CONGRUENCE PROPERTIES MODULO 5 AND 7 FOR THE POD FUNCTION

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Abstract. In this paper, we prove arithmetic properties modulo 5 and 7 satisfied by the function $pod(n)$ which denotes the number of partitions of n wherein odd parts must be distinct (and even parts are unrestricted). In particular, we prove the following: For all $n \geq 0$,

$$\begin{aligned} pod(135n+8) &\equiv 0 \pmod 5, \\ pod(135n+107) &\equiv 0 \pmod 5, \\ pod(135n+116) &\equiv 0 \pmod 5, \\ pod(675n+647) &\equiv 0 \pmod{25}, \\ pod(3375n+1997) &\equiv 0 \pmod{125}, \\ pod(3375n+3347) &\equiv 0 \pmod{125}, \\ pod(567n+260) &\equiv 0 \pmod 7, \end{aligned}$$

and

$$pod(567n+449) \equiv 0 \pmod 7.$$

1. Introduction

The focus of this paper is the function $pod(n)$ which denotes the number of partitions of n in which odd parts are distinct (and even parts are unrestricted). This function $pod(n)$ has been considered by many from a product–series point of view as well as from other directions. For example, $pod(n)$ appears in the works of Andrews [2, 3] and Berkovich and Garvan [6]. Moreover, Berkovich and Garvan note that Andrews [5] considered a restricted version of $pod(n)$ wherein each part was required to be larger than 1. In very recent work, Alladi [1] obtained a series expansion for the product generating function for $pod(n)$. It is significant to note that Hirschhorn and Sellers [7] appear to be the first to consider $pod(n)$ from an arithmetic viewpoint.

In contrast to the work of Hirschhorn and Sellers [7], in which $pod(n)$ was extensively studied modulo 3, we now wish to prove Ramanujan–like properties modulo 5 and 7 which are satisfied by $pod(n)$. In particular, we prove the following theorem:

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Theorem 1.1. *For all $n \geq 0$,*

$$pod(135n+8) \equiv pod(135n+107) \equiv pod(135n+116) \equiv 0 \pmod 5, \tag{1}$$

$$pod(675n+647) \equiv 0 \pmod{25}, \tag{2}$$

$$pod(3375n+1997) \equiv pod(3375n+3347) \equiv 0 \pmod{125}, \tag{3}$$

and

$$pod(567n+260) \equiv pod(567n+449) \equiv 0 \pmod 7. \tag{4}$$

For the proof of our congruences we need the following lemma.

Lemma 1.2. *Let p be a prime and α a positive integer. Then*

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^{p^\alpha}}{(1-q^{pn})^{p^{\alpha-1}}} \equiv 1 \pmod{p^\alpha}. \tag{5}$$

Proof. We note that for all primes p and X an indeterminate we have

$$X \equiv 1 \pmod{p^\alpha} \Rightarrow X^p \equiv 1 \pmod{p^{\alpha+1}}. \tag{6}$$

We see that (5) is true for $\alpha = 1$ because of the relation $(1-q^{np}) \equiv (1-q^n)^p \pmod p$. Next we prove that if (5) is true for $\alpha = N$ with $N \geq 1$, then (5) is true for $\alpha = N+1$. This follows by applying (6) with $X = \prod_{n=1}^{\infty} \frac{(1-q^n)^{p^N}}{(1-q^{pn})^{p^{N-1}}}$. □

By elementary partition theory we see that

$$\sum_{m=0}^{\infty} pod(m)q^m = \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n}}. \tag{7}$$

From here, we can prove some additional elementary generating function results which are critical to our proof of these congruences.

Lemma 1.3.

$$\sum_{m=0}^{\infty} pod(m)(-q)^m = \prod_{n=1}^{\infty} \frac{1-q^n}{(1-q^{2n})^2}.$$

Proof. By (7) we find

$$\begin{aligned} \sum_{m=0}^{\infty} pod(m)(-q)^m &= \prod_{n=1}^{\infty} \frac{1-q^{2n-1}}{1-q^{2n}} \\ &= \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{2n}} \frac{1}{1-q^{2n}} \\ &= \prod_{n=1}^{\infty} \frac{1-q^n}{(1-q^{2n})^2}. \end{aligned}$$

□

In order to prove the congruences (1)-(4) we could use Lemma 2.4 (below) directly. However, experiments show that a simple pre–processing of the congruences before the application of Lemma 2.4 gives us a proof where fewer computations are required. For this purpose we use the following related generating function and lemma to rewrite (1)-(4) in a form more convenient for us.

Definition 1.4. *For all positive integers α and primes p we define*

$$\sum_{m=0}^{\infty} pod_{\alpha,p}(m)q^m := \prod_{n=1}^{\infty} \frac{(1-q^n)^{p^\alpha+1}}{(1-q^{2n})^2(1-q^{pn})^{p^{\alpha-1}}}.$$

Lemma 1.5. *The congruences* (1)-(4) *are true iff, for all $n \geq 0$,*

$$pod_{1,5}(135n+8) \equiv pod_{1,5}(135n+107) \equiv pod_{1,5}(135n+116) \equiv 0 \pmod 5,$$
$$pod_{2,5}(675n+647) \equiv 0 \pmod{25},$$
$$pod_{3,5}(3375n+1997) \equiv pod_{3,5}(3375n+3347) \equiv 0 \pmod{125},$$

and

$$pod_{1,7}(567n+260) \equiv pod_{1,7}(567n+449) \equiv 0 \pmod 7.$$

Proof. The lemma follows immediately by observing that

$$pod_{\alpha,p}(n) \equiv (-1)^n pod(n) \pmod{p^\alpha},$$

which follows from Lemma 1.2, Lemma 1.3 and Definition 1.4. □

2. The Main Proof Machinery – Modular Forms

For M a positive integer let $R(M)$ be the set of integer sequences indexed by the positive divisors δ of M. Let $1 = \delta_1 < \cdots < \delta_k = M$ be the positive divisors of M and $r \in R(M)$. Then we will write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$.

For s an integer and m a positive integer we denote by $[s]_m$ the set of all elements congruent to s modulo m, in other words $[s]_m \in \mathbb{Z}_m$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subset \mathbb{Z}_m^*$ be the set of all squares in $\mathbb{Z}_m^*$.

Definition 2.1. *For $m, M \in \mathbb{N}^*$, $r = (r_\delta) \in R(M)$ and $t \in \{0, \ldots, m-1\}$ we define the map $\overline{\odot}_r : \mathbb{S}_{24m} \times \{0, \ldots, m-1\} \to \{0, \ldots, m-1\}$ with $([s]_{24m}, t) \mapsto [s]_{24m}\overline{\odot}_r t$ and the image is uniquely determined by the relation $[s]_{24m}\overline{\odot}_r t \equiv ts + \frac{s-1}{24}\sum_{\delta|M} \delta r_\delta \pmod m$. We define the set*

$$P_{m,r}(t) := \{[s]_{24m}\overline{\odot}_r t | [s]_{24m} \in \mathbb{S}_{24m}\}.$$

Let $a \in \mathbb{Z}$ and p an odd prime, then $\left(\frac{a}{p}\right)$ is the Legendre symbol.

Lemma 2.2. *Let $p \geq 5$ be a prime and α a positive integer. Let*

$$r^{(\alpha,p)} := (r_1^{(\alpha,p)}, r_2^{(\alpha,p)}, r_p^{(\alpha,p)}) = (1+p^\alpha, -2, -p^{\alpha-1}) \in R(2p).$$

Let a, b be positive integers, $m := 3^a p^b$ and $g := \gcd(m, 8t-1)$. Then if

$$3^{a-1}p^{b-1} | (8t-1)$$

we have

$$P_{m,r^{(\alpha,p)}}(t)=\left\{t'|\ g|(8t'-1), \left(\tfrac{(8t-1)/g}{p}\right)=\left(\tfrac{(8t'-1)/g}{p}\right)\ \textit{for each } p|\tfrac{m}{g},\ \begin{array}{c} \\ 0\le t'\le m-1\end{array}\right\}.$$

Proof. By Definition 2.1 we have

$$\begin{aligned}P_{m,r^{(\alpha,p)}}(t)&=\{t'|t'\equiv ts+\frac{s-1}{24}\sum_{\delta|m}\delta r_\delta^{(\alpha,p)}\pmod m, 0\le t'\le m-1, [s]_{24m}\in\mathbb{S}_{24m}\}\\
&=\{t'|t'\equiv ts+\frac{1-s}{8}\pmod m, 0\le t'\le m-1, [s]_{24m}\in\mathbb{S}_{24m}\}\\
&=\{t'|s(8t-1)\equiv 8t'-1\pmod m, 0\le t'\le m-1, [s]_{24m}\in\mathbb{S}_{24m}\}\\
&=\left\{t'|\ \begin{array}{c} g|(8t'-1), s(8t-1)/g\equiv(8t'-1)/g\pmod{m/g},\\ 0\le t'\le m-1, [s]_{m/g}\in\mathbb{S}_{m/g}\end{array}\right\}.\end{aligned}$$

The proof is finished by noting that the existence of $[s]_{m/g}\in\mathbb{S}_{m/g}$ such that

$$s(8t-1)/g\equiv(8t'-1)/g \mod m/g$$

is, for the case $\frac{m}{g}$ squarefree, equivalent to

$$\left(\frac{(8t-1)/g}{p}\right)=\left(\frac{(8t'-1)/g}{p}\right),$$

for each $p|\frac{m}{g}$. We also used the fact that the canonical homomorphism $\phi:\mathbb{S}_n\to\mathbb{S}_{n/d}$ is surjective for any positive integers n,d such that $d|n$. □

We now use Lemma 2.2 to compute $P_{m,r^{(\alpha,p)}}(t)$ for

$$\begin{aligned}(\alpha,p,m,t)=&(1,5,135,8),(1,5,135,107),(2,5,675,647),(3,5,3375,1997),\\
&(1,7,567,260)\text{ and }(1,7,567,449).\end{aligned}$$

$(\alpha,p,m,t)=(1,5,135,8)$: We see that $g=\gcd(135,8\cdot 8-1)=3^2$ and $\left(\frac{(8t-1)/g}{p}\right)$ is $\left(\frac{2}{5}\right)=-1$ for $p=5$ and $\left(\frac{1}{3}\right)=1$ for $p=3$. By Lemma 2.2 we need to solve the following equations for t':

$$\left(\frac{(8t'-1)/g}{5}\right)=\left(\frac{2}{5}\right)\quad\text{and}\quad\left(\frac{(8t'-1)/g}{3}\right)=\left(\frac{1}{3}\right).$$

We see that $\left(\frac{x}{5}\right)=-1$ has the solutions $x=2,3\pmod 5$ and $\left(\frac{x}{3}\right)=1$ has the solution $x\equiv 1\pmod 3$. By the Chinese Remainder Theorem we obtain $x\equiv 7,13\pmod{15}$. Consequently we need to solve the following congruences for t':

$$(8t'-1)/g\equiv 7,13\pmod{15},$$

which is equivalent to

$$(8t'-1)\equiv 7g,13g\pmod{15g},$$

and hence

$$t'\equiv(1+7g)/8,(13g+1)/8\pmod{15g}.$$

Finally using $g=9$ we obtain $t'\equiv 8,116\pmod{135}$. This shows that

$$P_{135,r^{(1,5)}}(8)=\{8,116\}. \tag{8}$$

$(\alpha, p, m, t) = (1, 5, 135, 107)$: We see that $g = \gcd(135, 8\cdot 107 - 1) = 45$ and $\left(\frac{(8t-1)/g}{3}\right) = \left(\frac{1}{3}\right)$. Note that the only prime which divides $m/g = 3$ is 3. By Lemma 2.2 we need to solve the following equation for t':

$$\left(\frac{(8t'-1)/g}{3}\right) = \left(\frac{1}{3}\right)$$

This gives

$$(8t'-1)/g \equiv 1 \pmod 3 \Rightarrow (8t'-1) \equiv g \mod 3g \Rightarrow t' \equiv (1+g)/8 \pmod{3g}.$$

Using $g = 45$ we obtain $t' \equiv 107 \pmod{135}$. We conclude

$$P_{135,r^{(1,5)}}(107) = \{107\}. \tag{9}$$

Applying Lemma 2.2 in analogous fashion we obtain:

$$P_{675,r^{(2,5)}}(647) = \{647\}, \tag{10}$$

$$P_{3375,r^{(3,5)}}(1997) = \{1997, 3347\}, \tag{11}$$

$$P_{567,r^{(1,7)}}(260) = \{260\}, \tag{12}$$

$$P_{567,r^{(1,7)}}(449) = \{449\}. \tag{13}$$

By using (8)-(13) and Lemma 1.5 we see that Theorem 1.1 can be rewritten as:

Lemma 2.3. *The congruences in Theorem 1.1 are true iff, for all $n \geq 0$,*

$$pod_{1,5}(135n+t) \equiv 0 \pmod 5, \quad t \in P_{135,r^{(1,5)}}(8), \tag{14}$$

$$pod_{1,5}(135n+t) \equiv 0 \pmod 5, \quad t \in P_{135,r^{(1,5)}}(107), \tag{15}$$

$$pod_{2,5}(675n+t) \equiv 0 \pmod{25}, \quad t \in P_{675,r^{(2,5)}}(647), \tag{16}$$

$$pod_{3,5}(3375n+t) \equiv 0 \pmod{125}, \quad t \in P_{3375,r^{(3,5)}}(1997), \tag{17}$$

$$pod_{1,7}(567n+t) \equiv 0 \pmod 7, \quad t \in P_{567,r^{(1,7)}}(260), \tag{18}$$

and

$$pod_{1,7}(567n+t) \equiv 0 \pmod 7, \quad t \in P_{567,r^{(1,7)}}(449). \tag{19}$$

For each $r \in R(M)$ we assign a generating function

$$f_r(q) := \prod_{\delta|M} \prod_{n=1}^{\infty} (1-q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} c_r(n) q^n.$$

Given p a prime, $m \in \mathbb{N}$ and $t \in \{0, \ldots, m-1\}$ we are concerned with proving congruences of the type $c_r(mn+t) \equiv 0 \pmod p, n \in \mathbb{N}$. The congruences we are concerned with here have some additional structure; namely $c_r(mn+t') \equiv 0 \pmod p, n \geq 0, t' \in P_{m,r}(t)$. In other words a congruence is a tuple (r, M, m, t, p) with $r \in R(M)$, $m \geq 1$, $t \in \{0, \ldots, m-1\}$ and p a prime such that

$$c_r(mn+t') \equiv 0 \pmod p, n \geq 0, t' \in P_{m,r}(t).$$

Throughout when we say that $c_r(mn+t) \equiv 0 \pmod p$ we mean that $c_r(mn+t') \equiv 0 \pmod p$ for all $n \geq 0$ and all $t' \in P(t)$.

In order to prove the congruences (1)-(4) we need a lemma ([8, Lemma 4.5]). We first state it and then explain the terminology.

Lemma 2.4. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $a = (a_\delta) \in R(N)$, n the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_a^*(\gamma_i) \geq 0$, $i \in \{1, \dots, n\}$. Let $t_{min} := \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m}.$$

Then if

$$\sum_{n=0}^{\lfloor \nu \rfloor} c_r(mn + t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$ then

$$\sum_{n=0}^{\infty} c_r(mn + t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$.

The lemma reduces the proof of a congruence modulo u to checking that finitely many values are divisible by u. We first define the set Δ^*. Let $\kappa = \kappa(m) = \gcd(m^2 - 1, 24)$ and $\pi(M, (r_\delta)) := (s, j)$ where s is a non-negative integer and j an odd integer uniquely determined by $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$. Then a tuple $(m, M, N, (r_\delta), t)$ belongs to Δ^* iff

- m, M, N are positive integers, $(r_\delta) \in R(M)$, $t \in \{0, \dots, m-1\}$;
- $p|m$ implies $p|N$ for every prime p;
- $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$;
- $\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv 0 \pmod{24}$;
- $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$;
- $\dfrac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} \mid N$;
- for $(s, j) = \pi(M, (r_\delta))$ we have ($4|\kappa N$ and $8|Ns$) or ($2|s$ and $8|N(1-j)$) if $2|m$.

Remark 2.5. We note that the condition $2|m$ in the last line is not in the definition of Δ^* in [8]. However every result in [8] holds with no modification having this extra condition. In fact this condition was somehow missed in [8] when Δ^* was defined and although the results hold without it, in some cases we obtain less optimality.

Next we need to define the groups $\Gamma, \Gamma_0(N)$ and Γ_∞:

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid N|c \right\}$$

for N a positive integer, and

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\}.$$

For the index we have

$$[\Gamma : \Gamma_0(N)] := N \prod_{p|N} (1 + p^{-1}) \tag{20}$$

(see, for example, [9]).

Finally for m, M, N positive integers, $r \in R(M)$, $a \in R(N)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}$$

and

$$p_a^*(\gamma) := \frac{1}{24} \sum_{\delta|M} \frac{a_\delta \gcd^2(\delta, c)}{\delta}.$$

Lemma 2.6. *Let N be a squarefree integer. Then*

$$\bigcup_{\delta|N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty = \Gamma.$$

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then if $h \in \mathbb{Z}$ such that

$$c + (ch - d) \gcd(c, N) \equiv 0 \pmod{N} \tag{21}$$

we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gcd(c, N) & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c + (ch - d) \gcd(c, N) & * \end{pmatrix} \in \Gamma_0(N).$$

This implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \gcd(c, N) & 1 \end{pmatrix} \Gamma_\infty.$$

In particular we have proven that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ implies

$$\gamma \in \bigcup_{\delta|N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty,$$

if for any $c, d \in \mathbb{Z}$ with $\gcd(c, d) = 1$ there exists a $h \in \mathbb{Z}$ such that (21) holds. Next observe that (21) is equivalent to

$$ch \equiv d - \frac{c}{\gcd(c, N)} \pmod{N/\gcd(c, N)},$$

which has a solution if $\gcd(c, N/\gcd(c, N)) = 1$. This is always true because N is squarefree. □

3. The Congruences

We start by proving (14). We apply Lemma 2.4 with

$$(m, M, N, t, r = (r_1, r_2, r_5)) = (135, 10, 30, 8, (6, -2, -1)) \in \Delta^*,$$

and

$$a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-7, 14, 2, 0, -4, 0, 0, 0).$$

For $\delta \in \mathbb{Z}$ let $\gamma_\delta := \left(\begin{smallmatrix} 1 & 0 \\ \delta & 1 \end{smallmatrix}\right)$. Then by Lemma 2.6 a complete set of double coset representatives is contained in the set

$$\{\gamma_\delta : \delta | N\}.$$

Hence verifying the condition

$$p_{m,r}(\gamma_\delta) + p_a^*(\gamma_\delta) \geq 0$$

for each $\delta | N$ is sufficient to fulfill the assumption of Lemma 2.4. This verification has been carried out using MAPLE. Next we obtain

$$\begin{aligned} \nu :&= \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m} \\ &= \frac{1}{24}((5+3)\cdot 72 - 3) + \frac{3}{24\cdot 135} - \frac{8}{135} \\ &= \frac{1429}{60}. \end{aligned}$$

Here we have used (20) to compute $[\Gamma : \Gamma_0(30)] = (1+2)(1+3)(1+5) = 72$. This gives $\lfloor \nu \rfloor = 23$. By Lemma 2.4 we obtain that

$$pod_{1,5}(135n + 8) \equiv pod_{1,5}(135n + 116) \equiv 0 \pmod 5 \text{ for each } 0 \leq n \leq 23 \tag{22}$$

implies

$$pod_{1,5}(135n + 8) \equiv pod_{1,5}(135n + 116) \equiv 0 \pmod 5 \text{ for all } n \geq 0.$$

We have verified (22) with MAPLE. This proves (14). In an analogous fashion, applying Lemma 2.4 we prove the congruences (15)-(19) below:

Congruence (15). We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_5)) = (135, 10, 30, 107, (6, -2, -1))$$

and

$$a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-7, 14, 2, 0, -4, 0, 0, 0).$$

We obtain $\lfloor \nu \rfloor = 23$ and $P(t) = \{107\}$.

Congruence (16). We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_5)) = (675, 10, 30, 647, (26, -2, -5))$$

and

$$a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-32, 64, 10, 6, -20, -12, -2, -4).$$

We obtain $\lfloor \nu \rfloor = 109$ and $P(t) = \{647\}$.

Congruence (17). We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_5)) = (3375, 10, 30, 1997, (126, -2, -25))$$

and

$$a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-159, 317, 54, 32, -106, -62, -11, 21).$$

We obtain $\lfloor \nu \rfloor = 554$ and $P(t) = \{1997, 3347\}$.

Congruence (18). We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_7)) = (567, 14, 42, 260, (8, -2, -1))$$

and

$$a = (a_1, a_2, a_3, a_7, a_6, a_{14}, a_{21}, a_{42}) = (-13, 26, 4, -8, 0, 0, 0, 0).$$

We obtain $\lfloor \nu \rfloor = 55$ and $P(t) = \{260\}$.

Congruence (19). We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_7)) = (567, 14, 42, 449, (8, -2, -1))$$

and

$$a = (a_1, a_2, a_3, a_7, a_6, a_{14}, a_{21}, a_{42}) = (-13, 26, 4, -8, 0, 0, 0, 0).$$

We obtain $\lfloor \nu \rfloor = 55$ and $P(t) = \{449\}$.

The above information is summarized in the following table:

Cong.	(m, M, N, t, r)	$\lfloor \nu \rfloor$	a	$P(t)$
(14)	$(135, 10, 30, 8, (6, -2, -1)$	23	$(-7, 14, 2, 0, -4, 0, 0, 0)$	$\{8, 116\}$
(15)	$(135, 10, 30, 107, (6, -2, -1))$	23	$(-7, 14, 2, 0, -4, 0, 0, 0)$	$\{107\}$
(16)	$(675, 10, 30, 647, (26, -2, -5)$	109	$(-32, 64, 10, 6, -20, -12, -2, -4)$	$\{647\}$
(17)	$(3375, 10, 30, 1997, (126, -2, -25))$	554	$(-159, 317, 54, 32, -106, -62, -11, 21)$	$\{1997, 3347\}$
(18)	$(567, 14, 42, 260, (8, -2, -1))$	55	$(-13, 26, 4, -8, 0, 0, 0, 0)$	$\{260\}$
(19)	$(567, 14, 42, 449, (8, -2, -1))$	55	$(-13, 26, 4, -8, 0, 0, 0, 0)$	$\{449\}$

In each of the cases, we used `MAPLE` to verify that the congruences (14)-(19) hold up to the bound $\lfloor \nu \rfloor$. Thus, by Lemma 2.4 we have proven (14)-(19). Hence, by Lemma 1.5, we have proven Theorem 1.1.

4. Notes On Computations

In our proofs above, we needed to check the divisibility by p^α of $pod_{\alpha,p}(n)$ for certain $\alpha, n \in \mathbb{N}$ and a prime p. However, we observe that

$$p^\alpha | pod_{\alpha,p}(n) \;\Leftrightarrow\; p^\alpha | pod(n) \;\Leftrightarrow\; p^\alpha | (-1)^n pod(n).$$

These facts simplify the check of divisibility because we can build a nice recurrence for $(-1)^n pod(n)$ which we deduce in the following way. ¿From Jacobi's Triple Product Identity [4, Theorem 2.8], we see that

$$1 + \sum_{n=1}^{\infty} q^{n(2n-1)} + q^{n(2n+1)} = \sum_{n \in \mathbb{Z}} q^{n(2n+1)} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n}.$$

Together with Lemma 1.3, we have

$$\left(1 + \sum_{n=1}^{\infty} q^{n(2n-1)} + q^{n(2n+1)}\right) \left(\sum_{n=0}^{\infty} (-1)^n pod(n) q^n\right) = 1.$$

Therefore, by using the formula for the Cauchy product of two sequences and simplifying, one obtains the following for all positive integers n:

$$(-1)^{n+1}pod(n) = \sum_{k\geq 1, k(2k-1)\leq n} pod(n-k(2k-1))(-1)^{n-k} + \sum_{k\geq 1, k(2k+1)\leq n} pod(n-k(2k+1))(-1)^{n-k}.$$

This provides an extremely efficient method for calculating the values of $pod(n)$ which are needed to complete our proofs.

5. Closing Thoughts

It is truly satisfying to prove these congruences modulo 5 and 7 for the *pod* function. However, our ultimate goal was to identify an infinite family of congruences modulo arbitrarily large powers of 5 or 7 satisfied by $pod(n)$. Unfortunately, we were unable to find such a family. We may return to this theme in the future.

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