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Complex Analysis Based Computer Algebra Algorithms for Proving Jacobi Theta Function Identities

Doctoral Thesis

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Linz, December 2016,

Liangjie Ye

Kurzfassung

Der Hauptgegenstand dieser Arbeit sind die Jacobi-Theta-Funktionen $\theta_j(z|\tau)$ mit $j = 1, \ldots, 4$; und die Klasse der Probleme, die wir betrachten, sind die algebraischen Relationen zwischen diesen. In vergangenen Jahrhunderten haben Forscher (einschließlich Mathematikern, Physikern und anderen) aufwändige, arithmetische Berechnungen verwendet, um selbst einfache Identitäten von Theta-Funktionen per Hand zu beweisen. Dies ist eine mühsame (vielleicht gar unmögliche) Aufgabe für komliziertere Identitäten.

In dieser Arbeit stellen wir Computeralgebraalgorithmen vor, die verschiedene, allgemeine Klassen von Identitäten von Theta-Funktionen mit Hilfe eines Computers behandeln. Die essenziellen, mathematischen Werkzeuge, die verwendet wurden, um diese Algorithmen zu entwickeln, sind die komplexe Analysis und insbesondere vor allem modulare Formen und die Theorie elliptischer Funktionen.

Unser algorithmischer Zugang kann verwendet werden, um Identitäten in sehr allgemeinen Funktionenklassen innerhalb weniger Minuten zu beweisen. Weiterhin können wir auch neue Identitäten in diesen Klassen mit Unterstützung des Computers entdecken. Wir haben diese Algorithmen im Mathematica-Paket “ThetaFunctions” implementiert.

Als Nebenprodukt finden wir (alte und neue) Relationen, die die Weierstraß’sche elliptische Funktion involvieren. Außerdem kann unser algorithmischer Ansatz auf andere Klassen von Identitäten ausgeweitet werden wie zum Beispiel auf einen wesentlichen Teil der Identitäten in Ramanujans verlorenen Notizbüchern sowie in anderen wissenschaftlichen Büchern und Artikeln.

Abstract

The main objects of this thesis are the Jacobi theta functions $\theta_j(z|\tau)$, $j = 1, \ldots, 4$, and the classes of problems we consider are algebraic relations between them. In the past centuries researchers, including mathematicians, physicists, etc., have been using some difficult arithmetic manipulations to prove even basic theta function identities by hand, a tedious (perhaps unfeasible) task for more complicated identities.

In this thesis we present computer algebra algorithms to deal with several general classes of theta function identities in computer-assisted manner. One essential mathematical tool used for developing these algorithms is complex analysis, in particular, mainly modular form techniques and the theory of elliptic functions.

Our algorithmic approaches can be used to prove identities from very general function classes within a few minutes; moreover, we can also discover identities from such classes in a computer-assisted way. We have implemented the algorithms into a Mathematica package "ThetaFunctions."

As a by-product, relations (old and new) involving the Weierstrass elliptic function are found. Moreover, our algorithmic approaches can be extended further to other classes of identities, for example a substantial amount of identities in Ramanujan's lost notebooks and in other research monographs and papers.

Acknowledgments

When I look back on the past four years of my PhD studies, I feel very lucky that I found this wonderful place and two extraordinary advisors. It is frightening to think how easily I could have missed out on this great opportunity. Before I entered RISC I knew nothing about symbolic computation or Jacobi theta functions. I owe Professor Peter Paule and Dr. Silviu Radu so much for all the time they devoted to me.

I took the first steps toward my thesis research when Professor Paule asked me to implement some algorithms from Silviu's lecture "Elliptic functions, theta series and modular forms" [25]. I finished that assignment quickly, and he then asked me to think about solving a challenging problem: to develop an algorithm for proving Jacobi theta function identities in a general class. His question opened a coffer full of treasure for me.

In the beginning I had some bad habits when writing mathematical papers. I sometimes jumped over important steps, using some "fancy" descriptions, or did not pay attention to details, and so on. During our meetings, Silviu and Professor Paule criticized my work strictly and carefully, discussing and arguing, never letting one single logical issue slip through. They have ensured that I become a clean-minded researcher, and even a better person. I wish to express my most sincere gratitude from the bottom of my heart.

The two of them are also philosophers. They tell me many meaningful stories about life, that lead me to think more wisely. For example, the story Professor Paule told me about fixing the door of his house inspired me a lot. Now whenever I encounter a recurring problem I do not try to go around it but take pains to fix it once and for all, and move on freely. When I was reluctant to crack tangled mathematical problems, Silviu cured me with his car repair stories. He told me "you can not repair a car by watching it, you have to get your hands greasy". In this spirit I am not afraid of getting my hands dirty with nasty computations. They are doctors of philosophy in the fullest sense.

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Gabriela Danter and Betina Curtis from the DK program and Tanja Gutenbrunner from RISC have helped me deal with paperwork and university administrative issues. Because of their help I could fully concentrate on my research. They deserve my sincere thanks. I would also like to thank all my colleagues at the DK program and at RISC for their help in the past four years. I especially want to thank Johannes Middeke for helping me understand German documents and translating files to German on many occasions.

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There are some friends whom I appreciate for showing up in my life and being fantastic company. In particular I want to thank Miriam Schussler, my former housemate. In my loneliest and most desperate times she was the one that I could always talk to. Special thanks go to Junpeng Deng for growing up with me, helping me become a considerate and conscientious person, and encouraging me when I was highly stressed out during my thesis-writing.

Last but not least I want to thank my parents, who have always been fully supportive of me. Although they do not understand the content of my studies they are respectful of my faith in mathematics and they always give me unconditional love.

Contents

Chapter 1

Introduction

The theory of theta functions is far from a finished polished topic. –D. Mumford

The overall objective of this thesis is to provide tools for the computer-assisted treatment of identities among Jacobi theta functions. In most of the books and papers about Jacobi theta functions or containing Jacobi theta functions there are many identities, for instance [9], [18], [23], [35], etc. Especially, in the book series "Ramanujan's Notebooks I-V" [3], [4], [5],[6], and [7] there is a substantial amount of identities involving theta functions, which are mostly in Ramanujan's notation but can be written in terms of Jacobi theta functions. For example the well-known identity

$$\theta_2(0,q)^4 + \theta_4(0,q)^4 \equiv^1 \theta_3(0,q)^4 \tag{1.1}$$

has the form

$$\varphi^4(q) \equiv \varphi^4(-q) + 16q\psi^4(q^2)$$

in Ramanujan's notation. Many researchers, e.g., B. C. Berndt, J. M. Borwein, P. B. Borwein, F. G. Garvan, etc., have been studying them and their applications for decades. In particular, in partition analysis Ramanujan's modular equations are widely applied, which are alternative expressions of Jacobi theta function identities. For instance, a form of the cubic modular equation [17, p. 218] is

$$\theta_3(q)\theta_3(q^3) - \theta_4(q)\theta_4(q^3) - \theta_2(q)\theta_2(q^3) \equiv 0.$$

[1] We use the notation $f_1(z_1,z_2,\dots) \equiv f_2(z_1,z_2,\dots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.

Since there was no global way of dealing with theta function identities, they usually use different manipulations to prove different identities. Then the motivation of this thesis came to us, namely, developing an algorithmic approach to systematically deal with theta function identities. More specifically, we expect to develop a package that whenever people want to prove an identity or acquire new relations among theta functions, they press on a button then wait a few seconds and the result will be shown on the screen.

In the first step of development, this amounts to zero-recognition of Taylor coefficients of the respective series expansions of theta functions, which are some combinations of $\theta_j^{(k)}(0|\tau)$. Hence in Chapter 3 we deal with identities among $\theta_j^{(k)}(0|\tau)$. The main content of Chapter 3 is also in our paper [34]. In Chapter 4 we involve another variable $z \in \mathbb{C}$, and deal with identities among $\theta_j^{(k)}(0|\tau)$ and $\theta_j(z|\tau)$. In Chapter 5 we extend the function space further and study identities among $\theta_j^{(k)}(0|\tau)$ and $\theta_j(az|\tau)$ with $a \in \mathbb{N}\backslash\{0\}$. Based on our decomposition theory in Chapters 3 and 4, we are also able to generate relations of any given degree, which is described in Chapter 6. In Chapter 7 we present some work that is ongoing or will be solved in the near future. In Chapter 8 we introduce the Mathematica package "ThetaFunctions." Since the programming of the package is not completely done, we mainly present some key features that are already available in the package.

Chapter 2

Preliminaries

Throughout the thesis $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{K} \subseteq \mathbb{C}$ is a field. We assume that $\mathbb{K}$ contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.). In algorithmic contexts, $\mathbb{K}$ is an effectively computable field. Throughout the thesis for $z = ce^{i\varphi}$ ($c > 0, 0 \leqslant \varphi < 2\pi$) we define $z^r := c^r e^{ir\varphi}$ for $r \in \frac{1}{2}\mathbb{Z}$.

2.1 Elliptic Functions

Definition 2.1. *[15, Def. V.1.1] A subset $L \subseteq \mathbb{C}$ is a lattice, if and only if there exist two "vectors" ω_1 and ω_2 in $\mathbb{C}$, which are linearly independent over $\mathbb{R}$, and generate L as an abelian group; i.e.,*

$$L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}.$$

Definition 2.2. *[15, Def. V.1.2] An elliptic function for the lattice L is a meromorphic function*

$$f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$$

with the property

$$f(z + \omega) \equiv f(z) \text{ for } \omega \in L \text{ and } z \in \mathbb{C}.$$

In this thesis we usually write an elliptic function without mentioning the corresponding lattice, when it is clear from the context.

Theorem 2.1. *[15, p. 253, Th. V.1.3 (The First Liouville Theorem, J. Liouville, 1847)] Any elliptic function without poles is constant.*

Definition 2.3. *Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. A period-parallelogram with periods ω_1 and ω_2 is denoted by*

$$P(\omega_1, \omega_2) := \{t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0, 1[\}.$$

Note. In this thesis we mainly use the case $\omega_1 = \pi$ and $\omega_2 = \pi\tau$ where $\tau\mathbb{H}$.

Definition 2.4. *Given a meromorphic function f on $\mathbb{C}$, we define*

$$\textit{poles } (f) := \{z \in \mathbb{C} : f \textit{ has a pole at } z\}$$

and

$$\textit{zeros } (f) := \{z \in \mathbb{C} : f \textit{ has a zero at } z\}.$$

Definition 2.5. *Given a lattice $L \subseteq \mathbb{C}$, two points $z_1, z_2 \in \mathbb{C}$ are equivalent with regard to L if $z_1 - z_2 \in L$, denoted by $z_1 \sim_L z_2$.*

Lemma 2.1. *Let L be the lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$. For every $z \in \mathbb{C}$ there exists one and only one point $z_1 \in P(\omega_1, \omega_2)$ such that $z_1 \sim_L z$.*

Proof. For any fixed $z = a\omega_1 + b\omega_2 \in \mathbb{C}$ with $a, b \in \mathbb{R}$, we can always find $m, n \in \mathbb{Z}$ and $t_1, t_2 \in [0, 1[$ such that $a = m + t_1$ and $b = n + t_2$. Let $z_1 := t_1\omega_1 + t_2\omega_2$, then $z_1 \in P(\omega_1, \omega_2)$ and $z_1 \sim_L z$. Assume there exists another point $z_2 \in P(\omega_1, \omega_2)$ with $z_2 \sim_L z$, then $z_2 \sim_L z_1$. Suppose $z_2 := t_3\omega_1 + t_4\omega_2$ with $t_3, t_4 \in [0, 1[$. Then $t_3 - t_1 \in \mathbb{Z}$ and $t_4 - t_2 \in \mathbb{Z}$. This implies $t_3 = t_1$ and $t_4 = t_2$, i.e., $z_2 = z_1$. □

The following theorem is crucial for zero-recognition later in Chapters 4 and 5.

Theorem 2.2. *For any nonzero elliptic function f with periods ω_1 and ω_2, one has*

$$\#\left(\textit{poles } (f) \cap P(\omega_1, \omega_2)\right) = \#\left(\textit{zeros } (f) \cap P(\omega_1, \omega_2)\right).$$

Note. poles $(f) \cap P(\omega_1, \omega_2)$ and zeros $(f) \cap P(\omega_1, \omega_2)$ are finite sets.

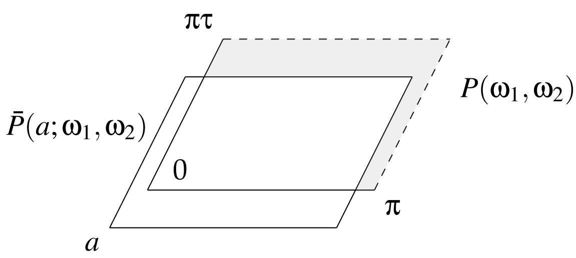
Proof of Theorem 2.2. Let $H := \{z \in P(\omega_1, \omega_2) : f$ has a pole or zero at $z\}$, $h_1 := \max\{t_1 : t_1\omega_1 + t_2\omega_2 \in H$ with $t_1, t_2 \in [0,1[\}$ and $h_2 := \max\{t_2 : t_1\omega_1 + t_2\omega_2 \in H$ with $t_1, t_2 \in [0,1[\}$. We define a closed period parallelogram by

$$\bar{P}(a; \omega_1, \omega_2) := \{a + b\omega_1 + c\omega_2 : b, c \in [0,1]\}$$

with

$$a := -\frac{1-h_1}{2}\omega_1 - \frac{1-h_2}{2}\omega_2.$$

The following image interprets the positions of $\bar{P}(a; \omega_1, \omega_2)$ and $P(\omega_1, \omega_2)$.



By the definition of $\bar{P}(a; \omega_1, \omega_2)$, one can easily check that for any $y \in P(\omega_1, \omega_2)$ if y is a zero (or a pole) of $f(z)$, then y is also in the interior of $\bar{P}(a; \omega_1, \omega_2)$; and $f(z)$ has poles or zeros neither in the gray area

$$\{z : z \in P(\omega_1, \omega_2) \text{ and } z \notin \bar{P}(a; \omega_1, \omega_2)\}$$

nor on the line segments where $P(\omega_1, \omega_2)$ intersects the boundary of $\bar{P}(a; \omega_1, \omega_2)$. Hence by Lemma 2.1, $f(z)$ does not have any zeros or poles on the whole boundary of $\bar{P}(a; \omega_1, \omega_2)$, and no zeros or poles in the region

$$\{z : z \in \bar{P}(a; \omega_1, \omega_2)\} \text{ and } z \notin P(\omega_1, \omega_2)\}.$$

Therefore the set of zeros and poles in $P(\omega_1, \omega_2)$ is equal to the set of zeros and poles in the interior of $\bar{P}(a; \omega_1, \omega_2)$.

By a classical argument, e.g. [11, p. 23, Th. 3] we complete the proof. □

Note. Usually in the literature Theorem 2.2 is stated in different ways, e.g. in [11, p. 23, Th. 3], [20, p. 75, Th. 3.6.4] and [31, p. 432].

Lemma 2.2. *[31, p. 434] The series*

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{(z-m\omega_1-n\omega_2)^2}-\frac{1}{(m\omega_1+n\omega_2)^2}\right)$$

converges absolutely and uniformly (with regard to z) in compact sets in $\mathbb{C}$ which contain no points in the lattice $L := \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$.

Definition 2.6. *(Weierstrass $\wp$ Function) For $z \in \mathbb{C}\setminus\{0\}$ we define*

$$\wp(z;\omega_1,\omega_2) := \frac{1}{z^2}+\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{(z-m\omega_1-n\omega_2)^2}-\frac{1}{(m\omega_1+n\omega_2)^2}\right).$$

One sees that $\wp$ is analytic in $\mathbb{C}\setminus L$ and has poles of order 2 at each point of L, and we can compute the Laurent expansion of $\wp$ at $z = 0$.

Proposition 2.1. *[15, p. 266, Prop. V.2.11]*

$$\wp(z;\omega_1,\omega_2) := \frac{1}{z^2}+\sum_{k=1}^{\infty}(2k+1)E_{2k+2}(\omega_1,\omega_2)z^{2k},$$

where $E_{2k+2}(\omega_1,\omega_2) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\omega_1+n\omega_2)^{-2k-2}$ is an Eisenstein series.

Note. The series E_{2k+2} converges absolutely when $k \geqslant 1$.

By Lemma 2.2, we can differentiate the series for $\wp(z;\omega_1,\omega_2)$ term-by-term and obtain

$$\wp'(z;\omega_1,\omega_2) \equiv -\frac{2}{z^3}+\sum_{k=1}^{\infty}2k(2k+1)E_{2k+2}z^{2k-1}. \tag{2.1}$$

It is important to note that the functions $\wp$ and $\wp'$ depend on the lattice generated by ω_1 and ω_2. However, in this thesis $\omega_1 = \pi$ and $\omega_2 = \pi\tau$, thus instead of writing $\wp(z;\pi,\pi\tau)$ and $\wp'(z;\pi,\pi\tau)$ everywhere, we use the abbreviations $\wp(z)$ and $\wp'(z)$.

2.2 Jacobi Theta Functions

The functions that are the building blocks of this thesis are the Jacobi theta functions defined as follows.

Definition 2.7. *[14, 20.2(i)] Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $q = e^{\pi i \tau}$, then*

$$\begin{aligned}
\theta_1(z,q) := \theta_1(z|\tau) &:= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z),\\
\theta_2(z,q) := \theta_2(z|\tau) &:= 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z),\\
\theta_3(z,q) := \theta_3(z|\tau) &:= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz),\\
\theta_4(z,q) := \theta_4(z|\tau) &:= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).
\end{aligned}$$

For fixed $\tau \in \mathbb{H}$, Definition 2.7 implies that the $\theta_j(z|\tau)$ $(j = 1,\ldots,4)$ are analytic functions on the whole complex plane with respect to z. For fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ $(j = 1,\ldots,4)$ are analytic functions of τ for all $\tau \in \mathbb{H}$, and correspondingly, analytic functions of q for $|q| < 1$.

Proposition 2.2. *[31, 21.12] For each $j \in \{1,2,3,4\}$, $\theta_j(z)$ has one and only one zero in $P(\pi,\pi\tau)$. The zeros of $\theta_1(z)$, $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$ are at the points congruent respectively to 0, $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{\pi\tau}{2}$, $\frac{\pi\tau}{2}$ modulo $\{m\pi + n\pi\tau : m,n \in \mathbb{Z}\}$.*

One can check by using Definition 2.7 that the following lemma holds.

Lemma 2.3. *[31, p. 465] Let $N := e^{-\pi i\tau - 2iz}$. For $j \in \{1,2,3,4\}$ we have $\theta_j(z+\pi\tau|\tau) = \varepsilon_1(j)\theta_j(z|\tau)$ and $\theta_j(z+\pi|\tau) = \varepsilon_2(j)\theta_j(z|\tau)$, where $\varepsilon_1(j)$ and $\varepsilon_2(j)$ are defined in Table 2.1.*

j	1	2	3	4
$\varepsilon_1(j)$	$-N$	N	N	$-N$
$\varepsilon_2(j)$	-1	-1	1	1

Table 2.1

For the other variable τ, the theta functions have the following transformations.

Lemma 2.4. *[31, p. 475] For the substitution $\tau \mapsto -1/\tau$ on $\theta_j(z|\tau)$ ($j = 1, \dots, 4$) we have*

$$\theta_1\left(z|-\frac{1}{\tau}\right) \equiv -i(-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_1(z\tau|\tau); \quad \theta_2\left(z|-\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_4(z\tau|\tau);$$
$$\theta_3\left(z|-\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_3(z\tau|\tau); \quad \theta_4\left(z|-\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_2(z\tau|\tau).$$

Directly from Definition 2.7 one can deduce the following:

Lemma 2.5. *For the substitution $\tau \mapsto \tau + 1$ on $\theta_j(\tau)$ ($j = 1, \dots, 4$) we have*

$$\theta_1(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_1(z|\tau); \quad \theta_2(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_2(z|\tau);$$
$$\theta_3(z|\tau+1) \equiv \theta_4(z|\tau); \quad \theta_4(z|\tau+1) \equiv \theta_3(z|\tau).$$

In Chapter 3 we will generalize Lemmas 2.4 and 2.5 to the derivatives $\theta_j^{(k)}(z|\tau)$, where

$$\theta_j^{(k)}(z|\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau), k \in \mathbb{N}.$$

2.3 Modular Forms

In the literature one finds several variants of definitions for modular forms. In this thesis we use the [15, p. 326, Def. VI.2.4] for analytic functions on $\mathbb{H}$.

Definition 2.8. *Let*

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \, and \, a, b, c, d \in \mathbb{Z} \right\},$$

$q = e^{\pi i \tau}$ and $\tau \in \mathbb{H}$. Given $k \in \mathbb{Z}$, a modular form of weight k is an analytic function f on $\mathbb{H}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) \equiv (c\tau + d)^k f(\tau) \quad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{2.2}$$

and where $f(\tau)$ can be written as a Taylor series in powers of q with complex coefficients; i.e.,

$$f(\tau) \equiv \sum_{n=0}^{\infty} a_n e^{\pi i \tau n} \equiv \sum_{n=0}^{\infty} a_n q^n. \tag{2.3}$$

We denote the corresponding analytic function by

$$\widetilde{f}(q) :\equiv \sum_{n=0}^{\infty} a_n q^n.$$

Note. Substituting $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in (2.2) we obtain $\widetilde{f}(-q) \equiv f(\tau+1) \equiv f(\tau) \equiv \widetilde{f}(q)$. Therefore $\widetilde{f}(-q) \equiv \sum_{n=0}^{\infty} a_n(-q)^n \equiv \sum_{n=0}^{\infty} a_n q^n \equiv \widetilde{f}(q)$, which implies by comparison of coefficients that $a_n = -a_n$ for all odd $n \in \mathbb{N}$. Consequently,

$$\widetilde{f}(q) \equiv \sum_{n=0}^{\infty} a_{2n} q^{2n}. \tag{2.4}$$

In the literature, as in [15], one often replaces (2.3) in Definition 2.8 by (2.4). We prefer to stay with (2.3) in order to keep

$$q = e^{\pi i \tau}, \tau \in \mathbb{H}$$

throughout the thesis.

Example 2.1. *When $k \geqslant 1$, the Eisenstein series $E_{2k+2}(\pi, \pi\tau)$ is a modular form of weight $2k+2$.*

Example 2.2. *Let $e_1 := \frac{1}{3}(\theta_3(0,q)^4 + \theta_4(0,q)^4)$, $e_2 := -\frac{1}{3}(\theta_2(0,q)^4 + \theta_3(0,q)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0,q)^4 - \theta_4(0,q)^4)$. Using Lemmas 2.4, 2.5 and 2.6 one can verify that the product $e_1e_2e_3$ is a modular form of weight 6.* [1]

Definition 2.9. *The $\mathbb{K}$-vector space of modular forms of weight $k \in \mathbb{N}$ is denoted by $M_k(\mathbb{H})$.*

Lemma 2.6. *[29, p. 78, Thm. 2] The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by*

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

[1] According to [11, p. 33], $e_1e_2e_3 = 35E_6(\pi, \pi\tau)$.

Definition 2.10. *[29, p. 84] Let f be a nonzero meromorphic function on $\mathbb{H}$, and a be a point of $\mathbb{H}$. The integer n such that $f/(\tau - a)^n$ is holomorphic and nonzero at a is called the order of f at a, and is denoted by $v_a(f)$.*

One observes that

$$v_a(f)\begin{cases} > 0, & \text{if } a \text{ is a zero of } f, \\ < 0, & \text{if } a \text{ is a pole of } f, \\ = 0, & \text{otherwise.} \end{cases}$$

Definition 2.11. *Let f be meromorphic on $\mathbb{H}$ satisfying*

$$f(\tau) \equiv \sum_{n=0}^{\infty} a_n e^{\pi i \tau n} \equiv \sum_{n=0}^{\infty} a_n q^n.$$

If $a_0 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, we define the order of f at ∞ by $\nu_\infty(f) := m$.

Note. In classical textbooks $q = e^{2\pi i \tau}$ is used, thus their order of f at ∞ is equal to our $\frac{1}{2}\nu_\infty(f)$.

Suppose f is a non-zero modular form of weight k, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ then by (2.2) we have

$$f(\tau) = (-1)^k f(\tau)$$

which means k must be an even number. Therefore in textbooks people often say a modular form of weight $2\tilde{k}$. But in this thesis for simplicity we prefer to say a modular form of weight k. Using this together with the Note above, we rewrite valence formula from [29, p. 85, Th. 3] as the following theorem.

Theorem 2.3. *Let f be a modular form of weight k, then*

$$\frac{1}{2}v_\infty(f) + \sum_{r \in \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} e_r v_r(f) = \frac{k}{12}, \tag{2.5}$$

where

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) := \left\{ z \in \mathbb{H} : -\frac{1}{2} \leqslant \mathrm{Re}(z) < \frac{1}{2} \textit{ and } |z| > 1, \textit{ or } z = e^{i\theta} \textit{ where } \frac{\pi}{2} \leqslant \theta \leqslant \frac{2\pi}{3} \right\},$$

$e_i = \frac{1}{2}$, $e_w = \frac{1}{3}$ with $w := -\frac{1}{2} + \frac{\sqrt{3}i}{2}$ and $e_r = 1$ when $r \neq i, w$.

Note. $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ gives a complete set of representatives of the group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. But we omit the proof here.

By Definition 2.8, f is holomorphic on $\mathbb{H}$, and every term on the left hand side of (2.5) is non-negative. Hence

$$v_\infty(f) \leqslant \frac{k}{6}$$

and we deduce the following corollary.

Corollary 2.1. *Let $q := e^{\pi i \tau}$ and $f \in M_k(\mathbb{H})$ with the q-expansion of f be $\sum\limits_{t=0}^{\infty} a_t q^t$.*

If $a_j = 0$ for $j \leqslant \lfloor \frac{k}{6} \rfloor$, then $f = 0$.

Chapter 3

Proving Identities among Powers of $\theta_j^{(k)}(0|\tau)$ (Class I)

To introduce the general idea and application domain of the method presented in this chapter, consider the following lemma that has been used in numerous papers like [8], [19] and [16] to prove relations between Jacobi theta series.

Lemma 3.1. *[2](Atkin and Swinnerton-Dyer Lemma) Given a non-zero meromorphic function f on $\mathbb{C}\backslash\{0\}$ with $f(wx) \equiv cx^n f(x)$ for some integer n and non-zero complex constants c and w with $0 < |w| < 1$, then*

$$\#\,poles(f) = \#\,zeros(f) + n$$

in $|w| < |x| \leqslant 1$. [1]

To do zero recognition of such $f(x) = f(x,q)$, where q is a parameter, the lemma classically is applied as follows: one cleverly chooses sufficiently many zeros $x_1,\ldots,x_m$ in the domain $|w| < |x| \leqslant 1$. According to the lemma the number m of such zeros needs to be greater than the number of poles of f minus n, in order to show that f is identically zero. By their clever choice of $x_1,\ldots,x_m$, $f(x_i,q)$ is a modular form when viewed as a function of q. And, zero-recognition of modular forms is algorithmic owing to methods using Sturm bounds or valence formula.

Our approach is different and streamlines the idea above by choosing *only one* evaluation point, namely $x_i = 1$ for all i, and by verifying that $f^{(j)}(1,q) = 0$ for $j \in \{0,\ldots,m-1\}$. In this way we prove that there is a zero of multiplicity at least m, which again implies that $f(x) \equiv 0$.

[1] By # poles(f), resp. # zeros(f), we count poles, resp. zeros, with multiplicity.

A crucial point is that, despite for $j \geqslant 1$ the Taylor coefficients, in general, are not modular forms anymore, the task of proving such relations like $f^{(j)}(1,q) = 0$ again can be carried out algorithmically for a large class of problems specified below. The functions that are the building blocks of this class are the Jacobi theta functions $\theta_j(z|\tau)$ ($j = 1, \ldots, 4$) and their derivatives evaluated at $z = 0$.

To illustrate our method of using Lemma 3.1, we consider the following classical example which generates identity (1.1).

Example 3.1. *[14] For* $q \in \mathbb{C}$ *with* $0 < |q| < 1$*, prove*

$$\theta_3(0,q)^2\theta_3(z,q)^2 - \theta_4(0,q)^2\theta_4(z,q)^2 - \theta_2(0,q)^2\theta_2(z,q)^2 \equiv 0. \tag{3.1}$$

Proof. Let $f_j(x) := \theta_j(z,q)$ with $x(z) = e^{2iz}$. Then using the series expansions in Definition 2.7 one can verify directly that $f_j^2(q^2x) = q^{-2}x^{-2}f_j^2(x)$. Define

$$g(x) := \theta_2(0,q)^2 f_2(x)^2 - \theta_3(0,q)^2 f_3(x)^2 - \theta_4(0,q)^2 f_4(x)^2.$$

Observing that $g(q^2x) = q^{-2}x^{-2}g(x)$, to prove the identity, by Lemma 3.1 it is sufficient to show that $g(x)$ has at least three more zeros than poles in $|q^2| < |x| \leqslant 1$. By Definition 2.7, $g(x)$ has no pole in $\mathbb{C}$. The Taylor expansion of $g(x)$ around $x = 1$ is

$$g(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2}(x-1)^2 + \frac{g^{(3)}(1)}{6}(x-1)^3 + O((x-1)^4).$$

We need to show

$$g(1) = 0, \quad g'(1) = 0 \quad \text{and} \quad g''(1) = 0. \tag{3.2}$$

Let $h(z) :=$ LHS of (3.1). Because $h(z) = g(e^{2iz}) = g(x)$, $h'(z) = 2ixg'(x)$ and $h''(z) = -4xg'(x) - 4x^2g''(x)$, to show (3.2), it is sufficient to show

$$h(0) = \theta_3(0,q)^4 - \theta_2(0,q)^4 - \theta_4(0,q)^4 \equiv 0, \tag{3.3}$$

$$h'(0) = 2\theta_3(0,q)^3\theta_3'(0,q) - 2\theta_2(0,q)^3\theta_2'(0,q) - 2\theta_4(0,q)^3\theta_4'(0,q) \equiv 0, \tag{3.4}$$

$$\text{and} \quad h''(0) = \theta_3(0,q)^2\theta_3'(0,q)^2 - \theta_2(0,q)^2\theta_2'(0,q)^2 - \theta_4(0,q)^2\theta_4'(0,q)^2 + \theta_3(0,q)^3\theta_3''(0,q) - \theta_2(0,q)^3\theta_2''(0,q) - \theta_4(0,q)^3\theta_4''(0,q) \equiv 0. \tag{3.5}$$

Note that identity (3.4) is trivial because $\theta_2'(0,q) \equiv \theta_3'(0,q) \equiv \theta_4'(0,q) \equiv 0$. The other two equalities will be treated below. In general, proving such identities can be done in a purely algorithmic fashion which will be explained in this chapter.

3.1 Problem Specification

For fixed $\tau \in \mathbb{H}$, Definition 2.7 implies that the $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ are analytic functions on the whole complex plane with respect to z. For fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ $(j = 1, \ldots, 4)$ are analytic functions of τ for all $\tau \in \mathbb{H}$, and correspondingly, analytic functions of q for $|q| < 1$. When $z = 0$, we often denote

$$\theta_j^{(k)}(\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\bigg|_{z=0} \left(= \frac{\partial^k \theta_j}{\partial z^k}(z,q)\bigg|_{z=0} \right), k \in \mathbb{N}.$$

Definition 2.7 also implies that $\theta_1^{(k_1)}(\tau) \equiv 0$ when $k_1 \in 2\mathbb{N}$, and $\theta_m^{(k_2)}(\tau) \equiv 0$ $(m = 2,3,4)$ when $k_2 \in 2\mathbb{N}+1$. Hence in the following setting we omit these cases.

Let $\{x_{j,k}\}_{k \in \mathbb{N}, j=1,\ldots,4}$ be a set of indeterminates. For convenience, we use $x_j^{(k)} := x_{j,k}$. Sometimes we write x_j for $x_j^{(0)}$ and x_j' for $x_j^{(1)}$. Define $R_\Theta := \mathbb{K}[\Theta]$, where

$$\Theta := \left\{\theta_1^{(2k+1)} : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)} : k \in \mathbb{N} \text{ and } j = 2,3,4\right\},$$

and $R_X := \mathbb{K}[X]$ where

$$X := \left\{x_1^{(2k+1)} : k \in \mathbb{N}\right\} \cup \left\{x_j^{(2k)} : k \in \mathbb{N} \text{ and } j = 2,3,4\right\}.$$

By a homomorphic extension we define the $\mathbb{K}$-algebra homomorphism[2]

$$\begin{aligned} \phi: \quad & R_X \to R_\Theta, \\ & x_j^{(k)} \mapsto \theta_j^{(k)}. \end{aligned}$$

In this chapter, we solve the following membership problem algorithmically:

[2]Here a $\mathbb{K}$-algebra homomorphism is a ring homomorphism and a $\mathbb{K}$-vector space homomorphism.

Problem 3.1. Given $p \in R_X$, decide whether $p \in \ker\phi$.

To solve this problem, we need to extend the $\mathbb{K}$-algebras and the map ϕ as follows:

$$\begin{aligned} \phi^* : \quad R_X[s^{\frac{1}{2}}] &\to R_\Theta[\delta^{\frac{1}{2}}], \\ x_j^{(k)} &\mapsto \theta_j^{(k)}, \\ s^{\frac{1}{2}} &\mapsto \delta^{\frac{1}{2}}, \end{aligned}$$

where for all $\tau \in \mathbb{H}$ and $r \in \frac{1}{2}\mathbb{N}$, $\delta^r(\tau) := \tau^r$. Since ϕ and ϕ^* are surjective, we have $R_X/\ker\phi \cong R_\Theta$ and $R_X[s^{\frac{1}{2}}]/\ker\phi^* \cong R_\Theta[\delta^{\frac{1}{2}}]$. Here we consider $s^{\frac{1}{2}}$ as a symbol for an indeterminate. We prefer to use $s^{\frac{1}{2}}$ instead of choosing a standard indeterminate like x or y as usual for polynomial rings. Later from Section 4.1 on, we will also need to go to the quotient field of $\mathbb{K}[\Theta]$, denoted by $\mathbb{K}(\Theta)$, which consists of all quotients $P[\Theta]/Q[\Theta]$ with $P[\Theta], Q[\Theta] \in \mathbb{K}[\Theta]$.

The chapter is structured as follows. In Section 3.2 we introduce a notion of degree in the $\mathbb{K}$-algebra R_X, and based on this we state a way to decompose any $p \in R_X$ into homogeneous polynomials in R_X. We prove that showing $p \in \ker\phi$ is equivalent to showing that the corresponding homogeneous polynomials are in $\ker\phi$. In Section 3.3 we develop a recursive algorithm to determine for a given homogeneous $g \in R_X$ whether $g \in \ker\phi$ or $g \notin \ker\phi$. In Section 3.4 we obtain a refined non-recursive algorithm which is more convenient to implement and with linear computational complexity in the length of g.

3.2 Homogeneous Decomposition of $p \in R_X$

We first extend Lemma 2.4 to derivatives.

Proposition 3.1. *Define $A := (-i\tau)^{\frac{1}{2}}$ and $E := e^{\frac{i\tau z^2}{\pi}}$. For $(u, v) \in \{(1,1), (2,4), (3,3), (4,2)\}$ and $k \in \mathbb{N}$ define*

$$g_u(k) := (EA)^{-1} \frac{\partial^k \theta_v}{\partial z^k}\left(z \middle| -\frac{1}{\tau}\right).$$

Then $g_u(k)$ can be written as

$$g_u(k) = p_{k,0}(z)\theta_u(z\tau|\tau) + p_{k,1}(z)\theta_u'(z\tau|\tau) + \cdots + p_{k,k}(z)\theta_u^{(k)}(z\tau|\tau) \qquad (*)$$

with $p_{k,j}(z) = \frac{k!}{j!}\left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \tau^{\frac{k+j}{2}} B_{k,j}(z)$ *and*

$$B_{k,j}(z) = \begin{cases} a_0(k,j) + a_2(k,j)z^2 + a_4(k,j)z^4 + \cdots + a_{k-j}(k,j)z^{k-j}, & k-j \text{ even;} \\ a_1(k,j)z + a_3(k,j)z^3 + a_5(k,j)z^5 + \cdots + a_{k-j}(k,j)z^{k-j}, & k-j \text{ odd,} \end{cases}$$

where for $\ell \in \{0,\ldots,k-j\}$ *when* $(u,v) = (1,1)$,

$$a_\ell(k,j) = -i\left(\frac{i\tau}{\pi}\right)^{\frac{\ell}{2}} \frac{2^\ell}{\ell!(\frac{k-j-\ell}{2})!}; \tag{$**$}$$

and when $(u,v) \in \{(2,4),(3,3),(4,2)\}$,

$$a_\ell(k,j) = \left(\frac{i\tau}{\pi}\right)^{\frac{\ell}{2}} \frac{2^\ell}{\ell!(\frac{k-j-\ell}{2})!}.$$

Proof. We prove the statement for $(u,v) = (1,1)$. The other three cases are analogous. We first prove by complete induction on k that for $k \in \mathbb{N}$ the relation $(*)$ holds where the $p_{k,j}(z)$ $(0 \leqslant j \leqslant k)$ are polynomials in z. Then we prove that $B_{k,j}(z)$ has the desired form.

For $k = 0$ we have $p_{0,0}(z) = -i$ by Lemma 2.4. Assume that $(*)$ holds for $k = n$ where the $p_{n,j}(z)$ $(0 \leqslant j \leqslant n)$ are polynomials in z.

Let $k = n+1$. We have

$$g_1(n+1) = (EA)^{-1}\frac{\partial^{n+1}\theta_1}{\partial z^{n+1}}\left(z\middle| -\frac{1}{\tau}\right) = \frac{2i\tau z}{\pi} g_1(n) + \frac{\partial g_1(n)}{\partial z}.$$

Since $\frac{\partial g_1(n)}{\partial z} = p'_{n,0}(z)\theta_1(z\tau|\tau) + (\tau p_{n,0}(z) + p'_{n,1}(z))\theta'_1(z\tau|\tau) + \cdots + \tau p_{n,n}\theta_1^{(n+1)}(z\tau|\tau)$, we obtain $g_1(n+1) = p_{n+1,0}(z)\theta_1(z\tau|\tau) + p_{n+1,1}(z)\theta'_1(z\tau|\tau) + \cdots + p_{n+1,n}(z)\theta_1^{(n+1)}(z\tau|\tau)$, where the $p_{n+1,j}(z)$ $(j = 0,\ldots,n+1)$ are polynomials in z.

Using the fact just proven we can exploit a recursive relation for $g_1(k)$ in the following way. On one hand, by

$$EA\, g_1(k+1) = \frac{\partial^{k+1}\theta_1}{\partial z^{k+1}}\left(z\middle| -\frac{1}{\tau}\right) = \frac{\partial(EA\, g_1(k))}{\partial z},$$

we obtain

$$\begin{aligned} g_1(k+1) =& \frac{2iz\tau}{\pi} g_1(k) + \frac{\partial g_1(k)}{\partial z} \\ =& \frac{2iz\tau}{\pi} \sum_{j=0}^{k} p_{k,j}(z)\theta_1^{(j)}(z\tau|\tau) + \sum_{j=1}^{k} \left(\frac{\partial p_{k,j}(z)}{\partial z} + \tau p_{k,j-1}(z) \right) \theta_1^{(j)}(z\tau|\tau) \\ & + \tau p_{k,k}(z)\theta_1^{(k+1)}(z\tau|\tau) + \frac{\partial p_{k,0}(z)}{\partial z} \theta_1(z\tau|\tau). \end{aligned}$$

On the other hand,

$$g_1(k+1) = \sum_{j=0}^{k+1} p_{k+1,j}(z)\theta_1^{(j)}(z\tau|\tau),$$

and by coefficient comparison, and defining $p_{k,-1}(z) := 0$ and $p_{k,k+1}(z) := 0$ we obtain,

$$p_{k+1,j}(z) = \frac{2iz\tau}{\pi} p_{k,j}(z) + \frac{\partial p_{k,j}(z)}{\partial z} + \tau p_{k,j-1}(z), \quad 0 \leqslant j \leqslant k. \tag{3.6}$$

Now we can prove that $B_{k,j}(z)$ has the desired form by induction on $k \in \mathbb{N}$. By definition we know $B_{0,0}(z) = -i$. Assume for $k = n$, $B_{k,j}(z)$ has the desired form. Let $k = n+1$. Applying (3.6) we have for $j = 0, \ldots, n+1$,

$$(n+1)B_{n+1,j}(z) = 2z \left(\frac{i\tau}{\pi} \right)^{\frac{1}{2}} B_{n,j}(z) + \left(\frac{i\tau}{\pi} \right)^{-\frac{1}{2}} \frac{\partial B_{n,j}(z)}{\partial z} + jB_{n,j-1}(z).$$

Case 1: $n - j$ is even. Then

$$\begin{aligned} (n+1)B_{n+1,j}(z) =& 2 \left(\frac{i\tau}{\pi} \right)^{\frac{1}{2}} (a_0(n,j)z + a_2(n,j)z^3 + \cdots + a_{n-j}(n,j)z^{n-j+1}) \\ & + \left(\frac{i\tau}{\pi} \right)^{-\frac{1}{2}} (2a_2(n,j)z + 4a_4(n,j)z^3 + \cdots + (n-j)a_{n-j}z^{n-j-1}) \\ & + j(a_1(n,j-1)z + a_3(n,j-1)z^3 + \cdots + a_{n-j+1}(n,j-1)z^{n-j+1}). \end{aligned}$$

We compare this to

$$B_{n+1,j}(z) = a_1(n+1,j)z + a_3(n+1,j)z^3 + a_5(n+1,j)z^5 + \cdots + a_{n-j+1}(n+1,j)z^{n-j+1},$$

and obtain for $1 \leqslant 2m+1 \leqslant n-j-1$,

$$a_{2m+1}(n+1,j) = \frac{1}{n+1} \left(2 \left(\frac{i\tau}{\pi} \right)^{\frac{1}{2}} a_{2m}(n,j) + \left(\frac{i\tau}{\pi} \right)^{-\frac{1}{2}} (2m+2)a_{2m+2}(n,j) + ja_{2m+1}(n,j-1) \right)$$

$$
\begin{aligned}
=&\frac{-i}{n+1}\left(\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+1}}{(2m)!(\frac{n-j}{2}-m)!}+\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+2}}{(2m+1)!(\frac{n-j}{2}-m-1)!}\right.\\
&\left.+\,j\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!}\right)\\
=&(-i)\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!};
\end{aligned}
$$

and for $2m+1 = n-j+1$,

$$
\begin{aligned}
a_{2m}(n+1,j) =&\frac{1}{n+1}\left(2\left(\frac{i\tau}{\pi}\right)^{\frac{1}{2}}a_{2m}(n,j)+ja_{2m+1}(n,j-1)\right)\\
=&\frac{-i}{n+1}\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+1}}{(2m)!(\frac{n-j}{2}-m)!}\left(1+\frac{j}{2m+1}\right)\\
=&(-i)\left(\frac{i\tau}{\pi}\right)^{m+\frac{1}{2}}\frac{2^{2m+1}}{(2m+1)!(\frac{n-j}{2}-m)!}.
\end{aligned}
$$

Thus for $1 \leqslant 2m+1 \leqslant n-j+1$, $a_{2m+1}(n+1,j)$ satisfies $(**)$.

Case 2: $n-t$ is odd. This case can be treated in the same way as Case 1, and the computation shows that for $0 \leqslant 2m \leqslant n-j+1$, $a_{2m}(n+1,j)$ satisfies $(**)$. Thus we have proven that $B_{n+1,j}(z)$ has the desired form. □

Applying Proposition 3.1 with $z=0$, we have:

Corollary 3.1. *For $k \in 2\mathbb{N}+1$,*

$$
\theta_1^{(k)}\left(-\frac{1}{\tau}\right) = (-i)^{\frac{3}{2}}\sum_{\substack{j=1\\ j\in 2\mathbb{N}+1}}^{k}\left(\frac{i}{\pi}\right)^{\frac{k-j}{2}}\frac{k!}{j!(\frac{k-j}{2})!}\tau^{\frac{k+j+1}{2}}\theta_1^{(j)}(\tau);
$$

for $k \in 2\mathbb{N}$ and $(u,v) \in \{(2,4),(3,3),(4,2)\}$,

$$
\theta_u^{(k)}\left(-\frac{1}{\tau}\right) = (-i)^{\frac{1}{2}}\sum_{\substack{j=0\\ j\in 2\mathbb{N}}}^{k}\left(\frac{i}{\pi}\right)^{\frac{k-j}{2}}\frac{k!}{j!(\frac{k-j}{2})!}\tau^{\frac{k+j+1}{2}}\theta_v^{(j)}(\tau).
$$

We are going to carry these analytic relations over to the symbolic algebra.

Definition 3.1. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $k \in \mathbb{Z}$ *and* $f : \mathbb{H} \to \mathbb{C}$, *we define* $f|_k\gamma : \mathbb{H} \to \mathbb{C}$ *by*

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

For instance, for the generators S and T, we have

$$(f|_kS)(\tau) \equiv \tau^{-k} f(-1/\tau) \text{ and } (f|_kT)(\tau) \equiv f(\tau+1).$$

Note. This action $f|_k\gamma$ of γ on f (for fixed k) is a group action. Hence knowing the action of generators (here S and T acting on the function space) gives the full action.

Definition 3.2. *We define σ to be the permutation on $\{1,2,3,4\}$ that transposes 2 and 4.*

Definition 3.3. *We define two $\mathbb{K}$-algebra homomorphisms:*

$$S_0 : R_\Theta \to R_\Theta[\delta^{\frac{1}{2}}]$$

by

$$(S_0 f)(\tau) :\equiv (f|_0 S)(\tau) \quad \left(\equiv f(-\frac{1}{\tau})\right);^{3}$$

and

$$S_X : \quad R_X \longrightarrow R_X[s^{\frac{1}{2}}]$$

by the homomorphic extension of

$$S_X(x_1^{(k)}) := (-i)^{\frac{3}{2}} \sum_{\substack{j=1 \\ j \in 2\mathbb{N}+1}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!} s^{\frac{k+j+1}{2}} x_1^{(j)},$$

if $k \in 2\mathbb{N}+1$; and of

$$S_X(x_u^{(k)}) := (-i)^{\frac{1}{2}} \sum_{\substack{j=0 \\ j \in 2\mathbb{N}}}^{k} \left(\frac{i}{\pi}\right)^{\frac{k-j}{2}} \frac{k!}{j!(\frac{k-j}{2})!} s^{\frac{k+j+1}{2}} x_{\sigma(u)}^{(j)},$$

if $k \in 2\mathbb{N}$ and $u \in \{2,3,4\}$.

[3] $S_0 f \in R_\Theta[\delta^{\frac{1}{2}}]$ owing to Corollary 3.1.

Lemma 3.2. *The following diagram commutes:*

$$\begin{array}{ccc} R_X & \xrightarrow{S_X} & R_X[s^{\frac{1}{2}}] \\ \phi \downarrow & & \downarrow \phi^* \\ R_\Theta & \xrightarrow[S_0]{} & R_\Theta[\delta^{\frac{1}{2}}] \end{array}$$

Proof. The way S_X was introduced in Definition 3.3 as a homomorphic extension satisfies exactly the required property. □

By Definition 3.3 we know the explicit form of $S_X(p)$ for any $p \in R_X$, and can set up the following convention.

Convention. Whenever for a non-zero $p \in R_X$ we write

$$S_X(p) = \sum_{j=1}^{n} s^{c_j} p_j,$$

we assume that

$$p_j \in R_X \backslash \{0\} \quad \text{and} \quad c_1 < \cdots < c_n \text{ with } c_j \in \frac{1}{2}\mathbb{N}.$$

For $c \in \frac{1}{2}\mathbb{N}$ the notation $\langle s^c \rangle q$ refers to the coefficient of s^c in $q \in R_X[s^{\frac{1}{2}}]$.

Example 3.2. *Let $p = x_2^{(4)} x_4''$. Then*

$$S_X(p) = p_4 s^7 + p_3 s^6 + p_2 s^5 + p_1 s^4,$$

where $p_4 := -i x_2'' x^{(4)}$, $p_3 := \frac{2}{\pi} x_2 x^{(4)} + \frac{12}{\pi} x_2'' x_4''$, $p_2 := \frac{12i}{\pi^2} x_4 x_2'' + \frac{24i}{\pi^2} x_2 x_4''$ and $p_1 := \frac{24}{\pi^3} x_2 x_4$.

Now we consider the action when $\tau \mapsto \tau + 1$. Also the relations in Lemma 2.5 are carried over to the algebraic side.

Definition 3.4. *For $k \in \mathbb{N}$ we define two $\mathbb{K}$-algebra homomorphisms*

$$T_0 : R_\Theta \to R_\Theta$$

by

$$(T_0 f)(\tau) :\equiv (f|_0 T)(\tau) \quad (\equiv f(\tau + 1));$$

and

$$T_X: \quad R_X \longrightarrow R_X,$$

by the homomorphic extension of

$$T_X(x_1^{(2k+1)}) := e^{\frac{\pi i}{4}} x_1^{(2k+1)}, \quad T_X(x_2^{(2k)}) := e^{\frac{\pi i}{4}} x_2^{(2k)},$$
$$T_X(x_3^{(2k)}) := x_4^{(2k)} \quad \text{and} \quad T_X(x_4^{(2k)}) := x_3^{(2k)}.$$

Analogous to Lemma 3.2 we have:

Lemma 3.3. *The following diagram commutes:*

$$\begin{array}{ccc} R_X & \xrightarrow{T_X} & R_X \\ \phi\downarrow & & \downarrow\phi \\ R_\Theta & \underset{T_0}{\longrightarrow} & R_\Theta \end{array}$$

Proof. By Lemma 2.5 and Definition 3.4 we have

$$\phi(T_X(x_1^{(k)}))(\tau) \equiv \phi(e^{\frac{\pi i}{4}} x_1^{(k)})(\tau) \equiv e^{\frac{\pi i}{4}} \theta_1^{(k)}(\tau) \equiv \theta_1^{(k)}(\tau+1) \equiv \phi(x_1^{(k)})(\tau+1) \equiv (T_0\phi(x_1^{(k)}))(\tau).$$

Analogously we have $\phi(T_X(x_j^{(k)}))(\tau) \equiv \phi(x_j^{(k)})(\tau+1)$ for $j = 2,3,4$. The rest follows from the fact that T_X is defined by a homomorphic extension. □

Example 3.3. *Let* $p = x_2^{(4)} x_4''$. *Then*

$$T_X(p) = e^{\frac{\pi i}{4}} x_2^{(4)} x_3''.$$

Note. Obviously, $T_X^8 = id$.

A non-trivial monomial in R_X is a finite product of elements in $\{x_j^{(k)} : k \in \mathbb{N}, j = 1, \ldots, 4\}$. The empty product gives $1 \in R_X$; it is considered to be the trivial monomial. Hence a polynomial in R_X is a $\mathbb{K}$-linear combination of monomials in R_X.

Definition 3.5. *We define the degree of a non-trivial monomial* $x_{j_1}^{(k_1)} x_{j_2}^{(k_2)} \cdots x_{j_n}^{(k_n)} \in R_X$ *where* $k_i \in \mathbb{N}$ *and* $j_i \in \{1, \ldots, 4\}$ *by*

$$\mathrm{Deg}(x_{j_1}^{(k_1)} x_{j_2}^{(k_2)} \cdots x_{j_n}^{(k_n)}) := \frac{n}{2} + \sum_{i=1}^{n} k_i,$$

and define the degree of the trivial monomial by $\mathrm{Deg}(1) := 0$. *For every polynomial* $p \in R_X$, *define* $\mathrm{Deg}(p) :=$ *highest degree of the monomials in its* $\mathbb{K}$*-linear representation. If all these monomials have the same degree, we say this polynomial is a homogeneous polynomial.*

Example 3.4. $\mathrm{Deg}(-x_1^{(3)}) = \frac{7}{2}$, $\mathrm{Deg}(2x_1^{(3)}x_4) = 4$, *and* $2x_1^{(3)}x_4 - 3x_4^{(2)}x_1'$ *is a homogeneous polynomial.*

Note. This definition is related to the weight of modular forms. See Definition 2.8 and Lemma 3.11.

According to Definition 3.5, we can write a polynomial $p \in R_X$ as a sum of homogeneous polynomials with pairwise different degrees. We are going to show that $p \in R_X$ is in $\ker\phi$ if and only if these homogeneous parts are all in $\ker\phi$. The key tool we use here is the S_X operation. We shall start by studying the patterns of the S_X action on monomials of R_X.

Lemma 3.4. *Let* $p \in R_X$ *be a non-trivial monomial and* $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$. *Then the* p_t *are homogeneous and*

$$\mathrm{Deg}\left(\langle s^{c_t}\rangle S_X(p)\right) = \mathrm{Deg}(p_t) = 2c_t - \mathrm{Deg}(p), \quad 1 \leqslant t \leqslant n.$$

Moreover, we have

$$c_n = \mathrm{Deg}(p)$$

and, if $p = x_{i_1}^{(k_1)} x_{i_2}^{(k_2)} \cdots x_{i_m}^{(k_m)}$,

$$\langle s^{c_n}\rangle S_X(p) = \langle s^{\mathrm{Deg}(p)}\rangle S_X(p) = (-i)^{\frac{m}{2}+c} x_{\sigma(i_1)}^{(k_1)} x_{\sigma(i_2)}^{(k_2)} \cdots x_{\sigma(i_m)}^{(k_m)},$$

where $c =$ *number of 1s in* $(i_1, i_2, \ldots, i_m)$.

Proof. Suppose $p = x_{i_1}^{(k_1)} x_{i_2}^{(k_2)} \cdots x_{i_m}^{(k_m)}$ with $x_{i_1} = x_{i_2} = \cdots = x_{i_c} = x_1$ and $x_{i_j} \neq x_1$ for $c+1 \leqslant j \leqslant m$. Then

$$\begin{aligned} S_X(p) =& S_X(x_{i_1}^{(k_1)}) S_X(x_{i_2}^{(k_2)}) \cdots S_X(x_{i_m}^{(k_m)}) \\ =& \left((-i)^{\frac{3}{2}} x_1^{(k_1)} s^{k_1+\frac{1}{2}} + \square x_1^{(k_1-2)} s^{k_1-\frac{1}{2}} + \cdots + \square x_1' s^{\frac{k_1}{2}+1}\right) \\ & \cdots \\ & \left((-i)^{\frac{1}{2}} x_{\sigma(i_{c+1})}^{(k_{c+1})} s^{k_{c+1}+\frac{1}{2}} + \square x_{\sigma(i_{c+1})}^{(k_{c+1}-2)} s^{k_{c+1}-\frac{1}{2}} + \cdots + \square x_{\sigma(i_{c+1})} s^{\frac{k_{c+1}}{2}+\frac{1}{2}}\right) \\ & \cdots \end{aligned}$$

[3]The boxes $\square$ stand for coefficients in $\mathbb{K}$ whose exact values are irrelevant for the proof.

$$\left((-i)^{\frac{1}{2}}x_{\sigma(i_m)}^{(k_m)}s^{k_m+\frac{1}{2}}+\square x_{\sigma(i_m)}^{(k_m-2)}s^{k_m-\frac{1}{2}}+\cdots+\square x_{\sigma(i_m)}s^{\frac{k_m}{2}+\frac{1}{2}}\right).^{4}$$

Hence $\langle s^{\mathrm{Deg}(p)}\rangle S_X(p)=\langle s^{c_n}\rangle S_X(p)=(-i)^{\frac{m}{2}+c}x_{\sigma(i_1)}^{(k_1)}x_{\sigma(i_2)}^{(k_2)}\cdots x_{\sigma(i_m)}^{(k_m)}$ and

$$c_n=\sum_{j=1}^{m}(k_j+\frac{1}{2})=\frac{m}{2}+\sum_{j=1}^{m}k_j=\mathrm{Deg}(p).$$

In the expansion of $S_X(p)$ each monomial has the form

$$\begin{aligned}\prod_{j=1}^{m}x_{\sigma(i_j)}^{(k_j-2a_j)}s^{k_j+\frac{1}{2}-a_j}&=s^{\sum\limits_{j=1}^{m}k_j+\frac{m}{2}-\sum\limits_{j=1}^{m}a_j}\prod_{j=1}^{m}x_{\sigma(i_j)}^{(k_j-2a_j)}\\&=s^{\mathrm{Deg}(p)-a}\prod_{j=1}^{m}x_{\sigma(i_j)}^{(k_j-2a_j)},\end{aligned}$$

where the a_j are integers with $0\leqslant a_j\leqslant\frac{k_j-1}{2}$ for $1\leqslant j\leqslant c$ and $0\leqslant a_j\leqslant\frac{k_j}{2}$ for $c+1\leqslant j\leqslant m$ and $a:=\sum\limits_{j=1}^{m}a_j$. Thus

$$\mathrm{Deg}\left(\langle s^{\mathrm{Deg}(p)-a}\rangle S_X(p)\right)=\mathrm{Deg}\left(\prod_{j=1}^{m}x_{i_j}^{(k_j-2a_j)}\right)=\frac{m}{2}+\sum_{j=1}^{m}(k_j-2a_j)=\mathrm{Deg}(p)-2a.$$

Substituting $\mathrm{Deg}(p)-a$ by c_t we obtain

$$\mathrm{Deg}\left(\langle s^{c_t}\rangle S_X(p)\right)=2c_t-\mathrm{Deg}(p),\quad 1\leqslant t\leqslant n.$$

□

For convenience we have:

Definition 3.6. *For monomials $p=x_{i_1}^{(k_1)}\ldots x_{i_m}^{(k_m)}\in R_X$ we define*

$$\begin{aligned}\mu(p)&:=m;\\\nu_1(p)&:=\textit{number of 1s in }(i_1,\ldots,i_m),\\\nu_2(p)&:=\textit{number of 2s in }(i_1,\ldots,i_m),\textit{ and}\\\sigma(p)&:=x_{\sigma(i_1)}^{(k_1)}\ldots x_{\sigma(i_m)}^{(k_m)}.\end{aligned}$$

Now we study the S_X operator on homogeneous polynomials.

Corollary 3.2. *Let $p \in R_X$ be homogeneous. Then $S_X(p) = 0$ if and only if $p = 0$.*

Proof. " $\Longleftarrow$ " is obvious. So we prove " $\Longrightarrow$ ". Assume $0 \neq p = a_1 p_1 + \cdots + a_n p_n$ with the $p_j \in R_X$ linearly independent monomials over $\mathbb{K}\backslash\{0\}$ with the same degree and the $a_j \in \mathbb{K}\backslash\{0\}$. Then the $\sigma(p_j)$ are also linearly independent monomials over $\mathbb{K}\backslash\{0\}$ because the involution σ is an automorphism on R_X, and

$$\begin{aligned}
\langle s^{\mathrm{Deg}(p)} \rangle S_X(p) &= \langle s^{\mathrm{Deg}(p)} \rangle (a_1 S_X(p_1) + \cdots + a_n S_X(p_n)) \\
&= a_1 \langle s^{\mathrm{Deg}(p)} \rangle S_X(p_1) + \cdots + a_n \langle s^{\mathrm{Deg}(p)} \rangle S_X(p_n) \\
&= a_1 (-i)^{\nu_1(p_1) + \frac{\mu(p_1)}{2}} \sigma(p_1) + \cdots + a_n (-i)^{\nu_1(p_n) + \frac{\mu(p_n)}{2}} \sigma(p_n).
\end{aligned}$$

Since the $(-i)^{\nu_1(p_j) + \frac{\mu(p_j)}{2}}$ are non-zero, we obtain $\langle s^{\mathrm{Deg}(p)} \rangle S_X(p) \neq 0$. Therefore $S_X(p) \neq 0$. $\square$

Lemma 3.5. *Given $p \in R_X$ homogeneous, and $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ with $p_t \in R_X$ and $c_t \in \frac{1}{2}\mathbb{N}$ such that $c_1 < \cdots < c_n$, then*
(i) $\mathrm{Deg}(p_n) = \mathrm{Deg}(p) = c_n$;
(ii) for $t \in \{1, \ldots, n\}$ the p_t are homogeneous ;
(iii) for $i, j \in \{1, \ldots, n\}$ with $i < j$ we have $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$.

Proof. Suppose

$$p = r_1 h_1 + \cdots + r_q h_q$$

with $r_\ell \in \mathbb{K}\backslash\{0\}$ and pairwise different monomials $h_\ell \in R_X$. By assumption on p we have $\mathrm{Deg}(h_\ell) = \mathrm{Deg}(p) =: d$ for all $\ell \in \{1, \ldots, q\}$. Suppose, for $1 \leqslant \ell \leqslant q$,

$$S_X(h_\ell) = \sum_{t=1}^{n_\ell} s^{c_{\ell,t}} p_{\ell,t},$$

where the $c_{\ell,t}$ and $p_{\ell,t}$ are as in Lemma 3.4. Then

$$S_X(p) = \sum_{\ell=1}^{q} r_\ell S_X(h_\ell) = \sum_{j=1}^{n} s^{c_j} \sum_{(\ell,t) \in C_j} r_\ell p_{\ell,t},$$

where $\{c_1,\ldots,c_n\} = \{c_{\ell,t} : 1 \leqslant \ell \leqslant q,\, 1 \leqslant t \leqslant n_\ell\}$ with the ordering $c_1 < c_2 < \cdots < c_n$, and

$$C_j := \{(\ell,t) \in \{1,\ldots,q\} \times \{1,\ldots,n_\ell\} : c_{\ell,t} = c_j\}.$$

Now the statements follow from observing that for $(\ell,t) \in C_j$ by Lemma 3.4

$$\mathrm{Deg}(p_{\ell,t}) = 2c_{\ell,t} - d = 2c_j - d,$$

and for $(\ell,t) \in C_n$ (i.e., $t = n_\ell$) again by Lemma 3.4

$$\mathrm{Deg}(p_{\ell,t}) = \mathrm{Deg}(p_{\ell,n_\ell}) = c_{\ell,n_\ell} = \mathrm{Deg}(h_\ell) = d = c_n.$$

$\square$

Remark. Note that Lemma 3.5 actually justifies the definition of Deg and also the Convention we introduced after Definition 3.3. Namely, the highest power of s in $S_X(p)$ is Deg(p).

Definition 3.7. *For each $q \in R_X[s^{\frac{1}{2}}]$ with $q = \sum_{t=1}^{n} s^{c_t} p_t$, using the Convention, we call p_n the leading coefficient of q, denoted by $\mathrm{lc}(q)$. We define $\mathrm{lc}(0) := 0$.*

Definition 3.8. *Let $R_X^d := \{p \in R_X : p \text{ homogeneous with } \mathrm{Deg}(p) = d\} \cup \{0\}$. We define the map*

$$\widetilde{S}: \quad R_X^d \longrightarrow R_X^d,$$

by $\widetilde{S}(0) := 0$ and if $p \neq 0$:

$$\widetilde{S}(p) := \mathrm{lc}(S_X(p)).$$

Example 3.5. $\widetilde{S}(x_1^{(3)}x_4 - x_4^{(2)}x_2') = -x_1^{(3)}x_2 + i\,x_2^{(2)}x_4'$ *by Lemmas 3.4 and 3.5.*

Proposition 3.2. *The map $\widetilde{S}$ is a $\mathbb{K}$-algebra homomorphism and $\widetilde{S}^8 = id$.*

Proof. The linearity of $\widetilde{S}$ is obvious by Lemma 3.5. From Definition 3.3 we see that $\widetilde{S}$ also preserves the multiplication. Thus $\widetilde{S}$ is a $\mathbb{K}$-algebra homomorphism. Let $p \in R_X^d$ be such that $p = \sum_{\ell=1}^{q} r_\ell h_\ell$ with $r_\ell \in \mathbb{K}\backslash\{0\}$ and with monomials $h_\ell \in R_X^d$. Then, by Lemma 3.4, for $\sigma = (2,4)$

$$\begin{aligned}\widetilde{S}^8(p) =& r_1\widetilde{S}^8(h_1) + \cdots + r_q\widetilde{S}^8(h_q)\\ =& r_1(-i)^{8(\frac{\mu(h_1)}{2}+\nu_1(h_1))}\sigma^8(h_1) + \cdots + r_q(-i)^{8(\frac{\mu(h_q)}{2}+\nu_1(h_q))}\sigma^8(h_q)\end{aligned}$$

$$=p.$$

□

According to Lemma 3.5, for any homogeneous $p \in R_X$, $S_X(p)$ has a presentation of the form

$$S_X(p) = \sum_{i=1}^{n} s^{c_i} p_i \in R_X[s^{\frac{1}{2}}] \tag{3.7}$$

with homogeneous $p_i \in R_X \backslash \{0\}$ and where

$$c_1 < \cdots < c_n \quad \text{and} \quad \mathrm{Deg}(p_1) < \cdots < \mathrm{Deg}(p_n);$$

moreover,

$$c_n = \mathrm{Deg}(p_n) = \mathrm{Deg}(p).$$

Definition 3.9. *A sum presentation of $S_X(p)$ as in (3.7) is called S-form presentation. We also say that $S_X(p)$ written as in (3.7) is in S-form.*

Lemma 3.6. *Suppose $p \in R_X$ with $p = \sum_{t=1}^{n} p_t$, where the p_t are homogeneous and $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$ if $i < j$. If $S_X(p) = \sum_{t=1}^{m} s^{c_t} q_t$ is in S-form, then $\widetilde{S}(p_n) = q_m$.*

Proof. First, by Lemma 3.5, we observe that

$$\mathrm{Deg}(p_i) = \text{ highest power of } s \text{ in } S_X(p_i). \tag{3.8}$$

One has,

$$q_m = \mathrm{lc}(S_X(p)) = \mathrm{lc}(S_X(p_1) + \cdots + S_X(p_n)) = \mathrm{lc}(S_X(p_n)) = \widetilde{S}(p_n),$$

where we used (3.8) together with $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_n)$ for $i \in \{1, \ldots, n-1\}$. □

For our context, a special case of the slash operator, introduced in Definition 3.1, is of special importance.

Recall S_0 from Definition 3.3.

Lemma 3.7. *Given $F(\tau) \in R_\Theta$, let $(S_0 F)(\tau) \equiv \sum_{t=1}^{n} \tau^{c_t} f_t(\tau)$ ($c_t \in \frac{1}{2}\mathbb{N}$) with $f_t(\tau) \in R_\Theta$ and $c_1 < c_2 < \cdots < c_n$. Then $F(\tau) \equiv 0$ if and only if $f_t(\tau) \equiv 0$ for all $t \in \{1, \ldots, n\}$.*

Proof. " $\Longleftarrow$ " is immediate.
" $\Longrightarrow$ ". If $F(\tau) \equiv 0$ then $(S_0 F)(\tau) \equiv 0$. Since $f_t(\tau) \equiv f_t(\tau+8)$, the rest can be done by using the same method as used to prove Lemma 1.1 in [26]. □

Applying Lemma 3.2, we carry Lemma 3.7 over to the symbolic world R_X.

Lemma 3.8. *Let $p \in R_X$ and $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ in S-form. Then $p \in \ker\phi$ if and only if $p_t \in \ker\phi$ for all $t \in \{1, \ldots, n\}$.*

Proof. The definitions of ϕ and ϕ^* imply that $\phi^*\big|_{R_X} = \phi$. Hence for $\tau \in \mathbb{H}$,

$$\phi^*(S_X(p))(\tau) \equiv \sum_{t=1}^{n} \phi^*(s^{c_t} p_t)(\tau) \equiv \sum_{t=1}^{n} \tau^{c_t} \phi(p_t)(\tau) \equiv S_0(\phi(p))(\tau),$$

where the last equality follows from Lemma 3.2. Using also Lemma 3.7, we have the following chain of equivalences:

$$p \in \ker\phi \iff \phi(p) = 0 \iff S_0(\phi(p)) = 0 \iff \forall t : \phi(p_t) = 0,$$

which completes the proof. □

Theorem 3.1. *Let $p \in R_X$ with $p = \sum_{t=1}^{n} p_t$, where the $p_t \in R_X$ are homogeneous and* $\mathrm{Deg}(p_i) < \mathrm{Deg}(p_j)$ *if $i < j$. Then $p \in \ker\phi$ if and only if $p_t \in \ker\phi$ for all $t \in \{1, \ldots, n\}$.*

Proof. " $\Longleftarrow$ " is immediate.
" $\Longrightarrow$ ". Suppose $p \in \ker\phi$ with $S_X(p) = \sum_{t=1}^{n_1} s^{c_{1,t}} p_{1,t}$ in S-form. By Lemma 3.6, $\widetilde{S}(p_n) = p_{1,n_1}$, and by Lemma 3.8, $p_{1,n_1} \in \ker\phi$. Next, if $S_X(p_{1,n_1}) = \sum_{t=1}^{n_2} s^{c_{2,t}} p_{2,t}$ in S-form, then $\widetilde{S}(p_{1,n_1}) = p_{2,n_2}$ and $p_{2,n_2} \in \ker\phi$. Iterating this process after k steps gives $\widetilde{S}^k(p_n) = p_{k,n_k}$ with $p_{k,n_k} \in \ker\phi$. For $k = 8$, Proposition 3.2 gives $p_n = \widetilde{S}^8(p_n) = p_{8,n_8} \in \ker\phi$. Because $p \in \ker\phi$ we conclude that $\sum_{t=1}^{n-1} p_t \in \ker\phi$. Applying the same procedure to this element we obtain $p_{n-1} \in \ker\phi$. Iterating we eventually obtain $p_t \in \ker\phi$ for all $t \in \{1, 2, \ldots, n\}$. □

Note. Theorem 3.1 is fundamental for our kernel membership test.

3.3 Membership Recognition for Homogeneous $p \in R_X$

Definition 3.10. *Given $p \in R_X$ homogeneous, define:*

$$\mathrm{LT}(p) := \{\widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots (p) : k_i \in \mathbb{N}\}.$$

We call $\mathrm{LT}(p)$ the leading term orbit of p.

Proposition 3.3. *For homogeneous $p \in R_X$, one has $|\mathrm{LT}(p)| \leqslant 2^7 \cdot 3$ and the bound is sharp.*

Proof. Since $p \in R_X$, p is a polynomial in infinitely many variables, that is $p = f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots)$. Assume $q \in \mathrm{LT}(p)$, then $q = \hat{\sigma} f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots)$ for some $\hat{\sigma} = \widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots \widetilde{S}^{k_{n-1}} T_X^{k_n}$. One can verify that

$$\hat{\sigma} f(x_1, \dots, x_4, x_1^{(1)}, \dots, x_4^{(1)}, \dots) = f(\hat{\sigma} x_1, \dots, \hat{\sigma} x_4, \hat{\sigma} x_1^{(1)}, \dots, \hat{\sigma} x_4^{(1)}, \dots).$$

Therefore the number of possible $\hat{\sigma} f$ is bounded by the number of possible infinite vectors of the form $(\hat{\sigma} x_1, \dots, \hat{\sigma} x_4, \hat{\sigma} x_1^{(1)}, \dots, \hat{\sigma} x_4^{(1)}, \dots)$. Such a vector is uniquely determined by the first four entries. We checked by computer that there are 384 possible values for the first four entries. Therefore there are at most $384 = 2^7 \cdot 3$ different $\hat{\sigma} f$. $\square$

Note. In fact, in view of $T_X^8 = id = \widetilde{S}^8$, $\mathrm{LT}(p)$ is the p-orbit of a corresponding group action. For instance,

$$\text{if } p_1 \in \mathrm{LT}(p) \text{ then } \mathrm{LT}(p_1) = \mathrm{LT}(p).$$

Lemma 3.9. *Suppose $p \in R_X$. If $p \in \ker\phi$, then $T_X(p) \in \ker\phi$.*

Proof. If $p \in \ker\phi$, then $\phi(p) = 0$. Hence $\phi(T_X(p))(\tau) \equiv \phi(p)(\tau + 1) \equiv 0$ by Lemma 3.3. Therefore $T_X(p) \in \ker\phi$. $\square$

Lemma 3.10. *Suppose $p \in R_X$ and $g \in \mathrm{LT}(p)$. Then $p \in \ker\phi$ if and only if $g \in \ker\phi$.*

Proof. " $\Longrightarrow$ ". Suppose $S_X(p) = \sum_{t=1}^{n} s^{c_t} p_t$ in S-form. From Lemma 3.8 we know that if $p \in \ker\phi$, then $\widetilde{S}(p) = p_n \in \ker\phi$. By Lemma 3.9, $T_X(p) \in \ker\phi$. According to Definition 3.10, for each $g \in \mathrm{LT}(p)$, there exists $k_j \in \mathbb{N}$ such that $g = \widetilde{S}^{k_1} T_X^{k_2} \widetilde{S}^{k_3} T_X^{k_4} \cdots \widetilde{S}^{k_{n-1}} T_X^{k_n}(p)$. Thus if $p \in \ker\phi$, then $g \in \ker\phi$.
" $\Longleftarrow$ ". Noting that $p \in \mathrm{LT}(p) = \mathrm{LT}(g)$ we can apply " $\Longrightarrow$ ". $\square$

Lemma 3.11. *Given $p \in R_X$ homogeneous, if $\phi(p) \in M_k(\mathbb{H})\backslash\{0\}$ then* $\mathrm{Deg}(p) = k$.

Proof. By Lemma 3.5, the highest power of s in the S-form of $S_X(p)$ is Deg(p). Thus by Lemma 3.2 we know that the highest power of τ in $(S_0\phi(p))(\tau)$ is $\mathrm{Deg}(p)$. If $\phi(p) \in M_k(\mathbb{H})\backslash\{0\}$, then $(S_0\phi(p))(\tau) = \phi(p)(-1/\tau) = \tau^k\phi(p)(\tau)$. Therefore $\mathrm{Deg}(p) = k$. □

Example 3.6. [5] *Let $p = -\frac{1}{27}(x_2^4+x_3^4)(x_3^4+x_4^4)(x_2^4-x_4^4)$. One can easily verify that p is homogeneous and* $\mathrm{Deg}(p) = 6$. *On the other hand, $\phi(p) = e_1e_2e_3$ where $e_1 := \frac{1}{3}(\theta_3(0,q)^4+\theta_4(0,q)^4)$, $e_2 := -\frac{1}{3}(\theta_2(0,q)^4+\theta_3(0,q)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0,q)^4-\theta_4(0,q)^4)$. One also verifies that the product $e_1e_2e_3$ is a modular form of weight 6.*

According to Lemma 2.1, to prove that $f \in R_\Theta$ is identically zero we follow two steps: first check if f is a modular form, then check if the first few coefficients of the q-expansion of f are zero.

But usually the given $f \in R_\Theta$ in our context is not a modular form in the sense of Definition 2.8. To be able to apply Lemma 2.1, instead of directly dealing with $f = \phi(p)$ (with homogeneous $p \in R_X$), we deal with $\prod_{u\in \mathrm{LT}(p)} \phi(u)$. We first check if this product is a modular form, and then we check whether the first few coefficients of the q-expansion of this product are zero. We will also show that if this product is zero then each single $\phi(u)$ is zero. This will imply $f = \phi(p) = 0$ because $p \in \mathrm{LT}(p)$.

Lemma 3.12. *Let $p \in R_X$ be homogeneous and $\mathrm{LT}(p) = \{p_1,\dots,p_m\}$ with $S_X(p_j) = \sum_{t=1}^{n_j} s^{c_{j,t}}p_{j,t}$ in S-form. If $p_{j,1}, p_{j,2},\dots,p_{j,n_j-1} \in \ker\phi$ for all $j \in \{1,\dots,m\}$ then*

$$\prod_{j=1}^{m}\phi(p_j)(\tau) \in M_{m\,\mathrm{Deg}(p)}(\mathbb{H}).$$

Proof. By Lemma 3.2 we have for $j \in \{1,\dots,m\}$,

$$(\phi(p_j)|_0S)(\tau) \equiv \phi^*(S_X(p_j))(\tau) \equiv \phi^*\left(\sum_{t=1}^{n_j} s^{c_{j,t}}p_{j,t}\right) \equiv \sum_{t=1}^{n_j}\tau^{c_{j,t}}\phi(p_{j,t})(\tau).$$

[5] Cf. Example 2.2.

Let $d = \mathrm{Deg}(p)$. Applying Lemma 3.5 we have $c_{1,n_1} = c_{2,n_2} \cdots = c_{m,n_m} = d$. Suppose $j \in \{1,\ldots,m\}$ is fixed. If $p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi$ then

$$(\phi(p_j)|_0 S)(\tau) \equiv \tau^d \phi(p_{j,n_j}).$$

Note that $p_{j,n_j} \in \mathrm{LT}(p)$ by Definitions 3.8 and 3.10. Thus

$$(\phi(p_j)|_0 S)(\tau) \equiv \tau^d \phi(p_i)$$

for some $p_i := \widetilde{S}(p_j) \in \mathrm{LT}(p)$. Moreover, by Definition 3.10 we have $T_X(p_j) = p_t$ for some $p_t \in \mathrm{LT}(p)$ and thus, by Lemma 3.3,

$$(\phi(p_j)|_0 T)(\tau) \equiv \phi^*(T_X(p_j))(\tau) \equiv \phi(p_t)(\tau).$$

Therefore

$$(\phi(p_j)|_d S)(\tau) \equiv \phi(p_i)(\tau) \quad \text{and} \quad (\phi(p_j)|_d T)(\tau) \equiv \phi(p_t)(\tau).$$

Thus for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

$$\left(\prod_{j=1}^{m} \phi(p_j)\bigg|_{dm} \gamma\right)(\tau) \equiv \prod_{j=1}^{m} \left(\phi(p_j)\bigg|_{d} \gamma\right)(\tau) \equiv \prod_{j=1}^{m} \phi(p_j)(\tau). \tag{3.9}$$

In fact the functions $\theta_i^{(k)}$ are analytic on $\mathbb{H}$, which can be seen from their q-expansion. Therefore the above product is analytic on $\mathbb{H}$. Again by Definition 2.7, each of the functions $\theta_i^{(k)}$ is a Taylor series in powers of $q^{1/4}$, therefore also the above product is a Taylor series in powers of $q^{1/4}$. Setting $\gamma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in (3.9) implies that the above product is invariant under the mapping $\tau \mapsto \tau + 2$. It is known that analytic functions with this property may be written as Laurent series in q; by the uniqueness of Laurent series the product is a Taylor series in q as required from the definition of modular form. □

Theorem 3.2. *Let $p \in R_X$ be homogeneous, $\mathrm{LT}(p) = \{p_1, \ldots, p_m\}$ with $S_X(p_j) = \sum_{t=1}^{n_j} s^{c_{j,t}} p_{j,t}$ in S-form. If for all $j \in \{1,\ldots,m\}$,*

$$p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi \quad \textit{and} \quad \mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)(\tau)\right) > \frac{m\mathrm{Deg}(p)}{6},$$

where ord *is the order of a power series in q in the usual sense, then $p \in \ker\phi$.*

Proof. If for all $j \in \{1,\ldots,m\}$, $p_{j,1}, p_{j,2}, \ldots, p_{j,n_j-1} \in \ker\phi$, then by Lemma 3.12 we have $\prod_{j=1}^{m} \phi(p_j)(\tau) \in M_{m\mathrm{Deg}(p)}(\mathbb{H})$. This together with $\mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)(\tau)\right) > \frac{m\mathrm{Deg}(p)}{6}$, by Lemma 2.1, we obtain $\phi\left(\prod_{j=1}^{m} p_j\right) = \prod_{j=1}^{m} \phi(p_j) = 0$. Thus for some j, $p_j \in \ker\phi$, which by Lemma 3.10 implies that for any $h \in \mathrm{LT}(p_j) = \mathrm{LT}(p)$, $h \in \ker\phi$. Therefore $p \in \ker\phi$. $\square$

Algorithm 3.1. *Let p, $\mathrm{LT}(p)$ and $S_X(p_j)$ be the same as in Theorem 3.2, and $d := \mathrm{Deg}(p)$. We have the following algorithm to prove or disprove $p \in \ker\phi$.*

Input: homogeneous $p \in R_X$. Output: True if $p \in \ker\phi$; False if $p \notin \ker\phi$.

$$\textit{Define } \mathrm{Prove}(p) := \begin{cases} \textit{True}, & \textit{if } p \in \ker\phi; \\ \textit{False}, & \textit{if } p \notin \ker\phi. \end{cases}$$

If $d = 0$ then $\mathrm{Prove}(p)$:=*True if $p = 0$; else* $\mathrm{Prove}(p)$:= *False.*

If $d > 0$ then

$$\begin{aligned} \mathrm{Prove}(p) := \textit{True if } \mathrm{ord}\left(\prod_{j=1}^{m} \phi(p_j)(\tau)\right) > \frac{dm}{6} \\ \textit{and} \quad \mathrm{Prove}(p_{j,1}) \textit{ and } \ldots \textit{ and } \mathrm{Prove}(p_{j,n_j-1}); \\ \textit{else} \quad \mathrm{Prove}(p) := \textit{False}. \end{aligned}$$

Theorem 3.3. *Algorithm 1 is correct.*

Proof. Suppose $p \in \ker\phi$. Using $p_j \in \mathrm{LT}(p)$ and Lemma 3.10 we have the equivalences

$$\begin{aligned} p \in \ker\phi &\iff p_j \in \ker\phi \text{ for all } j \in \{1,\ldots,m\} && \text{(a)} \\ &\iff p_j \in \ker\phi \text{ for some } j \in \{1,\ldots,m\} \\ &\iff \prod_{j=1}^{m} \phi(p_j)(\tau) \equiv 0. && \text{(b)} \end{aligned}$$

According to Theorem 3.2, (b) together with

(1) True=Prove($p_{j,1}$) $= \cdots =$Prove(p_{j,n_j-1}), $j = 1,\ldots,m$, gives Prove(p) =True; i.e., $p \in \ker\phi$. By (a) and Lemma 3.8 we have $p_{j,1},\ldots,p_{j,n_j} \in \ker\phi$ for $j \in \{1,\ldots,m\}$. We iterate the above procedure and note that owing to Lemma 3.5 the procedure terminates; namely

$$\mathrm{Deg}(p_{j,1}) < \cdots < \mathrm{Deg}(p_{j,n_j-1}) < \mathrm{Deg}(p_{j,n_j}) = d.$$

Suppose $p \notin \ker\phi$. This is equivalent to

(2) $p \notin \ker\phi$ for all $j \in \{1,\ldots,m\}$. In case (1) holds, then by Lemma 3.12,

$$f(\tau) :\equiv \prod_{j=1}^{m} \phi(p_j)(\tau) \in M_{dm}(\mathbb{H}).$$

Because of (2) we know that $f(\tau) \not\equiv 0$; thus $\mathrm{ord}(f(\tau)) \leqslant \frac{dm}{6}$ and Algorithm 3.1 returns Prove(p) =False. If at least one of the $p_{j,1},\ldots,p_{j,n_j-1}$ ($j = 1,\ldots,m$) is not in $\ker\phi$, the algorithm detects this in a base case (i.e., $p \in \mathbb{K}\backslash\{0\}$) when applying its steps recursively. □

Example 3.7. *Let us return to the task to do zero-recognition for* (3.5) *from Example 3.1. Since* $\theta_2'(0,q) \equiv \theta_3'(0,q) \equiv \theta_4'(0,q) \equiv 0$, *we need to prove the following identity.*

$$\theta_2(\tau)^3\theta_2''(\tau) - \theta_3(\tau)^3\theta_3''(\tau) + \theta_4(\tau)^3\theta_4''(\tau) \equiv 0.$$

Note that in Chapter 8 we will demonstrate how our Mathematica package can assist to prove this identity.

Proof. For $p := x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)} \in R_X^4$ we want to prove $p \in \ker\phi$. We compute

$$\mathrm{LT}(p) = \{p_1, p_2\} = \{x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)}, -(x_2^3x_2^{(2)} - x_3^3x_3^{(2)} + x_4^3x_4^{(2)})\}.$$

Since $\mathrm{Deg}(p) = 4$ and $|\mathrm{LT}(p)| = 2$, we need to show that $\phi(p_1p_2)(\tau)$ has the form $\sum_{t>\frac{8}{6}} a_tq^t$, which holds because

$$\begin{aligned}\phi(p_1p_2)(\tau) \equiv &(\theta_2(\tau)^3\theta_2''(\tau) - \theta_3(\tau)^3\theta_3''(\tau) + \theta_4(\tau)^3\theta_4''(\tau))\\ &(-\theta_2(\tau)^3\theta_2''(\tau) + \theta_3(\tau)^3\theta_3''(\tau) - \theta_4(\tau)^3\theta_4''(\tau))\\ \equiv &\,\square q^2 + \square q^3 + \ldots.\end{aligned}$$

Moreover we have

$$S_X(p_1) = (-x_2^3 x_2^{(2)} + x_3^3 x_3^{(2)} - x_4^3 x_4^{(2)})s^4 + \frac{2i}{\pi}(-x_2^4 + x_3^4 - x_4^4)s^3 = p_2 s^4 + \frac{2i}{\pi} p_{1,2} s^3$$

and

$$S_X(p_2) = (x_2^3 x_2^{(2)} - x_3^3 x_3^{(2)} + x_4^3 x_4^{(2)})s^4 + \frac{2i}{\pi}(x_2^4 - x_3^4 + x_4^4)s^3 = p_1 s^4 + \frac{2i}{\pi} p_{2,2} s^3.$$

According to Theorem 3.2, it is now left to show that $p_{1,2}, p_{2,2} \in \ker\phi$. We compute

$$\mathrm{LT}(p_{1,2}) = \mathrm{LT}(p_{2,2}) = \{-x_2^4 + x_3^4 - x_4^4, x_2^4 - x_3^4 + x_4^4\} = \{p_{1,2}, p_{2,2}\}.$$

Since $\mathrm{Deg}(p_{1,2}) = 2$ and $|\mathrm{LT}(p_{1,2})| = 2$, we need to show $\phi(p_{1,2}p_{2,2})(\tau)$ has the form $\sum_{t>\frac{4}{6}} a_t q^t$, which holds because

$$\phi(p_{1,2}p_{2,2})(\tau) \equiv (\theta_2(\tau)^4 - \theta_3(\tau)^4 + \theta_4(\tau)^4)(-\theta_2(\tau)^4 + \theta_3(\tau)^4 - \theta_4(\tau)^4) \equiv \square q + \square q^2 + \dots.$$

We also have

$$S_X(p_{1,2}) = (x_2^4 - x_3^4 + x_4^4)s^2 = p_1 s^2 \quad \text{and} \quad S_X(p_{2,2}) = (-x_2^4 + x_3^4 - x_4^4)s^2 = p_2 s^2.$$

Thus $p_{1,2}, p_{2,2} \in \ker\phi$. Consequently we obtain $p \in \ker\phi$. $\square$

Example 3.8. *As another example, we present an identity from the famous book by Rademacher, (93.22) in [24], which was used to derive the formula for the number of presentations of a natural number as a sum of 10 squares:*

$$\theta_3^{(4)}(\tau)\theta_3(\tau) - 3(\theta_3''(\tau))^2 - 2\theta_3(\tau)^2\theta_2(\tau)^4\theta_4(\tau)^4 \equiv 0.$$

The algorithmic effort to prove this identity is as simple as in Example 3.7. In Chapter 8 we will show the procedures of proving this identity in our Mathematica package.

3.4 A Refined Algorithm

Definition 3.11. *For any $\bar{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$ and $t \in \mathbb{N}$, we define*

$$D(\bar{k}, t) := \left\{ (b_1, \dots, b_m) \in \mathbb{N}^m : \sum_{i=1}^m b_i = \sum_{i=1}^m k_i - 2t,\ b_i \leqslant k_i \text{ and } b_i \equiv k_i \pmod 2 \right\}.$$

Lemma 3.13. *Let $p = x_{i_1}^{(k_1)} \cdots x_{i_m}^{(k_m)} \in R_X$ and $r := \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$. Then $r \in \mathbb{N}$ and*

$$S_X(p) = s^{\mathrm{Deg}(p)} p_0 + s^{\mathrm{Deg}(p)-1} p_1 + \cdots + s^{\mathrm{Deg}(p)-r} p_r$$

is in S-form, and for $0 \leqslant t \leqslant r$,

$$p_t = (-i)^{\nu_1(p)+\frac{m}{2}} \left(\frac{i}{\pi}\right)^t \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \frac{k_v!}{b_v! (\frac{k_v - b_v}{2})!} x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}.$$

Proof. Assume $S_X(p) = s^{c_n} g_n + \cdots + s^{c_1} g_1$ in S-form. Suppose $x_{i_1} = x_{i_2} = \cdots = x_{i_{\nu_1(p)}} = x_1$ and $x_{i_j} \neq x_1$ for $\nu_1(p) + 1 \leqslant j \leqslant m$. Then by Lemma 3.4, $c_n = \mathrm{Deg}(p)$, and from the proof of Lemma 3.4, we have

$$\begin{aligned}
c_n - c_1 &= \left(\sum_{j=1}^{\nu_1(p)} \left(k_j + \frac{1}{2}\right) + \sum_{j=\nu_1(p)+1}^{m} \left(k_j + \frac{1}{2}\right) \right) - \left(\sum_{j=1}^{\nu_1(p)} \left(\frac{k_j}{2} + 1\right) + \sum_{j=\nu_1(p)+1}^{m} \left(\frac{k_j}{2} + \frac{1}{2}\right) \right) \\
&= \sum_{j=1}^{\nu_1(p)} \left(\frac{k_j}{2} - \frac{1}{2}\right) + \sum_{j=\nu_1(p)+1}^{m} \frac{k_j}{2} \qquad (3.10) \\
&= \sum_{j=1}^{m} \frac{k_j}{2} - \frac{\nu_1(p)}{2} \\
&= \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2} \\
&= r.
\end{aligned}$$

Thus $c_1 = c_n - r = \mathrm{Deg}(p) - r$. Moreover, for (3.10), since k_j is odd when $1 \leqslant j \leqslant \nu_1(p)$ and is even when $\nu_1(p) + 1 \leqslant j \leqslant m$, we deduce that $r \in \mathbb{N}$.

By Definition 3.3, for every $x_\ell^{(k)}$, regardless that ℓ is even or odd, if we sort the power of s in $S_X(x_\ell^{(k)})$ from big to small, then the power of s decreases by 1 every time when the j in Definition 3.3 increases by 2 every time. This together with $c_1 = \mathrm{Deg}(p) - r$ implies that

$$S_X(p) = s^{\mathrm{Deg}(p)} p_0 + s^{\mathrm{Deg}(p)-1} p_1 + \cdots + s^{\mathrm{Deg}(p)-r} p_r$$

for some $r \in \mathbb{N}$, $p_j \in R_X$ and $p_r \neq 0$. Now we show that $p_j \neq 0$ for all $j \in \{0, \ldots, r\}$. By fully invoking Definition 3.3, for $0 \leqslant t \leqslant r$ we derive

$$\langle s^{\mathrm{Deg}(p)-t} \rangle S_X(p) = \langle s^{\mathrm{Deg}(p)-t} \rangle S_X(x_{i_1}^{(k_1)}) S_X(x_{i_2}^{(k_2)}) \cdots S_X(x_{i_m}^{(k_m)})$$

$$= (-i)^{\nu_1(p)+\frac{m}{2}} \sum_{\bar{b}\in D(\bar{k},t)} \left(\frac{i}{\pi}\right)^{\sum_{i=1}^{m}\frac{k_i-b_i}{2}} \prod_{\nu=1}^{m} \frac{k_\nu!}{b_\nu!(\frac{k_\nu-b_\nu}{2})!} x_{\sigma(i_1)}^{(b_1)}\cdots x_{\sigma(i_m)}^{(b_m)}$$

$$= (-i)^{\nu_1(p)+\frac{m}{2}} \left(\frac{i}{\pi}\right)^{t} \sum_{\bar{b}\in D(\bar{k},t)} \prod_{\nu=1}^{m} \frac{k_\nu!}{b_\nu!(\frac{k_\nu-b_\nu}{2})!} x_{\sigma(i_1)}^{(b_1)}\cdots x_{\sigma(i_m)}^{(b_m)},$$

where $\bar{b} = (b_1,\dots,b_m)$ and $\bar{k} = (k_1,\dots,k_m)$. Since $k_\nu \geqslant b_\nu \geqslant 0$, we have $\prod_{\nu=1}^{m} \frac{k_\nu!}{b_\nu!(\frac{k_\nu-b_\nu}{2})!} > 0$, which implies $\langle s^{\mathrm{Deg}(p)-t}\rangle S_X(p) \neq 0$. Therefore the expression of $S_X(p)$ in the statement is in S-form.

□

We shall see that the following refined sets of compositions of numbers play a crucial role. Throughout $\bar{b} \in \mathbb{N}^m$ has to be interpreted as $\bar{b} = (b_1,\dots,b_m)$.

Definition 3.12. *Given $\bar{d} \in \mathbb{N}^m$, $\bar{k} \in \mathbb{N}^m$, and $j,t \in \mathbb{N}$,*

$$B(\bar{d},\bar{k},t,j) := \left\{\bar{b} \in D(k,t) : \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} d_i + 2j, d_i \leqslant b_i \text{ and } d_i \equiv b_i \pmod 2\right\}.$$

Lemma 3.14. *Given $j,t \in \mathbb{N}$ and $\bar{d} \in \mathbb{N}^m$ and $\bar{k} \in \mathbb{N}^m$, then*

$$j \sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)} \prod_{\nu=1}^{m} \alpha(k_\nu,b_\nu)\beta(b_\nu,d_\nu) = (t+1) \sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)} \prod_{\nu=1}^{m} \alpha(k_\nu,e_\nu)\beta(e_\nu,d_\nu), \tag{3.11}$$

where

$$\alpha(k_\nu,e_\nu) := \frac{k_\nu!}{(\frac{k_\nu-e_\nu}{2})!} \text{ and } \beta(b_\nu,c_\nu) := \frac{1}{c_\nu!(\frac{b_\nu-c_\nu}{2})!}.$$

Proof. Let

$$M_1 := \{(\bar{b},\bar{b}-2z_i) : \bar{b} \in B(\bar{d},\bar{k},t,j), 1 \leqslant i \leqslant m \text{ and } b_i \geqslant d_i+2\}$$

and

$$M_2 := \{(\bar{e}+2z_i,\bar{e}) : \bar{e} \in B(\bar{d},\bar{k},t+1,j-1), 1 \leqslant i \leqslant m \text{ and } e_i \leqslant k_i-2\},$$

where $z_i = (a_1,\dots,a_m)$ with $a_i = 1$ and $a_j = 0$ $(j \neq i)$. Then

$$\text{LHS of (3.11)} = \sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)} \left(\sum_{i=1}^{m} \frac{b_i-d_i}{2} \prod_{\nu=1}^{m} \alpha(k_\nu,b_\nu)\beta(b_\nu,d_\nu)\right)$$

$$
\begin{aligned}
&= \sum_{\bar{b}\in B(\bar{d},\bar{k},t,j)} \prod_{v=1}^{m} \alpha(k_v,b_v) \sum_{i=1}^{m} \beta(b_i-2,d_i) \prod_{\substack{v=1\\ v\neq i}}^{m} \beta(b_v,d_v) \\
&= \sum_{(\bar{b},\bar{e})\in M_1} \prod_{v=1}^{m} \alpha(k_v,b_v)\beta(e_v,d_v)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{RHS of (3.11)} &= \sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)} \left(\sum_{i=1}^{m} \frac{k_i-e_i}{2} \prod_{v=1}^{m} \alpha(k_v,e_v)\beta(e_v,d_v) \right) \\
&= \sum_{\bar{e}\in B(\bar{d},\bar{k},t+1,j-1)} \prod_{v=1}^{m} \beta(e_v,d_v) \sum_{i=1}^{m} \alpha(k_v,e_i+2) \prod_{\substack{v=1\\ v\neq i}}^{m} \alpha(k_v,e_v) \\
&= \sum_{(\bar{b},\bar{e})\in M_2} \prod_{v=1}^{m} \alpha(k_v,b_v)\beta(e_v,d_v),
\end{aligned}
$$

where we define $\beta(b_i-2,d_i) := 0$ if $b_i = d_i$, and define $\alpha(k_i,e_i+2) := 0$ if $e_i = k_i$.

To prove the lemma we need to prove that $M_1 = M_2$.

Take $(\bar{b},\bar{e}) := (\bar{b},\bar{b}-2z_i) \in M_1$ for some $i \in \{1,\ldots,m\}$. Then $\bar{b} = \bar{e}+2z_i$, and we can write $(\bar{b},\bar{e}) = (\bar{e}+2z_i,\bar{e})$. Additionally, from the definition of M_1 we have $\bar{b} \in B(\bar{d},\bar{k},t,j)$ and $d_i+2 \leqslant b_i \leqslant k_i$, which implies $\bar{e}+2z_i \in B(\bar{d},\bar{k},t,j)$ and $d_i+2 \leqslant e_i+2 \leqslant k_i$. Then $d_i \leqslant e_i \leqslant k_i-2$ and by Definition 3.12 we have

$$\sum_{v=1}^{m} e_v + 2 = \sum_{v=1}^{m} k_v - 2t = \sum_{v=1}^{m} d_v + 2j.$$

Hence

$$\sum_{v=1}^{m} e_v = \sum_{v=1}^{m} k_v - 2(t+1) = \sum_{v=1}^{m} d_v + 2(j-1)$$

and $d_i \leqslant e_i \leqslant k_i-2$, which implies $\bar{e} \in B(\bar{d},\bar{k},t+1,j-1)$ and $d_i \leqslant e_i \leqslant k_i-2$. Therefore $(\bar{b},\bar{e}) = (\bar{e}+2z_i,\bar{e}) \in M_2$. The other direction goes analogously. □

Theorem 3.4. *Let $p = x_{i_1}^{(k_1)}\cdots x_{i_m}^{(k_m)} \in R_X$. According to Lemma 3.13 let $S_X(p) = s^{\mathrm{Deg}(p)}p_0 + \cdots + s^{\mathrm{Deg}(p)-r}p_r$ in S-form and $r := \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$. We have*

(1) $S_X(p_r) = s^{\mathrm{Deg}(p_r)}q$ with $q \in R_X\backslash\{0\}$; and

(2) for any adjacent pair (p_t, p_{t+1}), $t \in \{0, \ldots, r-1\}$,

$$S_X(p_{t+1}) = \frac{1}{t+1} \sum_{j=1}^{r-t} s^{\mathrm{Deg}(p_t)-j-1} j q_{t,j}$$

and

$$S_X(p_t) = \sum_{j=0}^{r-t} s^{\mathrm{Deg}(p_t)-j} q_{t,j}$$

with $q_{t,j} \in R_X$.

Proof. (1) According to Lemma 3.13, $p_r \neq 0$. Therefore the statement is implied by Definition 3.3.

(2) We first prove that the low degree of $S_X(p_t)$ with respect to s is $\mathrm{Deg}(p_t) - r + t$, then we prove that the coefficient relation

$$\frac{\langle s^{\mathrm{Deg}(p_t)-j} \rangle S_X(p_t)}{\langle s^{\mathrm{Deg}(p_t)-j-1} \rangle S_X(p_{t+1})} = \frac{t+1}{j}$$

is true for $j \in \{0, \ldots, r-t\}$. Suppose $x_{i_1} = x_{i_2} = \cdots = x_{i_a} = x_1$ and $x_{i_j} \neq x_1$ for $a+1 \leqslant j \leqslant m$. Let $C(p) := (-i)^{\nu_1(p)+\frac{m}{2}}$. Applying Lemma 3.13 we have

$$\begin{aligned} S_X(p_t) &= S_X\left(C(p) \left(\tfrac{i}{\pi}\right)^t \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)} \right) \\ &= C(p) \left(\tfrac{i}{\pi}\right)^t \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) S_X\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)} \right). \end{aligned} \tag{3.12}$$

Now let $d_t := \mathrm{Deg}(p_t)$. Concerning (3.12), for $\bar{b} \in D(\bar{k}, t)$ we apply Lemma 3.13 again and obtain

$$\begin{aligned} S_X\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)} \right) =& s^{d_t} C(p) x_{i_1}^{(b_1)} \cdots x_{i_m}^{(b_m)} \\ &+ s^{d_t-1} C(p) \left(\frac{i}{\pi}\right) \sum_{\bar{c} \in D(\bar{b},1)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\ &+ s^{d_t-2} C(p) \left(\frac{i}{\pi}\right)^2 \sum_{\bar{c} \in D(\bar{b},2)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\ &+ \ldots \\ &+ s^{d_t-r_t} C(p) \left(\frac{i}{\pi}\right)^{r_t} \sum_{\bar{c} \in D(\bar{b},r_t)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)}, \end{aligned} \tag{3.13}$$

where $r_t := \frac{d_t}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$ according to Lemma 3.13 and $\nu_1(p) = \nu_1\left(x_{\sigma(i_1)}^{(b_1)} \cdots x_{\sigma(i_m)}^{(b_m)}\right)$. Since $\bar{b} \in D(\bar{k},t)$, i.e., $\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} k_i - 2t$, we have $\sum_{i=1}^{m} b_i + \frac{m}{2} = \sum_{i=1}^{m} k_i + \frac{m}{2} - 2t$, which means $d_t = \mathrm{Deg}(p) - 2t$. This together with $r = \frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}$ implies

$$r_t = \frac{\mathrm{Deg}(p) - 2t}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2} = \left(\frac{\mathrm{Deg}(p)}{2} - \frac{m}{4} - \frac{\nu_1(p)}{2}\right) - t = r - t.$$

Plugging (3.13) into (3.12), we get

$$\begin{aligned}
S_X(p_t) = & C(p)^2 \left(\frac{i}{\pi}\right)^t \Bigg(s^{d_t} \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(b_1)} \cdots x_{i_m}^{(b_m)} \\
& + s^{d_t - 1} \left(\frac{i}{\pi}\right) \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) \sum_{\bar{c} \in D(\bar{b},1)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\
& + s^{d_t - 2} \left(\frac{i}{\pi}\right)^2 \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) \sum_{\bar{c} \in D(\bar{b},2)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)} \\
& + \dots \\
& + s^{d_t - r_t} \left(\frac{i}{\pi}\right)^{r_t} \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) \prod_{v=1}^{m} \beta(b_v, c_v) (x_1')^a x_{i_{a+1}} \cdots x_{i_m} \Bigg) \\
= & s^{d_t} h_0 + s^{d_t - 1} h_1 + \dots + s^{d_t - r_t} h_{r_t},
\end{aligned}$$

where for $j \in \{0, \dots, r_t\}$

$$h_j = C(p)^2 \left(\frac{i}{\pi}\right)^{t+j} \sum_{\bar{b} \in D(\bar{k},t)} \prod_{v=1}^{m} \alpha(k_v, b_v) \sum_{\bar{c} \in D(\bar{b},j)} \prod_{v=1}^{m} \beta(b_v, c_v) x_{i_1}^{(c_1)} \cdots x_{i_m}^{(c_m)}.$$

Analogously we have

$$\begin{aligned}
S_X(p_{t+1}) = & C(p)^2 \left(\frac{i}{\pi}\right)^{t+1} \Bigg(s^{d_t - 2} \sum_{\bar{e} \in D(\bar{k},t+1)} \prod_{v=1}^{m} \alpha(k_v, e_v) x_{i_1}^{(e_1)} \cdots x_{i_m}^{(e_m)} \\
& + s^{d_t - 3} \left(\frac{i}{\pi}\right) \sum_{\bar{e} \in D(\bar{k},t+1)} \prod_{v=1}^{m} \alpha(k_v, e_v) \sum_{\bar{u} \in D(\bar{e},1)} \prod_{v=1}^{m} \beta(e_v, u_v) x_{i_1}^{(u_1)} \cdots x_{i_m}^{(u_m)} \\
& + s^{d_t - 4} \left(\frac{i}{\pi}\right)^2 \sum_{\bar{e} \in D(\bar{k},t+1)} \prod_{v=1}^{m} \alpha(k_v, e_v) \sum_{\bar{u} \in D(\bar{e},2)} \prod_{v=1}^{m} \beta(e_v, u_v) x_{i_1}^{(u_1)} \cdots x_{i_m}^{(u_m)}
\end{aligned}$$

$$
\begin{aligned}
&+\dots \\
&\left. +s^{d_t-r_t-1}\left(\frac{i}{\pi}\right)^{r_t-1}\sum_{\bar{e}\in D(\bar{k},t+1)}\prod_{v=1}^{m}\alpha(k_v,e_v)\prod_{v=1}^{m}\beta(e_v,u_v)(x_1')^a x_{i_{a+1}}\cdots x_{i_m}\right) \\
=&s^{d_t-2}q_1+s^{d_t-3}q_2+\dots+s^{d_t-r_t-1}q_{r_t},
\end{aligned}
$$

where for $j\in\{1,\dots,r_t\}$

$$
q_j=C(p)^2\left(\frac{i}{\pi}\right)^{t+j}\sum_{\bar{e}\in D(\bar{k},t+1)}\prod_{v=1}^{m}\alpha(k_v,e_v)\sum_{\bar{u}\in D(\bar{e},j-1)}\prod_{v=1}^{m}\beta(e_v,u_v)x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}.
$$

Thus proving the statement to be true is equivalent to proving that

$$
\frac{h_j}{q_j}=\frac{t+1}{j}.
$$

For any fixed $\bar{c}=(c_1,\dots,c_m)\in\mathbb{N}^m$, the set of all possible $\bar{b}$ contributing to the coefficient of $x_{i_1}^{(c_1)}\cdots x_{i_m}^{(c_m)}$ in h_j is equal to $B(\bar{c},\bar{k},t,j)$, and for any fixed $\bar{u}=(u_1,\dots,u_m)\in\mathbb{N}^m$ the set of all possible $\bar{e}$ contributing to the coefficient of $x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}$ in q_j is equal to $B(\bar{u},\bar{k},t+1,j-1)$. Therefore

$$
h_j=C(p)^2\left(\frac{i}{\pi}\right)^{t+j}\sum_{\bar{c}\in\mathbb{N}^m}\left(\sum_{\bar{b}\in B(\bar{c},\bar{k},t,j)}\prod_{v=1}^{m}\alpha(k_v,b_v)\beta(b_v,c_v)\right)x_{i_1}^{(c_1)}\cdots x_{i_m}^{(c_m)}
$$

and

$$
q_j=C(p)^2\left(\frac{i}{\pi}\right)^{t+j}\sum_{\bar{u}\in\mathbb{N}^m}\left(\sum_{e_v\in B(\bar{u},\bar{k},t+1,j-1)}\prod_{v=1}^{m}\alpha(k_v,e_v)\beta(e_v,u_v)\right)x_{i_1}^{(u_1)}\cdots x_{i_m}^{(u_m)}.
$$

Now fix $(d_1,d_2,\dots,d_m)\in\mathbb{N}^m$. We need to prove that

$$
\frac{\langle x_{i_1}^{(d_1)}\cdots x_{i_m}^{(d_m)}\rangle h_j}{\langle x_{i_1}^{(d_1)}\cdots x_{i_m}^{(d_m)}\rangle q_j}=\frac{t+1}{j}.
$$

Applying Lemma 3.14 we immediately obtain the correctness of this equality.

□

According to Lemma 3.13, for any homogeneous $p \in R_X$, since S_X is a homomorphism, we have that the powers of s in $S_X(p)$ start from $\mathrm{Deg}(p)$ and decrease by integers; namely, $S_X(p) = s^{\mathrm{Deg}(p)}p_0 + \cdots + s^{\mathrm{Deg}(p)-r}p_r$ with the $p_j \in R_X$ and $p_r \neq 0$. At this moment, we guess that some of the $p_0, \ldots, p_{r-1}$ could be zero, but in Corollary 3.4 we will show that this is not the case.

Corollary 3.3. *Let $p \in R_X$ be homogeneous and $S_X(p) = s^{\mathrm{Deg}(p)}p_0 + \cdots + s^{\mathrm{Deg}(p)-r}p_r$ with $r \in \mathbb{N}$ and $p_r \in R_X \backslash \{0\}$. Then*

(1) $S_X(p_r) = s^{\mathrm{Deg}(p_r)}q$ with $q \in R_X \backslash \{0\}$; and

(2) for any neighboring pair (p_t, p_{t+1}), $t \in \{0, \ldots, r\}$, where $p_{r+1} := 0$, there exists $\gamma_t \in \mathbb{N}$ such that

$$S_X(p_{t+1}) = \frac{1}{t+1} \sum_{j=1}^{\gamma_t} s^{\mathrm{Deg}(p_t)-j-1} j p_{t,j}$$

and

$$S_X(p_t) = \sum_{j=0}^{\gamma_t} s^{\mathrm{Deg}(p_t)-j} p_{t,j}$$

with $p_{t,j} \in R_X$.

Proof. We first prove (2). Suppose

$$p = a_1 h_1 + \cdots + a_n h_n$$

where the h_j are monomials in R_X of the same degree, and the $a_j \in \mathbb{K} \backslash \{0\}$.

Let $d := \mathrm{Deg}(p) = \mathrm{Deg}(h_j)$. By Lemma 3.13, there exists an integer b_j such that

$$S_X(h_j) = s^d h_{j,0} + s^{d-1} h_{j,1} + \cdots + s^{d-b_j} h_{j,b_j}$$

in S-form. Let $r := \max_{j=1,\ldots,n} \{b_j\}$. Then

$$\begin{aligned} S_X(p) =& s^d(a_1 h_{1,0} + \cdots + a_n h_{n,0}) + \cdots + s^{d-r}(a_1 h_{1,r} + \cdots + a_n h_{n,r}) \\ =& s^d p_0 + \cdots + s^{d-r} p_r, \end{aligned}$$

where $p_t = a_1 h_{1,t} + \cdots + a_n h_{n,t}$ for $t = 0, \ldots, r$ and $h_{j,t} = 0$ when $t > b_j$.

Since the p_t are homogeneous by Lemma 3.5(2), we can define $d_t := \mathrm{Deg}(p_t)$. Hence we can suppose for $t \in \{0, \dots, r\}$ that

$$S_X(h_{j,t}) = s^{d_t} q_{j,0} + s^{d_t-1} q_{j,1} + \cdots + s^{d_t-b_{j,t}} q_{j,b_{j,t}} \tag{3.14}$$

where the $q_{j,i} \in R_X$ and $q_{j,b_{j,t}} \neq 0$. Therefore by letting $\gamma_t := \max_{j=1,\dots,m} \{b_{j,t}\}$ we obtain

$$S_X(p_t) = a_1 S_X(h_{1,t}) + \cdots + a_n S_X(h_{n,t}) = s^{d_t} q_0 + s^{d_t-1} q_1 \cdots + s^{d_t-\gamma_t} q_{\gamma_t}$$

where $q_i = a_1 q_{1,i} + \cdots + a_n q_{n,i}$ and $q_{j,i} = 0$ if $i > \gamma_t$. Furthermore, since the h_j are monomials, we immediately obtain from (3.14) by Theorem 3.4

$$S_X(h_{j,t+1}) = s^{d_t-2} \frac{1}{t+1} q_{j,1} + \cdots + s^{d_t-b_{j,t}-1} \frac{b_{j,t}}{t+1} q_{j,b_{j,t}}.$$

Hence

$$\begin{aligned}
S_X(p_{t+1}) =& a_1 S_X(h_{1,t+1}) + \cdots + a_n S_X(h_{n,t+1}) \\
=& a_1 \left(s^{d_t-2} \frac{1}{t+1} q_{1,1} + \cdots + s^{d_t-b_{1,t}-1} \frac{b_{1,t}}{t+1} q_{1,b_{1,t}} \right) \\
&+ \dots \\
&+ a_n \left(s^{d_t-2} \frac{1}{t+1} q_{n,1} + \cdots + s^{d_t-\gamma_t-1} \frac{b_{n,t}}{t+1} q_{n,b_{n,t}} \right) \\
=& s^{d_t-2} \frac{1}{t+1} q_1 + \cdots + s^{d_t-\gamma_t-1} \frac{\gamma_t}{t+1} q_{\gamma_t}.
\end{aligned}$$

It remains to prove (1). This follows immediately from (2). $\square$

Now we introduce a definition that will serve to increase readability.

Definition 3.13. *For half integers $a, b \in \frac{1}{2}\mathbb{Z}$, such that $a \leqslant b$ and $b - a \in \mathbb{N}$:*

$$\{a, \dots, b\} := \{a, a+1, a+2, \dots, b\}$$

and

$$\sum_{j=a}^{b} h(j) := h(a) + h(a+1) + \cdots + h(b).$$

Corollary 3.4. *Given $p \in R_X$ homogeneous, suppose $S_X(p) = \sum_{j=\gamma}^{\mathrm{Deg}(p)} s^j p_j$ with $p_j \in R_X$ and $p_\gamma \neq 0$. Then the sum is in S-form.*

Proof. Assume $p_j \neq 0$ for $j \geqslant \gamma$. Then $S_X(p_j) \neq 0$ by Corollary 3.2, which by Corollary 3.3(2) implies $S_X(p_{j+1}) \neq 0$, which again implies $p_{j+1} \neq 0$. $\square$

By Definition 3.13 and Corollary 3.4, for homogeneous $p \in R_X$, the notation of S-form $S_X(p) = \sum_{i=1}^{n} s^{c_i} q_i$ turns into $S_X(p) = \sum_{j=\gamma}^{\mathrm{Deg}(p)} s^j p_j$ where $\gamma \in \frac{1}{2}\mathbb{Z}$ such that $\gamma = c_1$.

The next theorem is crucial for refining Algorithm 3.1.

Theorem 3.5. *Let $p, g \in R_X$ be homogeneous and assume that both sums*

$$S_X(p) = \sum_{j=\gamma_p}^{\mathrm{Deg}(p)} s^j p_j \quad \textit{and} \quad S_X(g) = \sum_{j=\gamma_g}^{\mathrm{Deg}(g)} s^j g_j$$

are in S-form. If $g \in \mathrm{LT}(p)$ then $\mathrm{Deg}(p) = \mathrm{Deg}(g)$, $\gamma_p = \gamma_g$, and

$$g_j \in \mathrm{LT}(p_j), \quad j \in \{\gamma_p, \ldots, \mathrm{Deg}(p)\}.$$

Proof. By Definition 3.10, the LT orbit is built up by the powers of $\widetilde{S}$ and T_X. Since $\widetilde{S}$ and T_X both keep the degree, we deduce that if $g \in \mathrm{LT}(p)$ then $\mathrm{Deg}(p) = \mathrm{Deg}(g)$.

The proof of the remaining part proceeds by induction on the length of

$$g = S^{k_1} T^{\ell_1} \cdots S^{k_m} T^{\ell_m}(p).$$

For the induction step, it suffices to prove the statement for two neighboring situations:

$$g = \widetilde{S}(p) \quad \text{and} \quad g = T_X(p).$$

Assume $g = \widetilde{S}(p)$. Let $p = a_1 h_1 + a_2 h_2 + \cdots + a_n h_n$ where the h_t are monomials in R_X with the same degree and the $a_t \in \mathbb{K}\backslash\{0\}$. Suppose $S_X(h_t) = \sum_{j=r_t}^{d} s^j h_{t,j}$ in S-form with $d := \mathrm{Deg}(p)$.

We first prove that $S_X(\sigma(h_t)) = \sigma(S_X(h_t))$. Since σ and S_X are homomorphisms, it suffices to show this is true for the generators, which means we have to prove $S_X(\sigma(x_i^{(k)})) = \sigma(S_X(x_i^{(k)}))$ for any $i \in \{1, \ldots, 4\}$ and $k \in \mathbb{N}$. This is implied immediately by Definition 3.3.

Let $r := \max_{t=1,\ldots,n}\{r_t\}$ and $h_{t,j} := 0$ when $j < r_t$. Then by Lemma 3.13 we have

$$\begin{aligned} S_X(p) =& a_1 S_X(h_1) + \cdots + a_n S_X(h_n) \\ =& a_1 \sum_{j=r_1}^{d} s^j h_{1,j} + \cdots + a_n \sum_{j=r_n}^{d} s^j h_{n,j} \\ =& s^d(a_1 h_{1,d} + \cdots + a_n h_{n,d}) + \cdots + s^r(a_1 h_{1,r} + \cdots + a_n h_{n,r}). \end{aligned}$$

By Definition 3.3 and the linearity of $\widetilde{S}$ we also have

$$g = \widetilde{S}(p) = a_1\widetilde{S}(h_1) + \cdots + a_n\widetilde{S}(h_n) = a_1(-i)^{k_1}\sigma(h_1) + \cdots + a_n(-i)^{k_n}\sigma(h_n),$$

where the $k_t := \nu_1(h_t) + \frac{\mu(h_t)}{2}$. Then

$$\begin{aligned} S_X(g) =& a_1(-i)^{k_1} S_X(\sigma(h_1)) + \cdots + a_n(-i)^{k_1} S_X(\sigma(h_n)) \\ =& a_1(-i)^{k_1}\sigma\left(\sum_{j=r_1}^{d} s^j h_{1,j}\right) + \cdots + a_n(-i)^{k_n}\sigma\left(\sum_{j=r_n}^{d} s^j h_{n,j}\right) \\ =& s^d(a_1(-i)^{k_1}\sigma(h_{1,d}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,d})) \\ &+ \ldots \\ &+ s^r(a_1(-i)^{k_1}\sigma(h_{1,r}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,r})). \end{aligned}$$

Since for $j \in \{r, r+1, \ldots, d\}$,

$$\begin{aligned} \widetilde{S}(a_1 h_{1,j} + \cdots + a_n h_{n,j}) &= a_1\widetilde{S}(h_{1,j}) + \cdots + a_n\widetilde{S}(h_{n,j}) \\ &= a_1(-i)^{k_1}\sigma(h_{1,j}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,j}), \end{aligned}$$

we obtain

$$a_1(-i)^{k_1}\sigma(h_{1,j}) + \cdots + a_n(-i)^{k_n}\sigma(h_{n,j}) \in \mathrm{LT}(a_1 h_{1,j} + \cdots + a_n h_{n,j}).$$

Hence

$$g_j \in \mathrm{LT}(p_j).$$

Then $g_j = 0$ if and only if $p_j = 0$. Therefore $\gamma_p = \gamma_g$.

For $g = T_X(p)$ the proof is analogous. □

Applying Corollary 3.3 and Theorem 3.5 we can simplify Algorithm 3.1 substantially. The essence of the simplification is the following theorem.

Theorem 3.6. *Given $p \in R_X$ homogeneous and $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form, then*

$$p \in \ker\phi \text{ if and only if } \mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)(\tau)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6} \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}.$$

Proof. Assume $p \in \ker\phi$. By Lemma 3.8, $q_j \in \ker\phi$ for all $j \in \{r, \ldots, \mathrm{Deg}(p)\}$. Therefore, for any $g \in \mathrm{LT}(q_j)$, by Lemma 3.10 we have $g \in \ker\phi$. This implies $\prod_{g \in \mathrm{LT}(q_j)} \phi(g)(\tau) \equiv 0$. And hence

$$\infty = \mathrm{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)(\tau)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6}.$$

Assume $p \notin \ker\phi$. According to Lemma 3.8, at least one of the q_j is not in $\ker\phi$. Take $t \in \{r, \ldots, \mathrm{Deg}(p)\}$ such that $q_t \notin \ker\phi$ and $q_i \in \ker\phi$ when $i < t$. We prove that $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau)$ is a modular form.

Case 1: $t = r$. By Corollary 3.3.1, $S_X(q_t) = S_X(q_r) = s^{r_t} h$ in S-form, where $h \notin \ker\phi$ because $q_t \notin \ker\phi$. Hence for every $g \in \mathrm{LT}(q_t)$, by Theorem 3.5, there exists $q \in R_X$ such that $S_X(g) = s^{r_t} q$ in S-form and $q \notin \ker\phi$. By Lemma 3.12, $\prod_{g \in \mathrm{LT}(q_t)} \phi(g)(\tau) \in M(\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|)$.

Case 2: $t > r$. Suppose $S_X(q_t) = \sum_{j=r_t}^{\mathrm{Deg}(q_t)} s^j h_j$ in S-form. Since $q_t \notin \ker\phi$, at least one of the h_j is not in $\ker\phi$. By rewriting of Corollary 3.3.2,

$$S_X(q_{t-1}) = \sum_{j=r_t}^{\mathrm{Deg}(q_t)-1} s^{j-1} \frac{\mathrm{Deg}(q_t) - j - 1}{t+1} h_j \text{ in } S\text{-form},$$

where, again by Lemma 3.8, $h_j \in \ker\phi$ for $r_t \leqslant j \leqslant \mathrm{Deg}(q_t) - 1$ because $q_{t-1} \in \ker\phi$. Thus $h_{\mathrm{Deg}(q_t)} \notin \ker\phi$. Hence for $g \in \mathrm{LT}(q_t)$, applying Theorem 3.5 we have $S_X(g) = \sum_{j=r_t}^{\mathrm{Deg}(q_t)} s^j g_j$ in S-form with $g_j \in$

$\mathrm{LT}(h_j)$, which yields $g_j \in \ker\phi$ for $r_t \leqslant j \leqslant \mathrm{Deg}(q_t)-1$ and $g_{\mathrm{Deg}(q_t)} \notin \ker\phi$. Again by Lemma 3.12, $\prod_{g\in\mathrm{LT}(q_t)} \phi(g)(\tau) \in M(\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|)$.

In addition, $q_t \notin \ker\phi$ implies $\prod_{g\in\mathrm{LT}(q_t)} \phi(g)(\tau) \neq 0$. Therefore by Lemma 2.1 we obtain

$$\mathrm{ord}\left(\prod_{g\in\mathrm{LT}(q_t)} \phi(g)(\tau)\right) \leqslant \frac{\mathrm{Deg}(q_t)|\mathrm{LT}(q_t)|}{6}.$$

□

The algorithmic content of Theorem 3.6 is the following:

Algorithm 3.2. *Given* $p \in R_X$ *homogeneous and* $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ *in S-form, we have the following algorithm to prove or disprove* $p \in \ker\phi$.

Input: homogeneous $p \in R_X$. *Output: True if* $p \in \ker\phi$; *False if* $p \notin \ker\phi$.

If $\mathrm{Deg}(p) > 0$ *set* $j := r$. *While* $j \leqslant \mathrm{Deg}(p)$ *do*

$$\text{if } \mathrm{ord}\left(\prod_{g\in\mathrm{LT}(q_j)} \phi(g)(\tau)\right) > \frac{\mathrm{Deg}(q_j)|\mathrm{LT}(q_j)|}{6}$$

then $j := j+1$;

else return False;

exit;

end do;

return True.

If $\mathrm{Deg}(p) = 0$ *then True if* $p = 0$; *False if* $p \in \mathbb{K}\backslash\{0\}$.

One can connect our method to classical methods using "Sturm bounds". Namely, in Theorem 3.6 one can replace $|\mathrm{LT}(q_j)|$ with 384, owing to Proposition 3.3. Moreover, for every $g \in \mathrm{LT}(q_j)$, the q-expansion of $\phi(g)$ only contains non-negative powers of q. Thus to show that

$\operatorname{ord}\left(\prod_{g \in \mathrm{LT}(q_j)} \phi(g)(\tau)\right)$ is greater than a certain number, it suffices to show that $\operatorname{ord}(\phi(q_j)(\tau))$ is greater than this number. Summarizing, we have the following corollary.

Corollary 3.5. *Let $p \in R_X$ be homogeneous and $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form. Then*

$$p \in \ker\phi \text{ if and only if } \operatorname{ord}(\phi(q_j)(\tau)) > 2^6 \cdot \mathrm{Deg}(p) \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}.$$

We also present a modular form version.

Proposition 3.4. *Let $p \in R_X$ be homogeneous. If $\phi(p) \in M_k(\mathbb{H})\backslash\{0\}$ then*

$$\operatorname{ord}(\phi(p)(\tau)) \leqslant 2^6 \cdot k.$$

Proof. Let $S_X(p) = \sum_{j=r}^{\mathrm{Deg}(p)} s^j q_j$ in S-form. If $\phi(p) \neq 0$, by Corollary 3.5 we have

$$\operatorname{ord}(\phi(q_j)(\tau)) \leqslant 2^6 \cdot \mathrm{Deg}(p) \text{ for all } j \in \{r \ldots, \mathrm{Deg}(p)\}. \tag{3.15}$$

If $\phi(p) \in M_k(\mathbb{H})\backslash\{0\}$, by Definition 3.3 and Definition 2.8 we have

$$(S_0\phi(p))(\tau) = \phi(p)(-1/\tau) = \tau^k \phi(p)(\tau). \tag{3.16}$$

This together with Lemma 3.2 implies that $S_X(p) = s^k \widetilde{S}(p)$. Then (3.15) can be stated as

$$\begin{aligned} \operatorname{ord}(\phi(\widetilde{S}(p))(\tau)) &\leqslant 2^6 \cdot \mathrm{Deg}(p) \\ &= 2^6 \cdot k, \end{aligned}$$

where the last equality follows from Lemma 3.11. Finally we show that $\phi(\widetilde{S}(p)) = \phi(p)$. Again by using Lemma 3.2 we have

$$S_0\phi(p) = \phi^*(S_X(p)) = \phi^*(s^k \widetilde{S}(p)) = \tau^k \phi^*(\widetilde{S}(p)) = \tau^k \phi(\widetilde{S}(p)). \tag{3.17}$$

We plug (3.16) into (3.17) and complete the proof. □

Next we do the complexity analysis.

Definition 3.14. *For homogeneous $p \in R_X$ define $s(p)$ to be the number of S_X operations required to run Algorithm 1 on p.*

Definition 3.15. *For homogeneous $p \in R_X$ define $o(p)$ to be the number of* LT *operations required to run Algorithm 1 on p. An* LT *operation is a function that computes the elements of the leading term orbit of a given polynomial in R_X.*

Definition 3.16. *Let $p \in R_X$ be homogeneous with $S_X(p) = p_1 s^{\mathrm{Deg}(p)} + p_2 s^{\mathrm{Deg}(p)-1} + \cdots + p_r s^{\mathrm{Deg}(p)-r+1}$ in S-form. We define $\ell(p) := r$ to be the length of p.*

Lemma 3.15. *Let $p \in R_X$ be be homogeneous with $S_X(p) = p_1 s^{\mathrm{Deg}(p)} + p_2 s^{\mathrm{Deg}(p)-1} + \cdots + p_r s^{\mathrm{Deg}(p)-r+1}$ in S-form and $|\mathrm{LT}(p_j)| = 384$. Then the number of $o(\tilde{p})$ and $s(\tilde{p})$ applications on any polynomial $\tilde{p}$ appearing when running Algorithm 3.1 on p depends only on the length of $\tilde{p}$.*

Proof. Suppose $M = \{\tilde{p}_1, \dots, \tilde{p}_m\}$ are the polynomials appearing when running Algorithm 3.1 on p. Consider $M = M_1 \cup M_2 \cup \cdots \cup M_n$ where $M_j := \{\tilde{p} \in M : \ell(\tilde{p}) = j\}$. By Corollary 3.3.1 it is clear that for any $\tilde{p} \in R_X$ with $\ell(\tilde{p}) = 1$ we have $o(\tilde{p}) = 1$. Then by induction on j we prove that for every $f, g \in M_j : o(f) = o(g)$. Assume this is true for $j < k$. We prove that this is also true for $j = k$. Define $\tilde{o} : \{1, \dots, k-1\} \to \mathbb{N}$ by $\tilde{o}(j) := o(q)$ where $q \in M_j$, which by the induction hypothesis is well-defined. Let $\tilde{p} \in M_k$. Then by Theorem 3.5 we have $\tilde{p} \in \mathrm{LT}(p_j)$ for some $j \in \{1, \dots, r\}$, hence $|\mathrm{LT}(\tilde{p})| = 384$. Suppose $\mathrm{LT}(\tilde{p}) = \{q_1, \dots, q_{384}\}$ with $S_X(q_j) = \sum_{t=r_j}^{\mathrm{Deg}(q_j)} s^t q_{j,t}$ in S-form. Note that $\mathrm{Deg}(q_j) - r_j + 1 = k$. We know from Theorem 3.4 that $\ell(q_{j,r_j}) = 1$ and $\ell(q_{j,i}) = \ell(q_{j,i-1}) + 1$ for all $j \in \{1, \dots, 384\}$. Therefore running Algorithm 1 on $\tilde{p}$ results in one orbit computation on $\tilde{p}$ and triggers a running of Algorithm 1 on $q_{j,t}$ for all $j \in \{1, \dots, 384\}$ and for all $t \in \{1, \dots, \mathrm{Deg}(q_j) - 1\}$. For the operation count this means,

$$\begin{aligned} o(\tilde{p}) &= 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} o(q_{j,t}) = 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} \tilde{o}(\ell(q_{j,t})) \\ &= 1 + \sum_{j=1}^{384} \sum_{t=r_j}^{\mathrm{Deg}(p_j)-1} \tilde{o}(t - r_j + 1) = 1 + 384 \sum_{t=1}^{\mathrm{Deg}(p_j)-r_j} \tilde{o}(t) \\ &= 1 + 384 \sum_{t=1}^{k-1} \tilde{o}(t). \end{aligned}$$

Since this shows that $o(\tilde{p})$ is only dependent on $k = \ell(\tilde{p})$, this completes the induction proof for the $o(p)$ statement. The $s(p)$ statement is proven analogously. □

Corollary 3.6. *Let $N_1(p)$ and $N_2(p)$, respectively, be the total number of* LT *and S_X operations when running Algorithm 3.1 and Algorithm 3.2 on a given homogeneous $p \in R_X$. Let k be the length of p. Then in the worst case $N_1(p)$ is exponential and $N_2(p)$ is linear in k.*

Proof. According to Proposition 3.3, in the worst case $|\mathrm{LT}(\tilde{p})| = 384$ for every polynomial $\tilde{p}$ appearing when running Algorithm 3.1 on p. By Lemma 3.15 we have

$$o(p) = \tilde{o}(k) = 1 + 384 \sum_{t=1}^{k-1} \tilde{o}(t)$$

with $\tilde{o}(1) = 1$. Analogously we define $\tilde{s} : \{1, \dots, k-1\} \to \mathbb{N}$ by $\tilde{s}(j) := s(q)$ where $q \in M_j$. Then by doing the same induction steps as Lemma 3.15 one can prove that

$$s(p) = \tilde{s}(k) = 384 \sum_{t=1}^{k-1} \tilde{s}(t)$$

with $\tilde{s}(1) = 1$. Thus we obtain $o(p) = 385^{k-1}$ and $s(p) = 385^{k-1} - 385^{k-2}$ for $k \geqslant 2$. Therefore

$$N_1(p) = o(p) + s(p) = \begin{cases} 2 \cdot 385^{k-1} - 385^{k-2} & \text{if } k \geqslant 2 \\ 2 & \text{if } k = 1 \end{cases}.$$

For Algorithm 3.2, since only one S_X operation and k LT operations happen, we have $N_2(p) = 1 + k$. □

3.5 Examples

There are many identities that fit in this class, from which we list the following examples. For the examples below we denote $\theta_j^{(k)} := \theta_j^{(k)}(0, q)$.

Example 3.9. *The classical Jacobi's identity*

$$\theta_2^4 - \theta_3^4 + \theta_4^4 = 0,$$

which in Ramanujan's notation is written as

$$\varphi^4(q) \equiv \varphi^4(-q) + 16q\psi^4(q^2),$$

where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ *and* $\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$. *Because we can write* $\theta_3 = \varphi(q)$, $\theta_4 = \psi(-q)$ *and* $\theta_2 = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} = 2\sum_{n=0}^{\infty} q^{n^2+n+1/4} = 2q^{1/4}\psi(q^2)$.

Example 3.10. *[22, p. 22]*

$$\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0.$$

Example 3.11. *[22, p. 22]*

$$\frac{\theta_4''}{\theta_4} - \frac{\theta_3''}{\theta_3} - \theta_2^4 = 0,$$

Example 3.12. *[24, 93. 81]*

$$\frac{\theta_1}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0,$$

Example 3.13. *[24, 93. 7]*

$$\frac{\theta_\alpha^{(5)}}{\theta_1'} - 3\left(\frac{\theta_\alpha''}{\theta_\alpha}\right)^2 + 2\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\beta''}{\theta_\beta}\right)\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\gamma''}{\theta_\gamma}\right) = 0,$$

where $\alpha = 2, 3, 4$.

For Examples 3.10 – 3.13 we clear the denominators and turn the left hand side to an element in R_Θ.

Example 3.14. *The famous Eisenstein identity*

$$Q^3 - R^2 = 1728q^2\prod_{n=1}^{\infty}(1-q^{2n})^{24}, \tag{A}$$

where

$$Q := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}} \quad \textit{and} \quad R := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}}.$$

Note. In order to apply our algorithm, we need to rewrite both sides of relation (A) into quotients of $\theta_j^{(k)}$. Here $Q = \frac{\pi^4 E_4}{2\xi(4)}$ and $R = \frac{\pi^6 E_6}{2\xi(6)}$, [6] where the $\xi(s)$ is the Riemann Zeta function and the $E_k := E_k(\pi, \pi\tau)$ is the Eisenstein series defined in Chapter 2. By equation $(*)$ in Proposition 4.1, we have $g_2 = -4(e_2e_3 + e_3e_1 + e_1e_2)$ and $g_3 = 4e_1e_2e_3$, where $g_2 = 60E_4$, $g_3 = 140E_6$ and $e_1 := \frac{1}{3}(\theta_3(0,q)^4 + \theta_4(0,q)^4)$, $e_2 := -\frac{1}{3}(\theta_2(0,q)^4 + \theta_3(0,q)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0,q)^4 - \theta_4(0,q)^4)$. Hence

[6] See p. 174 of [1].

we can write Q and R in terms of quotients of $\theta_j^{(k)}$. In addition, the right hand side of (A) is equal to $27/4\theta_1'^8$. We plug these into identity (A), simplify and get

$$54(\theta_1')^8 - (\theta_2^8+\theta_3^8+\theta_4^8)^3 + 2(-3\theta_2^8\theta_3^4 - 3\theta_2^8\theta_4^4 + \theta_3^{12} + \theta_4^{12})^2 = 0. \tag{B}$$

Thus to prove (A), it is equivalent to proving (B), which fits in our algorithm. Another way to present Q and R in terms of quotients of $\theta_j^{(k)}$ is what we did for Example 4.5, where we can obtain different presentations than above. Nevertheless, Algorithm 3.2 can verify that those different presentations are in fact identically equal to the above rewriting for Q and R.

Example 3.15. *[12, p. 54] Let Q and R be the same as in Example 3.14. Let $z := \varphi^2(q)$ and $x := 16q\frac{\psi^4(q^2)}{\varphi^4(q)}$, where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ and $\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$. Then*

$$Q = z^4(1+14x+x^2)$$

and

$$R = z^6(1+x)(1-34x+x^2).$$

Since Q, R, $\varphi(q)$ and $\psi(q)$ can be written in forms of theta functions as in Examples 3.9 and 3.14, these two identities also fits in our algorithm.

Chapter 4

Proving Identities among Powers of $\theta_j^{(k)}(0|\tau)$ & $\theta_\ell(z|\tau)$ (Class II)

This chapter generalizes the algorithmic approach of Chapter 3 to the direction that deals with the argument $z \in \mathbb{C}$. The main content of this chapter is from our paper [32].

Recall that by $R_\Theta := \mathbb{K}[\Theta]$ we define the $\mathbb{K}$-algebra generated by

$$\Theta := \left\{\theta_1^{(2k+1)} : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)} : k \in \mathbb{N} \text{ and } j = 2,3,4\right\},$$

where $\mathbb{K} \subseteq \mathbb{C}$ is an effectively computable field which contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.), and where

$$\theta_j^{(k)} := \theta_j^{(k)}(0|\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\bigg|_{z=0}, \; k \in \mathbb{N}.$$

In Chapter 3 we have presented an algorithm to do zero-recognition for every $f \in R_\Theta$. Now we extend the function space R_Θ to

$$\hat{R}_\Theta := R_\Theta[\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)],$$

by which we define the R_Θ-algebra generated by $\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau)$ and $\theta_4(z|\tau)$. In this chapter, we solve the following problem algorithmically:

Problem 4.1: Given $f \in \hat{R}_\Theta$, decide whether $f = 0$.

In this chapter we should use the abbreviation

$$\theta_j(z) := \theta_j(z|\tau),\ j = 1,2,3,4.$$

The framework used to solve this problem is the theory of elliptic functions and modular forms. In particular, we have to use an essential tool, which is Algorithm 3.2 from Chapter 3. As a result, we provide Algorithm 4.1 for solving Problem 4.1.

Example 4.1. *Our algorithm will be used to prove* [1]

$$c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0,$$

where

$$c_1 := -8\theta_2^5\theta_3^2\theta_4^3 - 2\theta_2\theta_3^6\theta_4^3 - 2\theta_2\theta_3^2\theta_4^7 - 16\theta_3^2\theta_4^3\theta_2'' + 16\theta_2\theta_3^2\theta_4^2\theta_4'',$$

$$c_2 := 7\theta_2^5\theta_3^4\theta_4 + \theta_2\theta_3^8\theta_4 + \theta_2\theta_3^4\theta_4^5 + 8\theta_3^4\theta_4\theta_2'' - 8\theta_2\theta_3^4\theta_4'',$$

$$c_3 := \theta_2^5\theta_4^5 + \theta_2\theta_3^4\theta_4^5 + \theta_2\theta_4^9 + 8\theta_4^5\theta_2'' - 8\theta_2\theta_4^4\theta_4'',$$

and

$$c_4 := -6\theta_2^5\theta_3^2\theta_4^3.$$

However, we observe that in the literature most identities fitting into Problem 4.1 are also in a smaller class, in which the coefficient algebra R_Θ is replaced by a subalgebra of R_Θ:

$$\mathbb{K}[\widetilde{\Theta}]_h := \{p(\theta_2(0),\theta_3(0),\theta_4(0)) : p \in \mathbb{K}[x,y,z] \text{ homogeneous}\},$$

and we define

$$\hat{H}_{\widetilde{\Theta}} := \mathbb{K}[\widetilde{\Theta}]_h[\theta_1(z|\tau),\theta_2(z|\tau),\theta_3(z|\tau),\theta_4(z|\tau)].$$

Restricting $\hat{R}_\Theta$ to this subalgebra, we provide Algorithm 4.2 to solve the following problem algorithmically without invoking Algorithm 3.2. Algorithm 4.2 is faster than Algorithm 4.1 in our experiments. We will give some brief arguments concerning the speed comparison in Chapter 6. Moreover, working with this restricted class, we also found some classical mathematical insights, such as Proposition 4.2 and Lemma 4.2.

Problem 4.2: Given $g \in \hat{H}_{\widetilde{\Theta}}$, decide whether $g = 0$.

[1] See Chapter 6.

Example 4.2. *[14, 20.7.1] Our algorithm will be used to prove* [2]

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0.$$

This chapter is organized as follows. In Section 4.1 we present a theorem to decompose any $f(z|\tau) \in \hat{R}_{\Theta}$ into the set of quasi-elliptic components of $f(z|\tau)$, and prove that $f(z|\tau) \equiv 0$ if and only if its quasi-elliptic components are all equal to zero. In Section 4.2 we give an Algorithm to decide if a quasi-elliptic component of any function in $\hat{R}_{\Theta}$ is equal to zero or not, thus we achieve the goal to prove or disprove $f(z|\tau) \equiv 0$. In Section 4.3 we derive a theorem connecting the Weierstrass elliptic function and the theta functions in a new way, which plays an important role for solving Problem 4.2. Working in the restricted space $\hat{H}_{\tilde{\Theta}}$, in Section 4.4 we obtain a critical lemma about the finite-orbit weight. In Section 4.5 we give an Algorithm to decide if any function in $\hat{H}_{\tilde{\Theta}}$ is equal to zero or not, thus we achieve the goal of Problem 4.2; i.e., to prove or disprove $g(z|\tau) \equiv 0$.

Convention. (1) Given $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4$, we define

$$\theta^{\alpha}(z) := \theta^{\alpha}(z|\tau) := \theta_1(z|\tau)^{\alpha_1}\theta_2(z|\tau)^{\alpha_2}\theta_3(z|\tau)^{\alpha_3}\theta_4(z|\tau)^{\alpha_4} = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}.$$

(2) For two sets A and B, we use B^A to present the set of functions $\{f : A \to B\}$.

(3) For any $\alpha \in \mathbb{Z}^n$ we assume that $\alpha = (\alpha_1, \dots, \alpha_n)$ and define $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

4.1 Quasi-Elliptic Decomposition of $f^{\Psi} \in \hat{R}_{\Theta}$

For a given $f \in \hat{R}_{\Theta}$, similarly to Chapter 3 we first decompose it into certain "smaller components". However, here we do the decomposition with regard to z.

Definition 4.1. *Given $M \subseteq \mathbb{N}^4$ finite, define*

$$\begin{aligned} f_M : {R_{\Theta}}^M &\to \hat{R}_{\Theta} \\ \psi &\mapsto f_M(\psi) =: f_M^{\psi} \end{aligned}$$

[2] See Example 3.1 and Section 8.2.

where

$$f_M^{\psi}(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^{\alpha}(z|\tau).$$

Notation. If M is clear from the context, we write f instead of f_M, and f^{ψ} instead of f_M^{ψ}.

Sometimes, for convenience, we use $f^{\psi}(z)$ to present $f^{\psi}(z|\tau)$. As an illustration of Definition 4.1, let us look at the identity in Example 4.1. Here we have

$$M = \{(0,0,2,2),(0,0,0,4),(0,0,4,0),(2,2,0,0)\}, \tag{4.1}$$

$\psi((0,0,2,2)) = c_1$, $\psi((0,0,0,4)) = c_2$, $\psi((0,0,4,0)) = c_3$, $\psi((2,2,0,0)) = c_4$ and

$$f^{\psi}(z) = c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2.$$

Definition 4.2. *Given $M \subseteq \mathbb{N}^4$ finite, $a,b \in \{1,2\}$, and $t \in \mathbb{N}$, let*

$$X_{t,a,b}(M) := \{\alpha \in M : |\alpha| = t,\, \alpha_1+\alpha_4 \equiv a+1 (\mathrm{mod}\, 2),\, \alpha_1+\alpha_2 \equiv b+1 (\mathrm{mod}\, 2)\},$$

and define the following partition of M:

$$X(M) := \{X_{t,a,b}(M) \neq \varnothing : t \in \mathbb{N} \text{ and } a,b \in \{1,2\}\}.$$

Example 4.3. *(i) Let M be the the same as in expression* (4.1)*. Then*

$$X(M) = \{X_{4,1,1}(M)\} = \{M\}.$$

(ii) Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$. Then

$$\begin{aligned} X(M) &= \{X_{2,1,1}(M), X_{4,1,1}(M), X_{4,2,1}(M)\} \\ &= \{\{(0,0,2,0),(0,0,0,2)\},\{(2,0,2,0)\},\{(2,1,1,0)\}\}. \end{aligned}$$

We shall note that for a given $M \subseteq \mathbb{Z}^4$ finite, $X(M)$ is unique. One can check that if $X(M) = \{Y_1,\dots,Y_n\}$ then $Y_i \cap Y_j = \varnothing$ when $i \neq j$ and the disjoint union

$$Y_1 \dot\cup \cdots \dot\cup Y_n = M.$$

Definition 4.3 (set of quasi-elliptic components of ψ). *Given $M \subseteq \mathbb{N}^4$ finite, let $X(M) = \{M_1, \ldots, M_n\}$. For $\psi \in R_\Theta^M$ we define*

$$Q(\psi) := \{\psi_1, \ldots, \psi_n\}$$

where $\psi_j := \psi|_{M_j}$.

Definition 4.4. *[set of quasi-elliptic components of f^Ψ] Given $\psi \in R_\theta{}^M$, we define*

$$Q(f^\Psi) := \{f^{\Psi_1}, \ldots, f^{\Psi_n}\},$$

where $\{\psi_1, \ldots, \psi_n\} = Q(\psi)$.

Example 4.4. *Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$ as in Example 4.3 (ii) and*

$$f^\Psi = f_M^\Psi = c_1\theta_3(z)^2 + c_2\theta_4(z)^2 + c_3\theta_1(z)^2\theta_3(z)^2 + c_4\theta_1(z)^2\theta_2(z)\theta_3(z)$$

with the $c_j \in \mathbb{K}[\Theta]$. Then

$$Q(f^\Psi) = \{f_1, f_2, f_3\},$$

where $f_1 = c_1\theta_3(z)^2 + c_2\theta_4(z)^2$, $f_2 = c_3\theta_1(z)^2\theta_3(z)^2$ and $f_3 = c_4\theta_1(z)^2\theta_2(z)\theta_3(z)$.

Corollary 4.1. *If g^Ψ is a quasi-elliptic component of some f^Ψ, then it is the quasi-elliptic component of itself.*

Proof. The proof can be done by directly following Definitions 4.2 and 4.4. □

Definition 4.5. *If f^Ψ is the quasi-elliptic component of itself, we say that f^Ψ is quasi-elliptic.*

Theorem 4.1. *Let $f^\Psi = f_M^\Psi$ and $Q(f^\Psi) = \{f^{\Psi_1}, \ldots, f^{\Psi_n}\}$, then*

$$f^\Psi(z|\tau) \equiv 0 \text{ if and only if } f^{\Psi_j}(z|\tau) \equiv 0 \text{ for all } j \in \{1\ldots n\}.$$

Proof. " $\Longleftarrow$ " is immediate. We prove " $\Longrightarrow$ ". Write $f^\Psi(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z|\tau)$ and write M as a union of disjoint non-empty sets $X_1(M) \cup X_2(M) \cup \cdots \cup X_m(M)$ where for $t \in \{1\ldots m\}$,

$$X_t(M) := \{\alpha \in M : |\alpha| = d_t\}$$

with $d_1, \ldots, d_m$ pairwise distinct. In this proof we use $f(z)$ to present $f^\Psi(z|\tau)$. We can write $f(z) = \sum_{t=1}^{m} f_t(z)$ where $f_t(z) := \sum_{\alpha\in X_t(M)} \psi(\alpha)\theta^\alpha(z)$.

Next we write

$$0 \equiv f(z) \equiv \sum_{t=1}^{m} \left(f_{t,1}(z) + f_{t,2}(z)\right),$$

where

$$f_{t,1}(z) := \sum_{\alpha \in X_{t,1}(M)} \psi(\alpha)(\tau)\theta^{\alpha}(z) \quad \text{and} \quad f_{t,2}(z) := \sum_{\alpha \in X_{t,2}(M)} \psi(\alpha)(\tau)\theta^{\alpha}(z)$$

with $X_{t,1}(M) := \{\alpha \in X_t(M) : \alpha_1 + \alpha_4 \,\text{even}\}$ and $X_{t,2}(M) := \{\alpha \in X_t(M) : \alpha_1 + \alpha_4 \,\text{odd}\}$.

By employing Table 2.1, we obtain for $t \in \{1, \ldots, m\}$,

$$f_{t,1}(z + \pi\tau) \equiv N^{d_t} f_{t,1}(z) \quad \text{and} \quad f_{t,2}(z + \pi\tau) \equiv -N^{d_t} f_{t,1}(z).$$

Then for $k \in \{0, 1, \ldots, 2m-1\}$,

$$f_t(z + k\pi\tau) \equiv f_{t,1}(z + k\pi\tau) + f_{t,2}(z + k\pi) \equiv (N^{d_t})^k f_{t,1}(z) + (-N^{d_t})^k f_{t,2}(z).$$

Thus we have,

$$0 \equiv f(z) \equiv f(z + k\pi\tau) \equiv \sum_{t=1}^{m} (N^{d_t})^k f_{t,1}(z) + (-N^{d_t})^k f_{t,2}(z),$$

which can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ N^{d_1} & -N^{d_1} & \cdots & N^{d_m} & -N^{d_m} \\ (N^{d_1})^2 & (-N^{d_1})^2 & \cdots & (N^{d_m})^2 & (-N^{d_m})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (N^{d_1})^{2m-1} & (-N^{d_1})^{2m-1} & \cdots & (N^{d_m})^{2m-1} & (-N^{d_m})^{2m-1} \end{pmatrix} \begin{pmatrix} f_{1,1} \\ f_{1,2} \\ \vdots \\ f_{m,1} \\ f_{m,2} \end{pmatrix} = 0 \tag{4.2}$$

Since $N \neq 0$, the determinant of this Vandermonde matrix is nonzero. Therefore we can multiply both sides of (4.2) by the inverse of the Vandermonde matrix and obtain $f_{t,i} = 0$ for all $t \in \{1, \ldots, m\}$ and $i \in \{1, 2\}$.

Next we write

$$0 \equiv f_{t,1}(z) \equiv f_{t,1,1}(z) + f_{t,1,2}(z) \quad \text{and} \quad 0 \equiv f_{t,2}(z) \equiv f_{t,2,1}(z) + f_{t,2,2}(z) \tag{4.3}$$

where for $a \in \{1,2\}$

$$f_{t,a,1}(z) := \sum_{\alpha \in X_{t,a,1}(M)} \psi(\alpha)(\tau)\theta^\alpha(z) \quad \text{and} \quad f_{t,a,2}(z) := \sum_{\alpha \in X_{t,a,2}(M)} \psi(\alpha)(\tau)\theta^\alpha(z)$$

with $X_{t,a,1}(M) := \{\alpha \in X_{t,a}(M) : \alpha_1 + \alpha_2 \,\text{even}\}$ and $X_{t,a,2}(M) := \{\alpha \in X_{t,a}(M) : \alpha_1 + \alpha_2 \,\text{odd}\}$. Again by using Table 2.1 on the terms appearing in $f_{t,1}(z)$ and $f_{t,2}(z)$, we obtain for $a \in \{1,2\}$,

$$0 \equiv f_{t,a}(z) \equiv f_{t,a}(z+\pi) \equiv f_{t,a,1}(z+\pi) + f_{t,a,2}(z+\pi) \equiv f_{t,a,1}(z) - f_{t,a,2}(z).$$

This together with (4.3) implies $f_{t,a,1} = f_{t,a,2} = 0$ for all $t \in \{1,\ldots,m\}$ and $a \in \{1,2\}$.

In view of Definition 4.2 choose j such that $M_j = X_{t,a,b}(M)$. Then

$$f^{\Psi_j}(z) \equiv f_{t,a,b}(z) \equiv \sum_{\alpha \in M_j} \psi(\alpha)(\tau)\theta^\alpha(z) \equiv 0$$

for all $j \in \{1,\ldots,n\}$ where $n = |X(M)|$. □

4.2 Zero-Recognition for $f^\Psi \in \hat{R}_\Theta$

In this section we use elliptic function properties to decide whether any given $f^\Psi \in \hat{R}_\Theta$ is identically zero.

Theorem 4.2. *Let* $\sum_{\alpha \in Y} \psi(\alpha)\theta^\alpha(z|\tau) \in \hat{R}_\Theta$ *be quasi-elliptic. Then for all* $\alpha, \beta \in Y$, $\frac{\theta^\alpha(z|\tau)}{\theta^\beta(z|\tau)}$ *is elliptic with respect to z.*

Proof. Suppose that $\sum_{\alpha \in Y} \psi(\alpha)\theta^\alpha(z|\tau)$ is a quasi-elliptic component of some $f_M^\Psi(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^\alpha(z|\tau)$. By assumption Y is equal to some element in $X(M)$. Consequently $Y = X_{t,a,b}(M)$ for some $t \in \{1,\ldots,m\}$ and $a,b \in \{1,2\}$. Take an arbitrary $\alpha \in X_{t,a,b}(M)$. By Table 2.1 we have $\theta^\alpha(z+\pi\tau) \equiv (-1)^{a+1}N^{d_t}\theta^\alpha(z)$ and $\theta^\alpha(z+\pi) \equiv (-1)^{b+1}\theta^\alpha(z)$, which implies that for any $\alpha, \beta \in X_{t,a,b}(M)$,

$$\frac{\theta^\alpha(z+\pi\tau)}{\theta^\beta(z+\pi\tau)} \equiv \frac{\theta^\alpha(z)}{\theta^\beta(z)} \quad \text{and} \quad \frac{\theta^\alpha(z+\tau)}{\theta^\beta(z+\tau)} \equiv \frac{\theta^\alpha(z)}{\theta^\beta(z)}.$$

Therefore $\frac{\theta^\alpha(z)}{\theta^\beta(z)}$ is elliptic. □

Definition 4.6. *Given $M \in \mathbb{N}^4$ finite, we define*

$$\min(M) := \{(\beta_1,\beta_2,\beta_3,\beta_4) \in M : \beta_1 = \min\{\alpha_1 : (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in M\}\}.$$

The following theorem is a key of doing zero-recognition in $\hat{R}_\Theta$.

Theorem 4.3. *Let $f^\Psi(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z)$ be quasi-elliptic. For any $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \min(Y)$, let $g_\beta(z|\tau) := \frac{f^{\Psi|_Y}(z|\tau)}{\theta^\beta(z)}$. Then*

(1) the series expansion of $g_\beta(z|\tau)$ is of the form

$$g_\beta(z|\tau) = \sum_{j=0}^{\infty} d_j(\tau) z^j,$$

with $d_j(\tau) \in \mathbb{K}(\Theta)$;[3] and

(2) if $d_j(\tau) \equiv 0$ for $j = 0,\ldots,\beta_2+\beta_3+\beta_4$ then $f^\Psi(z|\tau) \equiv 0$.

Proof. From Definition 2.7 we know that for fixed $\tau \in \mathbb{H}$ the $\theta_j(z|\tau)$ ($j = 1,\ldots,4$) are analytic functions on the whole complex plane with respect to z, and for fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ ($j = 1,\ldots,4$) are analytic functions of τ for all $\tau \in \mathbb{H}$. By Proposition 2.2, only $\theta_1(z)$ has a zero at $z = 0$. Since $\theta_1(z)^{\beta_1}$ in the denominator of $g_\beta(z|\tau)$ cancels against each $\theta^\alpha(z)$ by the choice of β, we deduce that $g_\beta(z|\tau)$ is analytic at $z = 0$. Hence we have a Taylor expansion around $z = 0$.

By Theorem 4.2, $g_\beta(z|\tau)$ is an elliptic function with respect to z. We observe that the only possible poles of $g_\beta(z|\tau)$ in $P(\pi,\pi\tau)$ are $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{\pi\tau}{2}$ and $\frac{\pi\tau}{2}$. Thus $g_\beta(z|\tau)$ has at most $\beta_2+\beta_3+\beta_4$ poles including multiplicities in $P(\pi,\pi\tau)$. If $a_j(\tau) \equiv 0$ for $j = 0,\ldots,\beta_2+\beta_3+\beta_4$, then $g_\beta(z|\tau)$ has a zero at $z = 0$ with multiplicity at least $\beta_2+\beta_3+\beta_4+1$, which means the number of zeros of $g_\beta(z|\tau)$ in $P(\pi,\pi\tau)$ must be at least $\beta_2+\beta_3+\beta_4+1$. By Theorem 2.2, $g_\beta(z|\tau)$ must be zero, so $f^\Psi(z|\tau) \equiv 0$. □

We also obtain the algorithmic content of Theorem 4.3.

[3] $\mathbb{K}(\Theta)$ denotes the quotient field of $\mathbb{K}[\Theta]$ consisting of all quotients $P[\Theta]/Q[\Theta]$ with $P[\Theta], Q[\Theta] \in \mathbb{K}[\Theta]$.

Algorithm 4.1. *Given $f^{\Psi} \in \hat{R}_{\Theta}$ with $f^{\Psi} = f^{\Psi}_{M}$, we have the following algorithm to decide whether $f^{\Psi} = 0$.*

Input: $f^{\Psi} \in \hat{R}_{\Theta}$.

Output: True if $f^{\Psi} = 0$; False if $f^{\Psi} \neq 0$.

Write $f^{\Psi}(z|\tau) = \sum_{j=1}^{n} f^{\Psi_j}(z|\tau)$ where the $f^{\Psi_j}(z|\tau)$ are the quasi-elliptic components of $f^{\Psi}(z|\tau)$.

Set $j := 1$. While $j \leqslant n$ do

  Choose $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in M_j$ such that $\beta_1 = \min\{\alpha_1 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in M_j\}$;

  Let $g(z|\tau) := \frac{f^{\Psi_j}(z|\tau)}{\theta^{\beta}(z|\tau)}$;

  write $g(z|\tau) = \sum_{k=0}^{\infty} d_k(\tau) z^k$;

  Set $k := 0$. While $k \leqslant \beta_2 + \beta_3 + \beta_4$ do

    if $d_k(\tau) \equiv 0$;

      $k++$;

    otherwise return False;

   end do;

 $j++$;

 end do;

 return True;

Note. In Algorithm 4.1, we use Algorithm 3.2 of Chapter 3 to check whether $d_k(\tau) \equiv 0$.

Theorem 4.4. *Algorithm 4.1 is correct.*

Proof. According to Definition 4.2 we can always write any $f^\Psi \in \hat{R}_\Theta$ into a sum of quasi-elliptic components of f^Ψ for some finite set $M \subseteq \mathbb{N}^4$.

Assume $f^\Psi = 0$. Then by Theorem 4.1, $f^{\Psi_j} = 0$ for all $j \in \{1, \dots, n\}$. Hence the corresponding $g = 0$, which implies $d_k(\tau) \equiv 0$ for all $k \in \mathbb{N}$. Therefore Algorithm 4.1 returns True.

Assume $f^\Psi \neq 0$. By Theorem 4.1, there exists a $t \in \{1, \dots, n\}$ such that $f^{\Psi_t} \neq 0$. Then the corresponding g is nonzero. If g is a constant, then $d_0 \neq 0$ and Algorithm 4.1 returns False. Assume g is not a constant. By Theorem 4.2, $g(z|\tau)$ is an elliptic function. Since $g(z|\tau)$ has at most $\beta_2 + \beta_3 + \beta_4$ poles in $P(\pi, \pi\tau)$, by Theorem 2.2 we deduce that $g(z|\tau)$ has at most $\beta_2 + \beta_3 + \beta_4$ zeros in $P(\pi, \pi\tau)$. This means $d_0(\tau), \dots, d_{\beta_2+\beta_3+\beta_4}(\tau)$ cannot all be zero. Thus Algorithm 4.1 returns False. □

Example 4.1 (continued). Prove

$$f^\Psi(z) := c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0,$$

where the c_j are chosen as in Example 4.1.

Proof. One can check by Definition 4.2 that $f(z)$ is the quasi-elliptic component of itself, so in this case $f^\Psi(z) = f^{\Psi_1}(z)$. Following Algorithm 4.1, let

$$g(z) := \frac{f^\Psi(z)}{c_3\theta_4(z)^4} = c_1\frac{\theta_3(z)^2}{\theta_4(z)^2} + c_2 + c_3\frac{\theta_3(z)^4}{\theta_4(z)^4} + c_4\frac{\theta_1(z)^2\theta_2(z)^2}{\theta_4(z)^4}.$$

Then

$$g(z) = \sum_{k=0}^{\infty} d_k(\tau)z^k$$

with $d_0(\tau) = c_4\theta_1^2\theta_2^2 + c_3\theta_3^4 + c_1\theta_3^2\theta_4^2 + c_2\theta_4^4$ and $d_k(\tau)$ for $k = 1, \dots, 4$ of a form similar to $d_0(\tau)$. By Algorithm 3.2 we can prove that $d_0 = \cdots = d_4 = 0$. Thus by Algorithm 4.1 we have $g = 0$. □

Note. This identity contains only one quasi-elliptic component, and in general the identities we found in the literature are stated in their simplest form. Consequently, to produce an identity with more than one quasi-elliptic component, we need to take one identity containing one quasi-elliptic component (multiplied by an element of $\hat{R}_\Theta$) and add to it another identity containing one quasi-elliptic component (multiplied by an element of $\hat{R}_\Theta$).

4.3 Theta Functions and The Weierstrass ℘ Function

We are going to derive some connections between theta functions and the ℘ function. By applying them, we will obtain a faster algorithm on the restricted class $\hat{H}_{\tilde{\Theta}}$.

Definition 4.7 (elliptic theta-quotients)**.**

$$J := \{\theta^{\alpha}(z) : \alpha \in \mathbb{Z}^4 \textit{ such that } \theta^{\alpha}(z) \textit{ is elliptic}\}.$$

Lemma 4.1. *J forms a multiplicative group which is generated by*

$$j_1 := \frac{\theta_2(z)^2}{\theta_1(z)^2},\ j_2 := \frac{\theta_3(z)^2}{\theta_1(z)^2} \textit{ and } j_3 := \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3}.$$

In particular, for a given $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \in J$, the presentation in terms of the generators is

$$p = j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4}.$$

Proof. With the help of Table 2.1, one can verify that $j_1, j_2, j_3 \in J$ and that J is a multiplicative group. Suppose $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \in J$, then $p(z) = p(z+\pi\tau)$ and $p(z) = p(z+\pi)$, because every element in J is elliptic. On the other hand, by Table 2.1 we have

$$p(z+\pi\tau) = (-1)^{\alpha_1+\alpha_4} N^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} p(z) \text{ and } p(z+\pi) = (-1)^{\alpha_1+\alpha_2} p(z).$$

Hence $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0$, $\alpha_1+\alpha_4$ is even and $\alpha_1+\alpha_2$ is even. This implies that if α_2 is even then α_3 and α_4 must be even, and if α_2 is odd then α_3 and α_4 are also odd. Therefore $\frac{\alpha_2-\alpha_4}{2}$, $\frac{\alpha_3-\alpha_4}{2}$ and α_4 are all integers. Moreover,

$$\begin{aligned}
j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4} &= \theta_1(z)^{-\alpha_2+\alpha_4-\alpha_3+\alpha_4-3\alpha_4}\theta_2(z)^{\alpha_2-\alpha_4+\alpha_4}\theta_3(z)^{\alpha_3-\alpha_4+\alpha_4}\theta_4(z)^{\alpha_4} \\
&= \theta_1(z)^{-\alpha_2-\alpha_3-\alpha_4}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \\
&= \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \\
&= p.
\end{aligned}$$

□

Recall that in this chapter we denote $\theta_j^{(k)} := \theta_j^{(k)}(0|\tau)$.

Theorem 4.5. *The generators j_1, j_2 and j_3 of J satisfy*

$$j_1 = \frac{\theta_2^2}{\theta_1^2}(\wp(z)-e_1),^2$$
$$j_2 = \frac{\theta_3^2}{\theta_1^2}(\wp(z)-e_3),^2$$
$$j_3 = -\frac{1}{2{\theta'_1}^2}\wp'(z),$$

where $\wp(z) := \wp(z;\pi,\pi\tau)$ is the Weierstrass elliptic function with periods π and $\pi\tau$, $e_1 := \frac{1}{3}(\theta_3^4+\theta_4^4)$ and $e_3 := \frac{1}{3}(\theta_2^4-\theta_4^4)$.

Proof. Since $\frac{\theta_2(z)^2}{\theta_1(z)^2}$ is elliptic with a double pole at $z=0$ and is an even function, we can expand it as

$$\begin{aligned}\frac{\theta_2(z)^2}{\theta_1(z)^2} &= z^{-2}\left(\frac{\theta_2^2}{{\theta'_1}^2}+\left(-\frac{\theta_2^2\theta_1^{(3)}}{3{\theta'_1}^3}+\frac{\theta_2{\theta_2}''}{{\theta'_1}^2}\right)z^2+\cdots\right)\\ &= \frac{\theta_2^2}{{\theta'_1}^2}z^{-2}+\left(-\frac{\theta_2^2\theta_1^{(3)}}{3{\theta'_1}^3}+\frac{\theta_2{\theta_2}''}{{\theta'_1}^2}\right)+\cdots.\end{aligned}$$

By Proposition 2.1, $\wp(z)$ has a double pole at $z=0$. Thus $\wp(z)-\frac{{\theta'_1}^2}{\theta_2^2}\frac{\theta_2(z)^2}{\theta_1(z)^2}$ has no pole, which together with Theorem 2.1 implies that it has to be a constant function, i.e.,

$$\wp(z)-\frac{{\theta'_1}^2}{\theta_2^2}\frac{\theta_2(z)^2}{\theta_1(z)^2}=\frac{\theta_1^{(3)}}{3\theta'_1}-\frac{{\theta_2}''}{\theta_2}=\frac{1}{3}(\theta_3^4+\theta_4^4)=e_1,$$

where the second last equality is proven using Algorithm 3.2. Thus

$$\frac{\theta_2(z)^2}{\theta_1(z)^2}=\frac{\theta_2^2}{{\theta'_1}^2}(\wp(z)-e_1). \tag{4.4}$$

Analogously, we have

$$\wp(z)-\frac{{\theta'_1}^2}{\theta_3^2}\frac{\theta_3(z)^2}{\theta_1(z)^2}=\frac{\theta_1^{(3)}}{3\theta'_1}-\frac{{\theta_3}''}{\theta_3}=\frac{1}{3}(\theta_2^4-\theta_4^4)=e_2,$$

[2]See p.102 of [21].

where the second last equality is proven using Algorithm 3.2, and thus

$$\frac{\theta_3(z)^2}{\theta_1(z)^2} = \frac{\theta_3^2}{\theta_1^2}(\wp(z) - e_3). \tag{4.5}$$

One can verify that $j_3 = \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} \in J$ is an odd elliptic function, and we have the series expansion

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} = a_{-3}z^{-3} + a_{-1}z^{-1} + a_1 z + \cdots,$$

where

$$a_{-3} := \frac{\theta_2\theta_3\theta_4}{\theta_1'^3},$$

$$a_{-1} = \frac{1}{2\theta_1'^5}(\theta_3\theta_4\theta_1'^2\theta_2'' + \theta_2\theta_4\theta_1'^2\theta_3'' + \theta_2\theta_3\theta_1'^2\theta_4'' + \theta_2\theta_3\theta_1'\theta_1^{(3)}),$$

and a_1 is also in $\mathbb{K}(\Theta)$ but irrelevant to this proof. We have checked with Algorithm 3.2 that a_{-1} is zero.

By the series expression of $\wp'(z)$ in (2.1),

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} + \frac{1}{2}\frac{\theta_2\theta_3\theta_4}{\theta_1'^3}\wp'(z) \tag{4.6}$$

has no poles, which implies by Theorem 2.1 that (4.6) is constant. We take $z = 0$ and it turns out that the expression (4.6) is zero. Then

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} = -\frac{1}{2}\frac{\theta_2\theta_3\theta_4}{\theta_1'^3}\wp'(z) = -\frac{1}{2\theta_1'^2}\wp'(z),$$

where the last equality follows from the famous identity

$$\theta_1' \equiv \theta_2\theta_3\theta_4,$$

which can be also proven with Algorithm 3.2. □

Remark 4.1. *Replacing z by $\frac{\pi}{2}$ in* (4.4) *and using $\theta_2\left(\frac{\pi}{2}\right) = 0$, we obtain $\wp\left(\frac{\pi}{2}\right) = e_1$; substituting z by $\frac{\pi+\pi\tau}{2}$ in* (4.5) *and using $\theta_3\left(\frac{\pi+\pi\tau}{2}\right) = 0$ gives $\wp\left(\frac{\pi+\pi\tau}{2}\right) = e_3$. It can be verified that $\frac{\theta_3(z)^2}{\theta_1(z)^2}$ is also elliptic, and similarly we have*

$$\frac{\theta_4(z)^2}{\theta_1(z)^2} = \frac{\theta_4^2}{\theta_1^2}(\wp(z) - e_2),^3$$

where $e_2 := -\frac{1}{3}(\theta_2^4+\theta_3^4)$. *Moreover, by* $\theta_4(\frac{\pi\tau}{2})=0$ *we obtain* $\wp\left(\frac{\pi\tau}{2}\right)=e_2$.

The following proposition is a by-product of our reasoning, but will not be used to deduce the algorithm.

Proposition 4.1. *For any* $y\in J$, *there exist* $a,b,c\in\mathbb{Z}$, *such that*

$$y\in\left\{\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\cdot(\wp-e_1)^{\ell_1}(\wp-e_2)^{\ell_2}(\wp-e_3)^{\ell_3}d : \ell_i\in\mathbb{N}, d=1 \text{ or } d=-\wp'\right\}. \tag{4.7}$$

Proof. Suppose $y=\theta^\alpha(z)\in J$ with $\alpha\in\mathbb{Z}^4$. By Lemma 4.1 and Theorem 4.5 there exist integers a,b,c, such that

$$y=\left(-\frac{1}{2}\right)^c\cdot\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}p(z)$$

where $p(z):=(\wp(z)-e_1)^a(\wp(z)-e_3)^b\wp'(z)^c$. Let the set in (4.7) be denoted by G_1.

Case 1: $c>0$.

Assume $c=2n+1$ ($n\in\mathbb{N}$). By the classical differential equation $\wp'^2=4\wp^3-g_2\wp-g_3$, we get $\wp'^c=(4\wp^3-g_2\wp-g_3)^n\wp'$. Applying Remark 4.1 together with the fact that $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$ and $\frac{\pi+\pi\tau}{2}$ are the roots of $\wp'$, we can write

$$4\wp^3-g_2\wp-g_3=4(\wp-e_1)(\wp-e_2)(\wp-e_3). \tag{$*$}$$

Then

$$\begin{aligned}y&=(-\frac{1}{2})^{2n+1}\cdot\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}(\wp-e_1)^a(\wp-e_3)^b(4\wp^3-g_2\wp-g_3)^n\wp'\\&=-\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{2\theta_1'(0)^{2a+2b+2c}}(\wp-e_1)^{a+n}(\wp-e_2)^n(\wp-e_3)^{b+n}\wp'\quad\in G_1.\end{aligned}$$

Assume $c=2n$ ($n\in\mathbb{N}\backslash\{0\}$). We have

$$y=\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}(\wp-e_1)^{a+n}(\wp-e_2)^n(\wp-e_3)^{b+n}\quad\in G_1.$$

[3]See p.102 of [21].

Case 2: $c \leqslant 0$. We can write

$$\frac{1}{\wp'} = \frac{\wp'}{\wp'^2} = \frac{\wp'}{4\wp^3 - g_2\wp - g_3} = \frac{\wp'}{4(\wp-e_1)(\wp-e_2)(\wp-e_3)}.$$

If $c = -2n-1$ ($n \in \mathbb{N}$), we have

$$\begin{aligned}
\wp'^c &= \left(\frac{1}{\wp'}\right)^{2n+1} = \left(\frac{\wp'}{\wp'^2}\right)^{2n+1} = \left(\frac{\wp'}{4\wp^3 - g_2\wp - g_3}\right)^{2n+1} \\
&= \frac{\wp'^{2n}\wp'}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^{2n+1}} = \frac{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^n\wp'}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^{2n+1}} \\
&= \frac{\wp'}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^{n+1}}.
\end{aligned}$$

Then

$$\begin{aligned}
y &= (-\frac{1}{2})^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}} \cdot \frac{(\wp-e_1)^a(\wp-e_3)^b\wp'}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^{n+1}} \\
&= -\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{2\theta_1'(0)^{2a+2b+2c}} \cdot (\wp-e_1)^{a-n-1}(\wp-e_2)^{-n-1}(\wp-e_3)^{-n-1}\wp' \quad \in G_1.
\end{aligned}$$

If $c = -2n$ ($n \in \mathbb{N}$), we have

$$\wp'^c = \frac{1}{\wp'^{2n}} = \frac{1}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^n}.$$

Then

$$\begin{aligned}
y &= (-\frac{1}{2})^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}} \cdot \frac{(\wp-e_1)^a(\wp-e_3)^b}{(4(\wp-e_1)(\wp-e_2)(\wp-e_3))^n} \\
&= \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}} \cdot (\wp-e_1)^{a-n}(\wp-e_2)^{-n}(\wp-e_3)^{b-n} \quad \in G_1.
\end{aligned}$$

□

4.4 The Finite-Orbit Weight

This section will show the particularity of $\hat{H}_{\tilde{\Theta}}$, in terms of the finite-orbit weight, which will be used in the next section as a crucial property.

Definition 4.8. *Let* $M(\mathbb{H}) := \{g : g \text{ meromorphic on } \mathbb{H}\}$*. Define a group action*

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) \times M(\mathbb{H}) &\longrightarrow M(\mathbb{H}) \\ (\rho, g) &\mapsto g|_k\rho \end{aligned}$$

where $g|_k\rho(\tau) := (c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}\right)$ *for* $\rho := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\tau \in \mathbb{H}$*. For each* $k \in \mathbb{Z}$ *we define the k-orbit of g by* $G_k(g) := \{g|_k\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\}$*.*

Proposition 4.2. *For a nonzero* $g \in M(\mathbb{H})$ *and* $k \in \mathbb{Z}$*, if* $|G_k(g)|$ *is finite then k is unique with this property.*

Proof. Let k and t be integers such that $G_k(g)$ and $G_t(g)$ are both finite orbit sets. We need to prove that $k = t$. Let $s := k - t$. Take any $g|_t\rho \in G_t(g)$ with $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{aligned} g|_t\rho(\tau) &= (c\tau+d)^{-t} g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{s}(c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{s} \cdot g|_k\rho(\tau). \end{aligned}$$

Hence we can rewrite the set $G_t(g)$ as

$$\begin{aligned} G_t(g) &= \left\{ (c\tau+d)^{s} \cdot g|_k\rho : \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} \\ &= \left\{ (c\tau+d)^{s} \cdot g_{a,b,c,d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } g_{a,b,c,d} \in G_k(g) \right\}, \end{aligned}$$

where $g_{a,b,c,d} := g|_k\rho$ with $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Assume $s \neq 0$ and $G_k(g) = \{a_1, \ldots, a_n\}$, and define the map

$$\begin{aligned} \gamma \colon \mathrm{SL}_2(\mathbb{Z}) &\to G_k(g) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto g_{a,b,c,d}. \end{aligned}$$

Let $A_j := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : g_{a,b,c,d} = a_j \right\}$. By Definition 4.8, the map γ is surjective, thus $A_j \neq \varnothing$. Then we can write $\mathrm{SL}_2(\mathbb{Z}) = \bigcup_{j=1}^{n} A_j$ where $A_i \cap A_j = \varnothing$ if $i \neq j$. Let

$$B_j := \left\{ (c,d) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_j \right\}.$$

For every pair $(c,d) \in \mathbb{Z}^2$ with $\gcd(c,d) = 1$, there must exist some pairs $(a,b) \in \mathbb{Z}^2$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Hence there exists $r \in \{1, \ldots, n\}$ such that B_r is infinite; otherwise $\mathrm{SL}_2(\mathbb{Z}) \neq \bigcup_{j=1}^{n} A_j$. We also have

$$\left\{ (c\tau + d)^s a_r : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \subseteq \left\{ (c\tau + d)^s g_{a,b,c,d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = G_t(g),$$

which implies

$$N := \left| \left\{ (c\tau + d)^s a_r : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \right| \leqslant |G_t(g)|. \tag{4.8}$$

On the other hand

$$N = \left| \{ (c\tau + d)^s : (c,d) \in B_r \} \right|, \tag{4.9}$$

and the right hand side of (4.9) is equal to infinity because $c_1\tau + d_1 \neq c_2\tau + d_2$ when $(c_1,d_1) \neq (c_2,d_2)$, and because the set B_r is infinite. Thus N is equal to infinity, and by (4.8), $|G_t(g)| = \infty$, which contradicts the assumption that $G_t(g)$ is a finite orbit set. Therefore $s = 0$. □

Definition 4.9. *Given $g \in M(\mathbb{H})$ nonzero and $k \in \mathbb{Z}$ such that $|G_k(g)|$ is finite, we define the finite-orbit-weight of g by*

$$W(g) := k.$$

By using Definition 4.8 one can verify:

Proposition 4.3. *Given $g_1, \ldots, g_n \in M(\mathbb{H})$ with $W(g_j) = k_j$, then*
(1) $W(g_1 \cdots g_n) = k_1 + \cdots + k_n$,
(2) If $k_1 = \cdots = k_n = k$ and $g_1 + \cdots + g_n \neq 0$, then $W(g_1 + \cdots + g_n) = k$.

Note. According to Lemma 2.6, we have $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T\rangle$. Hence

$$G_k(g) = \{g|_k\rho : \rho \in \langle S, T\rangle\}.$$

Thus in our working frame, to compute $G_k(g)$, we compute $\{g|_k\rho : \rho \in \langle S, T\rangle\}$.

Lemma 4.2. *Let quasi-elliptic $f^\Psi(z|\tau) = \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z) \in \hat{H}_{\tilde{\Theta}}$ and $\beta \in \min(M)$. Suppose the series expansion of $\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)}$ around $z = 0$ is of the form $\sum_{n=0}^{\infty} d_n(\tau)z^n$ with $d_n(\tau) \in \mathbb{K}(\Theta)$. Then $W(d_n) = n$ when $d_n \neq 0$.*

Proof. By Theorem 4.3, $\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)}$ always has a Taylor expansion around $z = 0$. Since f^Ψ is a quasi-elliptic component of itself, by Theorem 4.2, $\frac{\theta^\alpha(z)}{\theta^\beta(z)}$ is elliptic for every $\alpha \in M$.

In view of $\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)} = \sum_{\alpha\in M} \frac{\psi(\alpha)\theta^\alpha(z)}{\psi(\beta)\theta^\beta(z)}$, we are going to show that the assertion is true for every $\frac{\psi(\alpha)\theta^\alpha(z)}{\psi(\beta)\theta^\beta(z)}$, and then we show the assertion is true for $\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)}$. For any fixed $\alpha \in M$, by Lemma 4.1 and Theorem 4.5 there exist integers a, b, c, such that

$$\frac{\theta^\alpha(z)}{\theta^\beta(z)} = \left(-\frac{1}{2}\right)^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\, p(z) \tag{4.10}$$

where $p(z) := (\wp(z) - e_1)^a(\wp(z) - e_3)^b\, \wp'(z)^c$. Applying Lemma 2.4 and Lemma 2.5 one can verify that $W(\theta_2(0)^2) = 1$ and

$$G_1(\theta_2(0)^2) = \{\pm\theta_2(0)^2, \pm i\theta_2(0)^2, \pm\theta_3(0)^2, \pm i\theta_3(0)^2, \pm\theta_4(0)^2, \pm i\theta_4(0)^2\}.$$

Similarly we have $W(\theta_3(0)^2) = 1$ and $W(\theta_1'(0)^2) = 3$. Then by Proposition 4.3.1 we obtain

$$\begin{aligned} W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) &= W(\theta_2(0)^{2a}\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= W(\theta_2(0)^{2a}) + W(\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= a + b - 3a - 3b - 3c \\ &= -2a - 2b - 3c. \end{aligned}$$

Next we compute $W([z^n]p(z))$, where by $[z^n]p(z)$ we mean the coefficient of z^n in the series expansion of $p(z)$ around $z = 0$. Let us first consider

$$p_1(z) := z^{2a+2b+3c}p(z) = z^{2a}(\wp(z) - e_1)^a z^{2b}(\wp(z) - e_3)^b z^{3c}\, \wp'(z)^c. \tag{4.11}$$

Let $g_1(z) := z^2(\wp - e_1)$. By Proposition 2.1 we have

$$g_1(z) = 1 - e_1 z^2 + \sum_{m=1}^{\infty}(2m+1)E_{2m+2}z^{2m+2}$$

where $E_{2m+2} := \sum_{\omega\in L,\omega\neq 0} \omega^{-(2m+2)}$ is an Eisenstein series and L is the lattice generated by π and $\pi\tau$. One can easily verify by using Definition 4.9 that $W(1) = 0$. Again using Lemma 2.4 and Lemma 2.5 one can verify that $W(e_1) = 2$. In addition, according to [29, p. 83] for $m \geqslant 1$,

$$W(E_{2m+2}) = W\left(\sum_{\omega\in L,\omega\neq 0} \omega^{-(2m+2)}\right) = 2m+2.$$

Therefore, for any $n \geqslant 0$, if $[z^n]g_1(z) \neq 0$ then

$$W([z^n]g_1(z)) = n. \tag{4.12}$$

Next we do a case distinction on the power of $g_1(z)$ in (4.11).

Case 1: $a \geqslant 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left(\sum_{n_1+n_2+\cdots+n_a=n} [z^{n_1}]g_1(z)\cdots[z^{n_a}]g_1(z)\right).$$

By (4.12) and by Proposition 4.3.1, for any combination $n_1,\ldots,n_a$ such that $n_1+\cdots+n_a = n$ we have

$$\begin{aligned} W([z^{n_1}]g_1(z)\cdots[z^{n_a}]g_1(z)) &= W([z^{n_1}]g_1(z)) + \cdots + W([z^{n_a}]g_1(z)) \\ &= n_1 + \cdots + n_a \\ &= n. \end{aligned}$$

Hence if $a \geqslant 0$, we find that

$$W\left([z^n]g_1(z)^a\right) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Case 2: $a < 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left([z^n]\left(\frac{1}{g_1(z)}\right)^{-a}\right)$$

$$= W\left(\sum_{n_1+n_2+\cdots+n_{-a}=n}[z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n_{-a}}]\left(\frac{1}{g_1(z)}\right)\right).$$

Assuming $g_1(z) = \sum_{j=0}^{\infty} v_j z^j$ we have $\frac{1}{g_1(z)} = \sum_{j=0}^{\infty} u_j z^j$, noting that $v_0 = u_0 = 1$. We have proven that for all $n \geqslant 0$, $W(v_n) = n$ when $v_n \neq 0$. Now we prove that $W(u_n) = n$ when $u_n \neq 0$ by induction on n. When $n = 0$ we have $W(u_0) = W(v_0) = 0$. Assume for $n \leqslant N$, $W(u_n) = n$. Let $n = N+1$. Using $\sum_{j=0}^{\infty} v_j z^j \cdot \sum_{j=0}^{\infty} u_j z^j = 1$ we obtain

$$u_{N+1} = -\frac{v_1 u_N + v_2 u_{N-1} + \cdots + v_N u_1 + v_{N+1} u_0}{v_0} = -v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}.$$

By Proposition 4.3.2, if $u_{N+1} \neq 0$, then

$$W(u_{N+1}) = W(-v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}) = N+1. \tag{4.13}$$

Hence $W(u_n) = n$ when $u_n \neq 0$. For any combination $n_1, \ldots, n_{-a}$ that $n_1 + \cdots + n_{-a} = n$ we have

$$\begin{aligned} W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right) &= W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\right) + \cdots + W\left([z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right) \\ &= n_1 + \cdots + n_{-a} \\ &= n. \end{aligned}$$

Again by Proposition 4.3.2 and by (4.13), for any $a < 0$ we find that

$$W([z^n]g_1(z)^a) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Analogously we deduce that for $b, c \in \mathbb{Z}$,

$$W\left([z^n]z^{2b}(\wp - e_3)^b\right) = n \quad \text{and} \quad W([z^n]z^{3c}\wp'(z)^c) = n$$

whenever the function to which W is applied is nonzero. Consequently we deduce that when $[z^n]p_1(z) \neq 0$,

$$\begin{aligned} W([z^n]p_1(z)) &= W\left([z^n]z^{2a}(\wp(z)-e_1)^a z^{2b}(\wp(z)-e_3)^b z^{3c}\wp'(z)^c\right) \\ &= W\left(\sum_{n_1+n_2+n_3=n}[z^{n_1}](\wp(z)-e_1)^a \cdot [z^{n_1}]z^{2b}(\wp(z)-e_3)^b \cdot [z^{n_3}]z^{3c}\wp'(z)^c\right) \\ &= n_1 + n_2 + n_3 \end{aligned}$$

$$= n,$$

where the second last equality follows from Proposition 4.3.1. This implies when $[z^n]p(z) \neq 0$,

$$W\left([z^n]p(z)\right) = W\left([z^{n+2a+2b+3c}]p_1(z)\right) = n + 2a + 2b + 3c.$$

Therefore if $[z^n]\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} \neq 0$, identity (4.10) implies

$$\begin{aligned} W\left([z^n]\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}\right) &= W\left((-\frac{1}{2})^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}p(z)\right) \\ &= W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) + W\left([z^{2n}]p(z)\right) \\ &= -2a - 2b - 3c + n + 2a + 2b + 3c \\ &= n. \end{aligned}$$

Moreover, since both $\psi(\alpha)$ and $\psi(\beta)$, by definition of β, are homogeneous polynomials in $\mathbb{K}[\widetilde{\Theta}]_h$ with the same degree, one can check, by using Lemma 2.4 and Lemma 2.5, that $W\left(\frac{\psi(\alpha)}{\psi(\beta)}\right) = 0$ for all $\alpha \in M$. Hence

$$W\left([z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}\right) = 0 + n = n \text{ when } [z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)} \neq 0$$

and

$$W(d_n) = \sum_{\alpha \in M}[z^n]\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)} = n \text{ when } d_n \neq 0.$$

□

4.5 Zero-Recognition for $f^{\Psi} \in \hat{H}_{\widetilde{\Theta}}$

Let us recall Definition 4.7. By Lemma 4.1 and Theorem 4.5, for any $\frac{\theta^{\alpha}}{\theta^{\beta}} \in J$ with $\alpha = (\alpha_1, \ldots, \alpha_4)$ and $\beta = (\beta_1, \ldots, \beta_4)$, we can write

$$\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} = \left(-\frac{1}{2}\right)^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}p(z), \tag{4.14}$$

where $p(z) := (\wp(z) - e_1)^a(\wp(z) - e_3)^b\,\wp'(z)^c$. The $p(z)$ has the following property.

Proposition 4.4. *Let $p(z)$ be the same as above and let g_n denote the coefficient of z^n in the series expansion of $p(z)$ around $z = 0$. Then when $g_n \neq 0$ we have*

$$|G_{w_n}(g_n)| \leqslant 3 = |G_2(e_1)|,$$

where w_n is the finite-orbit-weight of g_n.

Proof. From the proof of Lemma 4.2 we observe that g_n is a polynomial in e_1, e_3 and E_{2s+2} with some $s \geqslant 1$, and

$$W(g_n) = n' := n + 2a_j + 2b_j + 3c_j$$

when $g_n \neq 0$. Let $p_1, \ldots, p_t$ be the components of g_n, where each such component is a (finite) power product $e_1^{k_1} e_2^{k_2} E_4^{\ell_1} E_6^{\ell_2} \cdots$ with a coefficient in $\mathbb{K}$. One has

$$|G_{n'}(g_n)| = |\{(p_1 + \cdots + p_t)|_{n'}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|. \tag{4.15}$$

Additionally, from the proof of Lemma 4.2, the p_i in (4.15) are of the form

$$e_1^{k_1} e_3^{k_2} \prod_{s \in M_i} E_{2s+2}^{\ell_s}$$

where $k_1, k_2, \ell_s \in \mathbb{N}$, $M_i \subseteq \mathbb{N}$ and $2k_1 + 2k_2 + \sum_{s \in M_i} (2s+2)\ell_s = n'$. It can be verified by using Lemma 2.4 and Lemma 2.5 that $G_2(e_1) = G_2(e_3) = \{e_1, e_2, e_3\}$, thus $W(e_1) = W(e_3) = 3$.

By [29, p. 83] when $m \geqslant 1$, E_{2s+2} is a modular form of weight $2s+2$, which means

$$E_{2s+2}|_{2s+2}\rho = E_{2s+2} \text{ for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

Consequently,

$$E_{2s+2}^{\ell_s}|_{(2s+2)\ell_s}\rho = E_{2s+2}^{\ell_s} \quad \text{for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

By Proposition 4.3 (1),

$$W\left(\prod_{s \in M_i} E_{2s+2}^{\ell_s}\right) = \sum_{s \in M_i} (2s+2)\ell_s.$$

By Proposition 4.3 (2) we obtain $W(p_i) = 2k_1 + 2k_2 + \sum_{s \in M_i} (2s+2)\ell_s = n'$ for all $i \in \{1, \ldots, t\}$. Hence we continue (4.15) by

$$|G_{n'}(g_n)| = \{(p_1 + \cdots + p_t)|_{n'}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|$$

$$\begin{aligned} &\leqslant |\{\{p_1|_{n'}\rho,\dots,p_t|_{n'}\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}| \\ &= |\{\{(e_1^{k_{1,1}} e_3^{k_{1,2}}\gamma_1)|_{n'}\rho,\dots,(e_1^{k_{t,1}} e_3^{k_{t,2}}\gamma_t)|_{n'}\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}| \\ &= |\{\{(e_1^{k_{1,1}} e_3^{k_{1,2}})|_{2(k_{1,1}+k_{1,2})}\rho,\dots,(e_1^{k_{t,1}} e_3^{k_{t,2}})|_{2(k_{1,1}+k_{1,2})}\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|, \end{aligned} \tag{4.16}$$

where the γ_i are the corresponding $\prod_{m\in M_i} E_{2s+2}^{\ell_s}$ of p_i. On the other hand, for $k \in \mathbb{N}$,

$$G_{2k}(e_1^k) = \{e_1^k|_{2k}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{\underbrace{e_1|_2\rho\cdots e_1|_2\rho}_{k} : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_1^k, e_2^k, e_3^k\}$$

and analogously $G_{2k}(e_2^k) = G_{2k}(e_3^k) = \{e_1^k, e_2^k, e_3^k\}$. Then

$$\{e_1^{k_1} e_3^{k_2}|_{2(k_1+k_2)}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_1^{k_1}|_{2k_1}\rho \cdot e_2^{k_2}|_{2k_2}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_2^{k_1}e_1^{k_2}, e_3^{k_1}e_2^{k_2}, e_1^{k_1}e_3^{k_2}\},$$

which means there are only three possibilities when applying an arbitrary $\rho \in \mathrm{SL}_2(\mathbb{Z})$ on every $e_1^{k_{i,1}} e_3^{k_{i,2}}$ of (4.16). Note that the powers $k_{i,j}$ are irrelevant, i.e., we can choose three representatives ρ_1, ρ_2 and ρ_3 such that for all $i \in \{1,\dots,t\}$,

$$(e_1^{k_{i,1}} e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_1 = e_2^{k_{i,1}} e_1^{k_{i,2}},$$

$$(e_1^{k_{i,1}} e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_2 = e_3^{k_{i,1}} e_2^{k_{i,2}}$$

and

$$(e_1^{k_{i,1}} e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_3 = e_1^{k_{i,1}} e_3^{k_{i,2}}.$$

Hence the right hand side of (4.16) is equal to

$$\left\{\{e_2^{k_{1,1}} e_1^{k_{1,2}},\dots,e_2^{k_{t,1}} e_1^{k_{t,2}}\},\{e_3^{k_{1,1}} e_2^{k_{1,2}},\dots,e_3^{k_{t,1}} e_2^{k_{t,2}}\},\{e_1^{k_{1,1}} e_3^{k_{1,2}},\dots,e_1^{k_{t,1}} e_3^{k_{t,2}}\}\right\}.$$

Thus $|G_{n'}(g_n)| \leqslant 3$ when $g_n \neq 0$. □

Lemma 4.3. *Let quasi-elliptic* $f^\Psi(z|\tau) = \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z) \in \hat{H}_{\tilde{\Theta}}$ *and* $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \min(M)$. *Suppose*

$$\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)} = \sum_{n=0}^{\infty} d_n(\tau)z^n \text{ with } d_n(\tau) \in \mathbb{K}(\Theta).$$

Let $M = \{y^{(1)},\dots,y^{(m)}\}$ *with* $y^{(j)} = (y_1^{(j)}, y_2^{(j)}, y_3^{(j)}, y_4^{(j)})$. *For* $1 \leqslant j \leqslant m$ *let*

$$a_j := \frac{y_2^{(j)} - y_4^{(j)} - \beta_2 + \beta_4}{2},$$

$$\begin{aligned}
b_j :=& \frac{y_3^{(j)} - y_4^{(j)} - \beta_3 + \beta_4}{2},\\
c_j :=& y_4^{(j)} - \beta_4,\\
r_j :=& y_1^{(j)} - \beta_1,
\end{aligned}$$

and

$$t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$$

For all $n \geqslant 0$, if $d_n \neq 0$ then

$$|G_n(d_n)| \leqslant |\{\{t_1|_{r_1}\rho, \dots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|,$$

Proof. First of all we write

$$\frac{f^{\psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)} = h_1 + \dots + h_m$$

with $h_j := \frac{\psi(y^{(j)})\theta^{y^{(j)}}(z)}{\psi(\beta)\theta^{\beta}(z)}$. From the proof of Lemma 4.2 we see that for all $j \in \{1, \dots, m\}$,

$$W([z^n]h_j(z)) = n$$

and

$$W\left(\frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}\right) = -2a_j - 2b_j - 3c_j.$$

Then by Proposition 4.4 and expression (4.14) we deduce

$$|G_n([z^n]h_j(z))| \leqslant \left|\left\{\left\{t_j|_{r_j}\rho, e_1|_2\rho\right\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right|,$$

where $r_j := -2a_j - 2b_j - 3c_j = y_1^{(j)} - \beta_1$ following from the definition of a_j, b_j, c_j and where $t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$. Consequently, when $d_n \neq 0$ we have $W(d_n) = n$ by Lemma 4.2 and

$$\begin{aligned}
|G_n(d_n)| &= \left|\left\{[z^n](h_1(z) + \dots + h_m(z))\Big|_n \rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right|\\
&\leqslant \left|\left\{\left\{[z^n]h_1(z)\Big|_n \rho, \dots, [z^n]h_m(z)\Big|_n \rho\right\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\right\}\right|\\
&\leqslant |\{\{t_1|_{r_1}\rho, \dots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|.
\end{aligned}$$

□

Theorem 4.6. *Let $q := e^{\pi i \tau}$, $t_1, \ldots, t_m$, $r_1 \ldots, r_m$ and d_n be the same as in Lemma 4.3, and let*

$$\ell := |\{\{t_1|_{r_1}\rho, \ldots, t_m|_{r_m}\rho, e|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|.$$

For $n \geqslant 0$ suppose d_n has the q-expansion

$$\sum_{j=0}^{\infty} v_{n,j} q^j.$$

Then

$$d_n = 0 \text{ if and only if } v_{n,j} = 0 \text{ for } j \leqslant \lfloor \frac{n\ell}{6} \rfloor.$$

Proof. " $\Longrightarrow$ " If $d_n(\tau) \equiv \sum_{j=0}^{\infty} v_{n,j} q^j \equiv 0$, it immediately implies that all v_j are zero.

" $\Longleftarrow$ " Assume $v_{n,j} = 0$ for $j \leqslant \lfloor \frac{n\ell}{6} \rfloor$. If $d_n \neq 0$, by Lemma 4.2 we have $W(d_n) = n$ and by Lemma 4.3, $|G_n(d_n)| \leqslant \ell$. Suppose $G_n(d_n) = \{s_1, \ldots, s_{\ell_n}\}$ and $\ell_n \leqslant \ell$. Then for every $i \in \{1, \ldots, \ell_n\}$, there exists a unique $j \in \{1, \ldots, \ell_n\}$ such that $s_i|_n S = s_j$; and there exists a unique $k \in \{1, \ldots, \ell_n\}$ such that $s_i|_n T = s_k$. Then

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} S = \prod_{j=1}^{\ell_n} s_j \quad \text{and} \quad \left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} T = \prod_{j=1}^{\ell_n} s_j.$$

This yields

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} \rho = \prod_{j=1}^{\ell_n} s_j \quad \text{for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, we have proven in Lemma 3.12 that $\prod_{j=1}^{\ell_n} s_j$ is a Taylor series in q. Thus $\prod_{j=1}^{\ell_n} s_j$ is a modular form of weight $n\ell_n$.

Since $\ell_n \leqslant \ell$ we have $v_{n,j} = 0$ for $j \leqslant \lfloor \frac{n\ell_n}{6} \rfloor$. By Corollary 2.1, $\prod_{j=1}^{\ell_n} s_j = 0$. Because of the fact that for any meromorphic functions h and g on $\mathbb{H}$, if $(h|_n\rho)(\tau) = g(\tau)$ then $h(\tau) \equiv 0$ if and only if $g(\tau) \equiv 0$, we deduce that s_j must be zero for all $j \in \{1, \ldots, \ell_n\}$, otherwise $s_j \neq 0$ for all $j \in \{1, \ldots, \ell_n\}$ which contradicts $\prod_{j=1}^{\ell_n} s_i = 0$. As $d_n \in G_n(d_n) = \{s_1, \ldots, s_{\ell_n}\}$, we deduce that $d_n = 0$, which contradicts the earlier assumption $d_n \neq 0$. Therefore $d_n = 0$. $\square$

Algorithm 4.2. *Let $q = e^{\pi i \tau}$ and $f^{\Psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z|\tau) \in \hat{H}_{\tilde{\Theta}}$. We have the following algorithm to prove or disprove $f^{\Psi}(z|\tau) \equiv 0$.*

Input: $f^\Psi \in \hat{H}_{\tilde{\Theta}}$.

Output: True if $f^\Psi = 0$*; False if* $f^\Psi \neq 0$.

Write $f^\Psi(z|\tau) = \sum_{j=1}^{n} f^{\Psi_j}(z|\tau)$ *where the* $f^{\Psi_j}(z|\tau) := \sum_{\alpha\in M_j} \psi(\alpha)\theta^\alpha(z|\tau)$ *are the quasi-elliptic components of* $f^\Psi(z|\tau)$.

Set $i := 1$*. While* $i \leqslant n$ *do*

Let $m := |M_i|$ *and* $\{y^{(1)},\dots,y^{(m)}\} := M_i$*;*

Choose $\beta \in \min(M_i)$*;*

For $j \in \{1,\dots,m\}$,

compute $a_j := \frac{y_2^{(j)}-y_4^{(j)}-\beta_2+\beta_4}{2}$, $b_j := \frac{y_3^{(j)}-y_4^{(j)}-\beta_3+\beta_4}{2}$, $c_j := y_4^{(j)}-\beta_4$*;*

compute $r_j := y_1^{(j)} - \beta_1$*;*

let $t_j := \frac{\psi(y^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$*;*

Compute $\ell := |\{\{t_1|_{r_1}\rho,\dots,t_m|_{r_m}\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|$*;*

Let $g(z) := \frac{f^{\Psi_i}(z)}{\theta^\beta(z)}$*;*

Compute $g(z) = \sum_{k=0}^{\infty} d_k(\tau)z^k$*;*

Set $k := 0$*. While* $k \leqslant \beta_2+\beta_3+\beta_4$ *do*

if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$*;*

$k++$*;*

otherwise return False;

end do;

$j++$;

end do;

return True;

Theorem 4.7. *Algorithm 4.2 is correct.*

Proof. By Lemma 4.6, $d_k(\tau) \equiv 0$ if and only if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$. Since the only difference between Algorithm 4.1 and Algorithm 4.2 is the way in which we check $d_k(\tau) \equiv 0$, it follows that Algorithm 4.2 is correct. □

Example 4.2 continued.[14, 20.7.1] Prove

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0.$$

Proof. Let $\beta := (0,0,0,2)$ and

$$g(z) := \frac{\theta_2(0)^2\theta_2(z)^2}{\theta_4(0)^2\theta_4(z)^2} - \frac{\theta_3(0)^2\theta_3(z)^2}{\theta_4(0)^2\theta_4(z)^2} + 1.$$

Since $g(z)$ is an even analytic function we obtain

$$g(z) = \sum_{k=0}^{\infty} d_{2k}(\tau) z^{2k}$$

with

$$d_0(\tau) = \frac{\theta_2(0)^4 - \theta_3(0)^4 + \theta_4(0)^4}{\theta_4(0)^4},$$

$$d_2(\tau) = \frac{\theta_2(0)^3\theta_4(0)\theta_2''(0) - \theta_3(0)^3\theta_4(0)\theta_3''(0) - \theta_2(0)^4\theta_4''(0) + \theta_3(0)^4\theta_4''(0)}{\theta_4(0)^5}$$

and $d_k(\tau)(k > 1)$ are of a form similar to $d_0(\tau)$ and $d_2(\tau)$. According to Algorithm 4.2 we need to show that $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^{\frac{\ell}{3}+1})$ where

$$\ell = \left| \left\{ \frac{\theta_2(0)^4\theta_3(0)^2}{\theta_1'(0)^2}\Big|_0 \rho, \frac{\theta_2(0)^2\theta_3(0)^4}{\theta_1'(0)^2}\Big|_0 \rho, e_1|_2\rho : \rho \in \mathrm{SL}_2(\mathbb{Z}) \right\} \right|.$$

By implementing Algorithm 4.2 in Mathematica, we obtain that $\ell = 6$, and $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^3)$. $\square$

Remark 4.2. *Example 4.2 can also be solved by Algorithm 4.1. The speed comparison of Algorithms 4.1 and 4.2 can be found in Chapter 6.*

The following proposition shows that there is a further decomposition step that can be done before doing zero-recognition.

Proposition 4.5. *Given* $f^{\Psi}(z) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z) \in \hat{R}_{\Theta}$, *then* $f(z) \equiv 0$ *if and only if*

$$\sum_{\alpha \in M_i} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$$

for $i = 1, 2$, *where* $M_1 := \{(\alpha_1, \ldots, \alpha_4) \in M : \alpha_1$ *is odd*$\}$ *and* $M_2 := \{(\alpha_1, \ldots, \alpha_4) \in M : \alpha_1$ *is even*$\}$.

Proof. " $\Longleftarrow$ " is immediate. We show " $\Longrightarrow$ ". Let $f_i(z) \equiv \sum_{\alpha \in M_i} \psi(\alpha)\theta^{\alpha}(z)$. By Definition 2.7, $\theta_1(z)$ is an odd function while the other three are even functions, hence

$$0 \equiv f^{\Psi}(z) \equiv f^{\Psi}(-z) \equiv -f_1(z) + f_2(z).$$

This together with $f^{\Psi}(z) \equiv f_1(z) + f_2(z)$ implies $f_1(z) \equiv 0$ and $f_2(z) \equiv 0$.

$\square$

4.6 Examples

Example 4.5. *Prove the differential equation [11, p. 29, Thm. 3]*

$$\wp'(z)^2 \equiv 4\,\wp(z)^3 - g_2\,\wp(z) - g_3,$$

where $g_2 := 60E_4$ *and* $g_3 := 140E_6$ *with* $E_{2k+2} := E_{2k+2}(\pi, \pi\tau)$.

From the proof of Theorem 4.5 we see that

$$\wp(z) = \frac{\theta_1'^2}{\theta_2^2}\frac{\theta_2(z)^2}{\theta_1(z)^2} + \frac{1}{3}(\theta_3^4 + \theta_4^4) \tag{4.17}$$

and

$$\wp'(z) = -\frac{2\theta_1'^2\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3}.$$

In addition, we expand both sides of (4.17) around $z = 0$. The coefficients of z^n on both sides must be equal. Proposition 2.1 gives us the coefficients in the expansion of $\wp(z)$, which are E_{2k+2}. The coefficients in the expansion of the right hand side of (4.17) are some quotients of $\theta_j^{(k)}$. Hence we can express g_2 and g_3 in terms of some $\theta_j^{(k)}$. Therefore, we can write the identity in Example 4.5 in terms of theta quotients, and this example fits in the class studied in this chapter.

In the literature there are not many high degree (already in an expression like Examples 4.1 and 4.2, not in an implicit form like Example 4.5) identities found in $\hat{R}_\Theta$ and $\hat{H}_{\tilde{\Theta}}$. The one with the highest degree we are able to find in $\hat{R}_\Theta$ is

$$\theta_1(z)^4 + \theta_3(z)^4 \equiv \theta_2(z)^4 + \theta_4(z)^4$$

from [31, p. 462], whilst we have a way of producing all relations in $\hat{R}_\Theta$, which will be introduced in Chapter 6. Moreover, we are preparing a paper [28] that determines the generators of the ideal containing all relations in $\hat{H}_{\tilde{\Theta}}$.

On the other hand, based on this chapter, algorithmically dealing with other types of identities becomes possible. For instance, in Chapter 5 we will provide an algorithm to deal with identities like

$$\theta_2\,\theta_3\,\theta_4\,\theta_1(2z,q) - 2\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z) \equiv 0,$$

from [31, p. 485].

Chapter 5

Proving Identities among Powers of $\theta_j^{(k)}(0|\tau)$ & $\theta_\ell(az|\tau)$ (Class III)

In this chapter we extend $\mathbb{K}[\Theta]$ further to $\mathbb{K}[\Theta][G_1]$, by which we define the $\mathbb{K}[\Theta]$-algebra generated by

$$G_1 := \{\theta_j(az,q) : \ j = 1,2,3,4 \text{ and } a \in \mathbb{N}\backslash\{0\}\}.$$

We solve the following problem algorithmically:

Problem 5.1: Given $f \in \mathbb{K}[\Theta][G_1]$; decide whether $f = 0$.

Example 5.1. *[14, 20.7.10] Our algorithm will be used to prove*

$$\theta_2(0,q)\theta_3(0,q)\theta_4(0,q)\theta_1(2z,q) - 2\theta_1(z,q)\theta_2(z,q)\theta_3(z,q)\theta_4(z,q) \equiv 0.$$

5.1 Quasi-Elliptic Decomposition of $f \in \mathbb{K}[\Theta][G_1]$

First of all we examine any single theta function, say $\theta_3(az,q)$ with $a \in \mathbb{N}\backslash\{0\}$. Substituting $z \mapsto z+\pi$ and $z \mapsto z+\pi\tau$, we find that

$$\theta_3(a(z+\pi),q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nia(z+\pi)} = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niaz} = \theta_3(az,q)$$

and

$$
\begin{aligned}
\theta_3(a(z+\pi\tau),q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nia(z+\pi\tau)} = \sum_{n=-\infty}^{\infty} q^{n^2} q^{2na} e^{2niaz} \\
&= \sum_{n=-\infty}^{\infty} q^{(n+a)^2} q^{-a^2} e^{2(n+a)iaz} e^{-2ia^2 z} \\
&= q^{-a^2} e^{-2ia^2 z} \sum_{n=-\infty}^{\infty} q^{(n+a)^2} e^{2(n+a)iaz} \\
&= q^{-a^2} e^{-2ia^2 z} \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niaz} \\
&= N^{a^2} \theta_3(az,q),
\end{aligned}
$$

where $N := q^{-1}e^{-2iz}$. Carrying out the same steps for $\theta_1(az,q)$, $\theta_2(az,q)$ and $\theta_4(az,q)$ we obtain for $j \in \{1,2,3,4\}$, $\theta_j(a(z+\pi\tau)) = \varepsilon_1(j)\theta_j(az)$ and $\theta_j(a(z+\pi)) = \varepsilon_2(j)\theta_j(az)$, where $\varepsilon_1(j)$ and $\varepsilon_2(j)$ are defined in Table 5.1.

j	1	2	3	4
$\varepsilon_1(j)$	$(-1)^a N^{a^2}$	N^{a^2}	N^{a^2}	$(-1)^a N^{a^2}$
$\varepsilon_2(j)$	$(-1)^a$	$(-1)^a$	1	1

Table 5.1

Then we look at any product of the theta functions.

Definition 5.1. *Define $W := \{(j,a,\alpha) : j \in \{1,2,3,4\}$ and $a,\alpha \in \mathbb{N}\backslash\{0\}\}$. Given a finite subset $X \subseteq W$ and $c \in \mathbb{K}[\Theta]$, we define*

$$\theta_{X,c}(z) := c \prod_{(j,a,\alpha)\in X} \theta_j(az)^\alpha.$$

Given $\bar{X} = (X_1,\ldots,X_m)$ and $\bar{c} = (c_1,\ldots,c_m)$ with finite subsets $X_i \subseteq W$ and $c_i \in \mathbb{K}[\Theta]$, we define

$$\theta_{\bar{X},\bar{c}}(z) := \sum_{i=1}^{m} \theta_{X_i,c_i}(z)$$

By Definition 5.1, every element in $\mathbb{K}[\Theta][G_1]$ can be written in the form of $\theta_{\bar{X},\bar{c}}(z)$.

Definition 5.2. *Given a finite subset $X \subseteq W$ and $c \in \mathbb{K}[\Theta]$, we define*

$$\nu_j(\theta_{X,c}) := \sum_{(j,a,\alpha)\in X} a\alpha,$$

$$\mu_j(\theta_{X,c}) := \sum_{(j,a,\alpha)\in X} \alpha$$

and

$$\omega(\theta_{X,c}) := \sum_{(j,a,\alpha)\in X} a^2\alpha.$$

Example 5.2. *Let $p := \theta_4(0,q)\theta_2(2z,q)^2\theta_2(z,q)\theta_1(4z,q)$. Then $\nu_2(p) = 5$ and $\omega(p) = 25$.*

Definition 5.3. *Let $f := \sum_{i=1}^{m} f_i$ with $f_i := \theta_{X_i,c_i}$, where $X_i \subseteq W$ finite and $c_i \in \mathbb{K}[\Theta]$. We define*

$$Y_{t,b,s}(f) := \{i \in \{1,\dots,m\} : \omega(f_i) = t, \nu_1(f_i)+\nu_4(f_i) = b \,(\mathrm{mod}\, 2), \nu_1(f_i)+\nu_2(f_i) = s \,(\mathrm{mod}\, 2)\}.$$

Then we call

$$\left\{ \sum_{i\in Y_{t,b,s}(f)} f_i \neq \emptyset : t \in \mathbb{N}\backslash\{0\} \text{ and } b,s \in \{0,1\} \right\}$$

the set of quasi-elliptic components of f.

Theorem 5.1. *Let $\{g_1,\dots,g_r\}$ be the set of quasi-elliptic components of $f \in \mathbb{K}[\Theta][G_1]$. Then*

$$f = 0 \text{ iff } g_i = 0 \text{ for all } i \in \{1,\dots,r\}.$$

Proof. " $\Longleftarrow$ " is immediate. We prove " $\Longrightarrow$ ". Suppose $f := \sum_{i=1}^{m} f_i$ with $f_i := \theta_{X_i,c_i}$. Let

$$Y_{t,0}(f) := \{i \in \{1,\dots,m\} : \omega(f_i) = t, \nu_1(f_i)+\nu_4(f_i) \text{ even}\}$$

and

$$Y_{t,1}(f) := \{i \in \{1,\dots,m\} : \omega(f_i) = t, \nu_1(f_i)+\nu_4(f_i) \text{ odd}\}.$$

Assume $\{\omega(f_1),\dots,\omega(f_m)\} = \{d_1,\dots,d_\ell\}$. We can write

$$f = \sum_{t\in\{d_1,\dots,d_\ell\}} \left(\sum_{i\in Y_{t,0}(f)} f_i + \sum_{i\in Y_{t,1}(f)} f_i \right).$$

Then by using Table 5.1 and the definition of $d(f_i)$ we have

$$\begin{aligned}
0 \equiv f(z+\pi\tau) &\equiv \sum_{t\in\{d_1,\ldots,d_\ell\}} \left(\sum_{i\in Y_{t,0}(f)} f_i(z+\pi\tau) + \sum_{i\in Y_{t,1}(f)} f_i(z+\pi\tau) \right) \\
&\equiv \sum_{t\in\{d_1,\ldots,d_\ell\}} \left(\sum_{i\in Y_{t,0}(f)} (-1)^{\nu_1(f_i)+\nu_4(f_i)} N^t f_i(z) + \sum_{i\in Y_{t,1}(f)} (-1)^{\nu_1(f_i)+\nu_4(f_i)} N^t f_i(z) \right) \\
&\equiv \sum_{t\in\{d_1,\ldots,d_\ell\}} \left(\sum_{i\in Y_{t,0}(f)} N^t f_i(z) + \sum_{i\in Y_{t,1}(f)} (-N^t) f_i(z) \right).
\end{aligned}$$

Hence for $k \in \mathbb{N}$,

$$\begin{aligned}
0 \equiv f(z+k\pi\tau) &\equiv \sum_{t\in\{d_1,\ldots,d_\ell\}} \left(\sum_{i\in Y_{t,0}(f)} N^{tk} f_i(z) + \sum_{i\in Y_{t,1}(f)} (-N^t)^k f_i(z) \right) \\
&\equiv \sum_{t\in\{d_1,\ldots,d_\ell\}} \left(N^{tk} \sum_{i\in Y_{t,0}(f)} f_i(z) + (-N^t)^k \sum_{i\in Y_{t,1}(f)} f_i(z) \right), \qquad (5.1)
\end{aligned}$$

Let $g_{t,0}(z) = \sum_{i\in Y_{t,0}(f)} f_i(z)$ and $g_{t,1}(z) = \sum_{i\in Y_{t,1}(f)} f_i(z)$. Then (5.1) can be written as

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ N^{d_1} & -N^{d_1} & \cdots & N^{d_\ell} & -N^{d_\ell} \\ (N^{d_1})^2 & (-N^{d_1})^2 & \cdots & (N^{d_\ell})^2 & (-N^{d_\ell})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (N^{d_1})^{2\ell-1} & (-N^{d_1})^{2\ell-1} & \cdots & (N^{d_\ell})^{2\ell-1} & (-N^{d_\ell})^{2\ell-1} \end{pmatrix}}_{B} \begin{pmatrix} g_{d_1,1} \\ g_{d_1,2} \\ \vdots \\ g_{d_\ell,1} \\ g_{d_\ell,2} \end{pmatrix} = 0 \qquad (5.2)$$

Since $N \neq 0$, the determinant of this Vandermonde matrix B is nonzero. Therefore we can multiply both sides of (5.2) by the inverse of B and obtain $g_{t,i} = 0$ for all $t \in \{d_1,\ldots,d_\ell\}$ and $i \in \{0,1\}$.

Next we write

$$0 \equiv g_{t,0}(z) \equiv g_{t,0,0}(z) + g_{t,0,1}(z) \quad \text{and} \quad 0 \equiv g_{t,1}(z) \equiv g_{t,1,0}(z) + g_{t,1,1}(z), \qquad (5.3)$$

where for $b \in \{0,1\}$ we denote

$$g_{t,b,0}(z) := \sum_{i \in Y_{t,b,0}(f)} f_i(z) \quad \text{and} \quad g_{t,b,1}(z) := \sum_{i \in Y_{t,b,1}(f)} f_i(z)$$

with $Y_{t,b,0}(f) := \{i \in Y_{t,b}(f) : \nu_1(f_i) + \nu_2(f_i) \text{ even}\}$ and $Y_{t,b,1}(f) := \{i \in Y_{t,b}(f) : \nu_1(f_i) + \nu_4(f_i) \text{ odd}\}$. Again by using Table 5.1 on $f_i(z)$ we obtain for $s \in \{0,1\}$,

$$0 \equiv g_{t,b}(z) \equiv g_{t,b}(z+\pi) \equiv g_{t,b,0}(z+\pi) + g_{t,b,1}(z+\pi) \equiv g_{t,b,0}(z) - g_{t,b,1}(z).$$

This together with (5.3) implies $g_{t,b,0} = g_{t,b,1} = 0$ for all $t \in \{d_1, \ldots, d_\ell\}$ and $b, s \in \{0,1\}$.

In view of Definition 5.3 we have

$$\{g_{t,b,s} : t \in \{d_1, \ldots, d_\ell\} \text{ and } a, b \in \{0,1\}\} = \{g_1, \ldots, g_r\}.$$

Therefore $g_i = 0$ for all $i \in \{1, \ldots, r\}$. $\square$

5.2 Zero-Recognition for $f \in \mathbb{K}[\Theta][G_1]$

Lemma 5.1. *Let $f := \sum_{i=1}^{m} f_i$ with $f_i := \theta_{X_i,c_i}$, and let $\sum_{i \in V} f_i$ be a quasi-elliptic component of f, where $V \subseteq \{1, \ldots, m\}$. Then for all $i, j \in V$, $\frac{f_i(z)}{f_j(z)}$ is elliptic with respect to z.*

Proof. According to Definition 5.3, we suppose that V is equal to some fixed $X_{a,b,c}$ with $a \in \mathbb{N} \backslash \{0\}$ and $b, s \in \{0,1\}$. Take an arbitrary $i \in X_{a,b,c}$. By Table 5.1 we have $f_i(z+\pi) \equiv (-1)^{\nu_1(f_i)+\nu_2(f_i)} f_i(z)$ and $f_i(z+\pi\tau) \equiv (-1)^{\nu_1(f_i)+\nu_4(f_i)} N^a f_i(z)$, which together with the definition of $X_{a,b,c}$ implies that for any $i, j \in X_{a,b,c}$,

$$\frac{f_i(z+\pi)}{f_j(z+\pi)} \equiv \frac{f_i(z)}{f_j(z)} \quad \text{and} \quad \frac{f_i(z+\pi\tau)}{f_j(z+\pi\tau)} \equiv \frac{f_i(z)}{f_j(z)}.$$

Therefore $\frac{f_i(z)}{f_j(z)}$ is elliptic. $\square$

Theorem 5.2. *Let* $f := \sum_{i=1}^{m} f_i$ *with* $f_i := \theta_{X_i,c_i}$, *and let* $g := \sum_{i\in V} f_i$ *be a quasi-elliptic component of* f, *where* $V \subseteq \{1,\ldots,m\}$. *Take* $t \in V$ *such that* $\mu_1(f_t) = \min_{i\in V}\{\mu_1(f_i)\}$. *Then* $\frac{g}{f_t}$ *has the series expansion around* $z = 0$ *of the form* $\sum_{k=0}^{\infty} a_k(\tau)z^k$, *and*

$$g(z) \equiv 0 \text{ if and only if } a_k(\tau) \equiv 0 \text{ for } k = 0,1,\ldots,\mu_2(f_t)+\mu_3(f_t)+\mu_4(f_t).$$

Proof. According to Proposition 2.2, only θ_1 has a zero at $z = 0$. Then the choice of t can insure that $\frac{g}{f_t}$ has no pole at $z = 0$. Hence $\frac{g}{f_t}$ is analytic around $z = 0$, and then we have the Taylor expansion around $z = 0$. Moreover, $\frac{g}{f_t}$ is elliptic by Theorem 5.1, and has at most $\mu_2(f_t)+\mu_3(f_t)+\mu_4(f_t)$ poles in the period-parallelogram $P(\pi,\pi\tau)$ by Proposition 2.2. If $a_k(\tau) \equiv 0$ for $k = 0,1,\ldots,\mu(f_t)$, we deduce that $\frac{g}{f_t}$ has more than $\mu_2(f_t)+\mu_3(f_t)+\mu_4(f_t)$ zeros in $P(\pi,\pi\tau)$, which implies by Theorem 2.2 that $\frac{g}{f_t} = 0$. On the other hand if $g = 0$ then $\frac{g}{f_t} = 0$ and we have $a_k(\tau) \equiv 0$ for all $k \geqslant 0$.

□

Theorem 5.2 can also be stated as an algorithm.

Algorithm 5.1. *Given* $f \in \mathbb{K}[\Theta][G_1]$, *we have the following algorithm to decide whether* $f = 0$.

Input: $f \in \mathbb{K}[\Theta][G_1]$.

Output: True if $f = 0$; *False if* $f \neq 0$.

Let $f := \sum_{i=1}^{m} \theta_{X_i,c_i}$ *and let* $\{g_1,\ldots,g_r\}$ *be the set of quasi-elliptic components of* f.

Set $j := 1$. *While* $j \leqslant r$ *do*

Let $p = g_j$. *Write* $p = \sum_{i\in V_j} \theta_{X_i,c_i}$ *where* $V_j \subseteq \{1,\ldots,m\}$;

Choose $t \in V_j$ *such that* $\mu_1(\theta_{X_t,c_t}) = \min_{i\in V_j}\{\mu_1(\theta_{X_i,c_i})\}$;

Let $h := \frac{p}{\theta_{X_t,c_t}}$;

Write $h(z|\tau) = \sum_{k=0}^{\infty} a_k(\tau) z^k$;

Set $k := 0$. *While* $k \leqslant \mu_2(\theta_{X_t,c_t}) + \mu_3(\theta_{X_t,c_t}) + \mu_4(\theta_{X_t,c_t})$ *do*

if $a_k(\tau) \equiv 0$;

k*++;*

otherwise return False;

end do;

i++;

end do;

return True;

Note. In Algorithm 5.1, we use Algorithm 3.2 to check whether $a_k(\tau) \equiv 0$.

Theorem 5.3. *Algorithm 5.1 is correct.*

Proof. By Definition 5.3, we can always write f as the sum of its quasi-elliptic components. If $f = 0$, then by Theorem 5.1 every g_j is zero. Thus every a_k is zero, and Algorithm 5.1 returns True.

If $f \neq 0$, again by Theorem 5.1, there exists $\ell \in \{1, \ldots, r\}$ such that $g_\ell \neq 0$. Then in Algorithm 5.1 the corresponding h is nonzero. Assume h is constant. Then $a_0 \neq 0$ and Algorithm 5.1 returns False. Assume h is not constant. By Theorem 5.1, h is elliptic. Since h has at most $\ell := \mu_2(\theta_{X_t,c_t}) + \mu_3(\theta_{X_t,c_t}) + \mu_4(\theta_{X_t,c_t})$ poles in $P(\pi, \pi\tau)$, by Theorem 2.2, it has at most ℓ zeros in $P(\pi, \pi\tau)$. Therefore a_k can not be all zero for $k \in \{0, 1, \ldots, \ell\}$. Thus Algorithm 5.1 returns False. $\square$

Example 5.1. Prove

$$f(z) := \theta_2(0,q)\theta_3(0,q)\theta_4(0,q)\theta_1(2z,q) - 2\theta_1(z,q)\theta_2(z,q)\theta_3(z,q)\theta_4(z,q) \equiv 0.$$

Proof. One can check that $f(z)$ is a quasi-elliptic component of itself. Let

$$h(z) := \frac{\theta_2(0,q)\theta_3(0,q)\theta_4(0,q)\theta_1(2z,q)}{2\theta_1(z,q)\theta_2(z,q)\theta_3(z,q)\theta_4(z,q)} - 1.$$

We write $h(z) = \sum_{n=1}^{\infty} a_n z^n$. Then according to Algorithm 5.1 we need to check whether $a_0 = a_1 = a_2 = 0$. Let $\theta_j^{(k)} := \theta_j^{(k)}(0,q)$. When expanding $h(z)$ we find that a_0, a_1 and a_3 vanish by computation and

$$a_2 = \frac{1}{2}\left(\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4}\right).$$

By using Algorithm 3.2 we verify that $a_2 = 0$. □

5.3 Examples

We list some examples from [31].

Example 5.3. $\theta_1(z)^3\theta_1(3z) + \theta_4(z)^3\theta_4(3z) - \theta_4(2z)^3\theta_4 \equiv 0.$

Example 5.4. $\theta_2\theta_2(2z)\theta_4(z)^2 - \theta_2(z)^2\theta_4(z)^2 - \theta_1(z)^2\theta_3(z)^2 \equiv 0.$

Example 5.5. $\theta_4^3\theta_4(2z)^2 - \theta_3(z)^4 - \theta_2(z)^4 \equiv 0.$

Remark. A natural generalization is to extend $\mathbb{K}[\Theta][G_1]$ to $\mathbb{K}[\Theta][G_2]$, where

$$G_2 := \{\theta_j(a_1z_1 \cdots + a_nz_n, q) : j = \{1,\ldots,4\} \text{ and } (a_1,\ldots,a_n) \in \mathbb{N}^n \backslash \{\mathbf{0}\} \text{ with } n \in \mathbb{N}\backslash\{0\}\}.$$

We have a method to solve the following problem.

Problem 5.2: Given $f \in \mathbb{K}[\Theta][G_2]$; decide whether $f = 0$.

We tested our method on many examples including the ones below, and it worked fine. The main idea is to reduce the variables $z_1,\ldots,z_n$ once at a time. In the end we only have one variable, and then we can solve it by Algorithm 5.1. More details will be find in the paper [33] that we are currently writing.

Example 5.6. *[14, 20. 7. 6]Prove*

$$\theta_4(0,q)^2\theta_1(w+z,q)\theta_1(w-z,q)-\theta_3(w,q)^2\theta_2(z,q)^2+\theta_2(w,q)^2\theta_3(z,q)^2\equiv 0.$$

Example 5.7. *[23, p. 17]*

$$\sum_{j=1}^{4}(-1)^{j+1}\theta_j(x)\theta_j(y)\theta_j(u)\theta_j(v)-2\theta_3(x_1)\theta_3(y_1)\theta_3(u_1)\theta_3(v_1)\equiv 0,$$

where $x_1 := \frac{1}{2}(x+y+u+v)$ *and* $y_1 := \frac{1}{2}(x+y-u-v)$, $u_1 := \frac{1}{2}(x-y+u-v)$ *and* $v_1 := \frac{1}{2}(x-y-u+v)$.

Example 5.8. *[35] Let* α_i, β_i *(*$i \in \mathbb{Z}/3\mathbb{Z}$*) be six numbers satisfying* $\sum\limits_i \alpha_i = \sum\limits_i \beta_i = 0$. *Then*

$$\sum_i \theta_1(\alpha_i)\theta_1(\beta_i)\theta_1(\alpha_{i-1}+\beta_{i+1})\theta_1(\alpha_{i+1}-\beta_{i-1})\equiv 0.$$

Example 5.9. *[31, p. 480]*

$$\theta_4^2\theta_1(y+z)\theta_1(y-z)-\theta_3(y)^2\theta_2(z)^2+\theta_2(y)^2\theta_3(z)^2\equiv 0.$$

Example 5.10. *[22, p. 21]*

$$\begin{aligned}&\theta_1(x-y)\theta_1(x+y)\theta_1(z-w)\theta_1(z+w)-\theta_1(z-x)\theta_1(z+x)\theta_1(y-w)\theta_1(y+w)\\&\quad+\theta_1(y-z)\theta_1(y+z)\theta_1(x-w)\theta_1(x+w)\equiv 0.\end{aligned}$$

Example 5.11. *[13][Gosper and Schroeppel] Let* w_1, w_2, w_3, z_1, z_2 *and* z_3 *be complex variables, and consider the* 3×3 *matrix whose* j,k *entry is* $\theta_r(w_j - z_k|\tau)\theta_s(w_j+z_k|\tau)$, *where* $r,s \in \{1,2,3,4\}$. *Then*

$$\det(\theta_r(w_j-z_k|\tau)\theta_s(w_j+z_k|\tau))_{1\leqslant j,k\leqslant 3}=0.$$

Chapter 6

Ongoing Work and the "ThetaFunctions" Package

This chapter consists of some work that we are still working on. In Section 1 we illustrate the discovery procedures for new relations among Jacobi theta functions. In Section 2 we indicate our current step of generalizing the previous chapters, which will systematically prove Ramanujan's modular equations and will solve many problems in related research papers and monographs. In Section 3 we demonstrate our Mathematica package "ThetaFunctions".

6.1 Producing Identities

We mainly discuss how to produce relations in Classes I and II. The implementation is not finished yet. Thus we only present the main ideas and some important steps.

6.1.1 Class I

According to the homogeneous decomposition described in Theorem 3.1 of Chapter 3, to find relations in R_Θ, we need to find homogeneous polynomials $p \in R_X$ which map to identities $\phi(p) = 0$ in R_Θ. However, for a given degree d, there might be infinitely many homogeneous polynomials in R_X that map to identities in R_Θ, e.g., any linear combination of such polynomials still maps to an identity. Therefore, we restrict the recovery procedures only on the building-block-relations, which will be described below.

Following Definition 3.5, we can generate all monomials in R_X for a given degree. For instance, the set of all monomials of degree 3 is

$$\begin{aligned}
\{&x_2^6,\ x_2^5\,x_3,\ x_2^4\,x_3^2,\ x_2^3\,x_3^3,\ x_2^2\,x_3^4,\ x_2\,x_3^5,\ x_3^6,\ x_2^5\,x_4,\\
&x_2^4\,x_3\,x_4,\ x_2^3\,x_3^2\,x_4,\ x_2^2\,x_3^3\,x_4,\ x_2\,x_3^4\,x_4,\ x_3^5\,x_4,\\
&x_2^4\,x_4^2,\ x_2^3\,x_3\,x_4^2,\ x_2^2\,x_3^2\,x_4^2,\ x_2\,x_3^3\,x_4^2,\ x_3^4\,x_4^2,\ x_2^3\,x_4^3,\\
&x_2^2\,x_3\,x_4^3,\ x_2\,x_3^2\,x_4^3,\ x_3^3\,x_4^3,\ x_2^2\,x_4^4,\ x_2\,x_3\,x_4^4,\ x_3^2\,x_4^4,\\
&x_2\,x_4^5,\ x_3\,x_4^5,\ x_4^6,\ x_2^3\,x_1{}',\ x_2^2\,x_3\,x_1{}',\ x_2\,x_3^2\,x_1{}',\\
&x_3^3\,x_1{}',\ x_2^2\,x_4\,x_1{}',\ x_2\,x_3\,x_4\,x_1{}',\ x_3^2\,x_4\,x_1{}',\ x_2\,x_4^2\,x_1{}',\\
&x_3\,x_4^2\,x_1{}',\ x_4^3\,x_1{}',\ (x_1{}')^2,\ x_2\,x_2{}'',\ x_3\,x_2{}'',\ x_4\,x_2{}'',\\
&x_2\,x_3{}'',\ x_3\,x_3{}'',\ x_4\,x_3{}'',\ x_2\,x_4{}'',\ x_3\,x_4{}'',\ x_4\,x_4{}''\}
\end{aligned}\ .$$

We produce the building-block-relations of degree k in the following way.

Step 1: find all monomials of degree k in R_X, say $\{y_1,\ldots,y_n\}$.

Step 2: set $s := 1$, or choose s to be any positive value.

Step 3: make an ansatz of the form

$$c_1\phi(y_1)+\cdots+c_n\phi(y_n)=0, \tag{6.1}$$

where $c_i \in \mathbb{K}$ and ϕ is defined in Section 3.1. In view of Definition 2.7, every $\phi(y_j)$ can be written as a $q^{1/4}$-series. Hence from the left hand side of (6.1) we can extract the coefficients of

$$q^{1/4},\ldots,\left(q^{1/4}\right)^s$$

and obtain a system of equations in the unknowns $c_1,\ldots,c_n$.

Step 4: find a basis of the solution set for the above system of equations in Step 3, say

$$\{a^{(1)},\ldots,a^{(m)}\},$$

where $a^{(j)} = (a_1^{(j)}, \ldots, a_n^{(j)})$ with $a_i^{(j)} \in \mathbb{K}$ and $\sum_{i=1}^{n} a_i^{(j)} \phi(y_i) = 0$ for all $j \in \{1, \ldots, m\}$.

Step 5: apply Algorithm 3.2 on every $f_j := \sum_{i=1}^{n} a_i^{(j)} \phi(y_i)$ and if f_j is nonzero for some j, increase s by one and go back to step 4. If all f_j are zero, then go to step 6.

Step 6: let $A := \{f_1, \ldots, f_m\}$ and delete the polynomials which are multiples of relations with degree smaller than k. This step gives a reduced polynomial set

$$\{g_1, \ldots, g_t\} \subseteq A,$$

which is the desired set of building-block-relations of degree k.

Note. In the setting of our *ThetaFunction* package, we do not use the Deg, instead, we use another definition of degree which is equal to 2Deg. This makes the degree always be an integer and it is more convenient for implementation.

Example. Suppose that we want to find the building-block-relations of degree 3. We run our package and execute the command *identities[d,s]*, where d presents the degree and s is the number in Step 2. We first try $s = 9$.

In[24]:= **identities[6, 9]**

Out[24]=
$$\{\Theta_2^5\,\Theta_3 - \Theta_2\,\Theta_4^5 - 6\,\Theta_3^2\,\Theta_4\,\Theta_1{}' + 6\,\Theta_3\,\Theta_4^2\,\Theta_1{}' + 10\,\Theta_3\,\Theta_2{}'' - 11\,\Theta_4\,\Theta_2{}'' - 2\,\Theta_2\,\Theta_3{}'' + 3\,\Theta_2\,\Theta_4{}'',$$
$$\Theta_2^4\,\Theta_3^2 - 17\,\Theta_3^4\,\Theta_4^2 + 12\,\Theta_3^3\,\Theta_4^3 - 16\,\Theta_3^2\,\Theta_4^4 + 20\,\Theta_3\,\Theta_4^5 + \Theta_4^6 + 8\,\Theta_3\,\Theta_3{}'' - 18\,\Theta_4\,\Theta_3{}'' + 2\,\Theta_3\,\Theta_4{}'' + 8\,\Theta_4\,\Theta_4{}'',$$
$$\Theta_2\,\Theta_3^5 - \Theta_2\,\Theta_4^5 - 6\,\Theta_3^2\,\Theta_4\,\Theta_1{}' + 6\,\Theta_3\,\Theta_4^2\,\Theta_1{}' + 11\,\Theta_3\,\Theta_2{}'' - 11\,\Theta_4\,\Theta_2{}'' - 3\,\Theta_2\,\Theta_3{}'' + 3\,\Theta_2\,\Theta_4{}'',$$
$$\Theta_3^6 - 17\,\Theta_3^4\,\Theta_4^2 + 12\,\Theta_3^3\,\Theta_4^3 - 17\,\Theta_3^2\,\Theta_4^4 + 20\,\Theta_3\,\Theta_4^5 + \Theta_4^6 + 8\,\Theta_3\,\Theta_3{}'' - 18\,\Theta_4\,\Theta_3{}'' + 2\,\Theta_3\,\Theta_4{}'' + 8\,\Theta_4\,\Theta_4{}'',$$
$$\Theta_2^5\,\Theta_4 + \Theta_2\,\Theta_4^5 + \Theta_4\,\Theta_2{}'' - \Theta_2\,\Theta_4{}'',\ \Theta_2^4\,\Theta_3\,\Theta_4 + \Theta_4\,\Theta_3{}'' - \Theta_3\,\Theta_4{}'',$$
$$\Theta_2\,\Theta_3^4\,\Theta_4 + \Theta_4\,\Theta_2{}'' - \Theta_2\,\Theta_4{}'',\ \Theta_3^5\,\Theta_4 - \Theta_3\,\Theta_4^5 + \Theta_4\,\Theta_3{}'' - \Theta_3\,\Theta_4{}'',\ \Theta_2\,\Theta_3\,\Theta_4^4 + \Theta_3\,\Theta_2{}'' - \Theta_2\,\Theta_3{}'',$$
$$\Theta_2^3\,\Theta_1{}' + \Theta_4\,\Theta_3{}'' - \Theta_3\,\Theta_4{}'',\ \Theta_3^3\,\Theta_1{}' + \Theta_4\,\Theta_2{}'' - \Theta_2\,\Theta_4{}'',\ \Theta_4^3\,\Theta_1{}' + \Theta_3\,\Theta_2{}'' - \Theta_2\,\Theta_3{}''\}$$

We use Algorithm 3.2 to check the above set, and find that some are nonzero. So we try $s = 10$.

In[20]:= **identities[6, 10]**

Out[20]= $\{\Theta_2^5\,\Theta_3 - \Theta_2\,\Theta_3^5 - \Theta_3\,\Theta_2'' + \Theta_2\,\Theta_3'',\ \Theta_2^5\,\Theta_4 + \Theta_2\,\Theta_4^5 + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'',\ \Theta_2^4\,\Theta_3\,\Theta_4 + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'',$
$\Theta_2\,\Theta_3^4\,\Theta_4 + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'',\ \Theta_3^5\,\Theta_4 - \Theta_3\,\Theta_4^5 + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'',\ \Theta_2\,\Theta_3\,\Theta_4^4 + \Theta_3\,\Theta_2'' - \Theta_2\,\Theta_3'',$
$\Theta_2^3\,\Theta_1' + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'',\ \Theta_3^3\,\Theta_1' + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'',\ \Theta_4^3\,\Theta_1' + \Theta_3\,\Theta_2'' - \Theta_2\,\Theta_3''\}$

We use Algorithm 3.2 to check the above set, and it turns out that all elements are zero. Note that in our implementation, the above set is already reduced as in Step 5.

Thus the set of building-block-relations of degree 3 is

$$\begin{aligned}
\{&\Theta_2^5\,\Theta_3 - \Theta_2\,\Theta_3^5 - \Theta_3\,\Theta_2'' + \Theta_2\,\Theta_3'' = 0,\\
&\Theta_2^5\,\Theta_4 + \Theta_2\,\Theta_4^5 + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'' = 0,\\
&\Theta_2^4\,\Theta_3\,\Theta_4 + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'' = 0,\\
&\Theta_2\,\Theta_3^4\,\Theta_4 + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'' = 0,\\
&\Theta_3^5\,\Theta_4 - \Theta_3\,\Theta_4^5 + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'' = 0,\\
&\Theta_2\,\Theta_3\,\Theta_4^4 + \Theta_3\,\Theta_2'' - \Theta_2\,\Theta_3'' = 0,\\
&\Theta_2^3\,\Theta_1' + \Theta_4\,\Theta_3'' - \Theta_3\,\Theta_4'' = 0,\\
&\Theta_3^3\,\Theta_1' + \Theta_4\,\Theta_2'' - \Theta_2\,\Theta_4'' = 0,\\
&\Theta_4^3\,\Theta_1' + \Theta_3\,\Theta_2'' - \Theta_2\,\Theta_3'' = 0\}
\end{aligned}.$$

6.1.2 Class II

From Chapter 4 we know that to generate identities in $\hat{R}_\Theta$ we only need to generate quasi-elliptic functions which equal to zero. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4$ we define that the degree of the monomial $\theta^\alpha(z|\tau)$ is equal to $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. By Definition 4.2 we know that for any quasi-elliptic $f^\Psi \in \hat{R}_\Theta$, the summand monomials have the same degree.

Definition 6.1. *We define the degree of a quasi-elliptic $f^\Psi \in \hat{R}_\Theta$ to be the degree of its summands, denoted by* $\deg(f^\Psi)$.

For instance, the degree of the quasi-elliptic function $\theta_2(0|\tau)^2\theta_2(z|\tau)^2 + \theta_2(0|\tau)^2\theta_2(z|\tau)^2$ is 2.

Definition 6.2. *Given $k \in \mathbb{N}$ and $a, b \in \{1,2\}$, let*

$$Y_{a,b}(k) := \{\theta^\alpha : |\alpha| = k, \alpha_1 + \alpha_4 \equiv a + 1 (\mathrm{mod}\, 2), \alpha_1 + \alpha_2 \equiv b + 1 (\mathrm{mod}\, 2)\}$$

and define the following partition of monomials in $\hat{R}_\Theta$ of degree k:

$$Y(k) := \{Y_{a,b}(k) \neq \varnothing : a,b \in \{1,2\}\}.$$

We call $Y(k)$ the set of quasi-elliptic monomial sets of degree k.

Hence generating quasi-elliptic functions is equivalent to generating $Y(k)$ for $k \in \mathbb{N}$. Note that Definition 6.2 also gives us a way to compute $Y(k)$. For example, let $\theta_j(z) := \theta_j(z|\tau)$, then

$$Y(2) = \{\{\theta_1(z)^2, \theta_2(z)^2, \theta_3(z)^2, \theta_4(z)^2\}\}$$

and

$$\begin{aligned} Y(3) = \{&\{\theta_3(z)\theta_4(z)^2, \theta_3(z)^3, \theta_2(z)^2\theta_3(z), \theta_1(z)^2\theta_3(z)\},\\ &\{\theta_4(z)^3, \theta_3(z)^2\theta_4(z), \theta_1(z)^2\theta_4(z)\},\\ &\{\theta_1(z)\theta_4(z)^2, \theta_1(z)\theta_3(z)^2, \theta_1(z)\theta_2(z)^2, \theta_1(z)^3\}\}. \end{aligned}$$

Lemma 6.1. *Let $f^\Psi := f^\Psi_M$ be quasi-elliptic with $\deg(f^\Psi) = t$ and $f^\Psi(z) \equiv \sum_{j=0}^{\infty} v_j(\tau)z^j$. If $v_j(\tau) \equiv 0$ for $j \in \{0,\ldots,t\}$ then $f^\Psi = 0$.*

Proof. Let $g(z) := \frac{f^\Psi(z)}{\theta_3(z)^t} \equiv \sum_{k=0}^{\infty} d_k(\tau)z^k$ and $\theta_3(z)^t \equiv \sum_{\ell=0}^{\infty} u_\ell(\tau)z^\ell$. One can check that $u_0 \neq 0$. Then

$$\sum_{j=0}^{\infty} v_j(\tau)z^j \equiv \sum_{k=0}^{\infty} d_k(\tau)z^k \sum_{\ell=0}^{\infty} u_\ell(\tau)z^\ell$$

and

$$v_j = \sum_{i=0}^{j} u_i d_{j-i}. \tag{6.2}$$

Assume that $v_j(\tau) \equiv 0$ for $j \in \{0,\ldots,t\}$. First we let $j = 0$. By equation (6.2) and $u_0 \neq 0$ we have $d_0 = 0$. Next we let $j = 1$. Again by (6.2) and $u_0 \neq 0$ we have $0 = u_1 d_0 + u_0 d_1$ and $d_1 = -u_1 d_0/u_0 = 0$. Once more we let $j = 2$. We have $0 = u_2 d_0 + u_1 d_1 + u_0 d_2$ and $d_2 = (-u_2 d_0 - u_1 d_1)/u_0 = 0$. We continue this procedure and obtain that $d_k = (-\sum_{i=1}^{j} u_i d_{j-i})/u_0 = 0$ for all $k \in \{0,\ldots,t\}$. Therefore $g(z)$ has a zero at $z = 0$ with multiplicity at least $t+1$, i.e., $g(z)$ has at least $t+1$ zeros in the period-parallelogram $P(\pi, \pi\tau)$. On the other hand, $\theta_3(z)^t$ has exactly one zero with multiplicity t in $P(\pi, \pi\tau)$, which means, $g(z)$ has at most t poles in $P(\pi, \pi\tau)$. Thus $g(z)^2$ has at least $2t+2$ zeros and at most $2t$ poles in $P(\pi,\pi\tau)$. Since $g(z)^2$ is elliptic, by Theorem 2.2 we deduce that $g(z)^2 \equiv 0$. Therefore, $f^{\Psi} = 0$. $\square$

For any $k \in \mathbb{N}\backslash\{0\}$, to generate building-block quasi-elliptic functions f^{Ψ} such that $\deg(f^{\Psi}) = k$ and $f^{\Psi} = 0$, we follow the steps below.

Step 1: compute $Y(k)$.

Step 2: for each $x \in Y(k)$, divide out the common factor of elements in x and obtain the reduced set $Y_1(k)$. For instance

$$Y_1(2) = \{\{\theta_1(z)^2, \theta_2(z)^2, \theta_3(z)^2, \theta_4(z)^2\}\}$$

and

$$\begin{aligned} Y_1(3) = \{&\{\theta_4(z)^2, \theta_3(z)^2, \theta_2(z)^2, \theta_1(z)^2\}, \\ &\{\theta_4(z)^2, \theta_3(z)^2, \theta_1(z)^2\}\}. \end{aligned}$$

Step 3: for any $a, b \in Y_1(k)$, if $a \subseteq b$ then delete a from $Y_1(k)$ and obtain a new set $\tilde{Y}(k)$. For instance

$$\tilde{Y}(2) = \{\{\theta_4(z)^2, \theta_3(z)^2, \theta_2(z)^2, \theta_1(z)^2\}\}$$

and

$$\tilde{Y}(3) = \{\{\theta_4(z)^2, \theta_3(z)^2, \theta_2(z)^2, \theta_1(z)^2\}\}.$$

Note. We see from the above that $\tilde{Y}(3)$ generates the same identities as $\tilde{Y}(2)$. In this case, to produce identities of degree 3 we produce identities of degree 2 and then multiply every identity by a single $\theta_j(z)$. In other words, the identities of degree 3 are equivalent to identities of degree 2. But for a given degree k, if $\tilde{Y}(k) \neq \tilde{Y}(t)$ for all $0 < t < k$, we continue the next steps.

Step 4: choose an arbitrary but fixed $y \in \tilde{Y}(k)$; assume $y = \{y_1, \ldots, y_n\}$.

Step 5: make an ansatz of the form

$$c_1 y_1 + \cdots + c_n y_n = 0, \tag{6.3}$$

where $c_i \in R_{\Theta}$.

Step 6: expand the left hand side of (6.3) around $z = 0$, and deduce that for $\ell \geqslant 0$,

$$\sum_{i=0}^{n} c_i \frac{y_i^{(\ell)}(0)}{\ell!} = 0,$$

where $y_i^{(\ell)}(0) \in R_\Theta$. Hence

$$\sum_{i=0}^{n} c_i y_i^{(\ell)}(0) = 0. \tag{6.4}$$

By Lemma 6.1, we only need to make sure that the first $k+1$ coefficients in the expansion are zero. Thus we have:

Step 7: write the first $k+1$ equations into the form

$$\underbrace{\begin{pmatrix} y_1(0) & \cdots & y_n(0) \\ \vdots & \vdots & \vdots \\ y_1^{(k)}(0) & \cdots & y_n^{(k)}(0) \end{pmatrix}}_{D} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0. \tag{6.5}$$

Step 8: apply Algorithm 3.2 on every entry of D and replace by 0 when Algorithm 3.2 returns True; and obtain a simplified version of D, denoted by D_1.

Step 9: apply row reduction on D_1 to obtain a triangular matrix D_2.

Note. Here in the row reduction, Algorithm 3.2 is crucial to ensure that the pivot element chosen in each step of the row reduction is nonzero.

Step 10: compute the basis of the null space of the matrix D_2 and use the relations generated in Section 6.1 to simplify the solutions. Suppose the simplified solutions are $\{b_1, \ldots, b_m\}$, where $b_j = (b_j^{(1)}, \ldots, b_j^{(n)})$ with $b_j^{(t)} \in R_\Theta$, then the relations for this chosen y are

$$\left\{ \sum_{i=1}^{n} b_j^{(i)} y_i : j \in \{1, \ldots, m\} \right\}.$$

Step 11: repeat the above steps on every element of $\tilde{Y}(k)$, and take the union of all sets of relations, and obtain a desired set of relations of degree k.

Compared to Class I, the procedures of discovery for this Class in our package have far more tedious computations. So in the following example we omit the demonstration and only list the results.

For example, the building-block identities of degree 2 are

$$\theta_3(0)^2\theta_1(z)^2+\theta_4(0)^2\theta_2(z)^2-\theta_2(0)^2\theta_4(z)^2\equiv 0$$

and

$$\theta_4(0)^2\theta_1(z)^2+\theta_3(0)^2\theta_2(z)^2-\theta_2(0)^2\theta_3(z)^2\equiv 0.$$

In [14, 20.7] one finds four identities of degree 2, two more than our result. The other two in fact are combinations of the above two, thus are generated by the above two.

6.2 Generalizations to powers of q

Besides the ongoing generalizations we have mentioned at the end of each chapter, there are several further extensions and generalizations of our algorithmic approach. An interesting extension is from q to powers of q, as this is connected to Ramanujan's modular equations. Let us consider R_{Θ_1}, by which we define a $\mathbb{K}$-algebra generated by

$$\Theta_1 := \left\{\theta_1^{(2k+1)}(0,q^n) : k,n\in\mathbb{N}\right\}\cup\left\{\theta_j^{(2k)}(0,q^n) : k,n\in\mathbb{N}\text{ and } j=2,3,4\right\}.$$

We have a method to deal with the following problem.

Problem 6.1. Given $f\in R_{\Theta_1}$, decide whether $f=0$.

Remark. We are still working on some details to complete the algorithm. Therefore we do not present the method here. Nevertheless, our method succeeded on all of the examples listed below. One of the key ingredients to solve this problem is a variation of Algorithm 3.2.

Notation. For convenience, in this section we define $\theta_j(q^n):=\theta_j(0,q^n)$.

Example 6.1. *[17, p. 218] A form of the cubic modular equation is*

$$\theta_3(q)\theta_3(q^3)-\theta_4(q)\theta_4(q^3)-\theta_2(q)\theta_2(q^3)\equiv 0.$$

Example 6.2. *[9, p. 112] A form of the seventh-order modular equation is*

$$\sqrt{\theta_3(q)\theta_3(q^7)} - \sqrt{\theta_4(q)\theta_4(q^7)} - \sqrt{\theta_2(q)\theta_2(q^7)} \equiv 0.$$

Here one can move one term to the other side and square both sides to remove the square root and so on to make it fit into our function space.

Example 6.3. *The identity*

$$\frac{\theta_3(q^5)\theta_4(q^5)}{\theta_3(q)\theta_4(q)} + \frac{\theta_2(q^5)\theta_3(q^5)}{\theta_2(q)\theta_3(q)} - \frac{\theta_2(q^5)\theta_4(q^5)}{\theta_2(q)\theta_4(q)} \equiv 1.$$

In [5, p.276] it is written in the form

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) \equiv 1.$$

Example 6.4. *[9, p. 40] Entry 25(vi)*

$$2\varphi^2(q) \equiv \varphi^2(q^{\frac{1}{2}}) + \varphi^2(-q^{\frac{1}{2}}),$$

which can be written as

$$2\theta_3(q^2)^2 - \theta_3(q)^2 - \theta_4(q)^2 \equiv 0.$$

Example 6.5. *[10, 2.7] Let* $a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$*, then*

$$a(q) \equiv \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6). \tag{6.6}$$

From its definition, $a(q)$ can be written as $\theta_2(q)\theta_2(q^3) + \theta_3(q)\theta_3(q^3)$. If we also replace $\varphi(q)$ by $\theta_3(q)$ and $\psi(q^2)$ by $(2q^{\frac{1}{4}})^{-1}\theta_2(q)$, then (6.6) becomes trivial.

Example 6.6. *[10, p. 4182] Let* $c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}$*. Then*

$$1 - \frac{\varphi(q)^2}{\varphi(-q^3)^2} - 4\frac{c(q^4)}{c(q)} \equiv 0. \tag{6.7}$$

Note that $c(q)$ can be written as

$$\theta_2(q^{1/3})\theta_2(q) + \theta_3(q^{1/3})\theta_3(q) + \theta_2(q)\theta_2(q^3) + \theta_3(q)\theta_3(q^3).$$

The rewriting is very straight forward, by just comparing the series of theta functions. We plug this into (6.7) and substitute q by q^3; then the left hand side will be in R_{Θ_1} and we can run our algorithm.

Example 6.7. *[10] Let $a(q)$ and $c(q)$ be defined as in the previous examples, and let*

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2}$$

where $\omega := e^{2\pi i/3}$. Then

$$a(q)^3 = b(q)^3 + c(q)^3$$

and

$$a(q)a(q^2) = b(q)b(q^2) + c(q)c(q^2). \tag{6.8}$$

Since $b(q)$ can also be written as a combination of theta functions, we again can run our algorithm as for the previous examples.

Example 6.8. *[5, p. 285] Let $\eta(q) := q^{1/12} \prod_{k=1}^{\infty} (1 - q^{2k})$. Then*

$$\theta_3(q)^2\theta_3(q^5)^2 - \theta_2(q)^2\theta_2(q^5)^2 - \theta_4(q)^2\theta_4(q^5)^2 \equiv 8\eta(q^2)^2\eta(q^{10})^2.$$

In [5], this identity was considered to be with no direct proofs.

Remark. All the Dedekind eta function identities in Micheal Somos's data base [30] in principle can be proven by our algorithm, including his conjecture about a remarkable eta product identity of level 60. Some different ways to prove the identities in [30] can be found in [26] and [27].

6.3 The "ThetaFunctions" Package

We have indicated in Section 6.1 that our package can assist us in the discovery of identities. In this chapter we mainly demonstrate how the package helps to prove identities, based on Algorithms 3.2 and 4.2. However, the implementation is not finished yet. So, here we only show some key features or commands that are already available. The official package will be done soon and the reader will find it on RISC homepage under this link:

`http://www.risc.jku.at/research/combinat/software/`

6.3.1 Proving Identities in Class I

Recall Example 3.7: prove

$$\theta_2(0|\tau)^3\theta_2''(0|\tau)-\theta_3(0|\tau)^3\theta_3''(0|\tau)+\theta_4(0|\tau)^3\theta_4''(0|\tau)\equiv 0.$$

In Example 3.7 we used Algorithm 3.1 to prove this identity. Now we are using Algorithm 3.2 to prove it. According to Chapter 3, we need to show that

$$x_2^3x_2''-x_3^3x_3''+x_4^3x_4''\in\ker\phi.$$

In our package setting, we use the symbols a_k, respectively b_k, c_k, d_k, to present $x_1^{(k)}$, respectively $x_2^{(k)}$, $x_3^{(k)}$ and $x_4^{(k)}$. The input for the above example is

$$b_0^3b_2-c_0^3c_2+d_0^3d_2.$$

The command *ProveClass1* gives True if the input is in kerϕ, otherwise it gives False. For this example we have

```
In[43]:= ProveClass1[b0^3 b2 - c0^3 c2 + d0^3 d2]

Out[43]= True
```

If the users want to see more details of the proving procedures, some other commands are also available, which we describe as follows. According to Algorithm 3.2, we first have to compute the coefficients of the S_X transformation, which is done by the command *SXCoefficientList*.

In[89]:= **SXCoefficientList[b_0 ^ 3 b_2 - c_0 ^ 3 c_2 + d_0 ^ 3 d_2]**

Out[89]= $\left\{-\frac{2\,i\,b_0^4}{\pi}+\frac{2\,i\,c_0^4}{\pi}-\frac{2\,i\,d_0^4}{\pi},\ -b_0^3\,b_2+c_0^3\,c_2-d_0^3\,d_2\right\}$

This means

$$S_X(x_2^3x_2''-x_3^3x_3''+x_4^3x_4'')=g_1s^{c_1}+g_2s^{c_2},$$

where $g_1=-\frac{2ix_2^4}{\pi}+\frac{2ix_3^4}{\pi}-\frac{2ix_4^4}{\pi}$, $g_2=-x_2^3x_2''+x_3^3x_3''-x_4^3x_4''$ and $c_1\neq c_2$ are some half integers.

Next we need to compute $\mathrm{LT}(g_1)$ and $\mathrm{LT}(g_2)$. The computation for the leading term orbit is carried out by the command *LeadingtermOrb*.

In[90]:= **LeadingtermOrb$\left[-b_0^3\,b_2+c_0^3\,c_2-d_0^3\,d_2\right]$**

Out[90]= $\left\{-b_0^3\,b_2+c_0^3\,c_2-d_0^3\,d_2,\ b_0^3\,b_2-c_0^3\,c_2+d_0^3\,d_2\right\}$

In[91]:= **LeadingtermOrb$\left[-\frac{2\,i\,b_0^4}{\pi}+\frac{2\,i\,c_0^4}{\pi}-\frac{2\,i\,d_0^4}{\pi}\right]$**

Out[91]= $\left\{-\frac{2\,i\,b_0^4}{\pi}+\frac{2\,i\,c_0^4}{\pi}-\frac{2\,i\,d_0^4}{\pi},\ \frac{2\,i\,b_0^4}{\pi}-\frac{2\,i\,c_0^4}{\pi}+\frac{2\,i\,d_0^4}{\pi}\right\}$

This means

$$\mathrm{LT}(p_2)=\{-x_2^3x_2''+x_3^3x_3''-x_4^3x_4'',x_2^3x_2''-x_3^3x_3''+x_4^3x_4''\}$$

and

$$\mathrm{LT}(p_1)=\left\{-\frac{2ix_2^4}{\pi}+\frac{2ix_3^4}{\pi}-\frac{2ix_4^4}{\pi},\frac{2ix_2^4}{\pi}-\frac{2ix_3^4}{\pi}+\frac{2ix_4^4}{\pi}\right\}.$$

Finally we compute the order of the q-series expansions of $\prod_{g\in \mathrm{LT}(p_1)} \phi(g)$ and $\prod_{g\in \mathrm{LT}(p_2)} \phi(g)$. The command for this is *CheckTheSeries*, and *CheckTheSeries*(p_j) returns True if

$$\operatorname{ord}\left(\prod_{g\in \mathrm{LT}(p_j)} \phi(g)\right) > \frac{\mathrm{Deg}(p_j)|\mathrm{LT}(p_j)|}{6};$$

otherwise it returns False.

In[3]:= `CheckTheSeries`$\left[-\frac{2\,i\,b_0^4}{\pi}+\frac{2\,i\,c_0^4}{\pi}-\frac{2\,i\,d_0^4}{\pi}\right]$

Out[3]= `True`

In[4]:= `CheckTheSeries`$\left[-b_0^3\,b_2+c_0^3\,c_2-d_0^3\,d_2\right]$

Out[4]= `True`

Similarly, for Example 3.8 we have the input

$$c_4c_0-3c_2^2-2c_0^2b_0^4d_0^4,$$

and

In[45]:= **ProveClass1**$[c_0\,c_4-3\,c_2\,\hat{}\,2-2\,b_0\,\hat{}\,4\,c_0\,c_0\,d_0\,\hat{}\,4]$

Out[45]= `True`

We ask for the details of the proving procedures and it gives

In[99]:= **SXCoefficientList**$[c_4\,c_0-3\,c_2\,\hat{}\,2-2\,c_0\,\hat{}\,2\,b_0\,\hat{}\,4\,d_0\,\hat{}\,4]$

Out[99]= $\{3\,i\,c_2^2-i\,c_0\,c_4+2\,i\,b_0^4\,c_0^2\,d_0^4\}$

In[100]:= **LeadingtermOrb**$\left[3\,i\,c_2^2-i\,c_0\,c_4+2\,i\,b_0^4\,c_0^2\,d_0^4\right]$

Out[100]= $\{-3\,c_2^2+c_0\,c_4-2\,b_0^4\,c_0^2\,d_0^4,\ -3\,i\,c_2^2+i\,c_0\,c_4-2\,i\,b_0^4\,c_0^2\,d_0^4,\ 3\,i\,c_2^2-i\,c_0\,c_4+2\,i\,b_0^4\,c_0^2\,d_0^4,\ 3\,c_2^2-c_0\,c_4+2\,b_0^4\,c_0^2\,d_0^4,\ 3\,b_2^2-b_0\,b_4-2\,b_0^2\,c_0^4\,d_0^4,\ 3\,i\,b_2^2-i\,b_0\,b_4-2\,i\,b_0^2\,c_0^4\,d_0^4,\ -3\,i\,b_2^2+i\,b_0\,b_4+2\,i\,b_0^2\,c_0^4\,d_0^4,\ -3\,b_2^2+b_0\,b_4+2\,b_0^2\,c_0^4\,d_0^4,\ -2\,b_0^4\,c_0^4\,d_0^2+3\,d_2^2-d_0\,d_4,\ -2\,i\,b_0^4\,c_0^4\,d_0^2+3\,i\,d_2^2-i\,d_0\,d_4,\ 2\,i\,b_0^4\,c_0^4\,d_0^2-3\,i\,d_2^2+i\,d_0\,d_4,\ 2\,b_0^4\,c_0^4\,d_0^2-3\,d_2^2+d_0\,d_4\}$

In[6]:= CheckTheSeries$[3\,i\,c_2^2-i\,c_0\,c_4+2\,i\,b_0^4\,c_0^2\,d_0^4]$

Out[6]= True

6.3.2 Proving Identities in Class II

We have two algorithms for proving identities in Class II, namely Algorithms 4.1 and 4.2. For elements from $\hat{R}_\Theta$ we use Algorithm 4.1 and for elements from $\hat{H}_{\tilde{\Theta}}$ we use Algorithm 4.2. Note that Algorithm 4.1 is also suitable for $\hat{H}_{\tilde{\Theta}}$, but is slower compared to Algorithm 4.2.

Recall Example 4.1, which is suitable for Algorithm 4.1. The input for Class II is a set like the following *test1*.

In[32]:=
```
c1 = -8 dd2^5 dd3^2 dd4^3 - 2 dd2 dd3^6 dd4^3 -
    2 dd2 dd3^2 dd4^7 - 16 dd3^2 dd4^3 d2[2] + 16 dd2 dd3^2 dd4^2 d4[2];
c2 = 7 dd2^5 dd3^4 dd4 + dd2 dd3^8 dd4 + dd2 dd3^4 dd4^5 +
    8 dd3^4 dd4 d2[2] - 8 dd2 dd3^4 d4[2];
c3 = dd2^5 dd4^5 + dd2 dd3^4 dd4^5 + dd2 dd4^9 + 8 dd4^5 d2[2] - 8 dd2 dd4^4 d4[2];
c4 = -6 dd2^5 dd3^2 dd4^3;
test1 = {{c1, {0, 0, 2, 2}},
    {c2, {0, 0, 0, 4}}, {c3, {0, 0, 4, 0}}, {c4, {2, 2, 0, 0}}};
```

Every pair in the set, for example $\{c1,\{0,2,0,0\}\}$ represents the following combination:

- The first component of each pair represents a coefficient. The special notion $dd2$, respectively $dd3$ and $dd4$, stands for $\theta_2(0|\tau)$, respectively $\theta_3(0|\tau)$ and $\theta_4(0|\tau)$; and $d1[k]$, respectively $d2[k]$, $d3[k]$, $d4[k]$, stands for $\theta_1^{(k)}(0|\tau)$, respectively, $\theta_2^{(k)}(0|\tau)$, $\theta_3^{(k)}(0|\tau)$, $\theta_4^{(k)}(0|\tau)$.
- The tuple $\{a_1,a_2,a_3,a_4\}$ represents the monomial $\theta_1(z|\tau)^{a_1}\theta_2(z|\tau)^{a_2}\theta_3(z|\tau)^{a_3}\theta_4(z|\tau)^{a_4}$. For example $\{0,0,2,2\}$ stands for $\theta_3(z|\tau)^2\theta_4(z|\tau)^2$.

The command for automatic zero-recognition for $\hat{R}_\Theta$ is *ProveClass21*. It returns true if the tested function is zero, otherwise it returns False. For this example we have

```
In[37]:= ProveClass21[test1]
```

```
Out[37]= True
```

Recall Example 4.2: prove

$$\theta_2(0|\tau)^2\theta_2(z|\tau)^2 - \theta_3(0|\tau)^2\theta_3(z|\tau)^2 + \theta_4(0|\tau)^2\theta_4(z|\tau)^2 \equiv 0.$$

The input is:

```
In[29]:= test2 = {{dd2^2, {0, 2, 0, 0}},
        {-dd3^2, {0, 0, 2, 0}}, {dd4^2, {0, 0, 0, 2}}};
```

The command for automatic zero-recognition for $\hat{H}_{\tilde{\Theta}}$ is *ProveClass22*. It returns true if the tested function is zero, otherwise it returns False. We compare the time of Algorithm 4.1 and Algorithm 4.2 and get

```
In[13]:= Timing[ProveClass21[test2]]
```

```
Out[13]= {0.035143, True}
```

```
In[14]:= Timing[ProveClass22[test2]]
```

```
Out[14]= {0.025771, True}
```

For this example the time of running Algorithm 4.1 is slightly longer than the time of running Algorithm 4.2.

We consider another example. By applying the producing-identity feature for Class I, which we have introduced in Section 6.1, we can simplify the coefficients in Example 4.2 by looking at the tables of relations and plugging into the coefficients. We find that the left hand side of Example 4.2 is equal to

$$c_1\theta_3(z|\tau)^2\theta_4(z|\tau)^2 + c_2\theta_4(z|\tau)^4 + c_3\theta_3(z|\tau)^4 + c_4\theta_1(z|\tau)^2\theta_2(z|\tau)^2, \tag{6.9}$$

where $c_1 = \theta_2(0|\tau)^4 - 2\theta_3(0|\tau)^4$, $c_2 = c_3 = \theta_3(0|\tau)^2\theta_4(0|\tau)^2$ and $c_4 = \theta_2(0|\tau)^4$. To prove that expression (6.9) is identically zero, we have the input

```
In[38]:= test3 = {{dd2 ^ 4 - 2 dd3 ^ 4, {0, 0, 2, 2}},
       {dd3 ^ 2 dd4 ^ 2, {0, 0, 0, 4}},
       {dd3 ^ 2 dd4 ^ 2, {0, 0, 4, 0}}, {dd2 ^ 4, {2, 2, 0, 0}}};
```

Although the coefficients are more complicated than the previous example, this example still fits in both Algorithms 4.1 and 4.2. We compare the time and get

```
In[22]:= Timing[ProveClass21[test3]]

Out[22]= {1.43589, True}

In[23]:= Timing[ProveClass22[test3]]

Out[23]= {0.115965, True}
```

For this example the time of running Algorithm 4.1 is substantially longer than the time of running Algorithm 4.2.

We now analyze why Algorithm 4.2 is faster. The most difficult and time-consuming part in Algorithm 4.2 is to compute the orbit

$$\{t_1|_{r_1}\rho, \ldots, t_m|_{r_m}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\},$$

which is needed for the orbit length ℓ. The t_i do not contain any of $\theta_j^{(k)}$ $(k \geqslant 1)$, except for θ_1'. According to Lemmas 2.4 and 2.5, all of $\theta_2, \theta_3, \theta_4$ and θ_1' have very simple modular transformations. In contrast, Algorithm 4.1 uses Algorithm 3.2 and it directly computes the leading term orbits of certain coefficients in the series expansion $\sum_{k=0}^{\infty} d_k(\tau)z^k$, which contains $\theta_j^{(k)}$ $(k \geqslant 1)$. According to Corollary 3.1, the modular transformations for $\theta_j^{(k)}$ $(k \geqslant 1)$ are sophisticated. In addition, the coefficients become more complicated when the degree of z goes higher. Thus Algorithm 4.1 needs more time on the orbit computation.

The orbit computation in Algorithm 4.2 is done by the command *TheOrbit*.

For *test2* we know that $\theta_2(0|\tau)^2\theta_2(z|\tau)^2 - \theta_3(0|\tau)^2\theta_3(z|\tau)^2 + \theta_4(0|\tau)^2\theta_4(z|\tau)^2$ is a quasi-elliptic component of itself and hence can ask for the corresponding orbit. We have

In[41]:= `Timing[TheOrbitClass2[test2]]`

Out[41]=
$$\left\{0.023088, \left\{\left\{1, -\frac{dd3^4}{dd2^4}, \frac{dd4^2\, d1[1]^2}{dd2^6\, dd3^2}\right\}, \left\{1, -\frac{dd3^4}{dd4^4}, \frac{dd2^2\, d1[1]^2}{dd3^2\, dd4^6}\right\}, \left\{1, \frac{dd4^4}{dd2^4}, -\frac{dd3^2\, d1[1]^2}{dd2^6\, dd4^2}\right\}, \left\{1, -\frac{dd4^4}{dd3^4}, -\frac{dd2^2\, d1[1]^2}{dd3^6\, dd4^2}\right\}, \left\{1, \frac{dd2^4}{dd4^4}, -\frac{dd3^2\, d1[1]^2}{dd2^2\, dd4^6}\right\}, \left\{1, -\frac{dd2^4}{dd3^4}, -\frac{dd4^2\, d1[1]^2}{dd2^2\, dd3^6}\right\}\right\}\right\},$$

where the meanings of dd and $d1[1]$ are explained in the beginning of this sub-section.

Only 0.023 second was needed for the whole computation, given that this was done on a 4 year old 1.8 GHz Intel Core i5 laptop.

We can also ask for the orbit for *test3*.

In[42]:= `Timing[TheOrbitClass2[test3]]`

Out[42]=

$$\Big\{0.072315,\ \Big\{\Big\{1,\ \frac{dd4^2\,d1[1]^2}{dd2^2\,dd3^2\,(dd2^4-2\,dd3^4)},\ \frac{dd2^2\,dd3^6\,dd4^2}{(dd2^4-2\,dd3^4)\,d1[1]^2},\ \frac{dd2^8}{dd2^4-2\,dd3^4}\Big\},$$

$$\Big\{1,\ -\frac{dd2^2\,d1[1]^2}{2\,dd3^6\,dd4^2-dd3^2\,dd4^6},\ -\frac{dd2^2\,dd3^6\,dd4^2}{(2\,dd3^4-dd4^4)\,d1[1]^2},\ \frac{dd4^8}{2\,dd3^4-dd4^4}\Big\},$$

$$\Big\{1,\ -\frac{dd3^2\,d1[1]^2}{dd2^6\,dd4^2+2\,dd2^2\,dd4^6},\ -\frac{dd2^2\,dd3^2\,dd4^6}{(dd2^4+2\,dd4^4)\,d1[1]^2},\ -\frac{dd2^8}{dd2^4+2\,dd4^4}\Big\},$$

$$\Big\{1,\ \frac{dd4^2\,d1[1]^2}{dd2^6\,dd3^2-2\,dd2^2\,dd3^6},\ \frac{dd2^2\,dd3^6\,dd4^2}{(dd2^4-2\,dd3^4)\,d1[1]^2},\ \frac{dd2^8}{dd2^4-2\,dd3^4}\Big\},$$

$$\Big\{1,\ \frac{dd2^2\,d1[1]^2}{-dd3^6\,dd4^2+2\,dd3^2\,dd4^6},\ \frac{dd2^2\,dd3^2\,dd4^6}{(dd3^4-2\,dd4^4)\,d1[1]^2},\ -\frac{dd3^8}{dd3^4-2\,dd4^4}\Big\},$$

$$\Big\{1,\ -\frac{dd3^2\,d1[1]^2}{2\,dd2^6\,dd4^2+dd2^2\,dd4^6},\ -\frac{dd2^6\,dd3^2\,dd4^2}{(2\,dd2^4+dd4^4)\,d1[1]^2},\ \frac{dd4^8}{2\,dd2^4+dd4^4}\Big\},$$

$$\Big\{1,\ \frac{dd4^2\,d1[1]^2}{2\,dd2^6\,dd3^2-dd2^2\,dd3^6},\ -\frac{dd2^6\,dd3^2\,dd4^2}{(2\,dd2^4-dd3^4)\,d1[1]^2},\ \frac{dd3^8}{-2\,dd2^4+dd3^4}\Big\},$$

$$\Big\{1,\ -\frac{dd2^2\,d1[1]^2}{2\,dd3^6\,dd4^2-dd3^2\,dd4^6},\ \frac{dd2^2\,dd3^6\,dd4^2}{(-2\,dd3^4+dd4^4)\,d1[1]^2},\ \frac{dd4^8}{2\,dd3^4-dd4^4}\Big\},$$

$$\Big\{1,\ \frac{dd4^2\,d1[1]^2}{2\,dd2^6\,dd3^2-dd2^2\,dd3^6},\ \frac{dd2^6\,dd3^2\,dd4^2}{(-2\,dd2^4+dd3^4)\,d1[1]^2},\ \frac{dd3^8}{-2\,dd2^4+dd3^4}\Big\},$$

$$\Big\{1,\ -\frac{dd2^2\,d1[1]^2}{dd3^6\,dd4^2-2\,dd3^2\,dd4^6},\ \frac{dd2^2\,dd3^2\,dd4^6}{(dd3^4-2\,dd4^4)\,d1[1]^2},\ -\frac{dd3^8}{dd3^4-2\,dd4^4}\Big\},$$

$$\Big\{1,\ \frac{dd2^2\,d1[1]^2}{-2\,dd3^6\,dd4^2+dd3^2\,dd4^6},\ \frac{dd2^2\,dd3^6\,dd4^2}{(-2\,dd3^4+dd4^4)\,d1[1]^2},\ \frac{dd4^8}{2\,dd3^4-dd4^4}\Big\}\Big\}\Big\}$$

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