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L_2 error estimates for a nonstandard finite element method on polyhedral meshes

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Abstract — Recently, Hofreither, Langer and Pechstein have analyzed a nonstandard finite element method based on element-local boundary integral operators. The method is able to treat general polyhedral meshes and employs locally PDE-harmonic trial functions. In the previous work, the primal formulation of the method has been analyzed as an inexact Galerkin scheme, obtaining H^1 error estimates. In this work, we pass to an equivalent mixed formulation. This allows us to derive error estimates in the L_2-norm, which were so far not available. Many technical tools from our previous analysis remain applicable in this setting.

Keywords: non-standard FEM, polyhedral meshes, BEM-based FEM, mixed formulation

1. Introduction

In certain applications, it is advantageous to discretize partial differential equations (PDEs) on non-standard grids consisting of heterogeneous polyhedral elements and incorporating hanging nodes. For instance, in reservoir simulation, polygonal or polyhedral meshes are in common use (cf., e.g., [12]). In simulating drug diffusion through the human skin, tetrakaidecahedra (14-faced polyhedra) have been employed to model cells in the outermost skin layer, so-called corneocytes [7].

Previously established methods which are able to treat such generalized meshes are, among others, the Mimetic Finite Difference Method (see, e.g., [12] or [2]), special Mixed Finite Element Methods (see [10] and [11]), or the Discontinuous Galerkin Method (see, e.g., [6]). Copeland, Langer and Pusch have recently introduced a novel technique for treating boundary value problems on polyhedral meshes [5]. They have demonstrated that this new method works well for different classes of problems including diffusion problems, the Helmholtz equation and the Maxwell equations in the frequency domain (see also [4]). This approach employs locally PDE-harmonic trial functions, i.e., trial functions which satisfy the PDE locally on each element, and uses boundary element techniques to assemble the element stiff-

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ness matrices. For this reason, the new non-standard finite element method was also called BEM-based FEM.

First steps towards a rigorous analysis of this approach have been taken in [8], where the method was studied in the framework of a primal variational formulation with elementwise Dirichlet traces of the solution as its unknowns. The realization of this Galerkin method requires the inversion of the single layer potential operator in every element, which can typically only be done approximately. This implicates a “variational crime” in the form of an inexact bilinear form and introduces a consistency error to the numerical scheme, making L_2 error estimates hard to obtain via standard techniques. In the present work, we show an alternate approach to the analysis via a mixed formulation having both Dirichlet and Neumann traces as its unknowns. Building upon the technical tools developed in our previous work [8], we will be able to recover the error estimates in the H^1-norm obtained therein as well as derive previously unavailable L_2 error estimates.

The remainder of this paper is organized as follows. In Section 2, we derive both the primal variational formulation and the equivalent mixed variational formulation, and discretize the latter. In Section 3, we formulate regularity assumptions for general polyhedral meshes, and state an approximation result on the skeletons of such meshes. Section 4 is devoted to the derivation of mesh-independent error estimates for the BEM-based FEM in both the H^1- and the L_2-norms. In the final Section 5, we draw some conclusion.

2. Formulations of a BEM-based FEM

2.1. The primal skeletal variational formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma = \partial\Omega$ its boundary. We consider the pure Dirichlet boundary value problem for the Poisson equation,

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ on } \Gamma,$$

with $g \in H^{1/2}(\Gamma)$ and $f \in L_2(\Omega)$, as our model problem. The standard variational formulation is the following: find $u \in H^1(\Omega)$ such that the trace $\gamma_\Gamma^0 u$ of u on Γ equals g and the standard variational equation

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega). \tag{2.1}$$

holds.

We now consider a family of non-overlapping decompositions $(T_i)_{i=1}^N$ of Ω,

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{T}_i, \qquad T_i \cap T_j = \emptyset \quad \forall i \neq j,$$

into finite elements T_i which are assumed to be open Lipschitz polyhedra. Furthermore, we provide every element boundary $\Gamma_i = \partial T_i$ with a conforming triangulation $\mathscr{F}_i = \{\tau_{ij} \subset \Gamma_i\}_j$ composed of open triangles. We call such a decomposition $(T_i)_{i=1}^N$ a *polyhedral mesh* of Ω. We further assume that the elements are *matching* in the sense that, for all triangles $\tau_i \in \mathscr{F}_i$ and $\tau_j \in \mathscr{F}_j$, we have $\tau_i \cap \tau_j \neq \emptyset \Leftrightarrow \tau_i = \tau_j \in \mathscr{F}_i \cap \mathscr{F}_j$. In other words, boundary triangles from two neighboring elements should either be identical or not intersect at all.

For any suitable domain T, let

$$H^1_{\Delta,f}(T) := \left\{ u \in H^1(T) : \int_T \nabla u \cdot \nabla v \, dx = \int_T f v \, dx \quad \forall v \in H^1_0(T) \right\}$$

denote the manifold of weak local solutions of the Poisson equation.

Following McLean [13], we introduce the Dirichlet and Neumann trace operators

$$\gamma_i^0 = \gamma_{\Gamma_i}^0 : H^1(T_i) \to H^{1/2}(\Gamma_i) \quad \text{and} \quad \gamma_i^1 = \gamma_{\Gamma_i}^1 : H^1_{\Delta,f}(T_i) \to H^{-1/2}(\Gamma_i)$$

which satisfy, for all $u \in H^1_{\Delta,f}(T_i)$ and $v \in H^1(T_i)$, the Green's identity

$$\langle \gamma_i^1 u, \gamma_i^0 v \rangle = -\int_{T_i} f v \, dx + \int_{T_i} \nabla u \cdot \nabla v \, dx, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. Furthermore, we define the extension operators

$$\mathscr{H}_i^f : H^{1/2}(\Gamma_i) \to H^1_{\Delta,f}(T_i)$$

such that, for any $\varphi \in H^{1/2}(\Gamma_i)$, its image $\mathscr{H}_i^f(\varphi)$ is the uniquely defined element of $H^1_{\Delta,f}(T_i)$ having φ as its Dirichlet data. By a superposition argument, it is easy to see that $\mathscr{H}_i^f(\varphi) = \mathscr{H}_i^f(0) + \mathscr{H}_i^0(\varphi)$.

Finally, we introduce the *Dirichlet-to-Neumann maps*

$$S_i^f := \gamma_i^1 \circ \mathscr{H}_i^f \; : \; H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i),$$

and from the above we infer that

$$S_i^f(\varphi) = \gamma_i^1(\mathscr{H}_i^f(0) + \mathscr{H}_i^0(\varphi)) = S_i^f(0) + S_i^0(\varphi). \tag{2.3}$$

Note that $\mathscr{H}_i := \mathscr{H}_i^0$ and $S_i := S_i^0$ are linear operators.

Let $\Gamma_S := \bigcup_{i=1}^N \Gamma_i$ denote the *skeleton* of the mesh, and $H^{1/2}(\Gamma_S)$ the trace space of H^1-functions onto the skeleton. Furthermore, let $W = \{v \in H^{1/2}(\Gamma_S) : v|_\Gamma = 0\}$ be the space of all skeletal functions with vanishing boundary values. A discussion analogous to the one used to prove Proposition 2.1 in [8] convinces us that the following two variational problems are equivalent:

- *standard VF:* find $u_\Omega \in H^1(\Omega)$ such that $\gamma_\Gamma^0 u = g$ and

$$\int_\Omega \nabla u_\Omega \cdot \nabla v_\Omega = \int_\Omega f v_\Omega \, dx \qquad \forall v_\Omega \in H_0^1(\Omega);$$

- *skeletal VF:* find $u \in H^{1/2}(\Gamma_S)$ such that $u|_\Gamma = g$ and

$$\sum_{i=1}^N \langle S_i^f(u_i), v_i \rangle = 0 \qquad \forall v \in W. \tag{2.4}$$

(Here and henceforth we adopt the notational convention $v_i = v|_{\Gamma_i}$ for skeletal functions.) The equivalence is to be understood in the sense that $u_i = \gamma_i^0 u_\Omega$, and $u_\Omega = \mathscr{H}_i^f(u_i)$ on every element T_i. In other words, u is the skeletal trace of the solution u_Ω, and u_Ω can be locally reconstructed as the extension of u.

The Green's identity (2.2) with the choice $u = \mathscr{H}_i^f(0)$ and $v = \mathscr{H}_i \varphi$ for arbitrary $\varphi \in H^{1/2}(\Gamma_i)$ yields

$$\langle S_i^f(0), \varphi \rangle = -\int_{T_i} f \mathscr{H}_i \varphi \, dx + \int_{T_i} \nabla \mathscr{H}_i^f(0) \cdot \nabla \mathscr{H}_i \varphi \, dx = -\int_{T_i} f \mathscr{H}_i \varphi \, dx. \tag{2.5}$$

Using relations (2.3) and (2.5), we may rewrite the variational problem (2.4) as

$$\sum_{i=1}^N \langle S_i u_i, v_i \rangle = \sum_{i=1}^N \int_{T_i} f \mathscr{H}_i v_i \, dx \qquad \forall v \in W,$$

We introduce the shorthand notation $\mathscr{H}_S : H^{1/2}(\Gamma_S) \to H^1(\Omega)$ for the piecewise harmonic extension from the skeleton to each element T_i. Also, for convenience, we identify the given Dirichlet data g with a suitable skeletal extension $g \in H^{1/2}(\Gamma_S)$, which always exists. We thus have the variational problem: find $u \in g + W$ with

$$\sum_{i=1}^N \langle S_i u_i, v_i \rangle = \int_\Omega f \mathscr{H}_S v \, dx \quad \forall v \in W. \tag{2.6}$$

2.2. The mixed skeletal variational formulation

The Dirichlet-to-Neumann map S_i has the representations

$$S_i u_i = V_i^{-1}(\tfrac{1}{2} I + K_i) u_i = D_i u_i + (\tfrac{1}{2} I + K_i') V_i^{-1} (\tfrac{1}{2} I + K_i) u_i \tag{2.7}$$

in terms of the boundary integral operators

$$V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i), \quad K_i : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i),$$
$$K_i' : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i), \quad D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i).$$

The latter are called, in turn, the *single layer potential*, *double layer potential*, *adjoint double layer potential*, and *hypersingular* operators. Their definition requires the explicit knowledge of a fundamental solution of the differential operator in question. For details, we refer the reader to, e.g., McLean [13] or Steinbach [15].

We introduce the space of elementwise Neumann traces,

$$Z := \bigotimes_{i=1}^{N} H^{-1/2}(\Gamma_i).$$

In contrast to the space W, whose members are globally continuous on the skeleton, Z contains functions which are discontinuous and double-valued on inner triangles. In this space, we choose the auxiliary variable

$$t := (t_i)_{i=1}^{N} \in Z, \qquad t_i = V_i^{-1}(\frac{1}{2}I + K_i)u_i \quad \text{for } i = 1, 2, \ldots, N.$$

Equivalently, $t_i \in H^{-1/2}(\Gamma_i)$ is determined by the local variational equation

$$\langle z_i, V_i t_i \rangle = \langle z_i, (\tfrac{1}{2}I + K_i)u_i \rangle \qquad \forall z_i \in H^{-1/2}(\Gamma_i).$$

Note that $t_i = S_i u_i$ is just the Neumann trace belonging to u_i. With (2.7), we have $S_i u_i = D_i u_i + (\frac{1}{2}I + K_i')t_i$, and hence we can write the following equivalent mixed formulation for (2.6): find $(u,t) \in X := W \times Z$ such that

$$\begin{aligned} a(u,v) + b(v,t) &= \langle F, v \rangle \qquad \forall v \in W, \\ -b(u,z) + c(z,t) &= \langle G, z \rangle \qquad \forall z \in Z, \end{aligned}$$

where

$$a(u,v) = \sum_{i=1}^{N} \langle D_i u_i, v_i \rangle, \quad b(v,t) = \sum_{i=1}^{N} \langle t_i, (\tfrac{1}{2}I + K_i)v_i \rangle, \quad c(z,t) = \sum_{i=1}^{N} \langle z_i, V_i t_i \rangle,$$
$$\langle F, v \rangle = \int_\Omega f\, \mathscr{H}_S v\, dx - a(g,v), \qquad \langle G, z \rangle = b(g,z).$$

With the combined bilinear form

$$\mathscr{A}((u,t),(v,z)) := a(u,v) + b(v,t) - b(u,z) + c(z,t),$$

we may write more compactly: find $(u,t) \in X$ such that

$$\mathscr{A}((u,t),(v,z)) = \langle F, v \rangle + \langle G, z \rangle \quad \forall (v,z) \in X. \tag{2.8}$$

2.3. Discretization

Recall that the elements $\{T_i\}$ are equipped with boundary triangulations $\{\mathscr{F}_i\}$ which match across neighboring elements. Therefore, $\mathscr{F} := \bigcup_i \mathscr{F}_i$ describes a triangulation of the skeleton Γ_S. With this, we introduce the discretized trial spaces

$$\begin{aligned} W_h &:= \{v \in W : v|_\tau \in P^1(\tau)\ \forall \tau \in \mathscr{F}\}, \text{ and} \\ Z_h &:= \bigotimes_{i=1}^N Z_{h,i}, \text{ where } Z_{h,i} := \{z \in L_2(\Gamma_i) : z|_\tau \in P^0(\tau)\ \forall \tau \in \mathscr{F}_i\}. \end{aligned}$$

Here, $P^k(\tau)$ denotes the polynomial space of degree k on the triangle τ.

We discretize the variational formulation (2.8) by looking for some $(u_h,t_h) \in X_h := W_h \times Z_h \subset X$ such that

$$\mathscr{A}((u_h,t_h),(v_h,z_h)) = \langle F, v_h\rangle + \langle G, z_h\rangle \quad \forall (v_h,z_h) \in X_h. \tag{2.9}$$

In practice, the auxiliary variable t_h can be eliminated locally on each element, and only the primal unknowns u_h enter the linear system to be solved. In this way, we obtain the same numerical scheme as in the previous analysis [8], even though the variational formulation is now a mixed one. Indeed, the discrete variational formulation (2.9) is equivalent to a primal formulation where the Dirichlet-to-Neumann map S_i has been replaced with a symmetric approximation, leading to what is commonly called a variational crime. Based on Strang's Lemma, Hofreither, Langer and Pechstein provide a discretization error analysis of this inexact Galerkin scheme with respect to the H^1-norm in [8]. Now, the detour via the mixed variational reformulation leads to the conforming Galerkin discretization (2.9) of (2.8). In particular, we have the Galerkin orthogonality

$$\mathscr{A}((u-u_h,t-t_h),(v_h,z_h)) = 0 \quad \forall (v_h,z_h) \in X_h. \tag{2.10}$$

3. Mesh regularity

For general polyhedral meshes with arbitrary element shapes, we cannot use the standard technique of transforming to a reference element to obtain uniform approximation properties. In [8], Sect. 4.3, two generalized regularity assumptions on such meshes are given which substitute for more standard transformation-based regularity assumptions. For the sake of completeness we repeat these assumptions here.

Assumption 3.1. *We assume that the polyhedral mesh $(T_i)_{i=1}^N$ satisfies the following conditions.*

- *There is a small, fixed integer uniformly bounding the number of boundary triangles of every element.*

- *Every element T_i has an auxiliary conforming, quasi-regular, tetrahedral triangulation with regularity parameters which are uniform across all elements; cf.* [3].

Definition 3.1 (Uniform domain [9]). A bounded and connected set $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists a constant C_U such that any pair of points $x_1 \in D$ and $x_2 \in D$ can be joined by a rectifiable curve $\gamma(t) : [0,\, 1] \to D$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$, such that the arc length of γ is bounded by $C_U |x_1 - x_2|$ and

$$\min_{i=1,2} |x_i - \gamma(t)| \leqslant C_U \operatorname{dist}(\gamma(t), \partial D) \qquad \forall t \in [0,1].$$

If D is a uniform domain, we denote the smallest such constant C_U by $C_U(D)$ and call it the *Jones parameter* of D.

Any Lipschitz domain is a uniform domain. However, its Jones parameter may be arbitrarily large.

The second parameter we use is the constant in Poincaré's inequality. For a uniform domain D, let $C_P(D)$ be the smallest constant such that

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L_2(D)} \leqslant C_P(D) \operatorname{diam}(D)\, |u|_{H^1(D)} \qquad \forall u \in H^1(D). \tag{3.1}$$

For convex domains D, one can show that $C_P(D) \leqslant \pi^{-1}$, cf. [1]. Estimates for star-shaped domains can be found in [16,17].

Since each individual element T_i is Lipschitz, the Jones parameter $C_U(T_i)$ and the constant $C_P(T_i)$ in Poincaré's inequality are both bounded. Our notion of regularity for polyhedral meshes now demands that these constants, as well as the analogous parameters for a suitable exterior domain per element, are uniformly bounded across the whole family of meshes.

Assumption 3.2. *We assume that there are constants $C_U^* > 0$ and $C_P^* > 0$ such that, for all $i \in \{1, \ldots, N\}$,*

$$\begin{aligned} C_U(T_i) &\leqslant C_U^*, \qquad & C_U(B_i \setminus \overline{T}_i) &\leqslant C_U^*, \\ C_P(T_i) &\leqslant C_P^*, \qquad & C_P(B_i \setminus \overline{T}_i) &\leqslant C_P^*, \end{aligned}$$

where B_i is a ball (or a suitable Lipschitz domain) enclosing T_i which satisfies $\operatorname{dist}(\partial B_i, \partial T_i) \geqslant \frac{1}{2} \operatorname{diam}(T_i)$.

In the following, we will assume that all polyhedral meshes we work with satisfy Assumption 3.1 and Assumption 3.2. We will call such meshes *regular*. Furthermore, we will generically use C to refer to constants which depend only on the regularity parameters from the two assumptions, and call such constants *uniform*.

For the convergence and approximation results that follow, we equip the space $X = W \times Z$ with the norm

$$\|(v,z)\|_X^2 := |||v|||_S^2 + \|z\|_V^2 := \sum_{i=1}^N \langle S_i v_i, v_i \rangle + \sum_{i=1}^N \langle V_i z_i, z_i \rangle.$$

Let $h := \max_i \{\operatorname{diam} T_i\}$ denote the *mesh size*. On regular meshes, we have the following approximation theorem.

Theorem 3.1. *Assume that the mesh $(T_i)_{i=1}^N$ is regular, i.e., Assumptions* 3.1 *and* 3.2 *hold. If $w_\Omega \in H^2(\Omega)$ with piecewise linear boundary conditions g, and if $(\varphi, \eta) \in (g,0) + X$ denotes its skeletal Dirichlet and Neumann data, respectively, then*

$$\inf_{(\varphi_h,\eta_h)\in(g,0)+X_h} \|(\varphi - \varphi_h, \eta - \eta_h)\|_X \leqslant C\, h\, |w_\Omega|_{H^2(\Omega)} \tag{3.2}$$

with a uniform constant C.

Proof. This theorem subsumes results on approximation of both Dirichlet and Neumann traces which were originally derived in [8]. These results were therein stated for the case where the function w_Ω to be approximated is the exact solution of (2.1), but inspecting the proofs makes it clear that only the property $w_\Omega \in H^2(\Omega)$ is actually used. In particular, Theorem 4.8 from [8] asserts that, under the above assumptions,

$$\inf_{\varphi_h \in g + W_h} |||\varphi - \varphi_h|||_S \leqslant C\, h\, |w_\Omega|_{H^2(\Omega)}.$$

Analogously, for the Neumann traces, Theorem 4.11 from [8] states that on every element T_i,

$$\inf_{\eta_{h,i} \in Z_{h,i}} \|\eta_i - \eta_{h,i}\|_{V_i} \leqslant C\, (\operatorname{diam} T_i)\, |w_\Omega|_{H^2(T_i)}.$$

Obtaining the statement is then a simple matter of combining these results. □

4. Error estimates

In this section, we provide error estimates for the discretized problem (2.9). Error estimates in skeletal function spaces, while inherently mesh-dependent, are an important intermediate result in the derivation of mesh-independent estimates, and are given first. Next we provide an error estimate in the H^1-norm which was already given in [8], but is here rederived using our new mixed variational framework. Finally, we present an estimate in the L_2-norm which constitutes the main new result of this paper.

4.1. Convergence on the skeleton

Theorem 4.1. *Let Assumption* 3.1 *and Assumption* 3.2 *be fulfilled. Then the discrete solution* $(u_h, t_h) \in X_h$ *of* (2.9) *is a quasi-optimal approximation to the solution* $(u,t) \in X$ *of* (2.8). *That is,*

$$\|(u-u_h, t-t_h)\|_X \leqslant C \inf_{(v_h,z_h)\in X_h} \|(u-v_h, t-z_h)\|_X \tag{4.1}$$

with a uniform constant C.

Proof. The result is proved using Céa's Lemma. Hence, only uniform coercivity and boundedness of the bilinear form $\mathscr{A}$ need to be shown.

We take note of the spectral equivalence

$$\frac{1}{C}\langle S_i v_i, v_i\rangle \leqslant \langle D_i v_i, v_i\rangle \leqslant \langle S_i v_i, v_i\rangle \qquad \forall v \in H^{1/2}(\Gamma_i), \tag{4.2}$$

which is well-known in boundary integral operator theory [15]. Pechstein has shown in [14], Lemma 6.6, that $D_i \geqslant c^{\star}_{D,i} S_i$, where $c^{\star}_{D,i} = \frac{1}{2} C_E(B_i \setminus T_i)^{-2}(1 + C_P(B_i \setminus T_i)^2)^{-1}$, and the extension constant $C_E(B_i \setminus T_i)$ depends only on $C_U(B_i \setminus T_i) \leqslant C_U^*$. Therefore, the constant $C \geqslant 1$ in (4.2) can be bounded explicitly in terms of C_P^* and C_U^* and is thus uniform. Hence we obtain coercivity of the bilinear form $\mathscr{A}$ via

$$\begin{aligned}\mathscr{A}((v,z),(v,z)) &= \sum_i \langle D_i v_i, v_i\rangle + \sum_i \langle z_i, V_i z_i\rangle \\ &\geqslant \frac{1}{C}\sum_i \langle S_i v_i, v_i\rangle + \sum_i \langle z_i, V_i z_i\rangle \geqslant \frac{1}{C}\|(v,z)\|_X^2.\end{aligned}$$

In order to get upper bounds, we again use (4.2) as well as the Cauchy-Schwarz inequality for the symmetric and positive (semi-)definite forms $\langle \cdot, V_i \cdot\rangle$ and $\langle D_i \cdot, \cdot\rangle$ to see that

$$|a(u,v)| \leqslant |||u|||_S |||v|||_S, \qquad |c(t,z)| \leqslant \|t\|_V \|z\|_V.$$

By duality of the norms $\|\cdot\|_{V_i}$ and $\|\cdot\|_{V_i^{-1}}$, we get

$$\begin{aligned}b(v,t) &= \sum_i \langle t_i, (\tfrac{1}{2}I + K_i)v_i\rangle \leqslant \sum_i \|t_i\|_{V_i} \|(\tfrac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \\ &\overset{(*)}{\leqslant} C\sum_i \|t_i\|_{V_i} |v_i|_{S_i} \leqslant C\|t\|_V |||v|||_S.\end{aligned}$$

The inequality marked with $(*)$ stems from the relation $\|(\frac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \leqslant c_{K,i}(1 - c_{K,i})^{-1/2}|v_i|_{S_i}$ proved in [8], Equation (3.1). Pechstein [14] has shown that the contraction constants $c_{K,i}$ can be bounded explicitly in terms of C_P^* and C_U^*, and thus $C \geqslant 1$ is a uniform constant.

Combined, the above bounds yield

$$\begin{aligned}|\mathscr{A}((u,t),(v,z))| &\leqslant C\big(|||u|||_S|||v|||_S + \|t\|_V|||v|||_S + |||u|||_S\|z\|_V + \|t\|_V\|z\|_V\big)\\ &= C(|||u|||_S + \|t\|_V)(|||v|||_S + \|z\|_V)\\ &\leqslant 2C\,\|(u,t)\|_X\,\|(v,z)\|_X.\end{aligned}$$

□

While error estimates on the skeleton follow directly from this result and Theorem 3.1, they are inherently mesh-dependent and therefore of limited use. More interesting is the error within the domain with respect to the exact solution of (2.1), which will typically have additional regularity, say, $u_\Omega \in H^2(\Omega)$. Within a given element T_i, this error is given by

$$u_\Omega - \mathscr{H}_i^f(u_h+g) = \mathscr{H}_i^f(u+g) - \mathscr{H}_i^f(u_h+g) = \mathscr{H}_i(u-u_h),$$

and hence it suffices to bound the error $\mathscr{H}_S(u-u_h)$.

4.2. Convergence in the H^1-norm

From Green's identity, it is easy to see that

$$|\mathscr{H}_i\varphi|^2_{H^1(T_i)} = \langle S_i\varphi,\,\varphi\rangle \qquad \forall\varphi\in H^{1/2}(\Gamma_i).$$

Hence, with Theorem 4.1 and Theorem 3.1, it follows

$$|\mathscr{H}_S(u-u_h)|_{H^1(\Omega)} = |||u-u_h|||_S \leqslant \|(u-u_h,t-t_h)\|_X \leqslant C\,h\,|u_\Omega|_{H^2(\Omega)}.$$

4.3. Convergence in the L_2-norm

The proof of the error estimate in the L_2-norm proceeds by a standard Aubin-Nitsche duality argument. We assume that the adjoint to variational problem (2.1) is H^2-coercive and take the harmonic extension $\mathscr{H}_S(u-u_h)$ of the discretization error as the right-hand side in the adjoint variational problem. Then the solution $w\in H_0^1$ of the adjoint problem

$$\int_\Omega \nabla v\cdot\nabla w\,dx = \int_\Omega \mathscr{H}_S(u-u_h)\,v\,dx \qquad \forall v\in H_0^1(\Omega) \tag{4.3}$$

even lies in $H^2(\Omega)$ and satisfies the estimate

$$|w|_{H^2(\Omega)} \leqslant C\,\|\mathscr{H}_S(u-u_h)\|_{L_2(\Omega)}. \tag{4.4}$$

Because of the equivalence of the standard and the skeletal variational formulation, its skeletal traces (φ,η), where $\varphi_i := \gamma_i^0 w$, $\eta_i := \gamma_i^1 w$ for $i=1,\dots,N$, satisfy the (adjoint) mixed skeletal variational formulation (2.8), i.e.,

$$\mathscr{A}((v,z),(\varphi,\eta)) = \int_\Omega \mathscr{H}_S(u-u_h)\,\mathscr{H}_S v\,dx \qquad \forall(v,z)\in X.$$

In particular, with the choice $(v,z) = (u - u_h, t - t_h)$ and exploiting the Galerkin orthogonality (2.10) as well as uniform boundedness of $\mathscr{A}$, we get

$$\begin{aligned}
\|\mathscr{H}_S(u-u_h)\|^2_{L_2(\Omega)} &= \mathscr{A}((u-u_h, t-t_h), (\varphi, \eta)) \\
&= \mathscr{A}((u-u_h, t-t_h), (\varphi-\varphi_h, \eta-\eta_h)) \\
&\leqslant C\,\|(u-u_h, t-t_h)\|_X\,\|(\varphi-\varphi_h, \eta-\eta_h)\|_X
\end{aligned}$$

for arbitrary $(\varphi_h, \eta_h) \in X_h$. Taking the infimum over (φ_h, η_h) and applying Theorem 4.1 and Theorem 3.1, we obtain

$$\|\mathscr{H}_S(u-u_h)\|^2_{L_2(\Omega)} \leqslant C h^2\,|u_\Omega|_{H^2(\Omega)}\,|w|_{H^2(\Omega)}.$$

Using now estimate (4.4), we arrive at the L_2 error estimate

$$\|\mathscr{H}_S(u-u_h)\|_{L_2(\Omega)} \leqslant C h^2\,|u_\Omega|_{H^2(\Omega)}. \tag{4.5}$$

This proves our main theorem.

Theorem 4.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Furthermore, assume that the adjoint problem* (4.3) *is H^2-coercive. If the solution u of the variational problem* (2.1) *belongs to $H^2(\Omega)$, then the quasi-optimal L_2 discretization error estimate* (4.5) *holds.*

5. Conclusions

The detour via a mixed variational formulation allows us to establish a quasi-optimal L_2 discretization error estimate for the BEM-based FE discretization of the diffusion equation on polyhedral meshes that was introduced by Copeland, Langer and Pusch in [5] and whose H^1-convergence has been analyzed in [8]. Numerical results demonstrating the $O(h^2)$ behavior of the L_2 discretization error were already presented in [5] and [8] for two- and three-dimensional diffusion problems, respectively.

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