A Regularization Method for Computing Approximate Invariants of Plane Curves Singularities

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ABSTRACT

We approach the algebraic problem of computing topological invariants for the singularities of a plane complex algebraic curve defined by a squarefree polynomial with inexactly-known coefficients. Consequently, we deal with an ill-posed problem in the sense that, tiny changes in the input data lead to dramatic modifications in the output solution.

We present a regularization method for handling the ill-posedness of the problem. For this purpose, we first design symbolic-numeric algorithms to extract structural information on the plane complex algebraic curve: (i) we compute the link of each singularity by numerical equation solving; (ii) we compute the Alexander polynomial of each link by using algorithms from computational geometry and combinatorial objects from knot theory; (iii) we derive a formula for the delta-invariant and the genus. We then prove the convergence for inexact data of the symbolic-numeric algorithms by using concepts from algebraic geometry and topology.

Moreover we perform several numerical experiments, which support the validity for the convergence statement.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: algorithms—*algebraic algorithms*; G.4 [**Mathematics of Computing**]: Mathematical Software; G.1.2 [**Numerical Analysis**]: Approximation—*approximation of surfaces, piecewise polynomial approximation*; G.1.0 [**Numerical Analysis**]: General—*numerical algorithms, stability (and instability)*

General Terms

Algorithms, Design, Experimentation, Theory

Keywords

Plane curve singularity, ill-posed problem, regularization, symbolic-numeric algorithms, link of a singularity, Alexander polynomial, delta-invariant, genus

1. INTRODUCTION

In this paper, we treat the algebraic problem of computing topological invariants for each singularity of a plane complex algebraic curve defined by a squarefree polynomial with coefficients of limited accuracy, i.e. the coefficients are both exact and inexact data. The problem is ill-posed in the sense that tiny changes in the input data cause huge changes in the output solution. We employ an adapted regularization method based on [6, 18] to handle the ill-posedness of the problem. This regularization method allows us to construct approximate solutions to the ill-posed problem, which are stable under small changes in the initial data.

We first design symbolic-numeric algorithms for computing invariants for each singularity of a plane complex algebraic curve defined by a squarefree polynomial. We compute the link of each singularity by intersecting the curve with a sphere centered in the singularity and of a small radius, based on [3, 14]. The computation of the link of the singularity allows us to analyze the local topology of each singularity. We then compute the Alexander polynomial attached to the link of the singularity using algorithms from computational geometry [5] and combinatorial objects from knot theory, based on [4, 12]. The Alexander polynomial is a complete invariant for links of singularities, i.e. different links of singularities have different Alexander polynomials [22]. As applications, from the Alexander polynomial we derive formulas for the delta-invariant of each singularity and for the genus of the curve. In [1] a numerical method based on homotopy continuation for computing the genus of any one-dimensional irreducible component of an algebraic set is presented, while in [16] the authors provide a formula for the genus of an algebraic curve with all singularities affine and ordinary.

We implement the designed symbolic-numeric algorithms for invariants of plane curves singularities in the free library called GENOM3CK-GENus cOMputation of plane Complex algebraiC Curves using Knot theory-written in the free algebraic geometric modeler Axel [21] and in the free computer algebra system Mathemagix [10].

We sketch the proof for the convergence for inexact data property of the designed symbolic-numeric algorithms using concepts from algebraic geometry and topology. We perform several numerical experiments with the library GENOM3CK, which confirm the convergence for inexact data property.

We organize this paper as follows. In Section 2 we define the plane complex algebraic curves and their singularities. We also introduce invariants for each singularity of a plane complex algebraic curve: the link of each singularity, the Alexander polynomial attached to the link, and the delta-invariant of each singularity. In Section 3 we present the symbolic-numeric algorithms developed for the computation of the defined invariants. Section 4 contains regularization principles that we employ to handle the ill-posedness of the problem. We also sketch the proof for the convergence for inexact data property of the designed symbolic-numeric algorithms. In Section 5 we discuss implementation issues and we perform several test experiments. We give the conclusions in Section 6.

2. PLANE COMPLEX ALGEBRAIC CURVES

2.1 Singularities of Plane Complex Algebraic Curves

For our study, we define the (affine) plane complex algebraic curves following [17, 19]:

Definition 1. Let $\mathbb{C}$ be the algebraically closed field of complex numbers, and $\mathbb{A}^2(\mathbb{C}) = \{(z,w) \in \mathbb{C}^2\}$ the affine complex plane. Let $p(z,w) \in \mathbb{C}[z,w]$ be an irreducible polynomial in z and w with coefficients in $\mathbb{C}$ of degree m. An affine plane algebraic curve over $\mathbb{C}$ of degree m defined by $p(z,w)$ is the set of zeroes of the polynomial $p(z,w)$, i.e.

$$\mathcal{C} = \{(z,w) \in \mathbb{A}^2(\mathbb{C}) | p(z,w) = 0\}.$$

We define the singular points of a plane complex algebraic curve in the following way:

Definition 2. Let $\mathcal{C}$ be a plane complex algebraic curve defined by the irreducible polynomial $p(z,w) \in \mathbb{C}[z,w]$. We denote by $\partial_z p := \partial p(z,w)/\partial z$ and by $\partial_w p := \partial p(z,w)/\partial w$ the partial derivatives of $p(z,w)$ with respect to z and w. The set of singular points (or singularities) of $\mathcal{C}$ is defined as

$$Sing(\mathcal{C}) = \{(z_0,w_0) \in \mathbb{A}^2(\mathbb{C}) | p(z_0,w_0) = \partial_z p(z_0,w_0) = \partial_w p(z_0,w_0) = 0\}.$$

The points of a plane complex algebraic curve that are not singular are called nonsingular or regular points. An irreducible plane complex algebraic curve has at most finitely many singular points, and if it has none it is called nonsingular (or smooth). For simplicity reasons we denote the affine complex plane by $\mathbb{C}^2$. Since $\mathbb{C}^2$ is isomorphic with $\mathbb{R}^4$, we consider a plane complex algebraic curve $\mathcal{C} \subset \mathbb{C}^2$ as a real two-dimensional object in $\mathbb{R}^4$. For visualization purposes, we cannot draw this object in $\mathbb{R}^4$, but we sketch the equivalent curve in $\mathbb{R}^2$.

An important observation is that computing the singularities of a plane complex algebraic curve is an ill-posed problem, in the sense that small changes in the coefficients of the defining polynomial of the curve lead to dramatic changes in the topology (shape) of the curve itself.

Example 1. In Figure 1 the red inner curve represents the topology of $\mathcal{C} = \{(z,w) \in \mathbb{R}^2 : -z^3 - zw + w^2 = 0\}$ with a singularity in the origin $(0,0)$. The blue outer curve represents the topology of $\mathcal{D} = \{(z,w) \in \mathbb{R}^2 : -z^3 - zw + w^2 - 0.01 = 0\}$. We notice that for small perturbations of the defining polynomial of $\mathcal{C}$ the singularity of the curve disappears.

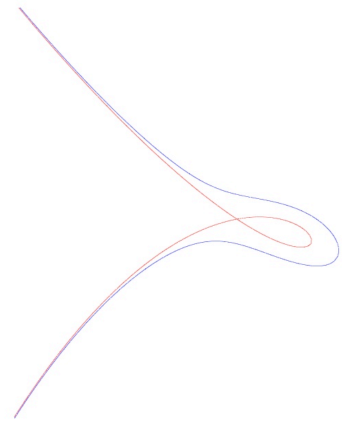

Figure 1: Example of ill-posedness of the singularity $(0,0)$ of $-z^3 - zw + w^2$. Picture produced with Axel, see Section 5 for more information.

2.2 Invariants of Plane Complex Algebraic Curves

First, we define an homeomorphism in the following way:

Definition 3. Two subsets $U \subset \mathbb{R}^k, V \subset \mathbb{R}^n$ are topologically equivalent or homeomorphic if there exists a bijective function $\varphi : U \to V$ such that both φ and its inverse are continuous. In this case, φ is called an homeomorphism.

A pair (X, A) of spaces is a topological space together with a subspace $A \subseteq X$. A mapping $\varphi : (X,A) \to (Y,B)$ of pairs is a continuous mapping $\varphi : X \to Y$ with $\varphi(A) \subseteq B$. A homeomorphism $\varphi : (X,A) \to (Y,B)$ of pairs is a mapping of pairs which is a homeomorphism $\varphi : X \to Y$ and induces a homeomorphism $\varphi/_A : A \to B$.

In this paper, the (topological) invariants of a plane complex algebraic curve $\mathcal{C}$ are those properties of $\mathcal{C}$ and its singularities that are unchanged under homeomorphism of small disks around 0 mapping the first curve onto the second curve.

We consider the stereographic projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ as a mapping that projects a sphere onto a plane. It is constructed as follows: we take a sphere; we draw a line from the north pole N of the sphere to a point $\hat{P}$ in the equator plane to intersect the sphere at a point P. The stereographic projection of $\hat{P}$ is P. The stereographic projection gives an explicit homeomorphism from the unit sphere minus the north pole to the Euclidean plane.

The stereographic projection may be applied to a n-sphere S^n in R^{n+1}: consider a n-sphere in $\mathbb{R}^{n+1}$, which we denote $S^n = \{(x_1, x_2, ..., x_{n+1}) \subset \mathbb{R}^{n+1} | x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$, and the north point of the n-sphere $Q(0, ..., 1) \in S^n$. If H is a hyperplane in $\mathbb{R}^{n+1}$ not containing Q, then the stereographic projection of the point $P \in S^n \setminus Q$ is the point P' of the intersection of the line QP with H. The stereographic projection is a homeomorphism from $S^n \setminus Q \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$. For our study, we use the stereographic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$ to project objects from $\mathbb{R}^4$ to $\mathbb{R}^3$ by preserving their topological properties.

Link of a Plane Curve Singularity

We introduce notions from knot theory, which are useful for the purpose of this paper. We define a knot and a link in the following way:

Definition 4. A knot is a piecewise linear or a differentiable simple closed curve in $\mathbb{R}^3$ and a link is a finite union of disjoint knots, see Figure 2. The knots that make up a link are called the components of the link, and thus a knot is a link with one component.

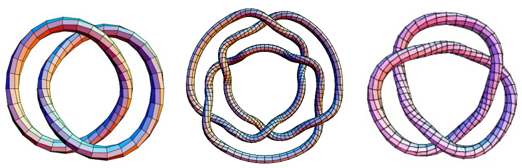

Figure 2: Examples of links/knot. From left to right: the Hopf link, the Borromean rings, the trefoil knot

We define the equivalence of two links as follows:

Definition 5. We say that two links are equivalent if there exists an orientation-preserving homeomorphism on $\mathbb{R}^3$ that maps one link onto the other. This equivalence is called (ambient) isotopy.

When we work with knots we actually work with their projections in $\mathbb{R}^2$. For our study, we work with a special type of projection called a diagram, which we introduce in the following way: (i) we consider that a regular projection is a linear projection for which no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot. A crossing point is an image of two knot points of such a regular projection to $\mathbb{R}^2$; (ii) then a diagram is the image under regular projection, together with the information on each crossing point telling which branch goes over and which goes under (Figure 3). Thus we speak about overcrossings and undercrossings; (iii) a diagram together with an arbitrary orientation of each knot in the link is called an oriented diagram.

We introduce the elements of a diagram as follows: (i) a crossing is called lefthanded (denoted with -1) if the underpass traffic goes from left to right or it is called righthanded (denoted with $+1$) if the underpass traffic goes from right to left; (ii) an arc is the part of a diagram between two undercrossings. Whether lefthanded or righthanded, each crossing is determined by three arcs and we denote the overgoing arc with i, and the undergoing arcs with j and k (Figure 4). The number of arcs in a link diagram is equal to the number of crossings in the same link diagram.

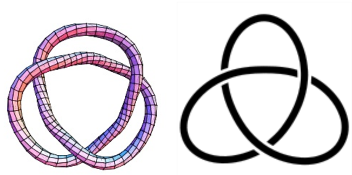

Figure 3: Example of a trefoil knot and its diagram

We employ the following theorem, which asserts that the equivalence class of a special type of link determines the homeomorphism type of the singularity:

Figure 4: Types of crossings: lefthanded (-1) and righthanded(+1), together with the labels for the 3 arcs of a crossing

THEOREM 1. *(Milnor[14]) Let $V \subset \mathbb{C}^{n+1}$ be a hypersurface in $\mathbb{C}^{n+1}$, i.e. an algebraic variety defined by a single polynomial f. Assume $\vec{0} \in V$ and $\vec{0}$ is an isolated singularity, i.e. there is no other singularity on a sufficiently small neighborhood of $(0,0)$; S_ϵ is the sphere centered in $\vec{0}$ and of radius ϵ; and D_ϵ is the disk centered in $\vec{0}$ of radius ϵ. Then, for sufficiently small ϵ, $L_\epsilon = S_\epsilon \cap V$ is a $(2n-1)$-dimensional nonsingular set and the pair $(D_\epsilon, D_\epsilon \cap V)$ is homeomorphic to the pair consisting of the cone over S_ϵ and the cone over $L_\epsilon = S_\epsilon \cap V$.*

For the case $n = 1$, Milnor's theorem says that there exists $\epsilon_0 \in \mathbb{R}_{>0}$ such that for any $\epsilon_1, \epsilon_2 \in \mathbb{R}_{>0}$ with $\epsilon_1 < \epsilon_0$ and $\epsilon_2 < \epsilon_0$ the links $L_{\epsilon_1} \subset S_{\epsilon_1}$ and $L_{\epsilon_2} \subset S_{\epsilon_2}$ are equivalent, i.e. $D_{\epsilon_1} \cap \mathcal{C}$ and $D_{\epsilon_2} \cap \mathcal{C}$ are homeomorphic. In addition, for any $0 < \epsilon < \epsilon_0$ the link $L = L_\epsilon$ is called *the link of the singularity* of f (or of $\mathcal{C}$) at $(0,0)$, and L is well-defined up to homeomorphism of pairs. In this case, the link $L_\epsilon \subset S_\epsilon$ determines the topological type of the singularity $(0,0)$ of $\mathcal{C}$. In theory, a link is called algebraic if it is equivalent to the link of a plane curve singularity.

Under the same hypotheses from Theorem 1 and considering S^1 the unit circle, Milnor fibration theorem states that the mapping $\phi : S_\epsilon \setminus L \to S^1, \phi(z,w) = f(z,w)/|f(z,w)|$ is a fibration, i.e. the complement $S_\epsilon \setminus L$ is a union of smooth surfaces, each being the preimage of one point.

Alexander Polynomial of a Plane Curve Singularity

An important result of Yamamoto [22] says that the Alexander polynomial is a complete invariant for the algebraic links, i.e. the Alexander polynomial uniquely defines all the algebraic links up to an (ambient) isotopy. In this way, we can use the Alexander polynomial of the link of a singularity to distinguish the topological type of the singularity itself. In [8] we present a straightforward algorithm to compute the Alexander polynomial attached to the link of a singularity by using combinatorial objects from knot theory such as the diagram of the link and the elements of the diagram. For introducing the Alexander polynomial, we need some preliminary definitions based on [12]:

Definition 6. Let $D(L)$ be an oriented link diagram with r components and n crossings $x_q : q \in \{1, ..., n\}$. We denote the arcs of $D(L)$ with the labels $\{1, ..., n\}$ and separately the crossings of $D(L)$ with $\{1, ..., n\}$. We denote the labeling matrix of $D(L)$ with $LM(L) \in \mathcal{M}(n, 4, \mathbb{Z})$. We define $LM(L) = (b_{ql})_{q,l}$ with $q \in \{1, ..., n\}, l \in \{1, ..., 4\}$ row by row for each crossing x_q as follows: (i) at b_{q1} store the type of the crossing x_q ($+1$ or -1); (ii) at b_{q2} store the label of the arc i of x_q in $D(L)$; (iii) at b_{q3} store the label of the arc

j of x_q in $D(L)$; (iv) at b_{q4} store the label of the arc k of x_q in $D(L)$.

Definition 7. Let $D(L)$ be an oriented link diagram with r components and n crossings $x_q : q \in \{1, ..., n\}$. We denote the arcs and the crossings of $D(L)$ as in Definition 6. We consider $LM(L)$ the labeling matrix of $D(L)$ as in Definition 6. We denote the prealexander matrix of L with $PM(L) \in \mathcal{M}(n, n, \mathbb{Z}[t_1, t_1, ..., t_r])$. We define $PM(L)$ row by row for each crossing x_q depending on $LM(L)$. For x_q we consider the variable t_s, where $s \in \{1, ..., r\}$ is the s-th knot component of $D(L)$, which contains the overgoing arc that determines the crossing x_q. Then: (i) if x_q is righthanded, i.e. $b_{q1} = +1$ in $LM(L)$, then at position b_{q2} of $PM(L)$ store the label $1 - t_s$, at position b_{q3} store -1 and at position b_{q4} store t_s; (ii) if x_q is lefthanded, i.e. $b_{q1} = -1$ in $LM(L)$, then at position b_{q2} of $PM(L)$ store the label $1 - t_s$, at position b_{q3} store t_s and at position b_{q4} store -1; (iii) if two or all of the positions b_{q2}, b_{q3}, b_{q4} have the same value, then store the sum of the corresponding labels at the corresponding position. All other entries of the matrix are 0.

We define the Alexander polynomial of $D(L)$ depending on the number of knot components in L:

Definition 8. Let $D(L)$ be an oriented link diagram with r components and n crossings, $LM(L)$ be its labeling matrix as in Definition 6 and $PM(L)$ be its prealexander matrix as in Definition 7. Then: (i) the univariate Alexander polynomial [12] $\Delta_L(t_1) \in \mathbb{Z}[t_1^{\pm 1}]$ is the normalized polynomial computed as the determinant of any $(n-1) \times (n-1)$ minor of the prealexander matrix of $D(L)$. A normalized polynomial is a polynomial in which the term of the lowest degree is a positive constant; (ii) the multivariate Alexander polynomial [4] $\Delta_L(t_1, ..., t_r) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_r^{\pm 1}]$ is the normalized polynomial computed as the greatest common divisor of all the $(n-1) \times (n-1)$ minor determinants of the prealexander matrix of $D(L)$.

Delta-Invariant of a Plane Curve Singularity

From the Alexander polynomial we derive a formula for the delta-invariant of the singularity of a plane complex algebraic curve in the following way:

Definition 9. (based on Milnor[14]) Let $\Delta_L(t_1, \ldots, t_r)$ be the Alexander polynomial of the link of the isolated singularity $P = (0, 0)$ of a plane complex algebraic curve. Let r be the number of variables in Δ_L and μ the degree of Δ_L. If $r = 1$, then the delta-invariant of P is computed as $\delta_P = \mu/2$, otherwise $\delta_P = (\mu + r)/2$.

We can derive a formula for the genus of a plane complex algebraic curve as described in [14]:

Definition 10. Let $\mathcal{C}$ be a plane complex algebraic curve in the projective plane as introduced in [20]. We denote by $Sing(\mathcal{C})$ the singularities of $\mathcal{C}$, and by $\delta_P \in \mathbb{N}$ the delta-invariant of the singularity P. The genus of $\mathcal{C}$, $genus(C) \in \mathbb{Z}$, is defined as: $genus(\mathcal{C}) = ((m-1)(m-2))/2 - \sum_{P \in Sing(\mathcal{C})} \delta_P$.

Approximate Invariants of a Plane Curve Singularity

We have previously introduced several invariants for a plane complex algebraic curve $\mathcal{C}$ with an isolated singularity, i.e. the Alexander polynomial attached to the link of the singularity, the delta-invariant of the singularity and the genus of the curve. We notice that the computation of these invariants is conditioned by the computation of the *link of the singularity* $L = L_\epsilon$ that depends on the parameter $\epsilon \in \mathbb{R}_+$.

Hence we are motivated to define the *ϵ-invariants* of a plane complex algebraic curve with an isolated singularity, which depend on a parameter $\epsilon \in \mathbb{R}_{>0}$:

Definition 11. Let $\mathcal{C}$ be a plane complex algebraic curve defined by the squarefree polynomial $p(z, w) \in \mathbb{C}[z, w]$. Let $P = (z_0, w_0) \in \mathbb{C}^2$ be an isolated singularity of $\mathcal{C}$ and let $S_\epsilon(P) = \{(z, w) \in \mathbb{C}^2 : |z - z_0|^2 + |w - w_0|^2 = \epsilon^2\}$ be the sphere centered in P of radius $\epsilon \in \mathbb{R}_{>0}$. We take $Y = \mathcal{C} \cap S_\epsilon(P)$. We consider $\pi_{(\epsilon,N)}$ the stereographic projection of the sphere $S_\epsilon(P)$ from its north pole N, which does not belong to $\mathcal{C}$ and which is defined as:

$$\begin{array}{c} \pi_{(\epsilon,N)} : S_\epsilon \setminus \{N\} \subset \mathbb{R}^4 \to \mathbb{R}^3 \\ (a, b, c, d) \to (x, y, z) = \left(\frac{a}{\epsilon - d}, \frac{b}{\epsilon - d}, \frac{c}{\epsilon - d}\right) \end{array} . \tag{1}$$

If $\pi_{(\epsilon,N)}(Y)$ has no singularities, then:

- we call $\pi_{(\epsilon,N)}(Y)$ the *ϵ-link of the singularity* of $p(z, w)$ (or of $\mathcal{C}$) at P. We call $\pi_{(\epsilon,N)}(Y)$ an ϵ-algebraic link.
- we define the *ϵ-Alexander polynomial* of $\mathcal{C}$ at P as the Alexander polynomial of $\pi_{(\epsilon,N)}(Y)$.
- we define the *ϵ-delta-invariant* of P as the delta-invariant of the ϵ-Alexander polynomial of $\mathcal{C}$ at P.

3. SYMBOLIC-NUMERIC ALGORITHMS FOR INVARIANTS OF PLANE CURVE SINGULARITIES

We shortly describe the symbolic-numeric algorithms we design for computing the ϵ-invariants of a plane complex algebraic curve as introduced in Subsection 2.2. For more information on these algorithms see [8, 9].

Problem 1. Given the following: (i) a squarefree polynomial $p(z, w) \in \mathbb{C}[z, w]$ that defines a plane complex algebraic curve $\mathcal{C} \subset \mathbb{C}^2$; (ii) a parameter $\epsilon \in \mathbb{R}_{>0}$ that determines the sphere S_ϵ centered in the origin $(0, 0)$ of radius ϵ.
our goal is: (1) to compute the singularities of $\mathcal{C}$ in $\mathbb{C}^2$; (2) to compute a set of ϵ-invariants of $\mathcal{C}$, i.e. the ϵ-algebraic link, the ϵ-Alexander polynomial, the ϵ-delta-invariant as introduced in Definition 11;

We compute the numerical singularities $Sing(\mathcal{C})$ of the plane complex algebraic curve $\mathcal{C}$ defined by the squarefree polynomial $p(z, w)$ by solving the system of polynomial equations $p(z_0, w_0) = \partial_z p(z_0, w_0) = \partial_w p(z_0, w_0) = 0$ with subdivision methods from [15]. These methods require two input parameters, i.e. a subset $B = [-a, a] \times [-b, b]$ of $\mathbb{R}^2$ and a positive real number $\sigma \in \mathbb{R}_{>0}$. These methods return as output a list of boxes $S \subset B$ smaller than σ and a list M containing the middle points of all the boxes from S s.t.: (i) the value of $p(z, w)$ and its derivatives in the points from M are small; (ii) every singularity from $Sing(\mathcal{C})$ is in one of the boxes from S.

We describe the algorithm $\texttt{APPROXLINK}(p, \mathcal{C}, P, \epsilon)$ for computing the ϵ-algebraic link L_ϵ of the singularity P of the plane complex algebraic curve $\mathcal{C}$ defined by the squarefree polynomial $p(z, w) \in \mathbb{C}[z, w]$. The parameter ϵ denotes the radius of the sphere $S_\epsilon \subset \mathbb{C}^2$ which we intersect with the zero set of $p(z, w)$, as described in Definition 11.

Algorithm 1 ϵ-link of the singularity P of the plane curve $\mathcal{C}$ defined by $p(z, w)$: `APPROXLINK`$(p, \mathcal{C}, P, \epsilon)$

Input: $p(z, w) \in \mathbb{C}[z, w]$ a squarefree complex polynomial
$\mathcal{C} = \{(z, w) \in \mathbb{C}^2 | p(z, w) = 0\}$ a plane algebraic curve
$P = (z_0, w_0)$ a numerical singularity of $\mathcal{C}$
$\epsilon \in \mathbb{R}_{>0}$ a positive real number
Output: $G,\ H \in \mathbb{R}[x, y, z]$
where the common zero set of G, H equals L_ϵ.

1. Substitute $z \leftarrow a+ib, w \leftarrow c+id$ in $p(z, w)$ and obtain
$$p(a, b, c, d) = R(a, b, c, d) + iI(a, b, c, d),$$
with $R, I \in \mathbb{R}[a, b, c, d]$.
2. Extract $R(a, b, c, d) = I(a, b, c, d) = 0$ which define
$$\mathcal{C} = \{(a, b, c, d) \in \mathbb{R}^4 : R(a, b, c, d) = I(a, b, c, d) = 0\}.$$
3. Compute the inverse of $\pi_{(\epsilon,N)}$ from Definition 1:
$$\pi^{-1}_{(\epsilon,N)} : \mathbb{R}^3 \to S_\epsilon \setminus \{N\} \subset \mathbb{R}^4$$
$$(x, y, z) \mapsto (a, b, c, d) = (\tfrac{2x\epsilon}{n}, \tfrac{2y\epsilon}{n}, \tfrac{2z\epsilon}{n}, \tfrac{-\epsilon+x^2\epsilon+y^2\epsilon+z^2\epsilon}{n}),$$
where $n = 1 + x^2 + y^2 + z^2$.
4. Define $\alpha =: (\frac{2x\epsilon}{n}, \frac{2y\epsilon}{n}, \frac{2z\epsilon}{n}, \frac{-\epsilon+x^2\epsilon+y^2\epsilon+z^2\epsilon}{n})$,
5. Substitute $(a, b, c, d) \leftarrow \alpha$ in $\mathcal{C}$ to get $R(\alpha) = I(\alpha) = 0$.
6. Eliminate the denominators in $R(\alpha) = I(\alpha) = 0$ to get $g_\epsilon(x, y, z) = h_\epsilon(x, y, z) = 0$, with $g_\epsilon, h_\epsilon \in \mathbb{R}[x, y, z]$, which for $Y = \mathcal{C} \cap S_\epsilon(P)$ define
$$\pi_{(\epsilon,N)}(Y) = \{(x, y, z) \in \mathbb{R}^3 : g_\epsilon(x, y, z) = h_\epsilon(x, y, z) = 0\}.$$
7. If $\pi_{(\epsilon,N)}(Y)$ has no singularities then
   - return $G =: g_\epsilon(x, y, z)$ and $H =: h_\epsilon(x, y, z)$.
   - else return "`failure`".

We implement the algorithm `APPROXLINK` in the Axel [21] system as Axel offers a wide range of algebraic and geometric functions for manipulating algebraic curves and surfaces.

We notice that the ϵ-link of the singularity L_ϵ computed by the algorithm `APPROXLINK` is an implicit smooth space algebraic curve given as the intersection of two implicit surfaces S_1, S_2 with defining equations $g_\epsilon, h_\epsilon \in \mathbb{R}[x, y, z]$. For visualization reasons, we also compute the surfaces defined by the sum $S_1 + S_2$ and the difference $S_1 - S_2$. Thus L_ϵ is at the intersection of any two of the surfaces $\{S_1, S_2, S_1 + S_2, S_1 - S_2\}$, that are all part of the Milnor fibration. We employ subdivision methods [11] from Axel to compute the certified piecewise linear approximation (topology) of the implicit smooth space algebraic curve L_ϵ. This approximation of L_ϵ is computed as a graph $Graph(L_\epsilon)$. The data structure $Graph(L_\epsilon)$ is given as a set of vertices V together with their Euclidean coordinates in $\mathbb{R}^3$, and a set of edges E connecting them. In addition $Graph(L_\epsilon) = \langle V, E\rangle$ is isotopic to L_ϵ.

In Figure 5 we visualize the link (trefoil knot) of the singularity $(0, 0)$ of the plane complex algebraic curve $\mathcal{C}$ defined by the polynomial $p(z, w) = z^3 - w^2$. By using subdivision methods, Axel computes the piecewise linear approximation of the trefoil knot as a graph data structure.

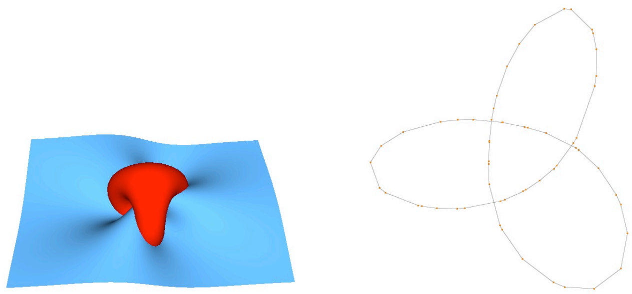

Figure 5: Piecewise linear approximation of the trefoil knot, computed as the intersection of two implicit surfaces with algorithm `APPROXLINK` in Axel

We next manipulate the approximation $Graph(L_\epsilon)$ symbolically to compute the ϵ-Alexander polynomial of L_ϵ. We first design an algorithm to compute the diagram $D(L_\epsilon)$ of the approximation $Graph(L_\epsilon)$, as defined in Subsection 2.2. We based this algorithm on computational geometry algorithms [5]. The algorithm requires as input the approximation $Graph(L_\epsilon) = \langle V, E\rangle$, and it returns as output the diagram $D(L_\epsilon)$, and that is: (1) the list of n crossings of $D(L_\epsilon)$ computed as all the intersections of the edges from E; (2) the list of n pairs of edges containing each intersection point. Each pair of edges (e_i, e_j) is ordered, i.e. e_i is under e_j in $\mathbb{R}^3$; (3) the r lists of edges from E for all the r knot components of $D(L_\epsilon)$; (4) the list of arcs of $D(L_\epsilon)$ and the type of each crossing. For more details on this algorithm see [9]. In Figure 6 we visualize the diagram $D(L_\epsilon)$ of the approximation $Graph(L_\epsilon)$ of the trefoil knot.

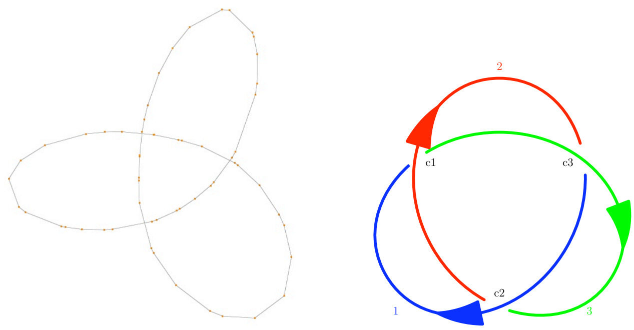

Figure 6: Diagram with 3 crossings and 3 arcs of the piecewise linear approximation of the trefoil knot

We now give the algorithm `APPROXALEXPOLY`$(D(L_\epsilon), r, n)$ for computing the ϵ-Alexander polynomial of the diagram $D(L_\epsilon)$ with r components and n crossings. We base this algorithm on Definition 8 from Subsection 2.2. For more details on this algorithm and an example see [8].

Algorithm 2 ϵ-Alexander polynomial of the diagram $D(L_\epsilon)$ of the APPROXALEXPOLY$(D(L_\epsilon), r, n)$

Input: $D(L_\epsilon)$ oriented algebraic link diagram of L_ϵ with r components, n crossings
Output: $\Delta_\epsilon(t_1, ..., t_r) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_r^{\pm 1}]$
where $\Delta_\epsilon(t_1, ..., t_r)$ is the ϵ-Alexander polynomial of L_ϵ with diagram $D(L_\epsilon)$.

1. Denote the arcs and separately the crossings of $D(L_\epsilon)$ with $\{1, ..., n\}$;
2. Compute $LM(L_\epsilon)$ the labelling matrix of $D(L_\epsilon)$;
3. Compute $PM(L_\epsilon)$ the prealexander matrix of $D(L_\epsilon)$;
4. If $r = 1$ then:
   (a) Compute M any $(n-1) \times (n-1)$ minor of $PM(L_\epsilon)$;
   (b) Compute D the determinant of the minor M;
   (c) Return $\Delta_\epsilon(t_1) = Normalize(D_\epsilon)$;
5. If $r \geq 2$ then:
   (a) Compute all the $(n-1) \times (n-1)$ minors of $PM(L_\epsilon)$;
   (b) Compute G the greatest common divisor of all the computed minors in 5.(a);
   (c) Return $\Delta_\epsilon(t_1, ...t_r) = Normalize(G)$.

We now present the algorithm APPROXDELTA$(\Delta_\epsilon, \mu, r)$ for computing the ϵ-delta-invariant from the ϵ-Alexander polynomial of degree μ and with r variables.

Algorithm 3 ϵ-delta-invariant of the singularity P of the plane curve $\mathcal{C}$ defined by $p(z, w)$: APPROXDELTA$(\Delta_\epsilon, \mu, r)$

Input: $\Delta_\epsilon(t_1, ..., t_m)$ the ϵ-Alexander polynomial of L_ϵ
L_ϵ the ϵ-algebraic link of the singularity $P = (z_0, w_0)$,
μ the degree of Δ_ϵ, r the number of variables in Δ_{L_ϵ}
Output: $\delta_\epsilon \in \mathbb{Z}_{>0}$
where δ_ϵ is the ϵ-delta-invariant of $P = (z_0, w_0)$.

1. If $r = 1$ then return $\delta_\epsilon = \mu/2$.
2. If $r \geq 2$ then return $\delta_\epsilon = (\mu + r)/2$.

4. REGULARIZATION PRINCIPLES

4.1 Basic Notations

We denote by I the set of coefficient vectors of all the squarefree polynomials from $\mathbb{C}[z, w]$ of fixed degree. The set $\mathcal{P} := \{\mathbb{Z}[t_1] \cup \mathbb{Z}[t_1, t_2] \cup ... \cup \mathbb{Z}[t_1, ..., t_i] \cup ...\}$ represents the set of all normalized Alexander polynomials either in the t_1 variable, or in the t_1, t_2 variables, or in the $t_1, t_2, ...t_i$ sequence of variables with $i \in \mathbb{N}$, etc. We denote by O the discrete set of integer coefficient vectors of all the polynomials from $\mathcal{P}$. For a polynomial $p(x, y)$ of fixed degree we denote with p its corresponding coefficient vector. The sets I, O are metric spaces by the Euclidean distance of coefficient vectors, denoted with $||\cdot||$. The notation $|\cdot|$ represents the absolute value function.

For $p(z, w) \in \mathbb{C}[z, w]$ we denote by:

$$M_p := \begin{pmatrix} \partial_z p(z, w) & \partial_w p(z, w) \\ \overline{z} & \overline{w} \end{pmatrix}$$

the two-by-two matrix formed by the partial derivatives of $p(z, w)$ with respect to z and w, and by the complex conjugates $\overline{z}, \overline{w}$. We denote by $Zeroes(p)$ the set of zeroes of the polynomial $p(z, w)$.

4.2 Definitions

First we establish a general framework for handling ill-posed algebraic problems using adapted regularization principles from [6, 18]. We then apply these principles to Problem 1 from Section 3, which we treat in this paper.

We define a well-posed problem as it was first formulated by J. Hadamard: a problem is said well-posed if: (i) there exists a solution to the problem (**existence**); (ii) the solution is unique (**uniqueness**); (iii) the solution depends continuously on the data in some given topological space (**stability**). Otherwise the problem is called ill-posed.

We consider the discontinuous function:

$$E : X \to Y, f \mapsto E(f), \tag{2}$$

on the metric spaces X, Y with metrics given by the Euclidean norm. The problem of computing $E(f) \in Y$ for given $f \in X$ is ill-posed as the computed output does not continuously depend on the input, i.e. the **stability** statement from the definition of well-posed problems does not hold. We define a perturbation function as follows:

Definition 12. A perturbation of $f \in X$ is defined as the function $f_- : \mathbb{R}_{>0} \to X, \delta \mapsto f_\delta$ with $||f - f_\delta|| \leq \delta$ for all $\delta \in \mathbb{R}_{>0}$. In this case f is called the exact data, f_δ the perturbed data and δ the noise level (error, tolerance).

In this framework we define a regularization as follows:

Definition 13. For any $\epsilon \in \mathbb{R}_{>0}$, let:

$$R_\epsilon : X \to Y, f \mapsto R_\epsilon(f)$$

be a continuous function. The function R_ϵ is called a regularization if there exists a bijective, monotonic function $\epsilon = \alpha(\delta), \alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with:

$$\lim_{\delta \to 0} \alpha(\delta) = 0, \tag{3}$$

such that for any $f \in X$ and for any perturbation function f_- with $||f - f_\delta|| \leq \delta$ for all $\delta \in \mathbb{R}_{>0}$, the following property holds:

$$\lim_{\delta \to 0} R_{\alpha(\delta)}(f_\delta) = E(f) \tag{4}$$

The function α is called a *parameter choice rule*, ϵ is called the *regularization parameter* and R_α is called the *regularized solution* of E. The equation (4) is called the *convergence for noisy data* property of R_α. The pair (R_α, α) is called a regularization method for solving the ill-posed problem E if the equations (3) and (4) hold.

For our problem, we consider X the set I of coefficient vectors of squarefree polynomials $p(z, w) \in \mathbb{C}[z, w]$ of fixed degree and Y the set O of integer coefficient vectors of normalized Alexander polynomials. In addition, we let:

$$E : I \to O, f \mapsto E(f) \tag{5}$$

be the exact algorithm for computing the Alexander polynomial of a plane curve singularity. Since O is a discrete set, the function E is discontinuous. Therefore, the problem of computing the Alexander polynomial $E(f) \in O$ for given $f \in I$ is ill-posed.

For every $\epsilon \in \mathbb{R}_{>0}$, we denote by:

$$A_\epsilon : U \subset I \to O, p \mapsto A_\epsilon(p) \tag{6}$$

the symbolic-numeric that computes the ϵ-Alexander polynomial $A_\epsilon(p)$ for given $(p, \epsilon) \in I \times \mathbb{R}_{>0}$, as described in Section 3. This polynomial arises as the intersection of the sphere S_ϵ with the curve $\mathcal{C}$ defined by p. We notice that A_ϵ is a partial function, because it is not defined in case the intersection $S_\epsilon \cap \mathcal{C}$ has singularities. Still the function A_ϵ is continuous in its domain of definition denoted by U.

We wish to show that A_ϵ is a regularization function for every $(p, \epsilon) \in U \subset I \times \mathbb{R}_{>0}$. Therefore, from Definition 13 we need to find a parameter choice rule $\epsilon = \alpha(\delta)$ with property (3) and that satisfies equation (4). Consequently, the pair (A_α, α) would be a regularization method for solving the ill-posed Problem 1.

4.3 Convergence Results

In this subsection, we include the lemmas and the theorems that we formulate to prove the convergence for noisy data property of the algorithm A_ϵ considered in (6). In this subsection, we sketch the main steps of the proofs. A complete proof would be beyond the scope of this submission.

First we set the general mathematical setting required for our study. Let $f(z, w)$ be arbitrary but fixed. For simplicity we denote $f_\delta(z, w) := g(z, w) \in \mathbb{C}[z, w]$ with $||g - f|| \le \delta$. Based on Theorem 1, we take $K > 0$ such that the system:

$$f(z, w) = \det(M_f)(z, w) = 0 \tag{7}$$

has no common solution except for $(0, 0)$ in the closed ball $B_K := \left\{(z, w) \in \mathbb{C}^2 : \left(|z|^2 + |w|^2\right)^{1/2} \le K\right\}$ of radius K around $(0, 0) \in \mathbb{C}^2$. Thus the following relation holds:

$$f(0, 0) = \det(M_f)(0, 0) = 0, \tag{8}$$

and the intersection $B_K \cap Zeroes(f)$ has no singularities except for $(0, 0)$.

To prove the convergence for noisy data property, we require a preliminary lemma.

LEMMA 1. *There exists $N > 0$ such that for all $\delta > 0$, and for all g with $||g - f|| \le \delta$ there exists no zero for the system of polynomial equations determined by $g(z, w) = \det(M_g)(z, w) = 0$ whose length is greater than $\delta^{1/N}$ and less than K.*

To prove Lemma 1 we prove the equivalent statement:

$$\begin{gathered} \exists N > 0 \; \forall \delta > 0 \; \forall g : ||g - f|| \le 0 \; \forall (z, w) : \\ g(z, w) = \det(M_g)(z, w) = 0 \text{ and} \\ \left(|z|^2 + |w^2|\right)^{1/2} \le K \Rightarrow \left(|z|^2 + |w^2|\right)^{1/2} \le \delta^{1/N}. \end{gathered} \tag{9}$$

We take $\delta > 0$ and g with $||g - f|| \le \delta$.

First, we define the set Z_δ of "special" zeroes of g :

$$\begin{aligned} Z_\delta = \Big\{ \Big((z, w), g\Big) &: ||g - f|| \le \delta, \\ & g(z, w) = \det(M_g)(z, w) = 0, \\ & \left(|z|^2 + |w|^2\right)^{1/2} \le K \Big\}. \end{aligned} \tag{10}$$

We introduce the function:

$$\begin{gathered} \tau : B_K \times I \to \mathbb{R}_{\ge 0} \\ \Big((z, w), g\Big) \mapsto \tau\Big((z, w), g\Big) = \left(|z|^2 + |w|^2\right)^{1/2}. \end{gathered} \tag{11}$$

By using the theorem on Euclidean extreme values of real-valued functions, we prove that τ attains its maximum and we define the monotonic, semialgebraic function:

$$\begin{gathered} \beta : \mathbb{R}_{>0} \to \mathbb{R}_{\ge 0} \\ \delta \mapsto \beta(\delta) = max \; \{\tau(a) : a \in Z_\delta\}. \end{gathered} \tag{12}$$

Secondly, we prove the convergence of β by using the theorem of Bolzano-Weierstrass on compact sets.

Finally, we show that the function β is bounded from above. We use the following theorem for estimating the rate of growth of a semialgebraic function of one variable:

THEOREM 2. *([2]) Let $f : (a, \infty) \to \mathbb{R}$ be a semialgebraic function (not necessarily continuous). There exists $b \ge a$ and an integer $N \in \mathbb{N}$ such that $|f(x)| \le x^N$ for all $x \in (b, \infty)$.*

Moreover, we use the following theorem for ensuring the piecewise continuity of a semialgebraic function:

THEOREM 3. *([13]) Let F be a real closed field and $f : F \to F$ be a semialgebraic function. Then, we can partition F into $I_1 \cup ... I_m \cup X$, where X is finite and I_j are pairwise disjoint open intervals with endpoints in $F \cup \{\pm\infty\}$ such that f is continuous on each I_j with $j \in \{1, ..., m\}$ and $m \in \mathbb{N}$.*

We get that there exists $N \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{R}_+$ such that:

$$\beta_r(\delta) \le \delta^{1/N},$$

for all $\delta < \eta = b^{-1}$, where β_r is the restriction of β to the first open interval.

We use Lemma 1 as a tool for proving the convergence for noisy data statement (4) and for ensuring the existence of a parameter choice rule (3) for A_ϵ. This convergence statement is given by the following theorem:

THEOREM 4. *There exists $N > 0$ and $\eta \in \mathbb{R}_{>0}$ such that for all $\delta > 0$ with $\delta < \eta$, for all g with $||g - f|| \le \delta$ and for all $\epsilon \in [\delta^{1/N}, K]$, the following property holds: $A_\epsilon(g) = E(f)$.*

We prove Theorem 4 by constructing the isotopy:

$$\begin{aligned} & g_t : \mathbb{C}^2 \times [0, 1] \to \mathbb{C} \\ & (z, w) \to g_t(z, w) = t f(z, w) + (1 - t) g(z, w), \end{aligned} \tag{13}$$

with g_t continuous function for all $0 \le t \le 1$, and $g_0 = g$, $g_1 = f$, and by showing that $A_\epsilon(g_t)$ is an ϵ-algebraic link based on Lemma 1.

From Theorem 4 it follows that $\epsilon = \delta^{1/N}$ is a parameter choice rule for A_ϵ, for which the convergence for noisy data statement (4) of A_ϵ holds. Still, this parameter choice rule depends on N which is unknown. The following lemma provides us with an upper bound for $\delta^{1/N}$ which is independent on N:

Lemma 2. *For all $N > 0$ there exists $\theta \in \mathbb{R}_+$ such that for all $\delta > 0$ with $\delta < \theta$, the inequality $\delta^{1/N} \leq \frac{1}{|ln\delta|}$ is true.*

We prove Lemma 2 by basic calculus and by using l'Hôpital rule. The preceding two lemmas allow us to formulate the following theorem concerning the existence of a parameter choice rule for A_ϵ which only depends on the given $\delta \in \mathbb{R}_+$:

Theorem 5. *The function $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \alpha(\delta) = \frac{1}{|ln\delta|}$ is a parameter choice rule, i.e.*

$$\lim_{\delta \to 0} A_{\alpha(\delta)}(f_\delta) = E(f) \tag{14}$$

The theorem is true based on Lemma 1, Theorem 4 and Lemma 2.

Remark 1. The parameter choice rule indicates that the "degree of ill-posedness" is rather high (cf. with linear regularization theory [18], where $\alpha(\delta) = \delta^{1/2}$ frequently occurs). For fixed input instance f, the smallest function $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that (noisy convergence) is true is equal to the function β from Lemma 1. The choice of α was done in order to ensure that α dominates β for every possible f. Here is a series of examples that show that a semi-algebraic parameter choice rule cannot be used as a choice rule.

Example 2. Let $n > 0$ be an integer. Let $f(z, w) = z^2 - w^{n+2}$. We consider the perturbation $g(z, w) = f_\delta(z, w) = z^2 - w^{n+2} + \delta w^2$, for $\delta \in (0, 1)$. Then we have a special zero of (g, M_g) at $(z, w) = (0, \delta^{1/n})$. A closer analysis shows that the ϵ-link of g is the Hopf link for every sphere with radius less than $\delta^{1/n}$, while the link of f is equal to the torus link $(2, n + 2)$. Consequently, $\beta(\delta) > \delta^{1/n}$ for this choice of f. Since n can be arbitrary, no function which is dominated by a function of the from $\delta \mapsto \delta^{1/m}$ for some m can be chosen as a parameter choice rule.

5. IMPLEMENTATION

5.1 A Library for Algebraic Curves

We implemented the symbolic-numeric algorithms for computing invariants of a plane complex algebraic curve described in Section 3 in the free library GENOM3CK [7]-GENus cOMputation of a plane Complex algebraiC Curve using Knot theory-written in the Axel free algebraic geometric modeler [21] and in the Mathemagix free computer algebra system [10], i.e. in C++ using Qt Script for Applications and OpenGL. By using Axel, we integrate symbolic, numeric and graphical capabilities into a single library. Together with its main functionality to compute the genus, the library performs operations in topology, algebraic geometry and knot theory. More information on GENOM3CK (including download, installation and complete documentation) can be found at: `http://people.ricam.oeaw.ac.at/m.hodorog/software.html`.

5.2 Test Experiments

We include several experiments performed with the library GENOM3CK in Axel. In Figure 7 we consider the plane complex algebraic curve defined by the squarefree polynomial $p(z, w) = z^3 - w^3$, with a singularity in the origin and the input parameter $\epsilon = 1.00$. From left to right, we visualize: (1) the link L_ϵ of the singularity computed as the intersection of two implicit surfaces S_1, S_2; (2) the two surfaces S_1, S_2; (3) the four surfaces $S_1, S_2, S_1 + S_2, S_1 - S_2$, which are all part of the Milnor fibration of the singularity.

The test experiments indicates the convergence for noisy data property of the regularization method as proved in Section 4. In Table 1 we consider several input curves defined by squarefree polynomials, which have the singularity in the origin or close to the origin. The first column indicates the defining polynomial of the curve, the second column indicates the value for ϵ and the next columns contain the computed values for the ϵ-link, the ϵ-Alexander polynomial and respectively the ϵ-delta-invariant of the singularity.

We emphasize that for the curves $\mathcal{C}$ and $\mathcal{D}$ defined by $p(z, w) = -z^3 - zw + w^2$ and $\tilde{p}(z, w) = -z^3 - zw + w^2 - 0.01$ from Example 1, the singularity $(0, 0)$ of $\mathcal{C}$ disappears under small perturbations of $p(z, w)$ and we obtain the nonsingular curve $\mathcal{D}$. By using the algorithm `APPROXLINK` we observe that for $\epsilon = 0.25$ the link of $(0, 0)$ of $\mathcal{C}$ coincide with the link of $(0, 0)$ of $\mathcal{D}$.

6. CONCLUSION

We presented symbolic-numeric algorithms for computing invariants for each singularity of a plane complex algebraic curve: the link of each singularity, the Alexander polynomial attached to each link, and the delta-invariant of each singularity. We implemented the symbolic-numeric algorithms in a free library which combines graphical, numerical and symbolic capabilities. We employed regularization principles to handle the ill-posedness of the problem.

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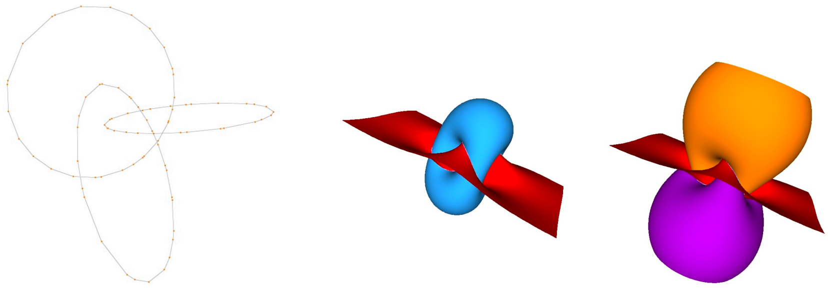

Figure 7: Link, Milnor fibration of the singularity $(0,0)$ of the plane complex algebraic curve defined by $z^3 - w^3$

Table 1: Evidence for the Convergence for Noisy Data Statement

Equation of the plane complex algebraic curve	$\epsilon \in \mathbb{R}_{>0}$	ϵ**-link**	ϵ**-Alexander polynomial**	ϵ**-delta-invariant**
$-z^3 - zw + w^2$	1.00	Trefoil knot	$\Delta(t_1) = t_1^2 - t_1 + 1$	$\delta = 1$
$-z^3 - zw + w^2$	0.25	Hopf link	$\Delta(t_1, t_2) = 1$	$\delta = 1$
$-z^3 - zw + w^2 - 0.01$	1.00	Trefoil knot	$\Delta(t_1) = t_1^2 - t_1 + 1$	$\delta = 1$
$-z^3 - zw + w^2 - 0.01$	0.25	Hopf link	$\Delta(t_1, t_2) = 1$	$\delta = 1$

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