bound of the degree of K_Π, we notice that we can compute such number by intersecting K_Π with the hyperplane H defining the boundary, and keeping in mind that if B is the boundary, namely $B = (X \cap H)_{\mathrm{red}}$, then we have $X \cap H = 2B$ as divisors in X. This means that in particular K_Π and H do not intersect tranversally, and so we cannot simply count the number of set-theoretical intersections. However, if $\widetilde{K}_\Pi$ denotes the top-dimensional part of K_Π — namely $\widetilde{K}_\Pi$ is an equidimensional scheme of dimension one — we have that:

$$\begin{aligned}
\deg K_\Pi = \deg \widetilde{K}_\Pi &= \deg\left(\widetilde{K}_\Pi \cap H\right) \\
&\leqslant \deg\left(K_\Pi \cap H\right) \\
&= \deg\left(\Lambda_\Pi \cap X \cap H\right) \\
&\leqslant 2 \deg\left(\Lambda_\Pi \cap B\right),
\end{aligned}$$

where Λ_Π is the linear space defined by the spherical conditions determined by the legs of Π.

Theorem 2.19. *Let Π be a hexapod of mobility one. Let $D_1 = f_{\vec{p}}(C)$ and $D_2 = f_{\vec{p}}(C)$ be the Möbius curves of its base and platform. Suppose that Π is non-planar, non-equiform and no 4 base or platform points are on a line. Then*

$$\deg K_\Pi \leqslant 2 \deg(D_1 \cap D_2),$$

where the intersection $D_1 \cap D_2$ is meant scheme-theoretically, namely the points of intersection are counted with multiplicity.

Proof. From the previous discussion we see that it is enough to prove that the degree of the intersection $\Lambda_\Pi \cap B$ is less than or equal to the degree of the intersection $D_1 \cap D_2$ of the two Möbius curves of Π. To do this, we revise the connection between B and the conic

$$C = \left\{ (x : y : z) \in \mathbb{P}^2_{\mathbb{C}} \,:\, x^2 + y^2 + z^2 = 0 \right\}$$

we started to establish at the end of Section 1.2. In particular, recall that we can associate to every inversion, similarity and butterfly point two elements $v, w \in C$, namely their left and right vectors (see Definition 1.15). Notice that our assumption on the hexapod Π rules out the existence of collinearity bonds, and so it is harmless to exclude them from the picture.

$$X \cap H = \left\{ (h : M : x : y : r) \,:\, \begin{array}{ll} h = 0, & MM^t = M^t M = 0 \\ M^t y = Mx = 0, & \langle x, x \rangle = \langle y, y \rangle = 0 \end{array} \right\}.$$

If we define

$$B' = \Big\{ \beta \in B \,:\, M \neq 0 \text{ or } (x \neq 0 \text{ and } y \neq 0) \Big\},$$

then the reduced structure of B' is given by

$$\left(B'\right)_{\text{red}} = B \setminus \Big(\{\text{collinearity points}\} \cup \{\text{vertex}\} \Big).$$

If $\beta \in B'$, then we know from Section 1.2 that there exist $v, w \in C$ and $\lambda, \mu, \alpha \in \mathbb{C}$ such that

$$M = \alpha v w^t, \quad x = \mu w, \quad y = \lambda x, \tag{24}$$

and from the definition of B' we get that either $\alpha \neq 0$ or $\lambda\mu \neq 0$. We reformulate the association of left and right vectors we described in Definition 1.15 by means of the following map:

$$\begin{array}{rccc} \delta\colon & B' & \longrightarrow & C \times C \\ & (0 : M : x : y : r) & \mapsto & (v, w) \end{array}$$

The fiber of δ over a point (v, w) is isomorphic to

$$\delta^{-1}(v, w) \cong \left\{ \vec{\omega} = (\alpha : \lambda : \mu : r) \in \mathbb{P}^3_{\mathbb{C}} \,:\, \alpha \neq 0 \text{ or } \lambda\mu \neq 0 \right\}.$$

As remarked before, our hypotheses imply that the scheme of bonds $B_\Pi = B \cap \Lambda_\Pi$ is a closed subscheme of B'.

Claim. *The map $\delta_{|B_\Pi}$ is an isomorphism.*

Recall that the pseudo spherical condition from Equation (17) imposed by a pair (P, p) of base and platform points on a boundary point $(0 : M : x : y : r)$ reads as

$$r - 2\,\langle Mp, P\rangle - 2\,\langle P, y\rangle - 2\,\langle p, x\rangle \;=\; 0.$$

Using Equation (24), and setting $W_i = P_i^t w$ and $V_i = p_i^t v$ yields

$$r - 2\alpha W_i V_i - 2\mu W_i - 2\lambda V_i \;=\; 0.$$

The scheme B_Π is cut out by these 6 pseudo spherical conditions for $i \in \{1, \ldots, n\}$. Hence, if we define the 4×6 matrix

$$N_\Pi(v, w) \;=\; \begin{pmatrix} 2\,W_1\,V_1 & 2\,W_1 & 2\,V_1 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 2\,W_6\,V_6 & 2\,W_6 & 2\,V_6 & -1 \end{pmatrix},$$

then the scheme B_Π is locally defined by

$$\Big\{(v, w, \vec{\omega}) \in \mathcal{V} \times \mathcal{V} \times \mathcal{W} \,:\, N_\Pi(v, w) \cdot \vec{\omega} = 0\Big\},$$

where $\mathcal{V}$ and $\mathcal{W}$ are suitable open subvarieties of C and $\mathbb{P}^3_{\mathbb{C}}$ respectively, and $\vec{\omega} = (\alpha : \lambda : \mu : r)$. Our claim then becomes:

Claim. *The map δ maps B_Π isomorphically to the scheme in $C \times C$ cut out by the 4×4 minors of N_Π.*

We prove that the rank of the matrix $N_\Pi(v, w)$ is always at least 3. In fact, if the rank is 1, then the collinearity hypothesis is violated (all W_i would be equal, and the same for the V_i); if the rank is 2, then the planarity condition is violated (this can be deduced by a direct computation, for example imposing that the second column of N_Π is a linear combination of the third and the fourth). Hence the rank of $N_\Pi(v, w)$ is greater than of equal to 3. Notice that column operations on N_Π correspond to projective transformations on the fibers of δ, while row operations on N_Π correspond to the choice of a different system of generators for the ideal of B_Π. Hence we can reduce to the case when N_Π has the form

$$N_\Pi(v, w) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J_1(v, w) \\ 0 & 0 & 0 & J_2(v, w) \\ 0 & 0 & 0 & J_3(v, w) \end{pmatrix},$$

where $J_i(v, w)$ are rational functions on $C \times C$. The local description of B_Π becomes

$$\left\{ (v, w, \vec{\omega}') \,:\, \begin{array}{l} \vec{\omega}' = (0 : 0 : 0 : r') \;\text{ for some } r' \neq 0, \;\text{ and} \\ J_1(v, w)\, r' = J_2(v, w)\, r' = J_3(v, w)\, r' = 0 \end{array} \right\},$$

while the zero locus of the 4×4 minors of N_Π is locally given by

$$\big\{(v, w) \,:\, J_1(v, w) = J_2(v, w) = J_3(v, w) = 0\big\},$$

and one sees that these two schemes are isomorphic.

The proof is then complete once we are able to show the following.

Claim. *The image of* B_Π *under* δ *is contained in the pullback of the two Möbius maps* $f_{\vec{p}}, f_{\vec{P}} \colon C \longrightarrow M_6$.

Notice that the fact that Π is supposed to be non-equiform implies that the Möbius curves of $\vec{p}$ and $\vec{P}$ are different. The pullback of the two maps is

$$\Big\{(v,w) \in C \times C \,:\, f_{\vec{p}}(v) = f_{\vec{P}}(w)\Big\}.$$

The coordinates of $f_{\vec{p}}(v)$ are obtained by substituting each term H_{ij} (defined as in Lemma 2.10) by $V_i - V_j$; for $f_{\vec{P}}(w)$ one just needs to consider $W_i - W_j$ instead. Then the pullback is the scheme cut out by the 2×2 minors of the following 2×5 matrix:

$$\begin{pmatrix} W_{12,36,45} & W_{14,23,56} & W_{16,25,34} & W_{16,23,45} & W_{12,34,56} \\ V_{12,36,45} & V_{14,23,56} & V_{16,25,34} & V_{16,23,45} & V_{12,34,56} \end{pmatrix},$$

where

$$W_{ij,kl,mn} = (W_i - W_j)(W_k - W_l)(W_m - W_n)$$

and similarly for $V_{ij,kl,mn}$. A direct computation shows that the ideal generated by such 2×2 minors is contained in the ideal of the 4×4 minors of N_Π. This settles the claim and hence concludes the proof. □

Remark 2.20. We cannot hope for equality in the last claim in the proof of Theorem 2.19. In fact, consider one of the 15 planes T_{st} in M_6 and suppose it parametrizes classes of tuples $(m_1, \ldots, m_6)$ where $m_s = m_t$; then the projection from such a plane maps M_6 to $\mathbb{P}^1_{\mathbb{C}}$, and the latter has a modular interpretation as M_4, namely the moduli space of 4-tuples $(m_1, \ldots, \widehat{m}_s, \ldots, \widehat{m}_t, \ldots, m_6)$ obtained by removing m_s and m_t. The images under the projection from T_{st} of the points $f_{\vec{p}}(v)$ and $f_{\vec{P}}(w)$ coincide if and only if

$$\begin{aligned}(W_i - W_j)(W_k - W_l)(V_i - V_l)(V_k - V_j) -& \\ - (V_i - V_j)(V_k - V_l)(W_i - W_l)(W_k - W_j) \;&=\; 0,\end{aligned} \tag{25}$$

where $\{i,j,k,l\} \cup \{s,t\} = \{1,\ldots,6\}$. On the other hand, the left hand side of Equation (25) is a 4×4 minor of N_Π. In fact, if we select the submatrix with rows of index i, j, k and l and we perform column operations we obtain

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ W_jV_j - W_iV_i & W_j - W_i & V_j - V_i & 1 \\ W_kV_k - W_iV_i & W_k - W_i & V_k - V_i & 1 \\ W_lV_l - W_iV_i & W_l - W_i & V_l - V_i & 1 \end{pmatrix},$$

and a direct computation proves the equality. As it is shown in the second claim of the proof, the zero locus of these minors is the image of B_Π under δ.

On the other hand, there are points in the pullback of $f_{\vec{p}}$ and $f_{\vec{P}}$ for which not all the cross-ratios of Equation (25) are equal: suppose in fact the points p_1, p_2 and p_3 are collinear along the direction v, and the points

P_1, P_2 and P_3 are collinear along the direction w; then $f_{\vec{P}}(v)$ and $f_{\vec{P}}(w)$ coincide with one of the nodes of M_6, independently of the projections of the other points; hence it is possible to have situations in which not all the 15 cross-ratios coincide, but still we have an intersection of the two Möbius curves.

Remark 2.21. In contrast to the negative outcome of Remark 2.20, one notices that if a configuration curve admits only distinct inversion points, then its degree is exactly twice as the number of such inversion points.

2.4 A NECESSARY CONDITION FOR PENTAPODS WITH MOBILITY TWO

We can finally apply the theory we developed so far to get necessary conditions for the mobility of pentapods. In fact, we can use the results of this chapter to refine Case (a) of Theorem 1.28 when $n = 5$.

Theorem 2.22. *Let $\Pi = (\vec{p}, \vec{P}, \vec{d})$ be a pentapod with mobility 2 or higher. Then one of the following conditions holds:*

(a) the platform and the base are similar;

(b) the platform and the base are planar and affine equivalent;

(c) there exists $m \leqslant 5$ such that $p_1, \ldots, p_m$ are collinear and $P_{m+1}, \ldots, P_5$ coincide, up to permutation of indices and interchange of platform and base;

(d) the points p_1, p_2, p_3 lie on a line $g \subseteq \mathbb{R}^3$ and p_4, p_5 lie on a line $g' \subseteq \mathbb{R}^3$ parallel to g, and P_1, P_2, P_3 lie on a line $G \subseteq \mathbb{R}^3$ and P_4, P_5 lie on a line $G' \subseteq \mathbb{R}^3$ parallel to G, up to permutation of indices.

(e) up to permutation of indices the following triples of points are collinear:

$$P_1, P_2, P_3 \qquad P_3, P_4, P_5 \qquad p_3, p_1, p_i \qquad p_3, p_j, p_k$$

with pairwise distinct $i, j, k \in \{2, 4, 5\}$. Moreover the points $P_1, \ldots, P_5$ are pairwise distinct as well as the points $p_1, \ldots, p_5$.

Proof. Since Π has mobility at least 2, then by Theorem 1.28 either we are in Case (c) or (d), or there are infinitely many pairs (L, R) of elements of S^2 such that the points $\pi_L(p_1), \ldots, \pi_L(p_5)$ and $\pi_R(P_1), \ldots, \pi_R(P_5)$ differ by an inversion or a similarity. Let us consider then this last case.

First of all, we can assume that base and platform points are pairwise distinct: in fact, let us suppose that $P_1 = P_2$, after a possible relabeling. Then the existence of infinitely many pairs of projections and corresponding inversions/similarities implies that $p_1 = p_2$, but then Π is a quadripod, and not a pentapod anymore. If three, four or five base or platform points coincide, then we are again in Case (c), so we can suppose that base and platform points are pairwise distinct. This allows us to make use of Möbius photogrammetry. Since we can suppose that no 4 point of the base or platform are aligned (otherwise we are in Case (c) or (d)), it follows that the Möbius maps $f_{\vec{p}}$ and $f_{\vec{P}}$ of base and platform points of Π are not constant. Re-interpreting the assumption in the language developed in this chapter, we have that $f_{\vec{p}}(C)$ and $f_{\vec{P}}(C)$ have infinitely points in common. Since both are irreducible algebraic curves, they must coincide, and we get Case (a) or (b) or (e) by Theorem 2.13 and by Remark 2.14. □

Remark 2.23. For quadripods the analogous statement of Theorem 2.22 does not hold. In fact, all quadripods have mobility at least 2, but the general quadripod does not fulfill any of the conditions (a)–(e) of the theorem. For tripods the statement is trivially true, since conditions (b) and (c) are always fulfilled.

Consider a twisted cubic $D \subseteq \mathbb{P}^3_{\mathbb{C}}$, namely the image of the Veronese morphism $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$. It is well-known that the ideal of D is generated by three quadrics, and not two as one could expect being D a curve. If we write $I(D) = (q_1, q_2, q_3)$, and we denote by Q_i the zero set of q_i, then the intersection $Q_1 \cap Q_2$ is a quartic curve Y containing the twisted cubic D. It follows that $Y = D \cup L$, where L is a line. In this case we say that D and L are *linked* via the curve Y.

The notion of *linkage* (or *liaison*), which has its roots in the works of Severi, Gaeta and others (see for example [MR10]) and was given a modern formulation in the fundational paper by Peskine and Szpiro (see [PS74]), is not restricted to curves in $\mathbb{P}^3_{\mathbb{C}}$, but can be considered for varieties of arbitrary dimension in any projective space: the key property is that their union gives a *complete intersection*, namely a variety defined by as many equations as its codimension in the ambient space, as happens in the previous example with the twisted cubic and the line. One of the main features of linkage is that it allows to transfer properties between linked curves, and so it permits to study complicated objects by means of simpler ones. In particular, in Chapter 3 we will consider linked curves in $\mathbb{P}^4_{\mathbb{C}}$ and we will infer properties of one of the two curves from the properties of the other.

To give a hint of what kind of relations may exist between linked curves, we consider the case of two smooth curves C and D in $\mathbb{P}^3_{\mathbb{C}}$ of degree c and d respectively, whose union Y is the complete intersection of two smooth surfaces F and G of degree f and g respectively (this example is taken from Dasaratha's thesis, see [Das13, Section 3.3]). Let us denote by $p_a(C)$ and $p_a(D)$ the genus of C and D, respectively. From the formula for the genus of a smooth curve on a surface, applied to C as a curve on F, we get

$$C^2 = 2\,p_a(C) - 2 - (f-4)c.$$

As divisors on F, we have by construction that $C + D = gH$, where H is the hyperplane divisor. So the intersection number of C and D is

$$C \cdot D = (f+g-4)c - \big(2\,p_a(C) - 2\big).$$

If we repeat the computation starting this time from D as a curve on G, we obtain

$$C \cdot D = (f+g-4)d - \big(2\,p_a(D) - 2\big).$$

It follows then that

$$(f+g-4)(c-d) = 2\big(p_a(C) - p_a(D)\big).$$

This means that the difference of the genera of C and D is proportional to the difference of their degrees. Such a result holds in greater generality, as one can find in [Mig98, Chapter 3, Corollary 5.2.14]. For our purposes, we just state the version of the previous result in the case of two curves in $\mathbb{P}^4_{\mathbb{C}}$, after having recalled the modern definition of linkage (see [Mig98, Chapter 3, Definition 5.1.1]).

Definition. Let V and W be two projective sets in $\mathbb{P}^n_{\mathbb{C}}$ such that no component of V is contained in a component of W and conversely. Then V is *geometrically linked* to W by a projective set Y if $V \cup W = Y$ and Y is a complete intersection. In terms of ideals, we have that $I(V) \cap I(W) = I(Y)$.

It follows that if V and W are linked via Y, then

$$\deg(V) + \deg(W) \;=\; \deg(Y),$$

and moreover both V and W are *equidimensional*, namely all their components have the same dimension. It also follows that the ideals of V and W can be reconstructed from each other (knowing the ideal of Y) via the colon operation:

$$I(Y) : I(V) \;=\; I(W) \quad \text{and} \quad I(Y) : I(W) \;=\; I(V).$$

Proposition. *Let D and D' be two projective curves in $\mathbb{P}^4_{\mathbb{C}}$ linked by a complete intersection $Y = F_1 \cap F_2 \cap F_3$ and let $p_a(D)$ and $p_a(D')$ be their arithmetic genera. Then*

$$p_a(D) - p_a(D') \;=\; \frac{1}{2}(t-5)(\deg D - \deg D'),$$

where $t = \deg(F_1) + \deg(F_2) + \deg(F_3)$.

This proposition implies in particular that if the curves D and D' have the same degree, then they have the same genus. This is what we are going to use in Proposition 3.5 and — except for a result about the canonical divisor of linked curves that will be used in the same proposition — it is everything one needs to know about linkage to be ready for reading Chapter 3.

3 LIAISON HEXAPODS

This chapter is based on the results from [GNS16]. In Section 3.1 we propose a method to construct a family of mobile hexapods that — to our knowledge — has never appeared in the literature. In Section 3.2 we determine a bound for the conformal degree of a mobile hexapod if we exclude some known cases, and this gives a strong indication that the family we constructed is maximal, namely cannot be obtained as a special case of a larger family.

3.1 CONSTRUCTION OF LIAISON HEXAPODS

This section is devoted to the construction of a family of mobile hexapods, that we call *liaison hexapods*. Studying the behavior of the Möbius map for a general 6-tuple of points in $\mathbb{R}^3$, we propose a technique to determine a candidate 6-tuple of platform, once a choice of 6 general base points has been made (Section 3.1.1). Then we compute a suitable dilation of the platform points, together with leg lengths, ensuring that the corresponding hexapod is mobile (Section 3.1.2).

3.1.1 *The platform*

We begin with the definition of the candidate platform, once we are given a base constituted of 6 general points.

We design the platform in order to maximize the number of intersections between the Möbius curves of base and platform. From Theorem 1.28 and from the theory developed in Chapter 2 we know that such intersections are indicators of the possibility of having a mobile hexapod, although they alone do not provide in general sufficient conditions for mobility. To construct such platform we start noticing that the Möbius curve of a general base is a component of a complete intersection of M_6 with two quadrics. Using basic results in liaison theory we are able to show that the residual curve to the Möbius curve of the base in the complete intersection shows several properties satisfied by Möbius curves (Proposition 3.5 and Remark 3.6). This led us to conjecture that there exists a 6-tuple of points in $\mathbb{R}^3$ whose Möbius curve is this residual curve (Conjecture 1). Unfortunately, we have not been able to prove this rigorously; however, every single concrete example we computed fulfills this requirement. We hence have a way that, at least in practice, allows to associate to a general base, a candidate for the platform points of a hexapod. Using again some basic tools in liaison theory, we are able to prove that there are 14 intersections between the Möbius curves of base and platform created in this way (Proposition 3.5).

Because of the nature of the Möbius map, the 6-tuple of platform points can be scaled by an arbitrary positive number without modifying the Möbius curve; as we are going to see in Section 3.1.2, it is possible to determine a suitable scaling and leg lengths leading to a mobile hexapod.

The process we are going to present can be applied only if the base points we choose are not in a "degenerate" configuration. Definition 3.1 specifies this condition.

Definition 3.1. A 6-tuple of points in $\mathbb{R}^3$ is called *Möbius-general* if its Möbius curve is smooth, the ideal of the Möbius curve contains only two linearly independent quadratic forms, and these two quadratic forms cut out a one-dimensional set from M_6 consisting of two smooth curves.

The discussion from Lemma 3.2 to Remark 3.4 clarifies that being Möbius general is a genereal condition, namely that Möbius general 6-tuples form an open subset of the variety of 6-tuples of different points in $\mathbb{R}^3$.

Lemma 3.2. *The Möbius curve of a general* 6*.tuple of points in* $\mathbb{R}^3$ *is a smooth sextic curve contained in the the complete intersection of* M_6 *and two quadrics.*

Proof. Let D be the Möbius curve of $\vec{A}$. Since $\vec{A}$ is general, we can suppose in particular that it is non-planar, so from Lemma 2.11 the degree of D is 6. We show that D is smooth. Since D is rational and of degree 6, then it defines a point $[D]$ in the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^4_{\mathbb{C}}, 6t+1)$ of subschemes of $\mathbb{P}^4_{\mathbb{C}}$ with Hilbert polynomial $6t+1$. Consider the map $\xi \colon \mathcal{V} \longrightarrow \mathrm{Hilb}(\mathbb{P}^4_{\mathbb{C}}, 6t+1)$ that associates to a 6-tuple $\vec{A}$ the point in the Hilbert scheme defined by the image of its Möbius map $f_{\vec{B}}$; the morphism ξ is defined on a suitable open set $\mathcal{V}$ of $\left(\mathbb{R}^3\right)^6$. Since smooth rational sextics form an open subset of $\mathrm{Hilb}(\mathbb{P}^4_{\mathbb{C}}, 6t+1)$, if we are able to show that for a particular 6-tuple $\vec{A}$ the curve D is smooth, then for all $\vec{A}$ belonging to some open set $\mathcal{W} \subseteq \mathcal{V}$ the Möbius curve of $\vec{A}$ will be smooth. By considering the following tuple $\vec{A}$

$$\begin{aligned} A_1 &= (0,0,0), & A_2 &= (2,0,0), & A_3 &= (3,2,0), \\ A_4 &= (2,3,1), & A_5 &= (1,2,2), & A_6 &= (3,1,3), \end{aligned} \tag{26}$$

we obtain that the curve D is a smooth sextic.

Since D is smooth and rational, by the Riemann-Roch theorem we know that $h^0(D, \mathscr{O}_D(2)) = 13$, hence there are at least two linearly independent quadrics in $\mathbb{P}^4_{\mathbb{C}}$ passing through D.

Moreover, one can check that for the concrete instance of $\vec{A}$ provided by Equation (26), there are exactly two quadrics passing through D, so this holds for a general $\vec{A}$. The same is true for the fact that the two quadrics form a complete intersection with M_6. □

Definition 3.3. Let $\vec{A}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Let D be the Möbius curve of $\vec{A}$ and let Y be the complete intersection of degree 12 whose existence is ensured by Lemma 3.2. The curve D' such that $D \cup D' = Y$ is called the *residual curve* of D.

Remark 3.4. The condition that also the residual curve D' is smooth is an open condition in the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^4_{\mathbb{C}}, 6t+1)$, and so we can proceed as in Lemma 3.2 to prove that a general 6-tuple $\vec{A}$ is Möbius-general.

Proposition 3.5. *Let* $\vec{A}$ *be a Möbius-general* 6*-tuple of points in* $\mathbb{R}^3$*. Let* D *be the Möbius curve of* $\vec{A}$ *and let* D' *be its residual curve. Then* D' *is rational and of degree* 6*. Moreover* D *and* D' *intersect in* 14 *points.*

Proof. Since $\vec{A}$ is Möbius-general, then by hypothesis D' is a sextic. Since, as we saw in the Intermezzo, linked curves of the same degree have the same arithmetic genus, we get that $p_a(D) = p_a(D') = 0$. Denoting by ω_D the canonical sheaf of D, from [Mig98, Remark 5.2.7] we know that the following sequence is exact:

$$0 \longrightarrow \omega_D(-2) \longrightarrow \mathscr{O}_{D \cup D'} \longrightarrow \mathscr{O}_{D'} \longrightarrow 0 \,.$$

Taking the associated long exact sequence in cohomology and using the fact that $p_a(D') = 0$, one proves that $h^0(D', \mathscr{O}_{D'}) = 1$, namely D' is connected. Since D' is smooth by the assumption of Möbius-generality, it is irreducible, and so rational.

To conclude the proof, we show that D and D' intersect in 14 points. By assumption the union $D \cup D'$ is a complete intersection whose ideal is generated by two quadrics and a cubic, hence the ideal sheaf $\mathscr{I}_{D \cup D'}$ admits a graded free resolution given by the Koszul complex:

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_{\mathbb{P}^4}(-7) \longrightarrow \mathscr{O}_{\mathbb{P}^4_{\mathbb{C}}}(-4) \oplus \mathscr{O}_{\mathbb{P}^4_{\mathbb{C}}}(-5)^2 \longrightarrow \\ \longrightarrow \mathscr{O}_{\mathbb{P}^4_{\mathbb{C}}}(-2)^2 \oplus \mathscr{O}_{\mathbb{P}^4_{\mathbb{C}}}(-3) \longrightarrow \mathscr{I}_{D \cup D'} \longrightarrow 0. \end{aligned}$$

The additivity of the Euler characteristic gives $\chi\left(\mathbb{P}^4_{\mathbb{C}}, \mathscr{I}_{D \cup D'}\right) = 13$. The proof that $D \cap D'$ is constituted of 14 points is concluded once we show that $\chi\left(\mathbb{P}^4_{\mathbb{C}}, \mathscr{O}_{D \cap D'}\right) = 14$, but this is a consequence of the exactness of the sequence

$$0 \longrightarrow \mathscr{O}_{D \cap D'} \longrightarrow \mathscr{O}_D \oplus \mathscr{O}_{D'} \longrightarrow \mathscr{O}_{D \cup D'} \longrightarrow 0 \,. \qquad \square$$

Remark 3.6. The residual curve D' of the Möbius curve D of a 6-tuple in $\mathbb{R}^3$ is a real variety. In fact, by construction D is a real curve, and so the complete intersection $D \cup D'$ is a real variety. Thus D' is a real curve.

The discussion we conducted so far shows that if we start from a Möbius-general 6-tuple, then the residual curve we obtain satisfies several properties shared by Möbius curves. Unfortunately, we cannot provide an argument ensuring that D' is a Möbius curve. However, all the experiments we conducted with the aid of computer algebra systems suggest that this is the case. We are therefore led to formulate the following conjecture.

Conjecture 1. *Let $\vec{A}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Let D be the Möbius curve of $\vec{A}$ and let D' be its residual curve. Then D' is a Möbius curve, namely there exists a 6-tuple $\vec{B}$ in $\mathbb{R}^3$ such that $f_{\vec{B}}(C) = D'$.*

Example 3.7. Consider $\vec{A}$ as in Equation (26). The residual curve D' to the image of $f_{\vec{A}}$ is a Möbius curve. We can prove this by a direct computation as follows. Since D' is rational and real without real points, we can compute a real isomorphism $f\colon C \longrightarrow D'$, where as always

$$C = \left\{(x : y : z) \in \mathbb{P}^2_{\mathbb{C}} \,:\, x^2 + y^2 + z^2 = 0\right\}.$$

The map f extends to a real morphism $F\colon \mathbb{P}^2_{\mathbb{C}} \longrightarrow M_6$. To compute a candidate 6-tuple $\vec{B}$ of points in $\mathbb{R}^3$ such that $f = f_{\vec{B}}$ we first identify the differences

$B_i - B_j$ for all $i, j \in \{1, \ldots, 6\}$ with $i \neq j$. An inspection of the structure of the Möbius map suggests that those differences are encoded as the normal vectors of the lines $F^{-1}(T_{ij})$, where T_{ij} is the plane in M_6 parametrizing tuples for which two points coincide. Once we know the values $B_i - B_j$, we can simply set B_1 to be the origin, thus obtaining:

$$
\begin{aligned}
B_1 &= (0,0,0), & B_2 &= \left(\tfrac{1397624}{806205}, -\tfrac{92216}{161241}, -\tfrac{437432}{806205}\right), \\
B_3 &= \left(\tfrac{340244}{161241}, \tfrac{82388}{53747}, -\tfrac{835486}{483723}\right), & B_4 &= \left(\tfrac{1341708}{1236181}, \tfrac{3724594}{1236181}, -\tfrac{922514}{1236181}\right), \\
B_5 &= \left(\tfrac{1125372}{2203627}, \tfrac{5582884}{2203627}, \tfrac{2416984}{2203627}\right), & B_6 &= \left(\tfrac{1719522}{591217}, \tfrac{824050}{591217}, \tfrac{1683982}{591217}\right).
\end{aligned}
$$

A direct computation ensures that $f = f_{\vec{B}}$, namely f is a Möbius map, hence D' is a Möbius curve.

If Conjecture 1 is true (and this is supported by the computations we conducted), then we can associate to every Möbius-general 6-tuple $\vec{P}$ of base points a candidate platform $\vec{p}$. Notice that, because of the definition and properties of the Möbius map, the candidate platform can by scaled by any non-zero factor without losing any of the properties ensured by Conjecture 1. Our next goal is to compute a scaling factor and leg lengths determining a mobile hexapod.

Noticing that, if we take into account Remark 2.20 and we enforce the Möbius-generality condition so that Möbius-general 6-tuples are not allowed to have three collinear points, then for every Möbius-general base $\vec{P}$ the platform $\vec{p}$, whose existence is predicated by Conjecture 1, satisfies the following property: every hexapod Π having $\vec{P}$ and $\vec{p}$ as base and platform admits 14 bonds, and has conformal degree equal to 28. In fact in this case the correspondence defined in the proof of Theorem 2.19 between bonds and intersections of the Möbius curves of base and platform is a bijection. From now on we will always consider this stronger notion of Möbius generality.

3.1.2 *The scaling factor and leg lengths*

The liaison technique described in Section 3.1.1 allows, at least in concrete examples, to construct a candidate platform starting from a Möbius-general base. This, however, is not enough to fully describe a hexapod, since we need to specify also the leg lengths; in particular, we do not have at the moment sufficient conditions ensuring that the hexapods we obtain are mobile. We are going to address both these issues in this section, and we will see that in order to obtain a mobile hexapod we also need to dilate the candidate platform by a suitable scaling factor.

The idea is to impose that the configuration sets K_Π of the pods Π we are trying to construct intersect "too much" the hyperplane H defining the boundary of X if we suppose that Π is not mobile. This will imply that K_Π must be at least a curve. To achieve this situation, we impose a contact of the second order between K_Π and H.

Given a Möbius-general base $\vec{P}$, as we remarked before we can scale the candidate platform $\vec{p}$ whose existence is predicted by Conjecture 1 by an arbitrary real number $\gamma \in \mathbb{R} \setminus \{0\}$ without changing the two Möbius curves of base and platform. However, the bonds of a hexapod having $\vec{P}$ and $\gamma\,\vec{p}$ as base and platform depend on γ — in particular, a direct computation shows that, as γ varies, each of them moves on a line contained in the boundary B — although their number is always equal to 14. We denote such bonds by $\{\beta_\gamma^k\}_{k=1}^{14}$.

From now one we will work as if Conjecture 1 were true, namely we will always assume that given a Möbius-general base $\vec{P}$, the liaison technique of Section 3.1.1 allows to construct a candidate $\vec{p}$. Recall from Definition 1.17 that the pseudo spherical condition is the restriction of the spherical condition to the hyperplane H defining the boundary. In contrast with the spherical condition, the pseudo spherical condition does not depend on the leg lengths.

Definition 3.8. Denote by $\widetilde{\Lambda}_\gamma$ the linear space defined by the 6 pseudo spherical conditions imposed by $\vec{P}$ and $\gamma\,\vec{p}$.

Fix a vector $\vec{d}$ of leg lengths and consider the hexapod $\Pi_{\gamma,\vec{d}} = (\vec{P}, \gamma\,\vec{p}, \vec{d})$. Then for all boundary points $\beta \in B$ we have

$$\beta \in \Lambda_{\gamma,\vec{d}} \cap X \quad \text{if and only if} \quad \beta \in \widetilde{\Lambda}_\gamma,$$

where $\Lambda_{\gamma,\vec{d}}$ is linear space cut out by the spherical conditions determined by $\Pi_{\gamma,\vec{d}}$. In fact, it holds

$$\Lambda_{\gamma,\vec{d}} \cap H = \widetilde{\Lambda}_\gamma \cap H.$$

In particular this holds for all 14 points β_γ^k.

Definition 3.9. Let $\vec{P}$ be a Möbius-general 6-tuple and let $\vec{p}$ be a candidate platform. Suppose that $\widetilde{\Lambda}_\gamma$ and X intersect properly, so that each of the boundary points β_γ^k is an irreducible component of $\widetilde{\Lambda}_\gamma \cap X$. We say that $\gamma \in \mathbb{R} \setminus \{0\}$ satisfies the tangency condition Tang_2 if and only if for each of the 14 points β_γ^k the intersection multiplicity $i(\widetilde{\Lambda}_\gamma, X; \beta_\gamma^k)$ of $\widetilde{\Lambda}_\gamma$ and X at β_γ^k is greater than or equal to 2.

We notice that, with the notation previously introduced, $i(\widetilde{\Lambda}_\gamma, X; \beta_\gamma^k) \geqslant 2$ if and only if $i(\Lambda_{\gamma,\vec{d}}, X; \beta_\gamma^k) \geqslant 2$ for every $\vec{d}$ for which $\Lambda_{\gamma,\vec{d}}$ and X intersect properly. To prove this it is enough to show that

$$i(\widetilde{\Lambda}_\gamma, X; \beta_\gamma^k) = 1 \quad \text{if and only if} \quad i(\Lambda_{\gamma,\vec{d}}, X; \beta_\gamma^k) = 1.$$

This follows from the fact that the projective tangent space $\mathbb{T}_{\beta_\gamma^k} X$ of X at β_γ^k is contained in H (see Definition 1.11) and that

$$i(\widetilde{\Lambda}_\gamma, X; \beta_\gamma^k) = 1 \quad \text{if and only if} \quad \mathbb{T}_{\beta_\gamma^k} X \cap \widetilde{\Lambda}_\gamma = \{\beta_\gamma^k\}$$

and similarly for $\Lambda_{\gamma,\vec{d}}$.

Lemma 3.10. *The condition* Tang_2 *is affine-linear in* γ.

Proof. Let β_γ be one of the 14 boundary points for $(\vec{P}, \gamma\,\vec{p})$. Requiring that γ satisfies the condition Tang_2 is equivalent to ask that the dimension of the intersection $\widetilde{\Lambda}_\gamma \cap \mathbb{T}_{\beta_\gamma} X$ is greater than or equal to 1. Since, as one can directly verify, the line in B on which β_γ moves as γ varies passes through the vertex, we can write, after a possible repametrization

$$\beta_\gamma = (0 : \alpha w v^t : \lambda w : \mu v : \gamma).$$

The projective tangent space $\mathbb{T}_{\beta_\gamma}X$ is spanned by the rows of the following matrix:

h	M	x	y	r
0	wv^t	0	0	0
0	$\alpha w' v^t$	$\lambda w'$	0	0
0	$\alpha w v'^t$	0	$\mu v'$	0
0	0	w	0	0
0	0	0	v	0
0	0	0	0	1
0	0	w'	v'	0

where w', together with w, span the tangent line in $\mathbb{P}^2_{\mathbb{C}}$ of C at w, and the same for v', and these vectors are subject to the condition $\langle w', w' \rangle = \langle v', v' \rangle$. In particular, the projective tangent space does not depend on γ. The 6 pseudo spherical conditions defining $\widetilde{\Lambda}_\gamma$ determine a linear map $\eta \colon \mathbb{C}^7 \longrightarrow \mathbb{C}^6$, where we identify $\mathbb{C}^7$ with the vector space associated to $\mathbb{T}_{\beta_\gamma}X$. Its kernel is the vector space associated to the intersection $\mathbb{T}_{\beta_\gamma}X \cap \widetilde{\Lambda}_\gamma$. We are going to show that the condition $\dim \ker \eta \geqslant 2$ is affine-linear in γ. To do this, we pick coordinates so that (see Proposition 1.13)

$$\begin{aligned}
&v = w = (1 : i : 0),\\
&v' = w' = (0 : 0 : 1),\\
&\lambda = \mu = 0,\\
&\alpha = 1.
\end{aligned}$$

Then a direct inspection of the matrix of η proves the statement. □

Example 3.11. Consider $\vec{P}$ and $\vec{p}$ as in Example 3.7: one can show that all the 14 equations for γ determined by imposing the condition Tang_2 are multiples of the equation $\gamma - 1 = 0$.

Definition 3.12. Let $\Pi_{\gamma,\vec{d}} = \big(\vec{P}, \gamma\,\vec{p}, \vec{d}\big)$ where $\vec{P}$ and $\vec{p}$ are as in Definition 3.9, and suppose that γ satisfies the condition Tang_2. Let $\Lambda_{\gamma,\vec{d}}$ be the linear space defined by the 6 spherical conditions determined by $\Pi_{\gamma,\vec{d}}$. Suppose that $\Lambda_{\gamma,\vec{d}}$ and X intersect properly. We say that $\vec{d}$ satisfies the tangency condition Tang_3 if and only if for each of the 14 points β^k_γ the intersection multiplicity $i\big(\Lambda_{\gamma,\vec{d}}, X; \beta^k_\gamma\big)$ is greater than or equal to 3.

Example 3.13. Recall Example 3.11. Naively one expects that the intersection $i\big(\Lambda_{\gamma,\vec{d}}, X; \beta^k_\gamma\big)$ is greater than or equal to 3 if we are able to find a solution of the system of equations given by $\Lambda_{\gamma,\vec{d}} \cap X$ in $\mathbb{C}[t]/(t^3)$. For a solution of the form $c_0 + c_1 t + c_2 t^2$, the coefficients c_0 and c_1 are determined by β^k_γ itself and by a tangent vector in $\mathbb{T}_{\beta^k_\gamma} \cap \Lambda_{\gamma,\vec{d}}$, which is unique up to scaling. For c_2, we obtain a system of affine-linear equations in $d_1^2, \ldots, d_6^2$ and c_2. The solvability of these equations with respect to c_2 is equivalent to another system of affine-linear equations in $d_1^2, \ldots, d_6^2$. These equations are:

$$\begin{aligned}
d_4^2 &= \tfrac{71}{92} d_1^2 - \tfrac{105}{92} d_2^2 + \tfrac{63}{46} d_3^2 - \tfrac{535801}{676062},\\
d_5^2 &= \tfrac{71}{41} d_1^2 - \tfrac{75}{41} d_2^2 + \tfrac{45}{41} d_3^2 - \tfrac{1908080}{1074159},\\
d_6^2 &= \tfrac{71}{44} d_1^2 - \tfrac{45}{44} d_2^2 + \tfrac{9}{22} d_3^2 - \tfrac{114265}{154638}.
\end{aligned} \tag{27}$$

Theorem 3.14. *Assume that $\vec{P}$ is a 6-tuple of points in $\mathbb{R}^3$. Assume that $\vec{p}$ is another 6-tuple such that the Möbius curves of $\vec{P}$ and of $\vec{p}$ intersect in 14 points and do not intersect in a node of M_6. Assume that γ and $\vec{d}$ satisfy the conditions* Tang_2 *and* Tang_3*. Then the hexapod $\Pi = \left(\vec{P}, \gamma\,\vec{p}, \vec{d}\right)$ is mobile.*

Proof. From the hypothesis that the two Möbius curves do not intersect in a node it follows taking into account Remark 2.20 that the hexapod Π admits 14 bonds. Denote by Λ_Π the linear space defined by the 6 spherical conditions determined by Π. Suppose that Π is not mobile, and so Λ_Π and X intersect in a finite number of points. By assumption, the intersection multiplicity of X and Λ_Π at each of the 14 bonds of Π is at least 3. Therefore we obtain at least $14 \cdot 3 = 42$ intersections between K_Π and H, but this is impossible by [Har77, Appendix A, Axiom A6] since the degree of X is 40. Therefore Π is mobile. $\square$

Remark 3.15. If $\vec{P}$ is a Möbius-general 6-tuple of points in $\mathbb{R}^3$, then Conjecture 1 predicts the existence of a 6-tuple $\vec{p}$ satisfying the hypothesis of Theorem 3.14.

Conjecture 2. *For a Möbius-general 6-tuple $\vec{P}$ of base points, consider the 6-tuple $\vec{p}$ of platform points predicted by Conjecture 1. There exists a real number γ satisfying* Tang_2*, and such number is unique. Moreover, there is a 3-dimensional set of leg length vectors $\vec{d}$ satisfying* Tang_3.

Both Conjecture 1 and 2 are fulfilled by every single example we constructed starting from 6-tuples with random coordinates. A picture of one of these examples is reported in Figure 11.

Definition 3.16. Let $\vec{P}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Then Conjecture 1 and 2 predict the existence of $\vec{p}$, γ and $\vec{d}$ satisfying the hypothesis of Theorem 3.14. We call the resulting hexapod Π a *liaison hexapod*. In particular, liaison hexapods have conformal degree equal to 28.

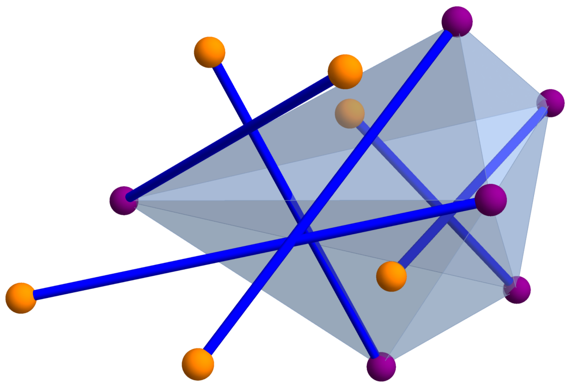


Figure 11: An example of a liaison hexapod, where the convex hull of the platform points is highlighted.

We conclude mentioning the fact that if the two Conjectures are correct, the family of liaison hexapods has dimension 21, since the possible bases of such hexapods form an open subset in $\left(\mathbb{R}^3\right)^6$ and for each base there exists exactly a 3-dimensional set of leg lengths.

3.2 LIAISON HEXAPODS HAVE MAXIMAL CONFORMAL DEGREE

The goal of this section is to give a strong indication that, if the two Conjectures of Section 3.1 hold, the family of liaison hexapods is maximal among mobile hexapods. In fact liaison hexapods have conformal degree equal to 28, and we are going to prove that, if we exclude some particular cases, this is the maximal value for the conformal degree of a mobile hexapod. In particular we will show that (the notion of *non-parallel* tuple of points will be defined later):

Theorem 3.17. *The conformal degree of a non-planar and non-equiform hexapod* Π *such that both base and platform are non-parallel is at most* 28*.*

Since the conformal degree is upper-semicontinuous, Theorem 3.17 implies that liaison hexapods cannot be the specialization of any family that has non-maximal conformal degree. At the moment, to the authors' knowledge no such family is known.

The strategy to prove Theorem 3.17 is to show that the number of intersections between the Möbius curves of the base and the platform of a hexapod as in the hypothesis is at most 14 (Proposition 3.36), and then use Theorem 2.19.

The road to Proposition 3.36 is quite long, and will involve several partial results. Recall that we denote by T_{ij} the plane in M_6 parametrizing the classes of 6-tuples $(m_1, \ldots, m_6)$ in $\mathbb{P}^1_\mathbb{C}$ such that $m_i = m_j$. There is a first condition that the Möbius curves of base and platform may or may not satisfy, namely being contained in a hyperplane of $\mathbb{P}^4_\mathbb{C}$. Lemma 3.18 clarifies that each of the two curves can be contained at most in one hyperplane.

From now on, by "Möbius curve" we will always mean "Möbius curve of a 6-tuple of different points".

Lemma 3.18. *A Möbius curve of degree* 6 *cannot be contained in a plane.*

Proof. Let D be such a curve, and suppose that D is contained in a plane E.

If E is completely contained in M_6, then $E = T_{ij}$ for some $i, j \in \{1, \ldots, 6\}$. Let $\vec{A}$ be a 6-tuple such that $D = f_{\vec{A}}(C)$. Then, recalling the modular interpretation of T_{ij}, the projections of the points A_i and A_j along any oriented direction in $\mathbb{R}^3$ coincide, and this is possible only if $A_i = A_j$, but we only consider 6-tuples of distinct points, so we get a contradiction.

If E is not contained in M_6, then $M_6 \cap E$ is curve of degree at most 3, contradicting the assumption on the degree of D. □

If only one of two Möbius curves lies in a hyperplane, we get easily an estimate for the number of their intersections.

Lemma 3.19. *Let* D_1 *and* D_2 *be two distinct degree* 6 *Möbius curves, and suppose that* D_1 *lies on a hyperplane* H*. If* D_2 *is not contained in* H*, then* $|D_1 \cap D_2| \leqslant 6$*.*

Proof. By assumption $|D_1 \cap D_2| \leqslant |H \cap D_2| \leqslant 6$. □

From Lemma 3.18 and 3.19 we know that if exactly one of the two Möbius curves is contained in a hyperplane, then the bound holds. Now we investigate the situation when both the two curves are contained in a hyperplane H. By Bertini's theorem, for a general H the intersection $H \cap M_6$ is an irreducible smooth cubic surface; however there are degenerate situations in which such surface is not irreducible, or not smooth, and we need to deal with them carefully. We start by excluding the possibility that a Möbius curve can lie on a surface of degree 2 in M_6.

Lemma 3.20. *A Möbius curve of degree* 6 *cannot be contained in a surface of degree* 2 *completely contained in* M_6.

Proof. Let $S \subseteq M_6$ be a surface of degree 2. Then S spans a hyperplane H in $\mathbb{P}^4_{\mathbb{C}}$, and by degree reasons $M_6 \cap H = S \cup E$, where E is a plane. Hence E is one of the 15 planes T_{ij} in M_6. Consider the projection $\mathbb{P}^4_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ from E: as we pointed out in Remark 2.20, its restriction to M_6 sends the class of a tuple $(m_1, \ldots, m_6)$ to the class of the tuple $(m_1, \ldots, \hat{m}_i, \ldots, \hat{m}_j, \ldots, m_6)$ — namely the points m_i and m_j are omitted — in $M_4 \cong \mathbb{P}^1_{\mathbb{C}}$. By construction the surface S is a fiber of such map. This means that if a Möbius curve $D = f_{\vec{A}}(C)$ of degree 6 is contained in S, then the cross-ratio of the orthogonal projections of the four points $A_1, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_6$ (omitting A_i and A_j) does not depend on the direction of the projection. In turn this implies that four points in $\vec{A}$ must be aligned, but this is in contradiction with Lemma 2.11. □

Lemma 3.21. *Let* D *be a degree* 6 *Möbius curve contained in a hyperplane* H*. Then* $H \cap M_6$ *is an irreducible cubic surface.*

Proof. Assume that the intersection $Y = H \cap M_6$ is reducible. Then either $Y = S \cup E$ where E is a plane and S is a surface of degree 2, or $Y = E_1 \cup E_2 \cup E_3$ where all E_i are planes. Since D is irreducible, if $D \subseteq Y$, then D must lie on one of the irreducible components of Y. This however contradicts Lemma 3.18 or Lemma 3.20. □

For a Möbius curve D contained in a hyperplane H we define $\ell_{ij} = H \cap T_{ij}$. We know that all ℓ_{ij} are lines (and not planes), because by Lemma 3.21 the hyperplane H cannot contain any of the planes T_{ij}.

If the surface $H \cap M_6$ is smooth, we know that there are exactly 27 lines on it, and so the lines ℓ_{ij} are some of these lines. We are going to use then the well-know intersection product on such a surface to determine the bound for the number of intersections of two Möbius curves. In order to do this, and to be able to treat also the special cases for the intersection $H \cap M_6$, we investigate more thoroughly the behavior of the lines ℓ_{ij}.

Consider a Möbius curve $D = f_{\vec{A}}(C)$ such that $D \cap T_{ij} \cap T_{kl} \neq \emptyset$ for pairwise distinct $i, j, k, l \in \{1, \ldots, 6\}$, i.e. $|\{i, j, k, l\}| = 4$. Then, by construction of the Möbius map, the lines $\overrightarrow{A_i A_j}$ and $\overrightarrow{A_k A_l}$ are parallel.

Definition 3.22. A 6-tuple $\vec{A}$ is called *non-parallel* if there do not exist pairwise distinct indices $i, j, k, l \in \{1, \ldots, 6\}$ such that the lines $\overrightarrow{A_i A_j}$ and $\overrightarrow{A_k A_l}$ are parallel. A Möbius curve $D = f_{\vec{A}}(C)$ is called *non-parallel* if $\vec{A}$ is so.

Lemma 3.23. *Let* D *be a degree* 6 *Möbius curve contained in a hyperplane* H*, and suppose that* D *is non-parallel. Then all* 15 *lines* $\{\ell_{ij}\}$ *are distinct.*

Proof. Suppose that for some $i, j, k, l \in \{1, \ldots, 6\}$ we have $\ell_{ij} = \ell_{kl}$. Observe that if $|i, j, k, l| = 3$ with $i \neq j$ and $k \neq l$, then T_{ij} and T_{kl} meet only in a point, so our hypothesis forces all i, j, k, l to be distinct. Then

$$\ell_{ij} = \ell_{kl} = T_{ij} \cap T_{kl}.$$

By construction, the curve $D = f_{\vec{A}}(C)$ intersects every plane T_{ij} in two points — the images under $f_{\vec{A}}$ of the two oriented directions $\overrightarrow{A_i A_j}$ and $\overrightarrow{A_j A_i}$. This implies that D must intersect both ℓ_{ij} and ℓ_{kl}, but then D meets $T_{ij} \cap T_{kl}$, and this is in contrast with the non-parallel assumption. $\square$

Lemma 3.24. *Let D be a degree 6 Möbius curve contained in a hyperplane H, and suppose that D is non-parallel. Then*

a. if $|\{i,j,k,l\}| = 4$, then $\ell_{ij} \cap \ell_{kl} = \{\text{point}\}$;

b. if $|\{i,j,k,l\}| = 3$ with $i \neq j$ and $k \neq l$, then $\ell_{ij} \cap \ell_{kl} = \{\text{point}\}$ if and only if the hyperplane H passes through the node of M_6 corresponding to configurations $(m_1, \ldots, m_6)$ for which $m_i = m_j = m_k = m_l$; otherwise $\ell_{ij} \cap \ell_{kl} = \emptyset$.

Proof. In Case a, the planes T_{ij} and T_{kl} intersect in a line and so the statement follows immediately. Instead, in Case b the planes T_{ij} and T_{kl} intersect in a point that is a node of M_6. $\square$

Remark 3.25. Let D be a degree 6 Möbius curve contained in a hyperplane H, and suppose that D is non-parallel. Then H can pass through at most one node of M_6. In fact, if H passes though two nodes, then H contains a line of the form $T_{ij} \cap T_{kl}$; this implies that D intersects $T_{ij} \cap T_{kl}$, which contradicts the hypothesis that D is non-parallel.

We now go through the possible cases for an irreducible cubic surface $S = H \cap M_6$, where H is a hyperplane containing a non-parallel Möbius curve of degree 6.

A short argument discards the situation when the surface S is singular and has non-isolated singularities. From [BW79, Case E] we know that in this case S is either a cone, or a projection of a cubic scroll in $\mathbb{P}^4_{\mathbb{C}}$. In the first case, all lines in S intersect in the vertex of the cone, but this contradicts Lemma 3.24, which implies that some lines in S do not intersect. In the second case there exists a pencil of pairwise disjoint lines, each of which intersects two other special lines on the surface; this case is again ruled out by Lemma 3.24, which implies the existence of three lines intersecting pairwise (for example, ℓ_{12}, ℓ_{34} and ℓ_{56}). Hence none of these cases can appear in our context.

Hence we only need to cover the case where S is smooth, or has isolated singularities. We start with the smooth situation.

Lemma 3.26. *Let D_1, D_2 be two distinct degree 6 non-parallel Möbius curves. Suppose that both D_1 and D_2 are contained in a hyperplane H and that $S = H \cap M_6$ is a smooth cubic surface. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. Being S smooth, we can express it as the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at 6 points $q_1, \ldots, q_6$ in general position. Then the Picard group of S is generated by $L, E_1, \ldots, E_6$ — where each E_i is the class of the exceptional divisor over q_i, and L is the class of the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$. The following relations hold:

$$L^2 = 1, \quad E_i^2 = -1, \quad E_i \cdot L = 0 \text{ for all } i, \quad E_i \cdot E_j = 0 \text{ if } i \neq j.$$

Moreover, denoting by $[\,\cdot\,]$ the class in $\mathrm{Pic}(S)$ of a divisor, we can suppose that

$$[\ell_{ij}] = L - E_i - E_j.$$

Let $D \subseteq S$ be a non-parallel Möbius curve of degree 6. We write

$$[D] = k\,L - (e_1\,E_1 + \cdots + e_6\,E_6)$$

for some integers k and $e_1, \ldots, e_6$. Since D intersects each ℓ_{ij} in 2 points,

$$k - e_i - e_j = 2 \qquad \forall i \neq j.$$

From this we deduce that

$$e_i = m \quad \forall i, \qquad k = 2m + 2 \qquad \text{for some integer } m.$$

Since D is effective, then $[D] \cdot E_i \geqslant 0$ for all i, and

$$[D] \cdot (2L - E_1 - \cdots - E_5) \geqslant 0,$$

because also $2L - E_1 - \cdots - E_5$ is effective, being the class of the strict transform of the conic passing through $q_1, \ldots, q_5$. It follows that $0 \leqslant m \leqslant 4$. We exclude now the cases $m = 0$ and $m = 4$. If $m = 0$, then $[D] = 2L$. Recall from Lemma 2.16 that D is contained in a projection of a Veronese surface, which is the complete intersection of M_6 with another cubic hypersurface U. Then $U \cap H$ is a cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ containing D, therefore $-3K_S - [D]$ is effective — where K_S is the canonical divisor on S; recall in fact that S is anticanonically embedded in $\mathbb{P}^3_{\mathbb{C}}$, so $[U] = -3K_S$ in $\mathrm{Pic}(S)$. In terms of the generators of $\mathrm{Pic}(S)$ we have

$$\begin{aligned} -3K_S - [D] &= 3\,(3L - E_1 - \cdots - E_6) - 2L \\ &= 7L - 3(E_1 + \cdots + E_6). \end{aligned}$$

On the other hand

$$\big(7L - 3(E_1 + \cdots + E_6)\big)(2L - E_1 - \cdots - E_5) < 0,$$

and this is absurd, since both divisors are effective. Similarly for $m = 4$. Hence $m \in \{1, 2, 3\}$.

We are ready to prove the statement. By what we said so far, for $i \in \{1, 2\}$ we have $[D_i] = (2m_i + 2)L - m_i(E_1 + \cdots + E_6)$ for some $m_i \in \{1, 2, 3\}$. Then

$$[D_1] \cdot [D_2] = -2(m_1 - 2)(m_2 - 2) + 12,$$

so it follows that $[D_1] \cdot [D_2] \leqslant 14$, which implies $|D_1 \cap D_2| \leqslant 14$. $\square$

The last situation to analyze is when the cubic surface $S = H \cap M_6$ has isolated singularities. By Lemma 3.23 we know that there are at least 15 lines on S, so from the classification of cubic surfaces provided in [BW79] we infer that the only possibilities for S are:

- being a cone over a cubic plane curve (infinitely many lines);
- having exactly one singularity of type A_1 (21 lines);
- having two singularities of type A_1 (16 lines);
- having exactly one singularity of type A_2 (15 lines).

We consider these cases one by one.

Lemma 3.27. *Let D_1, D_2 be two distinct degree 6 non-parallel Möbius curves. Then it cannot happen that both D_1 and D_2 are contained in a hyperplane H and that $S = H \cap M_6$ is a cone over a cubic plane curve.*

Proof. This case is ruled out as before by the existence of non-intersecting lines on S (see Lemma 3.24). □

Lemma 3.28. *Let D_1, D_2 be two distinct degree 6 non-parallel Möbius curves, both contained in a hyperplane H. Then it cannot happen that $S = H \cap M_6$ is a singular cubic surface with two singularities of type A_1.*

Proof. The main argument is that the number of lines passing through each of the singularities of this surface is not compatible with the configuration of the lines ℓ_{ij}.

We realize the surface S in the following way. Let $\widetilde{S}$ be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ at six points $q_1, \ldots, q_6$ such that both the triples q_1, q_2, q_3 and q_1, q_4, q_5 are collinear, but involving different lines, and q_6 is in general position with respect to the other points. Then S is obtained by blowing down the strict transforms in $\widetilde{S}$ of the two lines $\overrightarrow{q_1 q_2 q_3}$ and $\overrightarrow{q_1 q_4 q_5}$, whose image under the blow down map are the two A_1 singularities. If we use the same notation of Lemma 3.26 for the generators of $\mathrm{Pic}(\widetilde{S})$, then the classes of the strict transforms of the lines in S are:

$$\begin{aligned}
&E_1, \ldots, E_6, && \text{(6 lines)}\\
&L - E_1 - E_6, \ldots, L - E_5 - E_6, && \text{(5 lines)}\\
&L - E_2 - E_4, L - E_2 - E_5, L - E_3 - E_4, L - E_3 - E_5, && \text{(4 lines)}\\
&2L - E_2 - E_3 - E_4 - E_5 - E_6. && \text{(1 line)}
\end{aligned}$$

With this description, the number of lines passing through each singularity can be computed by counting how many of the previous classes has positive intersection with one of the (-2)-curves

$$L - E_1 - E_2 - E_3 \quad \text{or} \quad L - E_1 - E_4 - E_5,$$

and the result is 5.

On the other hand, in our case from Lemma 3.24 we deduce that either we have 6 lines passing through a singularity (and this happens when the hyperplane H passes though one of the nodes of M_6) or there are at most 2 lines passing through a point. Hence S cannot have two A_1 singularities. □

Lemma 3.29. *Let D_1, D_2 be two distinct degree 6 non-parallel Möbius curves, both contained in a hyperplane H such that $S = H \cap M_6$ is a singular cubic surface with one singularity of type A_1. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. As in the proof of Lemma 3.28 we realize S by first blowing up $\mathbb{P}^2_{\mathbb{C}}$ at 6 points $q_1, \ldots, q_6$ lying on a conic — thus obtaining a smooth surface $\widetilde{S}$ — and then blowing down the strict transform of such conic. Again, using the same notation as in Lemma 3.26 for the generators of $\mathrm{Pic}(\widetilde{S})$, the classes of the strict transforms in $\widetilde{S}$ of the lines in S are

$$\begin{aligned}
&E_1, \ldots, E_6, && \text{(6 lines)}\\
&L - E_i - E_j. && \text{(15 lines)}
\end{aligned}$$

The latter 15 lines are the classes of the lines ℓ_{ij}. The computation for $[D_1] \cdot [D_2]$ goes exactly as in the smooth case (Lemma 3.26) because none of the lines ℓ_{ij} passes through the singular point, as one can check by computing the intersection product of $L - E_i - E_j$ with the (-2)-curve $2L - (E_1 + \cdots + E_6)$. From this we conclude that $[D_k] \cdot [\ell_{ij}] = 2$ for all i, j and for $k \in \{1, 2\}$. However, we cannot directly infer from $[D_1] \cdot [D_2]$ the number of intersections of D_1 and D_2. In fact, if D_1 and D_2 intersect in the singular point of S, such intersection is counted as a contribution by $1/2$ — and not 1, as usual — in $[D_1] \cdot [D_2]$. Therefore the fact that $[D_1] \cdot [D_2] \leqslant 14$ does not imply that $|D_1 \cap D_2| \leqslant 14$. Luckily, in this situation we can exclude that m_1 or m_2 equals 3. In fact, for such a value we obtain the class $8L - 3(E_1 + \cdots + E_6)$, which should hence be effective; on the other hand also the class $2L - (E_1 + \cdots + E_6)$ is effective, since the points q_i are arranged on a conic. But

$$\left(8L - 3 \sum E_i\right)\left(2L - \sum E_i\right) = -2 < 0,$$

and this is absurd. Hence $m_i \in \{1, 2\}$. We analyze the possible cases.

a. Either $m_1 = 2$ or $m_2 = 2$, then $[D_1] \cdot [D_2] = 12$. By computing the intersection product with the (-2)-curve $2L - (E_1 + \cdots + E_6)$ we see that in this case either D_1 or D_2 does not pass through the singularity, so we can conclude that $|D_1 \cap D_2| \leqslant 12$.

b. Or $m_1 = m_2 = 1$, then $[D_1] \cdot [D_2] = 10$. In this case both D_1 and D_2 pass through the singular point, and moreover both have a node at that point. From this we conclude that $|D_1 \cap D_2| \leqslant 14$. □

Lemma 3.30. *Let D_1, D_2 be two distinct degree 6 non-parallel Möbius curves, both contained in a hyperplane H such that $S = H \cap M_6$ is a singular cubic surface with one singularity of type A_2. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. In this case we obtain S by blowing up $\mathbb{P}^2_{\mathbb{C}}$ at 6 points $q_1, \ldots, q_6$ such that q_1, q_2, q_3 are collinear and q_4, q_5, q_6 are collinear on another line, and then blowing down the (strict transforms of the) lines $\overleftrightarrow{q_1 q_2 q_3}$ and $\overleftrightarrow{q_4 q_5 q_6}$ — the latter get contracted to the unique A_2 singularity of S. In this case the classes of the (strict transforms of the) 15 lines of S, which coincide with the lines ℓ_{ij}, are

$$\begin{array}{ll}
E_1, \ldots, E_6, & \text{(6 lines)} \\
L - E_1 - E_4, \ldots, L - E_1 - E_6, & \text{(3 lines)} \\
L - E_2 - E_4, \ldots, L - E_2 - E_6, & \text{(3 lines)} \\
L - E_3 - E_4, \ldots, L - E_3 - E_6. & \text{(3 lines)}
\end{array}$$

Notice that the only lines ℓ_{ij} passing through the singular point are the ones whose class is an exceptional divisor E_i, as the computation of the intersection product with the two (-2)-classes $L - E_1 - E_2 - E_3$ and $L - E_4 - E_5 - E_5$ confirms. Here the computation of $[D_1] \cdot [D_2]$ does not work as in the smooth case, due to the fact that in the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at the points $\{q_i\}$ the (strict transforms of) the curves D_i may not intersect the (strict transforms of) the lines ℓ_{ij} — this depends whether or not the curves D_i pass through the singular point.

Let D be a Möbius curve with the properties of D_1 and D_2, and write

$$[D] = k\, L - (e_1\, E_1 + \cdots + e_6\, E_6).$$

Let us suppose that D does not pass through the singular point of S. Then we know that in the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at the points $\{q_i\}$ we have $[D] \cdot [\ell_{ij}] = 2$ for all $i \neq j$. Hence

$$\begin{aligned} [D] \cdot E_i &= 2 && \text{for all } i \in \{1, \ldots, 6\}, \\ [D] \cdot (L - E_i - E_j) &= 2 && \text{for all } i \in \{1,2,3\},\ j \in \{4,5,6\}. \end{aligned}$$

This forces $k = 6$ and $e_i = 2$ for all i, so $[D] = 6L - 2(E_1 + \cdots + E_6)$.

Suppose now that D passes through the singular point. Notice that the fact that both D and each ℓ_{ij} are real implies that their intersection is real; moreover, the fact that $D = f_{\vec{A}}(C)$ where $f_{\vec{A}}$ is a real map and C is a real variety without real points implies that it cannot happen that D intersects an ℓ_{ij} transversely at the singular point and then in another different point, or tangentially at the singular point. The only possibility is that D has an ordinary node at the singular point. Then (the strict transform of) D meets the two (-2)-curves in two points, thus

$$\begin{aligned} &[D] \cdot \big((L - E_1 - E_2 - E_3) + (L - E_4 - E_5 - E_6)\big) = 2, \\ &[D] \cdot (L - E_i - E_j) = 2 \ \text{ for all } i \in \{1,2,3\},\ j \in \{4,5,6\}. \end{aligned}$$

This forces $k = 4$ and $e_i = 1$ for all i, so $[D] = 4L - (E_1 + \cdots + E_6)$. Summing up, we have the following scenarios:

a. Both D_1 and D_2 do not pass through the singular point. Then $[D_1] \cdot [D_2] = 12$, so $|D_1 \cap D_2| \leqslant 12$.

b. Only D_1 (or D_2) passes through the singular point. Then $[D_1] \cdot [D_2] = 12$ and so $|D_1 \cap D_2| \leqslant 12$.

c. Both D_1 and D_2 pass through the singular point, and have a node there. Then $[D_1] \cdot [D_2] = 10$, and $|D_1 \cap D_2| \leqslant 10 + 4 = 14$. □

The results obtained so far guarantee that the statement of Proposition 3.36 is correct when at least one of the two Möbius curves is contained in a hyperplane. Therefore from this point on we can suppose that none of the Möbius curves is contained in a hyperplane. The Riemann-Roch theorem predicts that there are at least 2 quadrics in the ideal of a smooth rational curve of degree 6 in $\mathbb{P}^4_{\mathbb{C}}$ (see Lemma 3.2); this is true also in the case of a singular such curve D, because we have an injective homomorphism $H^0\big(D_i, \mathscr{O}_{D_i}(2)\big) \longrightarrow H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(12)\big)$.

Lemma 3.31. *Let D be a degree 6 non-parallel Möbius curve not contained in any hyperplane. Let $\mathcal{Q}$ be a pencil of quadrics containing D. Then the base locus of $\mathcal{Q}$ is a quartic surface.*

Proof. All the quadrics passing through D must be irreducible, because otherwise they would split into two hyperplanes, but by hypothesis D cannot be contained in any of them. This prevents the base locus of $\mathcal{Q}$ to be 3-dimensional. Then such base locus is a surface, namely the intersection of two quadrics in $\mathcal{Q}$, and so it has degree 4 by Bezout theorem. □

In Lemma 3.32 and Lemma 3.33 we discuss the situation when the base locus S of a pencil of quadrics passing through a Möbius curve is irreducible.

Lemma 3.32. *Let D_1 and D_2 be two distinct degree 6 non-parallel Möbius curves not contained in a hyperplane. Suppose that there is a pencil $\mathcal{Q}$ of quadrics containing D_1 whose base locus S is irreducible and not contained in M_6. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. Here D_1 is a component of the curve $Z = S \cap M_6$, which is a complete intersection of degree 12 by Lemma 3.31 and Bezout's theorem. If D_2 coincides with the other component of Z, then the second part of Proposition 3.5 proves that $|D_1 \cap D_2| = 14$. If this is not the case, then there exists at least a quadric Q passing through D_1, but not passing through D_2. Hence

$$|D_1 \cap D_2| \leqslant |Q \cap D_2| \leqslant 12. \qquad \square$$

Lemma 3.33. *Let D be a degree 6 non-parallel Möbius curve not contained in any hyperplane. Let $\mathcal{Q}$ be a pencil of quadrics containing D and suppose that its base locus S is contained in M_6. Then S is reducible.*

Proof. Suppose instead that S is irreducible. Pick any quadric Q in the pencil $\mathcal{Q}$: then the intersection $Q \cap M_6$ is the union of S and a surface S' of degree 2. We claim that it is always possible to choose Q such that S' splits in the union of two planes. In fact, each S' spans a hyperplane H, so that $H \cap M_6 = S' \cup E$, where E is a plane. Since the set of planes in M_6 is discrete, by continuity we obtain that the plane E is always the same, regardless of which $Q \in \mathcal{Q}$ we start with. Therefore the one-dimensional family of surfaces S' is obtained by cutting M_6 with the pencil of hyperplanes through the plane E. A direct computation — for example taking the plane $x_0 = x_4 = 0$, where the x_i are coordinates in $\mathbb{P}^4_{\mathbb{C}}$ — shows that in such one-dimensional family there are always reducible members. Hence we can select $Q \in \mathcal{Q}$ such that, after a possible rearrangement of the indices,

$$Q \cap M_6 = S \cup T_{12} \cup T_{34}.$$

We intersect both sides of the previous equality with the plane T_{56}: on the left we obtain either a conic (if T_{56} is not contained in Q) or the plane T_{56} itself (if $T_{56} \subseteq Q$), while on the right we get the union of $S \cap T_{56}$, $T_{12} \cap T_{56}$ (a line) and $T_{34} \cap T_{56}$ (another line). Since S is irreducible by assumption, then T_{56} cannot be contained in S, so $S \cap T_{56}$ can be either a curve, or a finite set of points. This forces the left hand side $Q \cap T_{56}$ to be a conic. In turn, this implies that $S \cap T_{56}$ has to be contained in the union of the two lines $T_{12} \cap T_{56}$ and $T_{34} \cap T_{56}$. On the other hand, since $D \subseteq S$ and D intersects T_{56} in two points, then D should intersect one of the two lines $T_{12} \cap T_{56}$ and $T_{34} \cap T_{56}$, but this contradicts the assumption that D is non-parallel. Hence S cannot be irreducible. $\square$

Hence we are left with the case when the base locus S is reducible. We notice that it cannot happen that S is contained in M_6 and at the same time splits into the union of two surfaces of degree 2, because by Lemma 3.20 no Möbius curve of degree 6 can lie on a degree 2 surface contained in M_6. The only remaining cases are when $S = S' \cup S''$ with both S' and S'' surfaces of degree 2, but $S \not\subseteq M_6$, or $S = E \cup S'$ where E is a plane and S' is a cubic surface.

Lemma 3.34. *Let D_1 and D_2 be two distinct degree 6 non-parallel Möbius curves not contained in any hyperplane. Suppose that there is a pencil $\mathcal{Q}$ of quadrics containing D_1 whose base locus S is not contained in M_6 and splits into the union $S = S' \cup S''$ of two surfaces of degree 2. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. Since D_1 is irreducible, then $D_1 \subseteq S'$ or $D_1 \subseteq S''$. From now on we will suppose $D_1 \subseteq S'$. Since D_1 cannot lie on a degree 2 surface contained in M_6, then S' is not contained in M_6 and hence by degree reasons $D_1 = S' \cap M_6$. If there exists a quadric $Q \in \mathcal{Q}$ not passing through D_2, then

$$|D_1 \cap D_2| \leqslant |Q \cap D_2| \leqslant 12.$$

Otherwise $D_2 \subseteq S$. It cannot happen that $D_2 \subseteq S'$, because otherwise we would have $D_1 = D_2$, contradicting the hypothesis. Thus $D_2 \subseteq S''$, and so $D_2 = S'' \cap M_6$. Therefore D_1 and D_2 are the two components of the degree 12 complete intersection $Z = S \cap M_6$, and then the second part of Proposition 3.5 concludes the proof. □

Lemma 3.35. *Let D_1 and D_2 be two distinct degree 6 non-parallel Möbius curves not contained in any hyperplane. Suppose that there is a pencil $\mathcal{Q}$ of quadrics containing D_1, whose base locus S splits into the union of a plane E and a cubic surface S'. Then $|D_1 \cap D_2| \leqslant 14$.*

Proof. By Lemma 3.18, the curve D_1 cannot be contained in the plane E, so $D_1 \subseteq S'$. Suppose that S' is not contained in M_6. There are several possibilities.

a. The intersection $Z = S' \cap M_6$ is a curve of degree 9. Then D_1 is a component of such curve. This implies that there is a quadric $Q \in \mathcal{Q}$ not passing through D_2, because otherwise we would have $D_1 = D_2$. Hence

$$|D_1 \cap D_2| \leqslant |Q \cap D_2| \leqslant 12.$$

b. The cubic S' splits into a plane E' and a surface S'' of degree 2, and S'' is contained in M_6. This case cannot happen, since D_1 neither can lie on a plane, nor on a degree 2 surface contained in M_6.

c. The cubic S' splits into a plane E' and a surface S'' of degree 2, and S'' is not contained in M_6. Then $D_1 = S'' \cap M_6$. This implies that there exists a quadric $Q \in \mathcal{Q}$ not passing through D_2, because otherwise we would have $D_1 = D_2$. Hence

$$|D_1 \cap D_2| \leqslant |Q \cap D_2| \leqslant 12.$$

The last case that needs to be treated is the one where S' is contained in M_6. Then S' is irreducible, because D' cannot lie on planes or surfaces of degree 2 contained in M_6 (see Lemma 3.18 and 3.20). Hence S' is a cubic scroll — maybe singular, namely a cone over a rational cubic plane curve. Thus S' admits a determinantal representation as the zero set of the 2×2 minors of a 2×3 matrix of linear forms. We consider the intersection of S' with the planes T_{ij}: first of all, each T_{ij} is not contained in S', because otherwise S' would be reducible. By restricting the determinantal representation of S' to T_{ij} we see that $\ell_{ij} = S' \cap T_{ij}$ is defined by three quadratic equations in T_{ij}, so it is either a finite set of points, or a line, or a conic. On the other hand if $\{i, j, k, l, m, n\} = \{1, \ldots, 6\}$ then there exists a hyperplane H in $\mathbb{P}^4_{\mathbb{C}}$ such that $H \cap M_6 = T_{ij} \cup T_{kl} \cup T_{mn}$. Therefore

$$H \cap S' = (S' \cap T_{ij}) \cup (S' \cap T_{kl}) \cup (S' \cap T_{mn}) = \ell_{ij} \cup \ell_{kl} \cup \ell_{mn}.$$

Since S' is not contained in H (because D_1 does not lie in any hyperplane) we have that $S' \cap H$ is a cubic curve. Suppose that one among ℓ_{ij}, ℓ_{kl} and ℓ_{mn}, say ℓ_{ij}, is a finite number of points. Then, by eventually rearranging the indices, ℓ_{kl} is a line and ℓ_{mn} is a conic. This implies that $\ell_{ij} \subseteq \ell_{kl} \cup \ell_{mn}$, but this contradicts the non-parallel hypothesis on D_1. Therefore we conclude that all ℓ_{ij} are lines. Moreover all lines ℓ_{ij} are distinct, because otherwise this would violate the non-parallel assumption. We consider now the mutual position of the lines ℓ_{ij}. First of all we notice that — analogously as in Lemma 3.24 — if $|\{i, j, k, l\}| = 3$ and ℓ_{ij} meets ℓ_{kl}, then they intersect at the node of M_6 given by the class of 6-tuples $(m_1, \ldots, m_6)$ for which $m_i = m_j = m_k = m_l$. This rules out the case where S' is a cone, because in that case all lines in S' meet in a single point, but on the other hand the points $\ell_{12} \cap \ell_{23}$ and $\ell_{12} \cap \ell_{24}$ are different, an absurd. So S' is smooth, then S' admits a pencil of mutually disjoint lines, all of which intersect one line ℓ. We distinguish two cases:

a. Suppose that ℓ does not appear among the lines ℓ_{ij}. Then in particular ℓ_{12}, ℓ_{34} and ℓ_{56} are disjoint. On the other hand, as mentioned before, $\ell_{12} \cup \ell_{34} \cup \ell_{56} = H \cap M_6$ for some hyperplane H; from [Har77, Chapter III, Corollary 7.9] we know that a hyperplane section of S' is connected, so this is absurd.

b. Suppose that ℓ appears among the lines ℓ_{ij}. After a possible rearrangement of the indices, we can suppose that $\ell = \ell_{12}$. Then ℓ_{23}, ℓ_{45} and ℓ_{16} are disjoint, but we can repeat the argument of Case a. and see that this leads to an absurd.

This concludes the proof of the statement. □

We sum up the previous discussion in the following proposition.

Proposition 3.36. *Let* D_1 *and* D_2 *be two distinct degree* 6 *non-parallel Möbius curves. Then* $|D_1 \cap D_2| \leqslant 14$.

Now we can prove Theorem 3.17.

Proof. Since Π is non-planar, then the Möbius maps of base and platform points are birational by Lemma 2.11; by the same result, using the non-parallel hypothesis we infer that no 4 points are on a line, and so we conclude that both the Möbius curves of base and platform D_1 and D_2 have degree 6. Since Π is non-equiform, then D_1 and D_2 are distinct. Theorem 2.19 states that the conformal degree of Π is bounded by $2\,|D_1 \cap D_2|$, and Proposition 3.36 asserts that $|D_1 \cap D_2| \leqslant 14$, so the statement is proved. □

Spectrahedra were introduced in [RG95] in the context of *semidefinite programming*. A short account on this topic is provided in [Vin14], while more detailed information can be found in [BPT13, Chapter 2]. We will use one particular result about spectrahedra in Chapter 4 to obtain real solutions of a system of equations related to pods with 20 legs.

In order to understand how the concept of spectrahedron arises, let us start from *linear programming*. Consider the following problem: suppose that various materials $M_1, \dots, M_n$ are sent to an incinerator to be burned; denote by x_i the quantity of the material M_i that is sent per day to the incinerator, and denote by c_i the amount of energy that be obtained by burning the material M_i; unfortunately several pollutants $P_1, \dots, P_k$ are produced during the combustion, and there are laws prescribing for each of them the maximal quantity that can be released in the atmosphere: let us denote by a_{ij} the amount of pollutant P_j per unit of material M_i, and by b_j the daily threshold for the pollutant P_j. The problem of determining the quantities $x = (x_1, \dots, x_n)$ maximizing the energy obtained from the materials according to the law can be formalized as follows:

$$\begin{aligned} \text{maximize:} \quad & \langle c, x \rangle \\ \text{subject to:} \quad & Ax \leqslant b \text{ and} \\ & x \geqslant 0, \end{aligned}$$

where $\langle \cdot, \cdot, \rangle$ is the standard Euclidean scalar product and the inequality $\geqslant$ is considered componentwise. This is an instance of a linear programming problem. In general, in a linear programming problem we are asked to find a vector $x \in \mathbb{R}^n$ maximizing the quantity $\langle c, x \rangle$ — where $c \in \mathbb{R}^n$ is a fixed vector — and at the same time fulfilling the conditions $x \geqslant 0$ and $Ax \leqslant b$ — where $A \in \mathbb{R}^{n \times n}$ is a fixed matrix and $b \in \mathbb{R}^n$ is a fixed vector.

Linear programming is a basic and fundamental subject in optimization, and several common problems can be interpreted as its instances. From a more geometric point of view, we can see a linear programming instance as the problem of maximizing a linear functional over a *polyhedron*. A polyhedron is a convex subset of $\mathbb{R}^n$ obtained by intersecting finitely many half-spaces, namely subsets of the form

$$\left\{ x \in \mathbb{R}^n \,:\, \langle a, x \rangle \leqslant \beta \right\}$$

for a given vector $a \in \mathbb{R}^n$ and a value β. Notice from the description of a linear programming instance that the condition $Ax \leqslant b$ prescribes that the vector x we are looking for lies on a polyhedron.

A generalization of linear programming is given by *semidefinite programming*, where the condition that the entries of the vector x are non-negative is substituted by the condition that a matrix X is positive semidefinite. Recall that a matrix $X \in \mathbb{R}^{n \times n}$ is called *positive semidefinite* if for all vectors $v \in \mathbb{R}^n$ we

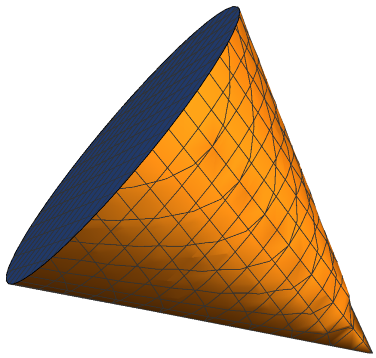


Figure 12: An example of a spectrahedron.

have $v^t X v \geqslant 0$. A semidefinite programming instance can be formalized by the following scheme:

$$\begin{aligned} \text{maximize:} \quad & \operatorname{trace}(CX) \\ \text{subject to:} \quad & \operatorname{trace}(A_j X) = b_j \text{ for all } j \in \{1, \dots, k\} \text{ and} \\ & X \text{ is positive semidefinite} \geqslant 0, \end{aligned}$$

where both C and all A_j are positive semidefinite matrices, and $b \in \mathbb{R}^k$. We notice that we can recover linear programming as a special case of semidefinite programming by considering matrices A, C and X that are diagonal. As we did for linear programming, we can consider semidefinite programming from a more geometric point of view as the problem of maximizing a linear functional (given by $\operatorname{trace}(CX)$) on the intersection of the cone of semidefinite matrices with an affine space (determined by the equations $\operatorname{trace}(A_j X) = b_j$). Such an intersection is called a *spectrahedron*, and so spectrahedra play for semidefinite programming the same role that polyhedra play for linear programming. As an example of a spectrahedron, we can consider the set of points (x, y, z) in $\mathbb{R}^3$ such that the matrix

$$\begin{pmatrix} x+y & z & 0 & 0 \\ z & x-y & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 2-x \end{pmatrix}$$

is positive semidefinite: this set is by construction a spectrahedron, and equals

$$\left\{ (x, y, z) \in \mathbb{R}^3 \, : \, x^2 - y^2 - z^2 \geqslant 0 \text{ and } 0 \leqslant x \leqslant 2 \right\}.$$

This is hence a cone in $\mathbb{R}^3$, and it is shown in Figure 12.

In Chapter 4 we are going to use a recent result regarding the boundary of a particular family of spectrahedra to construct an example of a mobile pod with 20 legs.

4 LINE-SYMMETRIC ICOSAPODS

The material for this chapter is taken from [GNSS16].

In this chapter we investigate an "extremal" case of mobile pods, the one of icosapods, namely pods with 20 legs. This case is extremal in the sense that it is known that if a mobile pod admits 21 legs, then it is possible to add an infinite number of legs without decreasing its mobility (we will reprove this in Proposition 4.4), so mobile icosapods are mobile pods with the maximal number of legs, when this number is finite.

Borel in [Bor08] proposed a construction for mobile icosapods, based on the following observation. Suppose that $\sigma \in SE_3$ is an isometry corresponding to a point in the configuration set of an n-pod Π. Suppose furthermore that σ is an involution, namely an element such that $\sigma^2 = id$; this means that σ is either the identity, or a rotation of 180° along some axis in $\mathbb{R}^3$. By assumption we have

$$\|\sigma(p_i) - P_i\| = d_i \qquad \text{for all } i \in \{1, \ldots, n\}, \tag{28}$$

where $\Pi = (\vec{P}, \vec{p}, \vec{d})$. Since σ is a direct isometry and at the same time an involution, Equation (28) is equivalent to

$$\|p_i - \sigma(P_i)\| = d_i \qquad \text{for all } i \in \{1, \ldots, n\}.$$

Hence if all isometries in the configuration set of Π are involutions, then it is possible to double the number of legs — adding $\vec{p}$ to the base points, and $\vec{P}$ to the platform points — without changing the configuration set. Therefore, if a mobile pod with 10 legs admits a configuration curve constituted of involutions, then we obtain a mobile icosapod. The idea of Borel consisted in getting such curves by intersecting the set of involutions of SE_3 with a linear space of suitable codimension. Unfortunately, Borel's original paper does not discuss the possibility to obtain real solutions for the base and platform coordinates using the equations coming from such approach.

We are going to prove that, under mild assumptions, all mobile icosapods arise as instances of Borel's construction (Theorem 4.14). Moreover, using some results from the theory of quartic spectrahedra, we show that it is possible to obtain a concrete example of a mobile icosapod (Example 4.21).

4.1 LEG SETS AND INVOLUTIONS

In this section we introduce two concepts that will be used in what follows: first, a variety Y playing a dual role to the conformal embedding X and allowing to handle algebraically the set of legs that admit a given curve as configuration set (Definition 4.2); second, a subvariety X_{inv} of X that is the complexification of the set of involutions in SE_3 (Definition 4.5).

4.1.1 *The leg variety*

We want to define a variety that parametrizes legs of a pod, namely pairs of points in $\mathbb{R}^3$ — whose coordinates can be given with respect to two different systems of reference — together with leg lengths. Given this description, one can expect that a compactification of such a variety is provided by a cone over the Segre variety $\Sigma_{3,3}$, since the latter parametrizes pairs of points in $\mathbb{P}^3_{\mathbb{C}}$. In order to introduce this variety with an explicit system of coordinates, we start by manipulating the spherical condition from Equation (8) to obtain a bilinear form.

Given two points $a, b \in \mathbb{R}^3$, and a real number d, one defines the quantity

$$l := \langle a, a \rangle + \langle b, b \rangle - d^2,$$

which we call the *corrected leg length*. In this way, the spherical condition imposed by the points a and b, together with the leg length d reads as:

$$l\,h + r - 2\,\langle a, x \rangle - 2\,\langle y, b \rangle - 2\,\langle M a, b \rangle \;=\; 0. \tag{29}$$

We consider the points $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ as points in $\mathbb{P}^3_{\mathbb{C}}$ by introducing two extra homogenization coordinates a_0 and b_0. In this way, the pair (a, b) becomes a point in the Segre variety $\Sigma_{3,3} \cong \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$. The Segre variety $\Sigma_{3,3}$ is embedded in $\mathbb{P}^{15}_{\mathbb{C}}$ as follows: if (z_{ij}) are coordinates on $\mathbb{P}^{15}_{\mathbb{C}}$, then points in $\Sigma_{3,3}$ have coordinates that satisfy $z_{ij} = a_i\, b_j$ for some $(a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}^3_{\mathbb{C}}$. One can check that (see [Har95, Example 2.11]) the equations of $\Sigma_{3,3}$ are given by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & \cdots & z_{03} \\ \vdots & & \vdots \\ z_{30} & \cdots & z_{33} \end{pmatrix}.$$

Homogenizing Equation (29) with respect to the coordinates (z_{ij}) and l yields

$$\begin{aligned} l\,h + z_{00} r - 2\,(z_{10} x_1 + z_{20} x_2 + z_{30} x_3) - \\ - 2\,(z_{01} y_1 + z_{02} y_2 + z_{03} y_3) - 2 \sum_{i,j=1}^{3} m_{ij}\, z_{ij} \;=\; 0. \end{aligned} \tag{30}$$

Notice that the left hand side of Equation (30) is a bilinear expression in the coordinates $(h : M : x : y : r)$ and $(z : l)$.

Definition 4.1. We call the expression on the left hand side of Equation (30) the *bilinear sphere condition* and we denote it by BSC.

The bilinear sphere condition determines a duality between $\mathbb{P}^{16}_{\mathbb{C}}$ with coordinates $(h : M : x : y : r)$ and $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ with coordinates $(z : l)$: a point $(h_0, M_0, x_0, y_0, r_0) \in \mathbb{P}^{16}_{\mathbb{C}}$ is associated to the hyperplane in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ of equation

$$\mathrm{BSC}(h_0, M_0, x_0, y_0, r_0, z, l) \;=\; 0,$$

while a point $(z_0, l_0) \in \check{\mathbb{P}}^{16}_{\mathbb{C}}$ is associated to the hyperplane in $\mathbb{P}^{16}_{\mathbb{C}}$ of equation

$$\mathrm{BSC}(h, M, x, y, r, z_0, l_0) \;=\; 0.$$

In this way, if we are given an n-pod $\Pi = (\vec{P}, \vec{p}, \vec{d})$, then we can assign to Π the n points $(Q_1, \ldots, Q_n)$ in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ obtained by setting

$$Q_k = \big((p^i_k\, P^j_k)_{ij} : l_k\big) \qquad \text{where} \quad \begin{cases} p_k = (p^1_k, p^2_k, p^3_k), \\ P_k = (P^1_k, P^2_k, P^3_k), \end{cases}$$

and $p^0_k = P^0_k = 1$ for all k, and l_k is the corrected leg length for the leg (p_k, P_k, d_k). Notice that all points Q_k belong to the cone Y over the Segre variety $\Sigma_{3,3}$ with vertex $(0 : \cdots : 0 : 1)$. By unraveling the definitions, one sees that the intersection of the variety X with the n hyperplanes in $\mathbb{P}^{16}_{\mathbb{C}}$ given by the equations

$$\mathrm{BSC}(h, M, x, y, r, z_k, l_k) = 0 \qquad \text{where} \quad Q_k = (z_k : l_k),\ k \in \{1, \ldots, n\}$$

is nothing but the configuration set K_Π.

From another perspective, given a point $(h_0 : M_0 : x_0 : y_0 : r_0) \in X$, one can consider the intersection between the hyperplane

$$\mathrm{BSC}(h_0, M_0, x_0, y_0, r_0, z, l) = 0$$

and the cone Y. Suppose that there are real points in this intersection that satisfy $z_{00} \neq 0$ and $a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l \geqslant 0$ where $z_{ij} = a_i\, b_j$ with $a_0 = b_0 = 1$. A direct inspection shows that those points correspond to legs of length $\sqrt{a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l}$, forming a pod whose configuration set contains the point $(h_0 : M_0 : x_0 : y_0 : r_0)$.

We see hence that the cone Y provides a way to deal with legs that is compatible with the setting we developed so far.

Definition 4.2. The variety Y defined above is called the *leg variety*. Since the Segre variety $\Sigma_{3,3}$ has dimension 6 and degree 20, it follows that Y has dimension 7 and degree 20.

We are particularly interested in sets of legs that contain a given curve in the configuration set.

Definition 4.3. Let $C \subseteq X$ be a curve. We define the *leg set* L_C of C as the set of all points $(z, l) \in Y$ such that the BSC — evaluated at (z, l) — holds for all points in C.

Rephrasing what we wrote earlier, we see that the leg set of a curve C in X is a compactification of the set of all triples $(a, b, d) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_{\geqslant 0}$ such that the image of a under any isometry determining a point on C has distance d from b.

Now we reprove in our setting the fact that the maximal number of legs for a mobile pod is 20, when this number is finite.

Proposition 4.4. *Let $C \subseteq X$ be a curve. If L_C has only finitely many points, then its cardinality is at most 20. If L_C has exactly 20 points, then the linear span of L_C in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$ is a projective subspace of dimension 9.*

Proof. By construction L_C is defined by linear equations as a subset of Y; in other words $L_C = Y \cap \operatorname{span}(L_C)$. Then the first statement follows from the characterization of the degree of a projective variety. Suppose that the cardinality of L_C is not 20 and that $\dim\bigl(\mathrm{span}(L_C)\bigr) < 9$. By taking a general linear superspace of $\mathrm{span}(L_C)$ of dimension 9 and considering its intersection with Y we get an absurd, so also the second statement holds true. □

Proposition 4.4 tells us that if we are given an icosapod Π of mobility one, then the linear span of its configuration set K_Π has dimension

$$\dim\bigl(\mathrm{span}(K_\Pi)\bigr) \;=\; 15 - 9 \;=\; 6.$$

We are going to use this to prove that all icosapods of mobility one arise from Borel's construction, namely considering configuration sets that are entirely formed by involutions.

4.1.2 *The subvariety of involutions*

We introduce the second variety of this section, namely a compactification X_{inv} of the set of involutions in SE_3. As we already remarked, involutions in SE_3 are rotations of 180° around a fixed axis. Since the family of lines in $\mathbb{R}^3$ is 4-dimensional, we have that X_{inv} is 4-dimensional subvariety of X.

If $(h : M : x : y : r)$ is a point in X corresponding to an involution, then we have that the map

$$v \;\mapsto\; M(Mv + y) + y \qquad \text{for all } v \in \mathbb{R}^3$$

equals the identity, and from this it follows that $M = M^t$ — that is to say, M is symmetric — and $y = x$. Hence we are led to consider the subvariety

$$\bigl\{(h : M : x : y : r) \in X \;:\; M = M^t \text{ and } x = y\bigr\}.$$

A direct computation shows that such variety has two irreducible components, namely the isolated point corresponding to the identity and another one of dimension 4, which is cut out by the further linear equation $m_{11} + m_{22} + m_{33} + h = 0$.

Definition 4.5. We define the *variety of involutions* X_{inv} to be the subvariety of X determined by the equations

$$\begin{aligned} M \;&=\; M^t, \qquad\qquad x \;=\; y, \\ m_{11} + m_{22} + m_{33} + h \;&=\; 0. \end{aligned}$$

Our goal in Section 4.2 is to show that all icosapods of mobility one admit a configuration curve that is projectively equivalent to a curve contained in X_{inv}. Because of this we introduce the following piece of notation.

Definition 4.6. Let $C \subseteq X$ be a curve. Then C is called an *involutory motion* if $C \subseteq X_{\mathrm{inv}}$.

4.2 MOBILE ICOSAPODS ARE LINE-SYMMETRIC

In this section we will show that if an icosapod Π of mobility one admits an irreducible configuration curve, then it is possible to find an isometry τ such that for every isometry σ in the configuration set of Π we have that $\tau \circ \sigma$ is an involution. As we saw in Section 1.1.2, every isometry in SE_3 determines a projective automorphism of $\mathbb{P}^{16}_{\mathbb{C}}$ leaving X invariant. This means that our goal can be restated by saying that we want to prove that for every icosapod Π of mobility one, whose configuration curve is irreducible, there exists an involutory motion $C \subseteq X_{\mathrm{inv}}$ that is projectively equivalent to K_{Π}, where the projective isomorphism is induced by an isometry. This is the content of Theorem 4.14.

As we pointed out at the end of Section 4.1.1, the configuration curve of a mobile icosapod spans a linear subspace of dimension 6. To understand how such configuration curves look like, we employ the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ sending X to a variety X_m isomorphic to a Veronese variety, as described in Remark 1.6. If C is a curve such that $\dim\big(\mathrm{span}(C)\big) = 6$, we denote by $C_m \subseteq X_m$ its projection, and by $C_e \subseteq \mathbb{P}^3_{\mathbb{C}}$ be inverse image of C_m under the Veronese isomorphism $\mathbb{P}^3_{\mathbb{C}} \xrightarrow{\cong} X_m$. In [Naw14b] it is proved that if C_m is a point, there exists a pod with infinitely many legs that admits C as configuration set, so from now on we do not consider such a case. Therefore from now on we suppose that C_m and C_e are curves.

Proposition 4.7. *If $C \subseteq X$ is an irreducible curve such that $\dim\big(\mathrm{span}(C)\big) = 6$, then C_e is either planar or a twisted cubic.*

Proof. The image of the linear span $\mathrm{span}(C)$ under the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ is a linear space, and by hypothesis its dimension is at most 6. This means that the ideal of C_m contains at least 3 linear polynomials, and taking into account the definition of the Veronese map this is equivalent to say that there are at least three linearly independent quadrics passing through C_e. If any of these quadrics is reducible, namely splits into the union of two planes, the statement follows because C_e, being irreducible, must be contained in one of those planes. Hence we can suppose that all three quadrics are irreducible; this means that the intersection of two of them gives a quartic curve E containing C_e. Since there are three, and not two, independent quadrics passing through C_e, the containment $C_e \subsetneq E$ is strict. This implies that $\deg(C_e) \in \{1, 2, 3\}$: if $\deg(C_e) = 1$ or $\deg(C_e) = 2$, then C_e is planar, otherwise it is a twisted cubic, so the statement is proved. $\square$

In order to proceed further and discard situations that are not interesting for us, we formulate a condition that is satisfied by most of the irreducible configuration curves C of mobile icosapods:

$$\begin{cases} C \text{ is an irreducible configuration curve of a pod,} \\ L_C \text{ consists of exactly 20 real finite points.} \end{cases} \tag{$\dagger$}$$

Here by the word "finite" we mean that the z_{00}-coordinates of such points are not zero; in other terms, such points determine pairs of base and platform points in $\mathbb{R}^3$ (and not at "infinity").

In principle, not all pods that could be called "mobile icosapods" satisfy condition (†), because for example there may exist configuration curves that have leg sets with infinitely many points of which only 20 are real and finite. On the other hand we believe that condition (†) captures "most" of the interesting examples (we are not aware of any example exhibiting the previous "pathological" behavior), and has the advantage to be concise.

Lemma 4.8. *Suppose that* $C \subseteq X$ *satisfies condition* (†). *Then* $\dim\bigl(\mathrm{span}(C)\bigr) = 6$ *and* C *does not pass through any butterfly or collinearity point.*

Proof. The condition about the dimension of $\mathrm{span}(C)$ is ensured by Proposition 4.4. Moreover, if C passed through a butterfly or a collinearity point, then the pod having C as configuration curve would have at least 10 collinear legs, and then it would be possible to add infinitely many other legs without modifying the configuration curve, in contrast with the hypothesis on L_C. □

Lemma 4.9. *Suppose that* $C \subseteq X$ *satisfies condition* (†). *Then* C_e *cannot be neither a twisted cubic, nor a plane cubic curve.*

Proof. Assume that C_e is a twisted cubic curve. This implies that the curve C_m, obtained from C_e by applying the Veronese embedding, is a rational normal sextic, and therefore spans a linear space of dimension 6. From Lemma 4.8 we know that also C spans a 6-dimensional linear space, so the restriction to $\mathrm{span}(C)$ of the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ is a projective isomorphism, thus in particular the degree of C is 6. Therefore the center of such projection must not intersect $\mathrm{span}(C)$, which implies by Remark 1.14 that C does not pass through any similarity point. Therefore C can only pass though inversion points, and so by Remark 2.21 we get a contradiction, since the degree of C should be twice the number of inversion points, hence we would get 3 of them, which is impossible, because they always come in conjugate pairs (see Remark 1.18).

Suppose now that C_e is a planar cubic. Then the curve C_m spans a linear space of dimension 5. Hence the center of the projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ intersects $\mathrm{span}(C)$ in a single point. Since similarity points occur in pairs (see Remark 1.18), such intersection cannot be a similarity point. It follows from Lemma 4.8 hence that C passes only through inversion points, and so we can argue as in the case of the twisted cubic. □

We focus therefore on curves C satisfying (†) such that C_e is planar of degree different from 3. First we rule out the case when C_e is a line or a conic.

Lemma 4.10. *Let* $C \subseteq X$ *and assume that* C *fulfills condition* (†). *Then the curve* C_e *cannot be a line.*

Proof. The situation when C_e is a line was studied by Husty and Karger in [HK02]. In this case the curve C cannot satisfy condition (†) because a pod having C as configuration curve would admit infinitely many legs. □

Lemma 4.11. *Let* $C \subseteq X$ *and assume that* C *fulfills condition* (†). *Then the curve* C_e *cannot be a conic.*

Proof. We prove that if C_e is a conic, then L_C cannot be constituted of 20 real finite points, because in this case there are always legs "at infinity". Consider in fact the subvariety $Y_\infty \subseteq Y$ composed of those pairs (a, b) of points with $a_0 = b_0 = 0$: we show that L_C intersects Y_∞ non-trivially, and this will prove the statement. The dimension of Y_∞ is 5, since it is a cone over the Segre variety $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. Consider the linear projection $\mathbb{P}^{16}_{\mathbb{C}} \dashrightarrow \mathbb{P}^{9}_{\mathbb{C}}$ onto the $(h : M)$ coordinates: by the BSC duality, this projection determines a subspace $\check{\mathbb{P}}^{9}_{\mathbb{C}} \subseteq \check{\mathbb{P}}^{16}_{\mathbb{C}}$. A direct inspection of Equation (30) clarifies that $Y_\infty \subseteq \check{\mathbb{P}}^{9}_{\mathbb{C}}$.

By hypothesis, the vector space of quadrics on $\mathbb{P}^3_{\mathbb{C}}$ vanishing on C_e has dimension 5: this implies that there are 5 linearly independent linear forms in $\mathbb{P}^9_{\mathbb{C}}$ vanishing on C_m. In this way we get a linear space $\check{\mathbb{P}}^4_{\mathbb{C}} \subseteq \check{\mathbb{P}}^9_{\mathbb{C}}$. The intersection $\check{\mathbb{P}}^4_{\mathbb{C}} \cap Y_\infty \subseteq L_C$ is non-empty for dimensional reasons, and so the proof is concluded. □

From Lemma 4.10 and 4.11 we conclude that the only possibility for a curve $C \subseteq X$ satisfying condition (†) is that C_e is a planar curve of degree greater than or equal to 4. In this case our goal is to determine an isometry τ such that the image of C under the corresponding projective automorphism is contained in X_{inv}.

We start taking advantage of the fact that C_e is planar: if $e_0, \ldots, e_3$ are coordinates in $\mathbb{P}^3_{\mathbb{C}}$, then we can suppose then that $e_0 = 0$, and this can be obtained by a suitable rotation of the coordinate frame of the platform. In terms of the coordinates of X, this means that we can apply an automorphism of $\mathbb{P}^{16}_{\mathbb{C}}$ induced by an isometry so that the points of C satisfy $m_{ij} = m_{ji}$ and $m_{11} + m_{22} + m_{33} + h = 0$: this follows from the relations between the variables $(h : M)$ and the variables $(e_0 : e_1 : e_2 : e_3)$, as explained in Proposition 1.4. It follows that after this change of coordinates we can suppose that C lies on the variety

$$\widetilde{X}_{\mathrm{sym}} := X \cap \big\{(h : M : x : y : r) \in \mathbb{P}^{16}_{\mathbb{C}} : \\ m_{ij} = m_{ji} \text{ and } m_{11} + m_{22} + m_{33} + h = 0\big\}.$$

Notice that we are not very far from our final goal, since what we are left to show is that we can apply another change of coordinates ensuring that also $x = y$ holds for C. Unfortunately, we need to undergo a series of changes of coordinates to make the structure that will allow to solve the problem emerge.

By forgetting the r-coordinate — namely by projecting $\widetilde{X}_{\mathrm{sym}}$ from the point $(0 : \cdots 0 : 1) \in \mathbb{P}^{16}_{\mathbb{C}}$ — we obtain the variety X_{sym}. If we express the coordinates h and m_{ij} for $i, j \in \{1, 2, 3\}$ in terms of the Euler coordinates e_1, e_2, e_3, we obtain a map $X_{\mathrm{sym}} \longrightarrow \mathbb{P}_{\mathbb{C}}(\vec{1}, \vec{2})$, where the codomain is the *weighted projective space* (see [Har95, Example 10.27] for a reference). Here we used the notation $\vec{1} = (1, 1, 1)$ and $\vec{2} = (2, 2, 2, 2, 2, 2)$ and we have coordinates e_1, e_2, e_3 of weight 1 and x_1, x_2, x_3, y_1, y_2, y_3 of weight 2. If we employ the change of variables

$$p_i = x_i + y_i, \quad q_i = x_i - y_i, \qquad \text{for } i \in \{1, 2, 3\},$$

then the image of X_{sym} under the previous map is the subvariety $Z \subseteq \mathbb{P}_{\mathbb{C}}(\vec{1}, \vec{2})$ defined by the equations

$$e_1\, p_1 + e_2\, p_2 + e_3\, p_3 = p_1\, q_1 + p_2\, q_2 + p_3\, q_3 = \\ e_1\, q_2 - e_2\, q_1 = e_1\, q_3 - e_3\, q_1 = e_2\, q_3 - e_3\, q_2 = 0.$$

Notice that in this new system of coordinates, the condition $x = y$ that we are trying to impose is replaced by the condition $q_1 = q_2 = q_3 = 0$.

Remark 4.12. Projective automorphisms of $\mathbb{P}^{16}_{\mathbb{C}}$ induced by translations leave $\widetilde{X}_{\mathrm{sym}}$ invariant, since the equations defining the latter inside X involve only the $(h : M)$-coordinates — which are responsible for the rotational part of isometries. Therefore translations act also on Z by changes of variables.

Lemma 4.13. *Suppose that* $C \subseteq X$ *satisfies condition* (†), *and suppose that after a suitable change of coordinates we have* $C \subseteq \widetilde{X}_{\mathrm{sym}}$. *Let* $C_z \subseteq Z$ *be the image of the curve* C *under the map* $\widetilde{X}_{\mathrm{sym}} \longrightarrow Z$ *defined before. Then there exists a translation of the platform such that the corresponding automorphism maps* C_z *to a curve* C'_z *whose points satisfy* $q_1 = q_2 = q_3 = 0$.

Proof. Since our focus is on the q-vector, we consider the subvariety $W \subseteq \mathbb{P}_{\mathbb{C}}(1,1,1,2,2,2)$ obtained by projecting Z keeping the e and q-coordinates. The variety W is the zero set of the following system:

$$e_1\, q_2 - e_2\, q_1 \;=\; e_1\, q_3 - e_3\, q_1 \;=\; e_2\, q_3 - e_3\, q_2 \;=\; 0. \tag{31}$$

Define C_w to be the image of C_z under the projection $Z \longrightarrow W$. By a direct inspection of the map $C \longrightarrow C_w$, one notices that forms of weighted degree 2 on C_w correspond to linear forms on C. It follows that the vector space of weighted degree 2 forms on C_w has dimension at most 7. There are 9 forms of weighted degree 2 on $\mathbb{P}_{\mathbb{C}}(1,1,1,2,2,2)$ and they are all linear independent as forms on W because the latter is defined by equations of weighted degree 3. Hence C_w satisfies at least 2 equations $E_1 = E_2 = 0$ of weighted degree 2.

The polynomials E_i are of the form $E_i = L_i(\vec{q}) + Q_i(\vec{e})$, where L_i is linear and Q_i is quadratic. The polynomial $L_1(\vec{q})\, L_2(\vec{e}) - L_1(\vec{e})\, L_2(\vec{q})$ vanishes on W, because it is a multiple of the polynomials in Equation (31). Therefore the following relation holds on C_w:

$$E_1(\vec{e},\vec{q})\; L_2(\vec{e}) - E_2(\vec{e},\vec{q})\; L_1(\vec{e}) \;=\; Q_1(\vec{e})\; L_2(\vec{e}) - Q_2(\vec{e})\; L_1(\vec{e}) \;=\; 0.$$

The latter is a cubic equation only in the variables $\vec{e}$, thus must be satisfied by C_e. On the other hand, C_e is a planar curve of degree greater than 3, so C_e cannot satisfy a cubic nontrivial equation. This implies that $Q_1(\vec{e})\; L_2(\vec{e}) - Q_2(\vec{e})\; L_1(\vec{e})$ is the zero polynomial. If L_1 and L_2 were proportional, we could get from E_1 and E_2 a non-trivial quadratic equation in $\vec{e}$ satisfied by C_e, which is absurd again for degree reasons. Therefore L_1 and L_2 are not proportional, so we obtain that

$$Q_i(\vec{e}) \;=\; L(\vec{e})\; L_i(\vec{e}) \quad \text{for } i \in \{1,2\}$$

for some linear polynomial L.

From Equation (31) we infer that $L_1(\vec{q})\; e_j = L_1(\vec{e})\; q_j$ for $j \in \{1,2,3\}$. Since E_1 is zero on C_w, we have $-L_1(\vec{q}) = L(\vec{e})\; L_1(\vec{e})$ on C_w. Multiplying by e_j the last equation yields:

$$-L_1(\vec{e})\; q_j \;=\; -L_1(\vec{q})\; e_j \;=\; L(\vec{e})\; L_1(\vec{e})\; e_j,$$

this implying that $q_j = L(\vec{e})\; e_j$ holds on C_w for $j \in \{1,2,3\}$.

One can verify that the automorphism corresponding to the translation by a vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ acts on the coordinates of $\mathbb{P}_{\mathbb{C}}(1,1,1,2,2,2)$ by sending

$$(\vec{e},\, \vec{q}) \;\mapsto\; (\vec{e},\, \vec{q} + \ell_{\vec{a}}\, \vec{e}),$$

where $\ell_{\vec{a}} = a_1\, e_1 + a_2\, e_2 + a_3\, e_3$. Hence, if $L(\vec{e}) = \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3$, it is enough to apply to C_w the automorphism corresponding to the translation by the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ to get that $q_1 = q_2 = q_3 = 0$ holds on C_w. This proves the statement. □

The discussion so far proves therefore the main result of this section.

Theorem 4.14. *Let* Π *be an icosapod of mobility one such that its configuration curve* K_Π *satisfies condition* ($\dagger$)*. Then* K_Π *is projectively equivalent — via an isomorphism induced by an isometry — to an involutory motion.*

4.3 REAL ICOSAPODS VIA SPECTRAHEDRA

Once translated in the notation of this chapter, Borel's idea to obtain mobile icosapods is to intersect the variety X_{inv} with a general linear space $T \subseteq S$ of dimension 7, where

$$S = \left\{ (h : M : x : y : r) \in \mathbb{P}^{16}_{\mathbb{C}} \,:\, M = M^t \text{ and } x = y \right\} \cong \mathbb{P}^{10}_{\mathbb{C}}.$$

Since X_{inv} is contained in S and has codimension 6, the intersection $X_{\mathrm{inv}} \cap T$ is an irreducible curve C of degree 12; its the leg set L_C is the intersection of the leg variety Y with a general linear space of codimension 7 in $\check{\mathbb{P}}^{16}_{\mathbb{C}}$, and so is constituted of 20 complex points. The goal of this sections is to show that it is possible to construct a curve C such that its leg set L_C is composed of only real points, namely points determining 20 legs for which the two endpoints have real coordinates. This gives therefore an example of a mobile icosapod, and to the author's knowledge this is the first such example. Our technique consists in reducing our problem to a question of convex algebraic geometry, and use recent results in this area.

In order to get advantage of the equations satisfied by S, we introduce the new variables $s_{ij} = z_{ij} + z_{ji}$ for $1 \leqslant i < j \leqslant 3$. With this choice, the restriction to S of the bilinear sphere condition from Equation (30) can be written as

$$l\,h + z_{00} r - 2 \sum_{i=1}^{3} s_{0i} x_i - 2 \sum_{i=1}^{3} z_{ii} m_{ii} - 2 \sum_{1 \leqslant i < j}^{3} s_{ij} m_{ij} \;=\; 0. \tag{32}$$

Definition 4.15. We call the left hand side of Equation (32) the *symmetric bilinear sphere condition*, and we denote it by SBSC. This bilinear form defines a duality between S and the linear subspace $\check{\mathbb{P}}^{10}_{\mathbb{C}} \cong \check{S} \subseteq \check{\mathbb{P}}^{16}_{\mathbb{C}}$ whose projective coordinates are $z_{00}, \ldots, z_{33}, s_{01}, \ldots, s_{23}$ and l. The intersection $Y \cap \check{S}$ parametrizes pairs of legs obtained by swapping the roles of the base and the platform points.

By unwrapping the definitions one can see that the following result holds:

Lemma 4.16. *Define* $\varpi \colon \check{S} \dashrightarrow \check{\mathbb{P}}^{9}_{\mathbb{C}}$ *to be the projection removing the* l*-coordinate and define* $\Psi \colon \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}} \longrightarrow \check{\mathbb{P}}^{9}_{\mathbb{C}}$ *to be the map sending*

$$\big((a_0 : \cdots : a_3), (b_0 : \cdots : b_3)\big) \mapsto \big(\underbrace{a_0\, b_0}_{z_{00}} : \cdots : \underbrace{a_3\, b_3}_{z_{33}} : \underbrace{a_0\, b_1 + a_1\, b_0}_{s_{01}} : \cdots : \underbrace{a_2\, b_3 + a_3\, b_2}_{s_{23}}\big).$$

Let Y_{inv} *be the image of* Ψ*. Then*

$$Y_{\mathrm{inv}} \;=\; \varpi\big(Y \cap \check{S}\big).$$

Remark 4.17. The map Ψ from Lemma 4.16 satisfies $\Psi(a, b) = \Psi(b, a)$ for $a, b \in \mathbb{P}^3_{\mathbb{C}}$. In particular, it can happen that two pairs of complex points are sent by Ψ to a real point of Y_{inv}.

Again by checking the definitions, one sees that

Lemma 4.18. *Let* $C \subseteq X_{\mathrm{inv}}$ *be a curve. Then the leg set* L_C *equals*

$$\varpi^{-1}\left(Y_{\mathrm{inv}}\right) \cap \check{\Gamma},$$

where $\check{\Gamma}$ *is the dual space — under the* SBSC *— of* $\Gamma = \operatorname{span}(C)$. *Notice in particular that if* $\operatorname{span}(C)$ *has dimension* 6*, then* $\check{\Gamma}$ *has dimension* 4*.*

Recall from Definition 4.5 that X_{inv} is contained in the hyperplane of equation $m_{11} + m_{22} + m_{33} + h = 0$. This implies that every linear space $\check{\Gamma}$ as in Lemma 4.18 passes through the point $p_e \in \check{S}$ that has coordinates

$$p_e = (\underbrace{0}_{z_{00}} : \underbrace{1:1:1}_{z_{11}, z_{22}, z_{33}} : \underbrace{0:\cdots:0}_{s_{ij}} : \underbrace{-2}_{l}).$$

Borel's construction can therefore be rephrased as follows: choose a general linear space $\check{\Gamma}$ such that $p_e \in \check{\Gamma}$ and consider the intersection $\varpi\left(\check{\Gamma}\right) \cap Y_{\mathrm{inv}}$. The latter is constituted of 10 points, that are in general complex, each of them determines a couple of legs. We want to be able to select $\check{\Gamma}$ so that all the 10 points are real, and come from 20 real legs (recall from Remark 4.17 that it is possible to have real points that arise because two complex conjugate legs are "collapsed" by Ψ).

At first, we tried a naive random search with 10000 examples to see if we were lucky enough to hit a linear space $\check{\Gamma}$ determining 20 real legs. The results we got are shown in Tables 1 and 2.

Table 1: Points in $\varpi\left(\check{\Gamma}\right) \cap Y_{\mathrm{inv}}$.

no. of real points	2	4	6	8	10
frequency	22	1067	3638	4035	1238

Table 2: Real legs determined by the points in $\varpi\left(\check{\Gamma}\right) \cap Y_{\mathrm{inv}}$.

no. of real legs	2	4	6	8	10	12	14	16	18	20
frequency	0	4107	0	5240	0	650	0	3	0	0

We try therefore a different approach, namely we link the notion of spectrahedron we encountered in the Intermezzo with the linear spaces that determine legs of icosapods.

Consider a 4-dimensional vector space $\mathcal{A}$ of symmetric 4×4-matrices over $\mathbb{R}$. Then the spectrahedron constituted of the positive semidefinite matrices in $\mathcal{A}$ is called the spectrahedron defined by $\mathcal{A}$. One can also consider the spectrahedron as a subset of the projective space $\mathbb{P}(\mathcal{A}) \cong \mathbb{P}^3_{\mathbb{R}} \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$. The boundary of the spectrahedron consists of the semidefinite matrices with determinant 0, and hence its Zariski closure is a quartic surface in $\mathbb{P}^3_{\mathbb{C}}$, called the *symmetroid* defined by $\mathcal{A}$. In general, a symmetroid has 10 double points, corresponding to matrices of rank 2.

Given a spectrahedron whose symmetroid has 10 complex double points, its *type* is defined to be the pair of integers (a, b), where a is the number of real double points of the symmetroid and b is the number of real double points of the symmetroid that are also contained in the spectrahedron.

Definition 4.19. A *Borel subspace* $\check{\Delta}$ is a linear subspace of $\check{\mathbb{P}}^9_{\mathbb{C}}$ of dimension 3 passing through $\varpi(p_e)$.

Proposition 4.20. *There is a correspondence between quartic spectrahedra obtained by cutting the cone of* 4×4 *positive semidefinite matrices with a linear subspace containing the matrix* $E := \left(\begin{smallmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{smallmatrix}\right)$ *and Borel subspaces. If a spectrahedron* $\mathcal{S}$ *and a Borel subspace* $\check{\Delta}$ *are in correspondence, then the following statement holds: if the spectrahedron* $\mathcal{S}$ *has type* (a, b)*, then* $\check{\Delta}$ *intersects* Y_{inv} *in* a *real points, and* $a - b$ *of them have real preimages under* Ψ.

Proof. We consider the following identification between the space $\check{\mathbb{P}}^9_{\mathbb{C}}$ where Y_{inv} lives and the projectivization of the vector space of symmetric 4×4 matrices: a point $(z_{00} : \cdots : s_{23})$ corresponds to the class of the matrix

$$\begin{pmatrix} 2z_{00} & s_{01} & s_{02} & s_{03} \\ s_{01} & 2z_{11} & s_{12} & s_{13} \\ s_{02} & s_{12} & 2z_{22} & s_{23} \\ s_{03} & s_{13} & s_{23} & 2z_{33} \end{pmatrix}.$$

A linear subspace of dimension 3 in the space of symmetric matrices containing the matrix E corresponds then to a Borel subspace.

Recalling the determinantal description of the ideal of the Segre variety one can check that under the previous identification the subvariety Y_{inv} corresponds to the subvariety of symmetric 4×4 matrices of rank 2. A real matrix of rank 2 does not lie on the spectrahedron if and only if the quadratic form defined by it is a product of two distinct real linear forms, and this is true if and only if the preimage under Ψ of the corresponding point is given by a pair of real points. $\square$

Degtyarev and Itenberg in [DI11] determined all possible types of quartic spectrahedra and from their work it follows that in particular spectrahedra of type $(10, 0)$ exist. Proposition 4.20 implies hence that there exist Borel subspaces intersecting Y_{inv} in 10 real points, each of them having two real preimages under the map Ψ. Therefore mobile icosapods exist.

Example 4.21. In order to construct a concrete example of a mobile icosapod we need to start from a concrete example of a spectrahedron of type $(10, 0)$. In [ORSV15], the authors provide such an example, which unfortunately does not contain the matrix E. It is however not difficult to adapt such example to one of the same type containing it. The Borel subspace corresponding to such

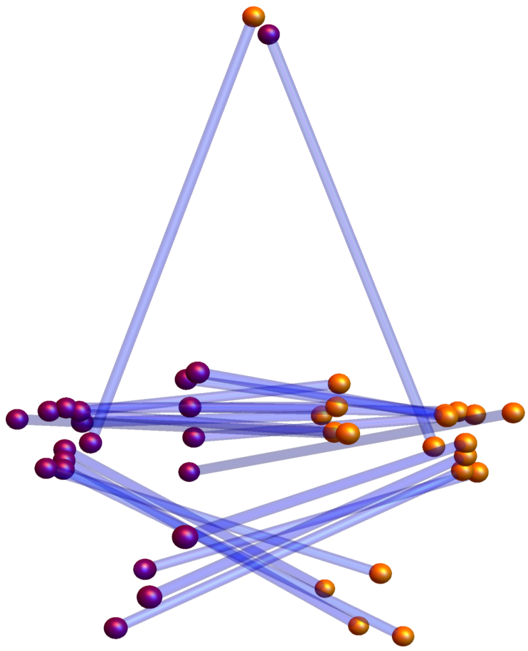


Figure 13: The icosapod determined by the choice of points in Example 4.21.

spectrahedron is spanned by $\varpi(p_e)$ and by the three points in Y_{inv} determined by the following three pairs of legs:

$$
\begin{aligned}
P_1 = p_4 &= \left(-\tfrac{19493}{142100}, -\tfrac{2088}{94325}, -\tfrac{24}{9625}\right), \\
p_1 = P_4 &= \left(-\tfrac{36411}{267844}, -\tfrac{1608}{177793}, \tfrac{504}{25399}\right), \\
P_2 = p_5 &= \left(-\tfrac{269}{5000}, \tfrac{39}{1000}, \tfrac{17}{500}\right), \\
p_2 = P_5 &= \left(-\tfrac{47}{368}, -\tfrac{12}{1771}, \tfrac{21}{1265}\right), \\
P_3 = p_6 &= \left(-\tfrac{1863}{14645}, -\tfrac{106851}{1555400}, \tfrac{2509}{222200}\right), \\
p_3 = P_6 &= \left(-\tfrac{15185}{112462}, -\tfrac{120}{149303}, \tfrac{48}{3047}\right).
\end{aligned}
$$

The mobile icosapod that is obtained by this choice of Borel subspace is shown in Figure 13.

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Matteo Gallet

Curriculum Vitae

Altenberger Straße 69
4040 Linz, Austria
✉ *matteo.gallet@ricam.oeaw.ac.at*
www.risc.jku.at/people/mgallet/

Personal data

Date of birth 24 February 1987

Place of birth Monfalcone (GO), Italy

Citizenship Italian

Education

2012–2016 **PhD in Mathematics**, *J. Kepler University, Linz (Austria)*, adviser Josef Schicho.

2009–2011 **Master studies in Mathematics**, *University of Trieste (Italy) (joint curriculum with SISSA-ISAS)*, Thesis: "Families of projective schemes and Hodge classes", adviser Dario Portelli.

2006–2009 **Bachelor studies in Mathematics**, *University of Trieste (Italy)*, Thesis: "Un'introduzione al mapping class group di una superficie", adviser Bruno Zimmermann.

Publications

Accepted

2016 **M. Gallet, J. Schicho, G. Nawratil**, *Liaison Linkages*, Journal of Symbolic Computation – Special Issue: MEGA 2015, [DOI:10.1016/j.jsc.2016.08.006, arXiv:1510.01127].

M. Gallet, G. Nawratil, J. Schicho, J.M. Selig, *Mobile Icosapods*, Accepted for publication in Advances in Applied Mathematics, [arXiv:1603.07304].

2015 **M. Gallet, C. Koutschan, Z. Li, G. Regensburger, J. Schicho, N. Villamizar**, *Planar Linkages Following a Prescribed Motion*, Mathematics of Computation, [DOI:10.1090/mcom/3120, arXiv:1502.05623].

J. Schicho, M. Gallet, *Ambiguities in a problem in planar geodesy*, SIGMA, [DOI:10.3842/SIGMA.2015.008, arXiv:1410.7630].

M Gallet, J. Schicho, G. Nawratil, *Möbius Photogrammetry*, Journal of Geometry, [DOI:10.1007/s00022-014-0255-x, arXiv:1408.6716].

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Preprint

2016 **M. Gallet, K. Ranestad, N. Villamizar**, *Varieties of apolar subschemes of toric surfaces*, [arXiv:1601.00694].

2016 **M. Gallet, E. Saini**, *The diffeomorphism type of small hyperplane arrangements is combinatorially determined*, [arXiv:1601.05705].

Talks

2015 *MEGA 2015*, 14–19 June, Trento (Italy) — Title: "*Liaison Hexapods*".

GAeL XXIII, 1–5 June, Leuven (Belgium) — Title: "*Liaison Hexapods*".

EACA 2014, 18–20 June, Barcelona (Spain) — Title: "*Bond theory for pentapods and hexapods*".

Conferences and schools

2014 *Summer School on An Interdisciplinary Approach to Tensor Decomposition*, 14–18 July, Trento (Italy).

GAeL XXII, 23–27 June, Trieste (Italy).

Vector Bundles Days II, 19–21 January, Trieste (Italy).

2013 *RICAM Special Semester on Applications of Algebra and Number Theory (Workshop 3: Computer Algebra and Polynomials)*, 25–29 November, Linz (Austria).

Gröbner Bases, Resultants and Linear Algebra, 3–6 September, Linz (Austria).

Combinatorial Algebraic Geometry, 10–15 June, Levico Terme (Italy).

Syzygies in Berlin, 27–31 May, Berlin (Germany).

2012 *Mini-courses on commutative rings, integer-valued polynomials and polynomial functions*, 16–18 December, Graz (Austria).

Mini-course on Macaulay2, September, Trieste (Italy).

Scuola Matematica Interuniversitaria, 30 July – 31 August, Perugia (Italy).

2012 *Clay Mathematics Institute Summer School on the Resolution of Singular Varieties*, 3–30 June, Obergurgl (Austria).

2011 *Cohomology theories: a road map*, 12–17 September, Bressanone (Italy).

Teaching

2016 Exercise instructor for the course "Algebra und Diskrete Mathematik", Johannes Kepler Universität (Linz).

2014 Series of 9 seminars on an introduction to Lie groups, Radon Institute for Computational and Applied Mathematics (Linz).

2013 Mini-course on the computer algebra software Sage, 10 hours of lectures for the students of the Mathematics department, University of Trieste.

2012 Series of 10 seminars on an introduction to cohomology of coherent sheaves, Research Institute for Symbolic Computation (Linz).

Awards

2012 "Marco Reni" prize from University of Trieste.

2009-2011 Fellowship for the joint curriculum between University of Trieste and SISSA-ISAS.

2006-2009 Fellowship from Collegio Universitario per le Scienze "Luciano Fonda".

Computer skills

Basic PYTHON, Sage, Linux, Git.

Intermediate LaTeX, OpenOffice.

Language skills

Italian **Mother tongue**

English **intermediate/advanced** *self-assessment*