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Rational general solutions of parametrizable AODEs

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Dedicated to Kálmán Győry, Attila Pethő, János Pintz, and András Sárközy on the occasions of their birthdays

Abstract. We describe a generic algorithm for deciding the existence of a rational general solution of a parametrizable algebraic ordinary differential equation (AODE) of order 1. In the positive case, we compute such a rational general solution. Our geometric approach depends heavily on rational parametrizations of algebraic curves and surfaces. We also relate our method to classical approaches.

1. Introduction

Ordinary differential equations (ODEs) come in two varieties: *linear* and *non-linear*. Of course, linear ODEs are just a special case of non-linear ones. However, the solutions of linear ODEs have a linear structure and they can be studied by linear algebra tools. Solutions of non-linear ODEs are more difficult to study and there is no general method for all those differential equations. While one can deal with linear ODEs of any order, the order of a non-linear ODE plays an essential role. In this paper we study algebraic ordinary differential equations (AODEs) of order 1.

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In the literature, e.g., [Pia33], [Mur60], AODEs of order 1 are usually studied in two classes: *differential equations of order 1 and of degree 1*, and *differential equations of order 1 and of degree higher than 1.* Typically, an AODE of the second class is manipulated in order to reduce it to an AODE of the first class, which one might know how to solve. Still, there is no general method for computing the solutions, unless the equation is of special type.

Here we describe a full algorithm in the generic case for deciding the existence of a rational general solution of a parametrizable AODE of order 1, i.e., we consider an AODE of the form $F(x, y, y') = 0$, where F is a trivariate polynomial and the algebraic surface $F(u, v, w) = 0$ admits a rational parametrization $\mathcal{P}(s, t)$. Having a rational parametrization of $F(u, v, w) = 0$ allows us to associate the differential equation $F(x, y, y') = 0$ with a planar system of autonomous differential equations of order 1 and of degree 1 in the derivatives of the two parameters s and t. If we can compute a rational general solution of this associated system, (indeed, we can do this in the generic case), then we obtain a rational general solution of $F(x, y, y') = 0$ via the parametrization mapping. Moreover, if the parametrization is proper, then the original AODE has a rational general solution if and only if its associated system w.r.t. $\mathcal{P}(s, t)$ has a rational general solution. This approach extends the geometric approach by R. FENG and X-S. GAO in the case of autonomous AODEs of order 1 ([FG04], [FG06]). Here we give a general outline of our approach. Although details of this method have been published in [NW10] and [NW11], we deem it necessary to combine them in order to obtain a complete algorithm.

Note that the associated system of $F(x, y, y') = 0$ can be transformed into a single differential equation of order 1 and of degree 1 in the derivative. In this sense, our approach reproduces results in the literature (Chapter1V in [Pia33]; Chapter A2, Part I in [Mur60]). Pointing out this correspondence is one of the goals of this paper. On the other hand, we also aim to demonstrate our approach by some complete examples, which are only partially presented in [NW10] and [NW11].

2. Rational general solutions of parametrizable AODEs

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $F(u, v, w)$ be an irreducible trivariate polynomial over $\mathbb{K}$. The AODE of order 1 defined by F is of the form

$$F(x, y, y') = 0,$$

where y is an indeterminate over the differential field of rational functions $\mathbb{K}(x)$ with the usual derivative $y' = \frac{dy}{dx}$.

We are interested in computing a rational general solution of $F(x, y, y') = 0$, i.e., a general solution of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}, \tag{1}$$

where a_i, b_j are constants in a transcendental extension field of $\mathbb{K}$. In the sequel, by an arbitrary constant we mean a transcendental constant over $\mathbb{K}$. A rigorous definition of a general solution of $F(x, y, y') = 0$ can be studied from the point of view of differential algebra, where the definition also applies for AODEs of higher orders. We refer to [Rit50], [Kol73], [Hub96] and [NW10] for a formal definition of a differential ring and a differential ideal. Roughly speaking, the solutions of $F(x, y, y') = 0$ can be identified with the zeros of the radical differential ideal generated by F, denoted by $\{F\}$. One can prove that

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where $S = \frac{\partial F}{\partial y'}$ is the separant of F. The decomposition implies that a solution of F could simultaneously be a solution of S. In that case, the solution is *singular*. However, this rarely happens. Most of solutions of F do not satisfy the equation of S and they are solutions of $\{F\} : S$. Since F is irreducible, the differential ideal $\{F\} : S$ is prime and its generic zero is defined as a *general solution* of $F(x, y, y') = 0$. A generic zero of $\{F\} : S$ of the form (1) is called a *rational general solution* of $F(x, y, y') = 0$.

Example 2.1. Consider differential equation $F \equiv y'^2 + 3y' - 2y - 3x = 0$. Its general solution is $y = \frac{1}{2}((x + c)^2 + 3c)$, where c is an arbitrary constant. The separant of F is $S = 2y' + 3$. So the singular solution of F is $y = -\frac{3}{2}x - \frac{9}{8}$. We can not get this singular solution by evaluating the general solution at a specific constant in $\mathbb{K}$ but that is the only one. Every other solutions of F is derived from the general one by evaluating the general solution at a specific constant in $\mathbb{K}$.

A general solution of $F(x, y, y') = 0$ can be described by giving a Gröbner basis of the differential ideal $\{F\} : S$. For this approach we refer to [Hub96]. In this paper we give a geometric approach to compute an explicit rational general solution of $F(x, y, y') = 0$ provided a certain assumption. To this end we consider the AODE $F(x, y, y') = 0$ whose *solution surface* $F(u, v, w) = 0$ admits a proper rational parametrization

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)),$$

where χ_1, χ_2, χ_3 are bivariate rational functions over $\mathbb{K}$ and the rational map $\mathcal{P}(s,t)$ has a generic rank 2. We call $\mathcal{P}$ proper if it has a rational inverse map. Such an AODE is called a *parametrizable AODE*. A rational solution $y = f(x)$ of $F(x,y,y') = 0$ generates a parametric curve $\mathcal{C}(x) = (x, f(x), f'(x))$ on the solution surface $F(u,v,w) = 0$. Here x is viewed as the parameter of the space curve. We call $\mathcal{C}(x)$ a *solution curve* of $F(x,y,y') = 0$. If $\mathcal{C}(x) \subset \mathrm{dom}(\mathcal{P}^{-1})$ and $(s(x),t(x)) = \mathcal{P}^{-1}(\mathcal{C}(x)) \subset \mathrm{dom}(\mathcal{P})$, then

$$\mathcal{P}(s(x),t(x)) = \mathcal{C}(x).$$

If the parametrization is not proper, a solution curve $\mathcal{C}(x)$ could be the image of a non-rational plane curve $(s(x),t(x))$. Otherwise, $(s(x),t(x))$ must be a rational curve because $\mathcal{P}^{-1}$ and $f(x)$ are rational. Since the solution curve $\mathcal{C}(x)$ satisfies the differential conditions on the coordinates, the plane curve $(s(x),t(x))$ must satisfy a certain differential condition. Indeed, if $\mathcal{P}(s,t)$ is proper and $f(x)$ is a rational general solution of $F(x,y,y') = 0$, then $(s(x),t(x))$ must be a rational general solution of the system

$$\begin{cases} s' = \dfrac{\chi_{2t} - \chi_3 \cdot \chi_{1t}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \\ t' = \dfrac{\chi_{1s} \cdot \chi_3 - \chi_{2s}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \end{cases} \tag{2}$$

provided that $\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s} \neq 0$ (which we can assume without loss of generality). The system (2) is called the *associated system* of $F(x,y,y') = 0$ w.r.t. $\mathcal{P}(s,t)$. It is constructed in such a way that if $(s(x),t(x))$ is a rational solution of the associated system and $(s(x),t(x)) \subset \mathrm{dom}(\mathcal{P})$, then

$$\mathcal{P}(s(x),t(x)) = (x + c,\ \chi_2(s(x),t(x)),\ \chi_2(s(x),t(x))')$$

for some constant c. Therefore,

$$y = \chi_2(s(x-c), t(x-c))$$

is a rational solution of the differential equation $F(x,y,y') = 0$. The following theorem is natural and its proof can be found in [NW10] (Theorem 3.14 and Theorem 3.15).

Theorem 2.1. *If the parametrization $\mathcal{P}(s,t)$ is proper, then there is a one-to-one correspondence between rational general solutions of the parametrizable AODE $F(x,y,y') = 0$ and rational general solutions of its associated system w.r.t. $\mathcal{P}(s,t)$.*

2.1. Invariant algebraic curves of the associated system. We still have to solve the associated system in order to complete the algorithm. Of course, one might ask why it would be easier to solve the associated system. Indeed, the associated system is an autonomous system of order 1 and of degree 1 w.r.t. the unknown parameters. These features enable us to solve the associated system.

In [NW11] we have proposed a method to compute explicit rational solutions of the associated system from its rational invariant algebraic curves. The method is based on the proper rational parametrization of these curves and on the fact that by linear reparametrizations we can find the rational solutions of the associated system. We will not go to details of the theory of invariant algebraic curves of a system, which applies for the associated system as a special case. Nevertheless, we state the definition of an invariant algebraic curve of a rational system to specify the meaning of the terminology and relate it to rational general solutions of the associated system. For further details on invariant algebraic curves we refer to [Lin88], [CLPZ02], [PS83].

Definition 2.1. An *invariant algebraic curve* of the rational system

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)},\ t' = \frac{M_2(s,t)}{N_2(s,t)} \right\}, \tag{3}$$

where M_1, M_2, N_1, N_2 are polynomials, is an algebraic curve $G(s,t) = 0$ such that

$$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K, \tag{4}$$

where K is some polynomial.

It follows that the set of invariant algebraic curves of the system (3) is the same as the one of the polynomial system

$$\begin{cases} s' = M_1(s,t) \cdot N_2(s,t), \\ t' = M_2(s,t) \cdot N_1(s,t). \end{cases} \tag{5}$$

Definition 2.2. An invariant algebraic curve of the system (3) is called a *rational invariant algebraic curve* iff it is also a rational curve, i.e., it has a rational parametrization.

A rational parametrization of a rational algebraic curve is called *proper* iff it has a rational inverse map. There is an exact degree bound for such a proper rational parametrization of a plane rational algebraic curve in terms of the degree of the curve ([SW01]). Note that this degree bound has been used in the termination of those algorithms in [FG04] and [FG06] for the case of autonomous AODEs of order 1. We refer to [SWPD08] for further details of rational algebraic curves over an algebraically closed field and of methods to rational parametrization.

A rational general solution of the system (3) parametrizes a rational invariant algebraic curve of the system (3). Moreover, this rational invariant algebraic curve is defined by an irreducible monic polynomial whose coefficients contain a transcendental constant over $\mathbb{K}$ (cf. Lemmas 3.12 and 3.13 in [NW10] or Lemma 5.1 in [NW11]).

Definition 2.3. A rational invariant algebraic curve $G(s,t) = 0$ of the system (3) is called a *rational solution curve* of the system iff there exists a rational parametrization of $G(s,t) = 0$ being a rational solution of the system (3).

Assume that we have found an irreducible invariant algebraic curve of the system (3). Then we show how to obtain a rational solution of the system (3) from a proper rational parametrization of that invariant algebraic curve.

Theorem 2.2. *Let $G(s,t) = 0$ be a rational invariant algebraic curve of the system (3) such that $G \nmid N_1$ and $G \nmid N_2$. Let $(s(x), t(x))$ be a proper rational parametrization of $G(s,t) = 0$. Then $G(s,t) = 0$ is a rational solution curve of the system (3) if and only if one of the following differential equations has a rational solution $T(x)$:*

(1) $T' = \dfrac{1}{s'(T)} \cdot \dfrac{M_1(s(T),t(T))}{N_1(s(T),t(T))}$ *when* $s'(x) \neq 0$,

(2) $T' = \dfrac{1}{t'(T)} \cdot \dfrac{M_2(s(T),t(T))}{N_2(s(T),t(T))}$ *when* $t'(x) \neq 0$.

If there is such a rational solution $T(x)$, then the rational solution of the system (3) corresponding to $G(s,t) = 0$ is given by

$$(s(T(x)), t(T(x))).$$

We refer to [NW11] (Theorem 3.5) for the proof of this theorem. Note that if we apply this theorem for a *general rational invariant algebraic curve*, the monic rational invariant algebraic curve containing a transcendental constant over $\mathbb{K}$ in its coefficients, then we will obtain a rational general solution of the system (3).

In order to compute an irreducible invariant algebraic curve of the system (3) or (5) we need an upper bound for the degree of the curve. As a corollary of Darboux's Theorem ([Sin92]), the degree of irreducible invariant algebraic curves of the system (5) is bounded. If the polynomial system (5) has no *dicritical singularities*, which is the generic case, then there is an explicit upper bound for the degree of the irreducible invariant algebraic curves in terms of the degree of the polynomial system ([Car94]). Using this upper bound we can find all irreducible invariant algebraic curves of the system (5) by undetermined coefficient method. There is a discussion on the efficiency of implementing such a procedure in [Man93].

Note that we can derive from the associated system (2) a single differential equation of order 1 and of degree 1:

$$\frac{ds}{dt} = \frac{\chi_{2t} - \chi_3\chi_{1t}}{\chi_{1s}\chi_3 - \chi_{2s}}. \tag{6}$$

Sometimes, this differential equation is separable, then we can integrate the equation in order to obtain the general invariant algebraic curve without using the undetermined coefficient method.

We summarize our procedure in the following algorithm.

=== **Algorithm** `RATSOLVE` ===

Input: a parametrizable AODE $F(x, y, y') = 0$;

Output: a rational general solution of $F(x, y, y') = 0$ in the generic case, if there is one.

(1) Compute a proper rational parametrization $\mathcal{P}(s, t)$ of $F(x, y, z) = 0$;

(2) Compute the associated system (2) w.r.t. $\mathcal{P}(s, t)$;

(3) Compute the set $\mathcal{I}$ of irreducible invariant algebraic curves of the associated system if it has no dicritical singularities;

(4) If $\mathcal{I}$ contains an irreducible general invariant algebraic curve $G(s, t) = 0$, then check whether $G(s, t) = 0$ is a rational curve;

(5) If $G(s, t)$ is a rational curve, then parametrize this curve to find a rational general solution $(s(x), t(x))$ of the associated system;

(6) Compute $c = \chi_1(s(x), t(x)) - x$;

(7) Return $y = \chi_2(s(x - c), t(x - c))$.

For executing Step 3 we can use the upper bound in [Car94] when the associated system has no dicritical singularities. Note that Step 5 of the algorithm `RATSOLVE` is based on Theorem 2.2 and it is also described in [NW11]. In Step 4, if $\mathcal{I}$ contains no general invariant algebraic curve, then the algorithm is still valid and applicable for other irreducible invariant algebraic curves. In this case it returns some rational solutions but not the general one.

Example 2.2. We demonstrate this algorithm by considering the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0. \tag{7}$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ can be parametrized by

$$\mathcal{P}(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, -\frac{1}{s} - \frac{2s+t^2}{s^2}, \frac{t}{s}\right).$$

This is a proper parametrization and its associated system is

$$\{s' = st,\ t' = s + t^2\}.$$

We compute the set of irreducible invariant algebraic curves of the system:

$$\{s,\ t^2 + 2s,\ s^2 + ct^2 + 2cs \mid c \text{ is an arbitrary constant}\}.$$

(1) The first invariant algebraic curve $s = 0$ can be parametrized by $\mathcal{Q}(x) = (0, x)$. Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = T^2$. Hence $T(x) = -\frac{1}{x}$. Therefore, $s(x) = 0$, $t(x) = \frac{1}{x}$. However, this solution does not belong to the domain of $\mathcal{P}(s,t)$. Therefore, it is not corresponding to any solution of $F(x, y, y') = 0$ parametrized by $\mathcal{P}(s,t)$.

(2) The second invariant algebraic curve $t^2 + 2s = 0$ can be parametrized by $\mathcal{Q}(x) = \left(-\frac{x^2}{2}, x\right)$. Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = \frac{1}{2}T^2$. Hence $T(x) = -\frac{2}{x}$. Therefore, $s(x) = -\frac{2}{x^2}$, $t(x) = -\dfrac{2}{x}$. The parametrization $\mathcal{P}(s,t)$ maps this solution to the solution $y(x) = \frac{1}{2}x^2$ of $F(x, y, y') = 0$.

(3) The third invariant algebraic curve $s^2 + ct^2 + 2cs = 0$ can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2}\right).$$

Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = 1$. Hence $T(x) = x$. Therefore, the rational solution in this case is

$$s(x) = -\frac{2c}{1+cx^2}, \quad t(x) = -\frac{2cx}{1+cx^2}.$$

Since $G(s,t)$ contains a transcendental constant, the above solution is actually a rational general solution of the associated system. Therefore, the rational general solution of (7) is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}(x^2 + 2cx + c^2 + 3c).$$

In fact, the algebraic curve $s^2 + ct^2 + 2cs = 0$ can be identified with the curve $cs^2 + t^2 + 2s = 0$ in the space of curves, where c is an arbitrary constant. In this way, we can obtain the invariant algebraic curve $t^2 + 2s = 0$ by specializing $c = 0$ in the curve $cs^2 + t^2 + 2s = 0$.

We now apply the algorithm `RATSOLVE` to an autonomous AODE $F(y, y') = 0$ such that the algebraic curve $F(y, z) = 0$ is rational. This reproduces Lemma 4 and Theorem 5 in [FG04] or Lemma 3.1 in [FG06].

Example 2.3. Consider the autonomous AODE $F(y, y') = 0$, where $F(y, z)$ is a bivariate polynomial defining a rational algebraic curve. Suppose that $(f(t), g(t))$ is a proper rational parametrization of the curve $F(y, z) = 0$. Then

$$\mathcal{P}(s,t) = (s,\ f(t),\ g(t))$$

is a proper rational parametrization of the cylindrical surface $F(y, z) = 0$ and the associated system w.r.t. $\mathcal{P}(s, t)$ is

$$\left\{ s' = 1,\ \ t' = \frac{g(t)}{f'(t)} \right\}. \tag{8}$$

A possible rational general solution of this system must be of the form

$$s(x) = x + C, \quad t(x) = \frac{ax + b}{cx + d},$$

where a, b, c, d are constants and C is an arbitrary constant. Therefore, the rational general solution of $F(y, y') = 0$ is $y(x) = f(t(x - C))$.

2.2. Rational first integrals of the associated system.

Definition 2.4. A *rational first integral* of the system (5) is a rational function $W(s, t) \in \mathbb{K}(s, t)$ such that

$$W_s \cdot M_1 N_2 + W_t \cdot M_2 N_1 = 0, \tag{9}$$

where W_s and W_t are the partial derivatives of $W(s, t)$ w.r.t. s and t.

Rational first integrals of the system (3) are defined as rational first integrals of the system (5). There is a close relation between irreducible invariant algebraic curves of the system (5) and its rational first integrals. As a corollary of Darboux's Theorem ([Sin92]), either there are finitely many irreducible invariant algebraic curves or there is a rational first integral of the polynomial system (5). Suppose that $W = \frac{U}{V}$ is a rational first integral of the system (5). Then every irreducible invariant algebraic curve of the system (5) is a factor of $c_1U - c_2V$ for some constants c_1, c_2. Therefore, we can also find all irreducible invariant algebraic curves via computing a rational first integral if it exists. Based on that we have the following theorem connecting a rational general solution of the system (3) with a rational first integral of the system.

Theorem 2.3. *The system* (3) *has a rational general solution if and only if it has a rational first integral* $\frac{U}{V} \in \mathbb{K}(s,t)$ *with* $\gcd(U,V) = 1$ *and any irreducible factor of* $U - cV$ *determines a rational solution curve for a transcendental constant* c *over* $\mathbb{K}$.

For a proof of the theorem we refer to Theorem 5.6 in [NW11].

Example 2.4. Consider the associated system in Example 2.2. It has a rational first integral as follows

$$W(s,t) = \frac{s^2}{2s + t^2}.$$

Therefore, every irreducible invariant algebraic curves of the system is a factor of $c_1s^2 - c_2(t^2 + 2s)$ for some constants c_1, c_2. Therefore, the set of all irreducible invariant algebraic curves of the associated system is

$$\{G(s,t) = s,\ G(s,t) = t^2+2s,\ G(s,t) = s^2+ct^2+2cs \mid c \text{ is an arbitrary constant}\}.$$

One can compute a rational first integral of the system (5) by the undetermined coefficients method, setting up an upper bound on the degree of a rational first integral, then solving an algebraic system of equations. In particular, if one looks for a polynomial first integral, then the system is linear. For a general discussion on polynomial first integrals of autonomous systems we refer to [Sch85]. There are also bigger classes of first integrals of the system: elementary first integrals and Liouvillian first integrals ([PS83, Sin92]).

2.3. Independence of the proper parametrization. We know that proper rational parametrizations of a rational surface are not unique. Indeed, let

$$\phi(s_1,t_1) = (\phi_1(s_1,t_1), \phi_2(s_1,t_1))$$

be a birational map of the plane and $\psi(s_2,t_2) = \phi^{-1}(s_2,t_2)$. If $\mathcal{P}(s_1,t_1)$ is a proper parametrization of $F(u,v,w) = 0$, then $(\mathcal{P} \circ \psi)(s_2,t_2)$ is a new proper rational parametrization of $F(u,v,w) = 0$. Suppose that

$$\{s_1' = R_1(s_1,t_1),\ \ t_1' = R_2(s_1,t_1)\}$$

is the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}(s_1, t_1)$. Then the associated system of $F(x, y, y') = 0$ w.r.t. $(\mathcal{P} \circ \psi)(s_2, t_2)$ is

$$\begin{pmatrix} s_2' \\ t_2' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} s_1' \\ t_1' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} R_1(s_1, t_1) \\ R_2(s_1, t_1) \end{pmatrix}\Bigg|(s_1, t_1) = (\psi_1(s_2, t_2),\ \psi_2(s_2, t_2))$$

where

$$J_\phi = \begin{pmatrix} \phi_{1s_1} & \phi_{1t_1} \\ \phi_{2s_1} & \phi_{2t_1} \end{pmatrix}$$

is the Jacobian matrix of the map ϕ. The two associated systems have the same rational solvability although they look differently.

It is known that every birational map of the line is of the form $\phi(x) = \frac{ax+b}{cx+d}$, where $ad - bc \neq 0$. Unfortunately, it is not known what are the forms of a birational map of the plane. Therefore, any description on the birational maps of the plane could help us to simplify the associated system and perhaps find the simplest one.

Example 2.5. Consider again the differential equation (7). We can also use the proper parametrization

$$\mathcal{Q}_2(s, t) = \left(s,\ \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8},\ t - \frac{3}{2}\right).$$

Then the associated system is $\{s' = 1,\ \ t' = 1\}$. It is simpler than the previous associated system.

3. Solving AODEs by pattern matching

In [Pia33], Chapter V; and in [Mur60], Chapter A2, Part I, the ordinary differential equations of order 1 and of degree higher than 1, $F(x, y, y') = 0$, can sometimes be solved if they belong to one of the special types: *those solvable for* y'; *those solvable for* y; and *those solvable for* x. The observation is that one can transform the original differential equation to a new differential equation of order 1 and of degree 1 and then try to solve the latter one. In this section, we stress the fact that those differential equations are in the set of parametrizable AODEs and they can be uniformly treated by the parametrization method. In fact, the new differential equation in the classical method can be derived from the associated system w.r.t. a certain parametrization of the solution surface.

3.1. Equations solvable for y'. If the differential equation is solvable for y', i.e., $y' = G(x, y)$, then we need not change the variable because it is already in a special type.

3.2. Equations solvable for y. If the differential equation is of the form $y = G(x, y')$, one can differentiate the equation w.r.t. x to obtain

$$y' = G_x(x, y') + G_{y'}(x, y') \cdot y'',$$

where G_x and $G_{y'}$ are the partial derivatives of $G(x, y')$ w.r.t. x and y'. Denoting $\tilde{y} = y'$, one can rewrite the above differential equation in the form

$$\tilde{y} = G_x(x, \tilde{y}) + G_{\tilde{y}}(x, \tilde{y}) \cdot \frac{d\tilde{y}}{dx},$$

or equivalently,

$$\frac{dx}{d\tilde{y}} = \frac{G_{\tilde{y}}(x, \tilde{y})}{\tilde{y} - G_x(x, \tilde{y})}. \tag{10}$$

Therefore, one has transformed the differential equation $y = G(x, y')$ to a new differential equation of order 1 and of degree 1 in the derivatives. What has been done for this class is that one tries to find x and y' from the relation (10) between dx and dy', which is derived by differentiation. Then the equation $y = G(x, y')$ simply returns the solution in terms of the two parameters x and y'.

3.3. Equations solvable for x. In the last case, if the differential equation is of the form $x = G(y, y')$, one can differentiate the equation w.r.t. y to obtain

$$\frac{dx}{dy} = G_y(y, y') + G_{y'}(y, y') \cdot \frac{dy'}{dy}.$$

Let $\tilde{y} = y'$, then we have

$$\frac{1}{\tilde{y}} = G_y(y, \tilde{y}) + G_{\tilde{y}}(y, \tilde{y}) \cdot \frac{d\tilde{y}}{dy}.$$

Therefore, one also has transformed the differential equation $x = G(y, y')$ to a new differential equation of order 1 and of degree 1 in the derivatives:

$$\frac{dy}{d\tilde{y}} = \frac{\tilde{y} G_{\tilde{y}}(y, \tilde{y})}{1 - \tilde{y} G_y(y, \tilde{y})}. \tag{11}$$

In this case, one gets an implicit relation between y and y' via the new differential equation (11).

3.4. Relation to our geometric method. As we have mentioned above the three classes are in the set of parametrizable AODEs. Let us interpret these methods in the light of our algebraic geometric approach. Of course, in order for our approach to be applicable, we have to assume that the function G is rational. We choose an appropriate parametrization for the corresponding special solution surface and derive the associated system w.r.t. that parametrization.

	Solvable for y'	Solvable for y	Solvable for x
D.E.	$y' = G(x,y)$	$y = G(x,y')$	$x = G(y,y')$
P.P.	$(s,t,G(s,t))$	$(s,G(s,t),t)$	$(G(s,t),s,t)$
A.S.	$\begin{cases} s' = 1 \\ t' = G(s,t) \end{cases}$	$\begin{cases} s' = 1 \\ t' = \dfrac{t - G_s(s,t)}{G_t(s,t)} \end{cases}$	$\begin{cases} s' = t \\ t' = \dfrac{1 - tG_s(s,t)}{G_t(s,t)} \end{cases}$
New D.E.	$\dfrac{dt}{ds} = G(s,t)$	$\dfrac{dt}{ds} = \dfrac{t - G_s(s,t)}{G_t(s,t)}$	$\dfrac{dt}{ds} = \dfrac{1 - tG_s(s,t)}{tG_t(s,t)}$

Example 3.1 (*Clairaut's equation*). Let f be a smooth function of one variable. Consider the Clairaut differential equation

$$y = y'x + f(y').$$

We can solve this equation using the algorithm `RATSOLVE`. Indeed, this is a differential equation solvable for y and it can be parametrized by

$$\mathcal{P}(s,t) = (s, st + f(t), t).$$

If f is rational, then $\mathcal{P}(s,t)$ is a rational parametrization of the differential equation. The associated system w.r.t. $\mathcal{P}(s,t)$ is $\{s' = 1, t' = 0\}$. The set of irreducible invariant algebraic curves is

$$\{t - c = 0 \mid c \text{ is an arbitrary constant}\}.$$

Hence $s(x) = x$ and $t(x) = c$, where c is an arbitrary constant, is the rational general solution of the associated system. So we have the general rational solution of the Clairaut's differential equation $y = cx + f(c)$.

4. Conclusion

We have presented a generic decision algorithm for the existence of a rational general solution of an AODE of order 1. In the positive case, we actually compute such a rational general solution. The difficulties of the problem lie in the two main steps of the method: parametrizing the solution surface and solving the associated system. The first subproblem captures the geometric aspect of the problem, while the second deals with the differential aspect. We believe that our method is a fruitful approach to the solution of parametrizable AODEs of order 1 and probably also of higher orders.

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