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Using Gröbner Bases for Finding the Logarithmic Part of the Integral of Transcendental Functions

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Abstract

We show that Czichowski's algorithm for computing the logarithmic part of the integral of a rational function can be carried over to a rather general class of transcendental functions.

1 Introduction

A standard approach in symbolic integration is to use differential fields for modeling the integrand. When integrating in terms of elementary functions one tries to construct an elementary extension of the field such that an indefinite integral can be found there. In view of Liouville's Theorem, terms of the integral not lying in the field originally given are called the "logarithmic part" of the integral. In the 1970's Rothstein and Trager [8, 9] gave a formula for the logarithmic part without factoring the denominator using the minimal algebraic extension of the field necessary. Later Lazard and Rioboo [6] presented an algorithm avoiding any calculations in algebraic extensions for computing the logarithmic part of the integral of rational functions based on subresultants, according to [1] Trager also discovered that algorithm but did not publish it. In [2] Czichowski observed that alternatively Gröbner bases can be used for the same purpose. An extension of the Lazard-Rioboo-Trager algorithm from the rational case to certain transcendental integrands can be found in [1]. In this note we will show that also Czichowski's algorithm can be carried over to the same class of transcendental functions. In contrast to [2] the proofs given here are more detailed and make explicit use of Lazard's structure theorem [5] instead of reproving the relevant parts. Indeed, that theorem plays a key role in our presentation of Czichowski's method and we find it remarkable that a structure theorem, established in the context of primary decomposition of ideals, finds a direct connection to symbolic integration. For algebraic integrands, first attempts have been made by Kauers [4].

In section 2 we give the definitions used and for the convenience of the reader necessary preliminary results are stated and some context is provided as well. Section 3 contains the

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main result Theorem 8 and some remarks on efficiency. In particular no postprocessing of the logands is necessary once the Gröbner basis has been computed in contrast to the necessity in the Lazard-Rioboo-Trager algorithm that has been pointed out in [7]. After that some examples are presented in section 4.

All fields are implicitly understood to be of characteristic zero. In addition, we need to define $\operatorname{res}(0, b) := 1$ for the special case $\deg(b) = 0$ in order to simplify the statements, since then a vanishing resultant corresponds to common roots of polynomials.

2 Preliminaries

Recall the following definitions needed to formulate the main result.

Definition 1. *Let (F, D) be a differential field, K a differential subfield and $t \in F$ then t is called a monomial over (K, D) if*

1. *t is transcendental over K and*
2. *$Dt \in K[t]$.*

For such t we define $K\langle t\rangle := \left\{ \frac{a}{b} \;\middle|\; a, b \in K[t], b|Db \right\}$ the set of reduced elements of $K(t)$.

Note that in this case $K[t]$ and $K\langle t\rangle$ are differential rings and $K[t] \subseteq K\langle t\rangle$ since $b = 1$ trivially satisfies $b|Db$.

Definition 2. *A differential field $(F(t_1, \ldots, t_n), D)$ is called an elementary extension of F if each t_i is elementary over $F_i := F(t_1, \ldots, t_{i-1})$, i.e.*

- *t_i is algebraic over F_i, or*
- *$Dt_i = \frac{Df}{f}$ for some $f \in F_i$ (i.e. t_i is a logarithm of f), or*
- *$\frac{Dt_i}{t_i} = Df$ for some $f \in F_i$ (i.e. t_i is an exponential of f).*

We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ such that

$$Dg = f.$$

Using these definitions we can think of the following variant of integration in finite terms for providing the context for the algorithm presented later, which can be used as a subroutine for solving it.

Given $(F, D) = (C(t_1)\ldots(t_n), D)$ a tower of successive differential field extensions of $C = \operatorname{Const}(F)$ by monomials and $f \in F$. Decide in finitely many steps whether f has an elementary integral over (F, D), and compute one if it exists.

Recursive Risch-type algorithms (in contrast to Risch-Norman-type algorithms) proceed through the field extensions one by one. Integrands from $F =: K(t_n)$ are reduced to integrands from the differential subfield $K = C(t_1)\ldots(t_{n-1})$ and at the same time parts of the integral are computed. At each step of the recursion part of the logarithmic part of the integral can be computed relying on the following theorem, which is a corrected and stronger version of Theorem 5.6.1 from [1]. All necessary proof ingredients can be adapted in a straightforward way, so no proof is given here. Following preceding results it relies on the Rothstein-Trager resultant, see [8, 9] for example. In statement 2 it gives a necessary criterion on when the integral of an element $f \in K(t)$ is elementary over $K(t)$.

Theorem 3. *Let t a monomial over the differential field (K, D) and assume that $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Let $a, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ and let z be an indeterminate over $K[t]$. Define*

$$r := \mathrm{res}_t(a - zDb, b) \in K[z] \tag{1}$$

and

$$g := \sum_{r(\alpha)=0} \alpha \frac{Dg_\alpha}{g_\alpha} \in \overline{K}(t), \tag{2}$$

where $g_\alpha := \gcd(a - \alpha Db, b) \in K(\alpha)[t]$ for each root $\alpha \in \overline{K}$ of r.

1. *Then $g \in K(t)$ and $\frac{a}{b} - g \in K[t]$.*
2. *If there exists $h \in K\langle t\rangle$ such that $h + \frac{a}{b}$ has an elementary integral over $(K(t), D)$, then all roots $\alpha \in \overline{K}$ of r are in $\overline{C}$.*
3. *If E is an algebraic extension of C such that there are $h \in K\langle t\rangle$, $v \in K(t)$, $c_1, \ldots, c_n \in E$, and $u_1, \ldots, u_n \in EK(t)$ with $h + \frac{a}{b} = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}$, then E contains all roots $\alpha \in \overline{K}$ of r.*

The algorithm which we are going to present relies on the following structure theorem for bivariate lexicographic Gröbner bases, see [5, Thm 1]. Since we will use it a lot in the next section, it is restated here for the reader's convenience.

Theorem 4. *(Structure Theorem) Let K be any field, consider the commutative polynomial ring $K[x, y]$ with lexicographic ordering $x < y$.*

1. *Let $\{P_0, \ldots, P_m\} \in K[x, y]$ be a minimal Gröbner basis of an ideal in $K[x, y]$ such that $\mathrm{lt}(P_{i-1}) < \mathrm{lt}(P_i)$ for all $i \in \{1, \ldots, m\}$. Then*

   $$\forall i \in \{0, \ldots, m\} : P_i = Q_{i+1} \cdot \ldots \cdot Q_{m+1} \cdot R \cdot S_i,$$

   where $Q_1, \ldots, Q_{m+1} \in K[x]$, $Q_{m+1} = \mathrm{cont}_y(P_m)$, $R = \mathrm{pp}_y(P_0) \in K[x, y]$, $S_0 = 1$, and $S_1, \ldots, S_m \in K[x, y]$ such that for all $i \in \{1, \ldots, m\}$:

   (a) S_i is monic w.r.t. y,

   (b) $\deg_y(S_{i-1}) < \deg_y(S_i)$, and

   (c) $S_i \in \langle Q_{j+1} \cdot \ldots \cdot Q_{i-1} \cdot S_j \mid j \in \{0, \ldots, i-1\}\rangle$.

2. *Every set of polynomials which satisfies the preceding conditions is a Gröbner basis; it is minimal if and only if $\forall i \in \{1, \ldots, m\} : Q_i \notin K$.*

3 Computing the Logarithmic Part

For computing elementary integrals using Theorem 3 we would need to compute all the $g_\alpha = \gcd(a - \alpha Db, b)$ as gcd's in various algebraic extensions $K(\alpha)[t]$. There are two methods for avoiding gcd computation in algebraic extensions at this point. In [1] it is shown how the idea of Lazard, Rioboo and Trager of using the subresultant PRS for computing the Rothstein-Trager resultant (1) to obtain the g_α can be carried over from rational functions to this general setting of monomials t. We do not discuss this here. Instead we show how Czichowski's idea of using a bivariate Gröbner basis to obtain the g_α carries over from rational functions to this general setting as well. For the following three lemmas the field K does not need to carry a differential structure. These are generalizations of the Lemmas 2.1, 2.2.iii, and 2.3 from [2] with essentially the same proofs. The proofs given here are more detailed and make explicit use of Theorem 4 instead of reproving the relevant parts.

Lemma 5. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$. Then the ideal $I := \langle a - zc, b\rangle \subseteq K[t, z]$ is zero-dimensional and radical. Moreover, $\{b, z - pa\}$ is a minimal Gröbner basis of I w.r.t. lexicographic ordering $t < z$ for $p \in K[t]$ such that $pc \equiv 1 \pmod{b}$.*

Proof. First, we show that $\{b, z-pa\}$ is a minimal Gröbner basis of I w.r.t. lexicographic ordering $t < z$. Since $\gcd(b, c) = 1$ such a $p \in K[t]$ always exists and let $q \in K[t]$ such that $pc + qb = 1$. Hence we have $(-p)\cdot(a - zc) + (zq)\cdot b = -pa + zpc + zqb = z - pa$, i.e., $z - pa \in I$. On the other hand $(qa)\cdot b + (-c)\cdot(z - pa) = a - zc$. Thus $\{b, z - pa\}$ is a minimal Gröbner basis of I w.r.t. the lexicographic ordering $t < z$.

Now, for proving zero-dimensionality we show that the corresponding algebraic variety of the ideal I is a finite set. To this end, let $\beta_1, \ldots \beta_d \in \overline{K}$ be the roots of $b \in K[t]$. From $\gcd(b, c) = 1$ it follows that $c(\beta_i) \neq 0$ for all $i \in \{1, \ldots, d\}$. Hence for each β_i there is exactly one $\alpha_i \in \overline{K}$ such that $a(\beta_i) - \alpha_i c(\beta_i) = 0$. So the system of equations $a(t) - z\cdot c(t) = 0, b(t) = 0$ has only finitely many solutions $(t, z) \in \overline{K}^2$.

Next, we show that the radical ideal $\mathrm{Rad}(I)$ is contained in I. Let $r \in \mathrm{Rad}(I)$ and reduce it by $\{b, z - pa\}$ as follows: $r(t, z)$ is reduced by $z - pa$ to $r(t, p(t)a(t))$, which in turn is reduced by b to some $\tilde{r} \in K[t]$ with $\deg(\tilde{r}) < \deg(b)$. In addition, $\tilde{r}$ vanishes on the $\deg(b)$ distinct roots (in $\overline{K}$) of b because of $\tilde{r} \in \mathrm{Rad}(I)$. Altogether this implies $\tilde{r} = 0$, i.e., $r \in I$. □

Lemma 6. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$, and let $\{P_0, \ldots, P_m\} \subseteq K[z, t]$ be a minimal Gröbner basis of the ideal $I := \langle a - zc, b\rangle \subseteq K[z, t]$ w.r.t. lexicographic ordering $z < t$ such that $\mathrm{lt}(P_0) < \mathrm{lt}(P_i)$ for all $i \in \{1, \ldots, m\}$.*
Then $P_0 \in K[z]$ is the squarefree part of $r(z) := \mathrm{res}_t(a - zc, b) \in K[z]$.

Proof. By the elimination property $\{P_0, \ldots, P_m\} \cap K[z]$ is a Gröbner basis of $I \cap K[z]$. Since by Lemma 5 the ideal I is zero-dimensional $\{P_0, \ldots, P_m\} \cap K[z]$ is not empty. Since P_0 is the basis element with smallest leading term we obtain $P_0 \in K[z]$. From the minimality of the Gröbner basis we conclude $\{P_0, \ldots, P_m\} \cap K[z] = \{P_0\}$. So the roots of $P_0 \in K[z]$ are those $\alpha \in \overline{K}$ such that the polynomials $\{P_0(\alpha, t), \ldots, P_m(\alpha, t)\} \subseteq \overline{K}[t]$ have a common root in $\overline{K}$. In addition, by Lemma 5 the ideal I is radical, hence also

$I \cap K[z] = \langle P_0 \rangle$ is radical. This implies that P_0 is squarefree.
The roots of $r \in K[z]$ are those $\alpha \in \overline{K}$ such that $a - \alpha c \in \overline{K}[t]$ and b have a common root in $\overline{K}$. Now, $\{a - zc, b\}$ and $\{P_0, \dots, P_m\}$ generate the same ideal (in $K[z,t]$) so by the evaluation homomorphism $z \mapsto \alpha$ also $\{a - \alpha c, b\}$ and $\{P_0(\alpha, t), \dots, P_m(\alpha, t)\}$ generate the same ideal (in $\overline{K}[t]$). Hence the roots of r and P_0 are the same. $\square$

Lemma 7. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$, and let $\{P_0, \dots, P_m\} \subseteq K[z,t]$ be a minimal Gröbner basis of the ideal $I := \langle a - zc, b \rangle \subseteq K[z,t]$ w.r.t. lexicographic ordering $z < t$ with $\operatorname{lt}(P_{i-1}) < \operatorname{lt}(P_i)$ for all $i \in \{1, \dots, m\}$. Furthermore, let $Q_1, \dots, Q_{m+1} \in K[z]$ and $R, S_0, \dots, S_m \in K[z,t]$ be as in Theorem 4.*
Then for any $\alpha \in \overline{K}$ root of $r(z) := \operatorname{res}_t(a - zc, b) \in K[z]$ there is a unique $i \in \{1, \dots, m\}$ such that $Q_i(\alpha) = 0$. With this i we have

$$S_i(\alpha, t) = \gcd(a - \alpha c, b) \in K(\alpha)[t].$$

Proof. From Lemma 6 we know that $R = 1$ and $P_0 = Q_1 \cdot \ldots \cdot Q_{m+1}$ is squarefree and has the same roots as r. So there is a unique $i \in \{1, \dots, m+1\}$ such that $Q_i(\alpha) = 0$. Since by Lemma 5 I is zero-dimensional we have $\deg(Q_{m+1}) = 0$, otherwise for the roots $\tilde{\alpha} \in \overline{K}$ of Q_{m+1} all $P_j(\tilde{\alpha}, t)$ would vanish on all $t \in \overline{K}$. So $i \neq m+1$.

Next, using this i we prove $\forall k \in \{0, \dots, m\} : P_i(\alpha, t) | P_k(\alpha, t)$ by induction on k. For $k < i$ we have $Q_i | P_k$ and hence $P_k(\alpha, t) = 0$; for $k = i$ we have $P_k(\alpha, t) \neq 0$ by the uniqueness of i and the statement is trivial. For $k \in \{i+1, \dots, m\}$ Theorem 4 implies $S_k \in \langle Q_{j+1} \cdot \ldots \cdot Q_{k-1} \cdot S_j \mid j \in \{0, \dots, k-1\} \rangle$. Multiplication by $Q_k \cdot \ldots \cdot Q_{m+1}$ yields $Q_k P_k \in \langle P_j \mid j \in \{0, \dots, k-1\} \rangle$. Hence we obtain $Q_k P_k = \sum_{j=0}^{k-1} T_j P_j$ for some $T_j \in K[z,t]$. Evaluation at $z = \alpha$ yields

$$Q_k(\alpha) P_k(\alpha, t) = \sum_{j=0}^{k-1} T_j(\alpha, t) P_j(\alpha, t) \in K(\alpha)[t].$$

By the induction hypothesis each summand of the right hand side is divisible by $P_i(\alpha, t)$. Dividing by $Q_k(\alpha) \in K(\alpha)^*$ concludes the induction step.

Now, from this it follows that $\gcd(P_k(\alpha, t) \mid k \in \{0, \dots, m\}) = S_i(\alpha, t)$, note that $S_i(\alpha, t)$ is monic by Theorem 4. But we also have $\gcd(P_k(\alpha, t) \mid k \in \{0, \dots, m\}) = \gcd(a - \alpha c, b)$, since by the evaluation homomorphism $z \mapsto \alpha$ we know that $\{P_k(\alpha, t) \mid k \in \{0, \dots, m\}\}$ and $\{a - \alpha c, b\}$ generate the same ideal in $K(\alpha)[t]$. $\square$

The algorithm that can be read off from the proof of the following result may be used as a building block in a recursive reduction strategy for finding elementary integrals of elements from $K(t)$. In short it provides a way to reduce simple integrands to polynomial integrands and at the same time logarithms appearing in the integral are found. Note that "simple" refers to a certain property of the denominator and "polynomial" refers to elements from $K[t]$ here.

Theorem 8. *Let t a monomial over the differential field (K, D) and assume that $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$. Let $a, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ and let z be an indeterminate. Then using modular inversion in $K[t]$ and linear systems over K we can compute $Q_1, \dots, Q_m \in K[z]$ and $S_1, \dots, S_m \in K[z,t]$ such that*

1. *$r \in K[z]$ as defined in (1) has all its roots $\alpha \in \overline{K}$ lying in $\overline{C}$ if and only if $Q_1, \dots, Q_m \in C[z]$ and*

2. *$Q_1, \dots, Q_m$ are squarefree, $S_1, \dots, S_m$ are monic w.r.t. t and*

$$\frac{a}{b} - \sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DS_i(\alpha, t)}{S_i(\alpha, t)} \in K[t].$$

Proof. First, in $K[t]$ we compute $p \in K[t]$ such that

$$pDb \equiv 1 \pmod{b}.$$

Then $\{b, z - pa\} \subseteq K[z, t]$ is a Gröbner basis of $\langle a - zDb, b\rangle$ w.r.t. lexicographic ordering $t < z$ by Lemma 5. From this, by the FGLM-algorithm [3], we compute a monic minimal Gröbner basis $\{P_0, \dots, P_m\} \subseteq K[z, t]$ for the same ideal but w.r.t. lexicographic ordering $z < t$, with $\mathrm{lt}(P_{i-1}) < \mathrm{lt}(P_i)$ for all $i \in \{1, \dots, m\}$. By finding solutions of linear systems over K and Lemma 5 we can do this. Next, for $i \in \{0, \dots, m\}$ we extract

$$R_i := \mathrm{lc}_t(P_i) \in K[z]$$

and finally we compute for $i \in \{1, \dots, m\}$

$$Q_i := \frac{R_{i-1}}{R_i} \in K(z) \quad \text{and} \quad S_i := \frac{P_i}{R_i} \in K(z)[t].$$

Now we verify the desired properties. By construction $S_1, \dots, S_m$ are monic w.r.t. t. Additionally, since the ideal is zero-dimensional we have $\mathrm{lc}_t(P_m) = \mathrm{lc}(P_m) = 1$ and $\deg_t(P_0) = 0$, hence $\mathrm{cont}_t(P_m) = 1$ and $\mathrm{pp}_t(P_0) = 1$. So by Theorem 4 we get $Q_1, \dots, Q_m \in K[z]$, $S_1, \dots, S_m \in K[z, t]$ and $P_0 = Q_1 \cdot \ldots \cdot Q_m$. Now Lemma 6 implies that $\{\alpha \in \overline{K} \mid r(\alpha) = 0\}$ is the disjoint union of $\{\alpha \in \overline{K} \mid Q_i(\alpha) = 0\}$ for $i \in \{1, \dots, m\}$ and that $Q_1, \dots, Q_m$ are squarefree. From this assertion 1 follows trivially since by construction $\mathrm{lc}(Q_i) = \frac{\mathrm{lc}(P_{i-1})}{\mathrm{lc}(P_i)} = 1$. Also assertion 2 follows immediately using Theorem 3.1 and Lemma 7. □

Remark Regarding the algorithmic efficiency in the proof of Theorem 8 note the following:

1. The Gröbner basis $\{b, z - pa\}$ of I is minimal. Computing $p \in K[t]$ with $\deg(p) < \deg(b)$ such that $pDb \equiv a \pmod{b}$ instead, we would obtain $\{b, z - p\}$ as a reduced Gröbner basis for I, which shortens computation of normal forms in the FGLM-algorithm.

2. During execution of the FGLM-algorithm $P_0 \in K[z]$ is the first element of the Gröbner basis that is computed. In view of Theorems 3.2 and 8.1 this can be used as a necessary criterion whether $h + \frac{a}{b}$ can have an elementary integral over $(K(t), D)$ without computing the full Gröbner basis $\{P_0, \dots, P_m\}$.

3. It can be shown that $\deg(b) = \dim_K(K[z, t]/I) = \sum_{i=1}^{m} \deg(Q_i) \deg_t(S_i)$. This can be exploited during the FGLM-algorithm in the following way. When computing P_k

we consider all partitions of $\deg(b) - \sum_{i=1}^{k-1} \deg(Q_i)\deg_t(S_i)$ into $m_0 := \deg_z(\mathrm{lt}(P_{k-1}))$ parts where each part is greater than $\deg_t(S_{k-1})$. By looking at the size m_1 and multiplicity m_2 of the smallest part in each of those partitions we obtain restrictions on the possible leading terms $z^{m_0-m_2}t^{m_1}$ of P_k. Thereby we can identify some steps in the FGLM-algorithm where the linear system will not have a solution. More explicitly, exactly the terms $1, t, \ldots, t^{\deg_t(P_m)-1}$ can be dropped from the candidates for leading terms.

4. Defining $S_i := P_i \in K[z,t]$ instead of computing the quotient $\frac{P_i}{R_i}$ we would retain all necessary properties (except monicity) since $\gcd(Q_i, P_i) = 1$. In this case we still have $\sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DS_i(\alpha,t)}{S_i(\alpha,t)} - g = \sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DR_i(\alpha)}{R_i(\alpha)} \in K$, where g is as in (2).

4 Examples

In the examples we keep the notation of Theorem 8. For simplicity all examples were chosen such that $C = \mathbb{Q}$ and $\sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DS_i(\alpha,t)}{S_i(\alpha,t)} = \frac{a}{b}$.

Example 1 Let $(K, D) = (\mathbb{Q}, 0)$, then the function $\tanh(x)$ can be represented by t with $Dt = -t^2 + 1 \in K[t]$. Now let $a = t^3 - t$, $b = \frac{2}{27}t^3 - t + 1$, then a monic minimal Gröbner basis of $\langle a - zDb, b\rangle \subseteq K[z,t]$ w.r.t. $z <_{lex} t$ is given by

$$\{(z+3)(z-\tfrac{3}{2}), (z-\tfrac{3}{2})(t-3), t^2+3t+3z-9\}.$$

Since in this example $K = C$ the necessary condition for having an elementary integral $P_0 \in C[z]$ is satisfied automatically. We read off $Q_1(z) = z + 3$, $Q_2(z) = z - \frac{3}{2}$ and $S_1(z,t) = t - 3$, $S_2(z,t) = t^2 + 3t - \frac{9}{2}$. So we successfully computed the following integral:

$$\int \frac{\tanh(x)^3 - \tanh(x)}{\frac{2}{27}\tanh(x)^3 - \tanh(x) + 1}\,dx = -3\log\left(\tanh(x) - 3\right) + \frac{3}{2}\log\left(\tanh(x)^2 + 3\tanh(x) - \frac{9}{2}\right).$$

Example 2 Let $(K, D) = (\mathbb{Q}(x), D)$ with $Dx = 1$, then the function $\frac{\mathrm{Bi}'(x)}{\mathrm{Bi}(x)}$, where $\mathrm{Bi}(x)$ is an Airy function, can be represented by t with $Dt = -t^2 + x \in K[t]$. Now let $a = t^3 - xt, b = t^3 + t^2 + 1$, then a monic minimal Gröbner basis of $\langle a - zDb, b\rangle \subseteq K[z,t]$ w.r.t. $z <_{lex} t$ is given by

$$\{z^3 - \tfrac{3}{31}z - \tfrac{1}{31}, t + \tfrac{31}{3}z^2 - \tfrac{1}{3}\}.$$

So in this simple case we have $Q_1(z) = P_0(z) = z^3 - \frac{3}{31}z - \frac{1}{31}$ and $S_1(z,t) = t + \frac{31}{3}z^2 - \frac{1}{3}$. Hence we successfully computed the following integral, where Mathematica, Maple, and Maxima currently do not succeed to find an integral:

$$\int \frac{\mathrm{Bi}'(x)^3 - x\mathrm{Bi}'(x)\mathrm{Bi}(x)^2}{\mathrm{Bi}'(x)^3 + \mathrm{Bi}'(x)^2\mathrm{Bi}(x) + \mathrm{Bi}(x)^3}\,dx = \sum_{31\alpha^3-3\alpha-1=0} \alpha\log\left(\frac{\mathrm{Bi}'(x)}{\mathrm{Bi}(x)} + \frac{31}{3}\alpha^2 - \frac{1}{3}\right).$$

Example 3 Let $(K, D) = (\mathbb{Q}(x), D)$ with $Dx = 1$, then the function $\log(x)$ can be represented by t with $Dt = \frac{1}{x} \in K[t]$. Now let $a = (x+1)t^2 + x, b = xt(t^2+1)$, then a monic minimal Gröbner basis of $\langle a - zDb, b \rangle \subseteq K[z, t]$ w.r.t. $z <_{lex} t$ is given by

$$\{(z-x)(z-\tfrac{1}{2}), (z-\tfrac{1}{2})t, t^2 - \tfrac{2}{2x-1}z + \tfrac{2x}{2x-1}\}.$$

So we have $Q_1(z) = z - x, Q_2(z) = z - \frac{1}{2}$ and $S_1(z,t) = t, S_2(z,t) = t^2 - \frac{2}{2x-1}z + \frac{2x}{2x-1}$. Since $P_0(z) = (z-x)(z-\frac{1}{2}) \notin C[z]$ we know that $\frac{a}{b}$ does not have an elementary integral over $(K(t), D)$. Nevertheless, due to $Q_2(z) \in C[z]$ we at least can write

$$\int \frac{(x+1)\log(x) + x}{x\log(x)(\log(x)^2+1)}\,dx = \int x\frac{\frac{d}{dx}\log(x)}{\log(x)}\,dx + \frac{1}{2}\log\left(\log(x)^2+1\right).$$

As a matter of fact the logarithmic integral $\int \frac{1}{\log(x)}\,dx = \mathrm{li}(x)$ is not an elementary function.

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