# ISSAC 2010 Software Presentation GENOM3CK- A library for genus computation of plane complex algebraic curves using knot theory

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**Abstract**

We report on a library for computing the genus of a plane complex algebraic curve using knot theory. The library also computes other type of information about the curve, such as for instance: the set of singularities of the curve, the topological type (algebraic link) of each singularity, the Alexander polynomial of each algebraic link, the delta-invariant of each singularity, etc. Using the algebraic geometric modeler called Axel [1], we integrate symbolic, numeric and graphical capabilities into a single library, which we call GENOM3CK [3].

## 1 Describing the library

GENOM3CK is a library designed mainly for computing the genus of a plane complex algebraic curve defined by a squarefree polynomial with coefficients of limited accuracy, i.e. the coefficients may be exact data (i.e. integer or rational numbers) or inexact data (i.e. real numbers).

### Method and algorithm specifications

The proposed symbolic-numeric algorithm from GENOM3CK is based on knot theory and on the topology analysis of the singularities of the curve. More precisely, we use Milnor's theory [5] and Yamamoto's result [7], who showed that the Alexander polynomial is a complete invariant for all the algebraic links up to an ambient isotopy, i.e. it distinguishes all algebraic links up to ambient isotopy. We notice that the computation of the genus reduces to the computation of the delta-invariant of each singularity of the plane complex algebraic curve. Using the algebraic geometric modeler Axel, we compute a piecewise linear approximation of each algebraic link as a graph data structure, which is isotopic to the exact algebraic link of each singularity. We design new computational geometry and combinatorial algorithms for the computation of the Alexander polynomial of each algebraic link, i.e. an adapted version of the Bentley-Ottmann algorithm [2]. From the Alexander polynomial of each algebraic link we compute the delta-invariant of each singularity. The algorithm takes as input parameters, a positive real number $\epsilon \in \mathbb{R}_+^*$, and a domain $B = [-a, a] \times [-b, b] \times [-c, c]$, with $a, b, c \in \mathbb{N}^*$, for the $x, y$ and $z$ coordinates of the three-dimensional space. Together with its main functionality to compute the genus, the library computes other important information about the plane complex algebraic curve: the set of distinct singularities of the curve, the 3-dimensional visualization of the algebraic link of each singularity of the curve, the Milnor fibration of each singularity, the Alexander polynomial of each algebraic link and the delta-invariant of each singularity.

### Short history

GENOM3CK is implemented in the free algebraic geometric modeler Axel [1, 6] written in C++ and in the free computer algebra system Mathemagix [4]. For our purpose Axel provides an easy-to-use interface and unique algebraic tools for the visualization of implicit curves and surfaces in the three-dimensional space. Moreover, the existence of plugins in Axel allows us to reuse and combine all of its computational power with the proposed symbolic-numeric algorithm into one library. We use techniques from Mathemagix, such as, for instance, subdivision techniques for the computation of the singularities of the curve. More information about the library can be found at: http://people.ricam.oeaw.ac.at/m.hodorog/software.html.

### Interface

The interface of GENOM3CK is part of Axel's interface, generated with Qt Script for Applications. All the computational operations performed with the library GENOM3CK are incorporated into a main menu, which is called `Complex Invariant`, and they are divided into geometric, invariant and algebraic properties as shown in Figure 1. The proposed symbolic-numeric algorithm proves to be efficient, as both its theoretical and its practical complexity analysis shows it.

## 2 Examples

We performed several tests experiments with GENOM3CK both on symbolic and on numeric input. In Figure 1 we give a symbolic example. As we notice, the *geometric properties* refer to: the algebraic link of each singularity, the Milnor fibration of each singularity, and some information on the diagram of each algebraic link. The *invariant properties* are represented by the Alexander polynomial of each algebraic link and the delta-invariant of each singularity. The *algebraic properties* contain the set of singularities of the input curve both in the affine and in the projective plane. The library also displays the computational time needed for performing each operation on the input curve. At ISSAC 2010, for the Software Presentations Session, we presented several symbolic and numeric examples performed with GENOM3CK. We also gave more information about setting the input parameters and the input data of the algorithm.

## 3 Conclusion

By computing the topology of each singularity with the generalized stereographic projection method based on Milnor's research, the library allows the analysis of the singularities of the plane complex algebraic curve in the 4-dimensional space. The main advantage of the generalized stereographic

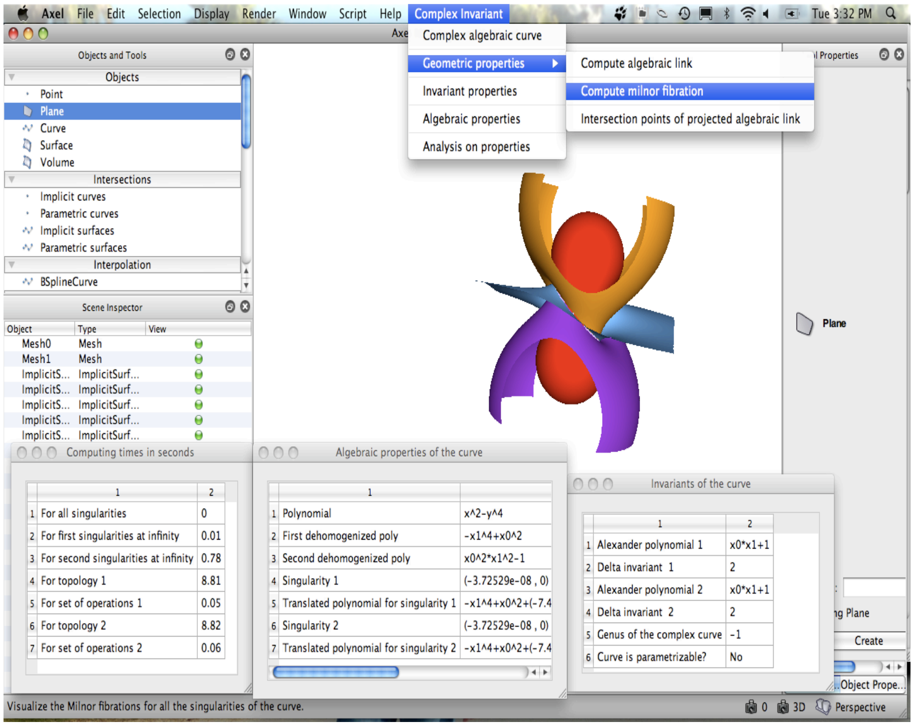


Figure 1: GENOM3CK on the input curve defined by $F(x, y) = x^2 - y^4 \in \mathbb{C}^2$

projection method is the stability with respect to the numeric coefficients of the defining polynomial of the curve. In addition, the library computes one important invariant from knot theory, the Alexander polynomial. Moreover, in the symbolic-numeric algorithm from the library, we basically derive a general formula for the delta-invariant of each singularity, which does not distinguish between the type of singularity (i.e ordinary or nonordinary).

## Acknowledgements

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