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Classification of algebraic ODEs with respect to rational solvability

L. X. Châu Ngô, J. Rafael Sendra, and Franz Winkler

Abstract. In this paper, we introduce a group of affine linear transformations and consider its action on the set of parametrizable algebraic ODEs. In this way the set of parametrizable ODEs is partitioned into classes with an invariant associated system, and hence of equal complexity in terms of rational solvability. We study some special parametrizable ODEs: some well-known and obviously parametrizable classses of ODEs, and some classes of ODEs with special geometric shapes, whose associated systems are characterized by classical ODEs such as separable or homogeneous ones.

Contents

1. Introduction

Solving algebraic ordinary differential equations (AODEs) is still a challenge in symbolic computation. After the work by J.F. Ritt [**Rit50**] and later by E.R. Kolchin [**Kol73**] in differential algebra, the theory of differential equations has been rapidly developed from the algebraic point of view. In particular, most of the studies of AODEs can be seen as a differential counterpart of the one of algebraic equations.

In this paper, we first recall the notion of a general solution of an AODE from the point of view of differential algebra. Then we describe a geometric approach to decide the existence of a rational general solution of a parametrizable ODE of

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order 1, i.e., an AODE whose solution surface is rational. In the affirmative case this decision method can be turned into an algorithm for actually computing such a rational general solution. A rational parametrization of the solution surface allows us to reduce the given differential equation to a system of autonomous AODEs of order 1 and of degree 1 in the derivatives. This often turns out to be an advantage because the original differential equation is typically of higher degree in the derivative. In fact, we can solve the associated system in a generic situation and therefore in most cases.

Obviously, some equations (or their associated systems) are easier to solve than others. So, the natural question is whether a given equation can be transformed into an easier one, and thus is of the same low complexity. Such a classification is the main goal of this paper. Since we are interested in rational solutions, the natural transformations are birational maps (i.e., invertible rational maps with rational inverse). However, since we are working in a differential frame, we cannot expect all birational transformations to be suitable. Indeed, we investigate birational transformations preserving certain characteristics of the rational solutions of the corresponding equations.

In this paper, which is the starting point of our strategy, we focus on linear transformations preserving rational solvability. We characterize them showing that they form a group whose orbits yield a decomposition of the set of parametrizable ODEs into classes with an invariant associated system, and hence of equal complexity in terms of rational solvability. Now, intuitively speaking, the easiest solvable AODE in a class will be seen as a normal representative. The goal is then twofold: on the one hand to find interesting classes in this quotient set, on the other to determine normal representatives; all from a computational point of view. We demonstrate this strategy by treating some special and interesting parametrizable ODEs in Section 4 and Section 5.

2. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $F(u, v, w)$ be a trivariate polynomial over $\mathbb{K}$. The *algebraic ordinary differential equation* (AODE) of order 1 defined by F is of the form

$$F(x, y, y') = 0, \tag{1}$$

where y is an indeterminate over the differential field of rational functions $\mathbb{K}(x)$ with the derivation $' = \frac{d}{dx}$.

Let $\{F\}$ be the radical differential ideal generated by F in the differential ring $\mathbb{K}(x)\{y\}$. Then one can prove ([**Rit50**], II, Section 14) that

$$\{F\} = (\{F\} : S) \cap \{F, S\}, \tag{2}$$

where $S = \dfrac{\partial F}{\partial y'}$ is the separant of F. $(\{F\} : S)$ is a prime differential ideal. So the set of solutions of $F = 0$ consists of two components: the general component on which the separant does not vanish, and the singular component which also requires vanishing of S. Of course, almost all the solutions of $F = 0$ belong to the general component. This decomposition is valid for differential polynomials of any order.

Definition 2.1. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of $F(x, y, y') = 0$. A common zero of F and S is called a *singular solution* of $F(x, y, y') = 0$.

We are interested in computing a *rational general solution* of $F(x,y,y')=0$, i.e., a general solution of the form

$$y = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0}, \tag{3}$$

where a_i, b_j are constants in a transcendental extension field of $\mathbb{K}$. In the sequel, by an arbitrary constant we mean a transcendental constant over $\mathbb{K}$.

We now give a geometric approach to compute an explicit rational general solution of $F(x,y,y')=0$ provided that the *solution surface* in $\mathbb{A}^3(\mathbb{K})$, defined by

$$F(u,v,w)=0, \tag{4}$$

is rationally parametrizable; that is, it admits a rational parametrization

$$\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)), \tag{5}$$

where χ_1, χ_2, χ_3 are bivariate rational functions over $\mathbb{K}$ and the Jacobian of $\mathcal{P}(s,t)$ has generic rank 2.

Definition 2.2. An AODE $F(x,y,y')=0$ is called a *parametrizable ODE* if it admits a rational parametrization of the form (5).

In the sequel, we denote by $\mathcal{AODE}$ the set

$$\mathcal{AODE} = \{F(x,y,y')=0 \mid F \in \mathbb{K}[x,y,z]\}$$

and by $\mathcal{PODE}$ the set

$$\mathcal{PODE} = \{F \in \mathcal{AODE} \mid \text{the surface } F=0 \text{ is rationally parametrizable}\}.$$

A solution $y=f(x)$ of $F(x,y,y')=0$ generates a curve $\mathcal{C}(x) = (x, f(x), f'(x))$ on the solution surface $F(u,v,w)=0$. Here x is viewed as the parameter of the space curve. If $f(x)$ is a rational function, the parametric curve $\mathcal{C}(x)$ is then rational.

Definition 2.3. Let $y=f(x)$ be a rational solution of $F(x,y,y')=0$. The curve $\mathcal{C}(x) = (x,f(x),f'(x))$ is called a *rational solution curve* of $F(x,y,y')=0$. The rational solution curve generated by a rational general solution of $F(x,y,y')=0$ is called a *rational general solution curve*.

Assume that the solution surface parametrization $\mathcal{P}(s,t)$ in (5) is proper, i.e., it has an inverse and its inverse is also rational or, equivalently,

$$\mathbb{K}(\mathcal{P}(s,t)) = \mathbb{K}(s,t).$$

Then a rational general solution curve can be determined by computing $(s(x), t(x))$ such that

$$\mathcal{P}(s(x), t(x)) = \mathcal{C}(x).$$

In order to satisfy this condition, it turns out that $(s(x),t(x))$ must be a *rational general solution* of the system

$$\begin{cases} s' = \dfrac{\chi_{2t} - \chi_3 \cdot \chi_{1t}}{\chi_{1s}\cdot\chi_{2t} - \chi_{1t}\cdot\chi_{2s}}, \\ t' = \dfrac{\chi_{1s}\cdot\chi_3 - \chi_{2s}}{\chi_{1s}\cdot\chi_{2t} - \chi_{1t}\cdot\chi_{2s}}, \end{cases} \tag{6}$$

provided that $\chi_{1s}\cdot\chi_{2t} - \chi_{1t}\cdot\chi_{2s} \neq 0$. Here χ_{is}, χ_{it} denote the partial derivatives of χ_i w.r.t. s and t, respectively.

Definition 2.4. The system (6) is called the *associated system* of the AODE $F(x,y,y')=0$ w.r.t. the parametrization $\mathcal{P}(s,t)$.

The associated system (6) is constructed in such a way that if $(s(x), t(x))$ is a rational solution of the associated system and $\mathcal{P}(s(x), t(x))$ is well defined, then

$$\mathcal{P}(s(x), t(x)) = (x + c,\ \chi_2(s(x), t(x)),\ \chi_2(s(x), t(x))')$$

for some constant c. Therefore,

$$y = \chi_2(s(x - c), t(x - c))$$

is a rational solution of the corresponding differential equation $F(x, y, y') = 0$. In fact, the correspondence also holds for rational general solutions. Of course, we have to specify the notion of a general solution of the associated system (6) in the differential algebra context. Hence we have the following theorem, whose proof can be found in [**NW10**].

THEOREM 2.1. *If the parametrization $\mathcal{P}(s, t)$ is proper, then there is a one-to-one correspondence between rational general solutions of the parametrizable ODE $F(x, y, y') = 0$ and rational general solutions of its associated system w.r.t. $\mathcal{P}(s, t)$.*

The associated system (6) is an autonomous system in two differential indeterminates s and t; and the degrees w.r.t. s' and t' are 1. Beside these advantages, in [**NW10**] and [**NW11**] the authors provide an algorithm for determining the rational general solution of the associated system in a generic case; later in the next paragraphs, we clarify the meaning of generality. Note that one can derive from the associated system a single rational ODE of order 1 and of degree 1 in the derivative, namely:

$$\frac{dt}{ds} = \frac{\chi_{1s} \cdot \chi_3 - \chi_{2s}}{\chi_{2t} - \chi_3 \cdot \chi_{1t}}. \tag{7}$$

This type of differential equation is well-known in the literature [**Jou79, PS83, Lin88, Sin92, Car94, CLPZ02**]. In fact, Darboux's theory on invariant algebraic curves studies the algebraic solutions of this type of differential equations and we apply that theory to the associated system (6) in order to find a rational solution.

DEFINITION 2.5. Let M_1, M_2, N_1, N_2 be polynomials in $\mathbb{K}[s, t]$ and $\gcd(M_i, N_i) = 1$ for $i = 1, 2$. An *invariant algebraic curve* of the rational system

$$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)}, \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)}, \end{cases} \tag{8}$$

is an algebraic curve $G(s, t) = 0$ such that

$$G_s M_1 N_2 + G_t M_2 N_1 = GK,$$

where G_s and G_t are the partial derivatives of G w.r.t. s and t, and K is some polynomial. An invariant algebraic curve of the system is called a *general invariant algebraic curve* if it contains an arbitrary constant in its coefficients.

One can think of a general invariant algebraic curve as an infinite family of invariant algebraic curves over $\mathbb{K}$. A rational general solution of the system (8) parametrizes a general invariant algebraic curve of the system. Assume that we have found an irreducible invariant algebraic curve $G(s, t) = 0$ of the system (8) containing an arbitrary constant c in its coefficients, and assume that it is rational when seen as a curve over the algebraic closure of $\mathbb{K}(c)$. Then we can obtain a rational general solution of the system (8) from a proper rational parametrization of that general invariant algebraic curve. Namely, we take any proper rational parametrization of the invariant algebraic curve and use system (8) to define a reparametrization for the invariant algebraic curve itself. This new parametrization is a rational solution of the system (8). For a complete description of this step we refer to [**NW11**].

Of course, the main problem is computing an irreducible invariant algebraic curve of the system; for that we use the undetermined coefficients method based on the degree bound given by [**Car94**] for systems having no *dicritical singularities*, which is the generic case.

EXAMPLE 2.1. We illustrate this approach by considering the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0. \tag{9}$$

The corresponding algebraic surface $z^2+3z-2y-3x=0$ can be parametrized by

$$\mathcal{P}_0(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, -\frac{1}{s} - \frac{2s+t^2}{s^2}, \frac{t}{s}\right).$$

This is a proper parametrization and the corresponding associated system is

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases}$$

We compute the set of irreducible invariant algebraic curves of the system and obtain

$$\{s=0, t^2+2s=0, s^2+ct^2+2cs=0 \mid c \text{ is an arbitrary constant}\}.$$

The general invariant algebraic curve $s^2+ct^2+2cs=0$ can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2cx^2}{x^2+c}, -\frac{2cx}{x^2+c}\right).$$

By the algorithm RATSOLVE in [**NW11**], we have to solve an auxiliary differential equation for the reparametrization, namely:

$$T' = \frac{1}{\mathcal{Q}_1(T)'}\mathcal{Q}_1(T)\mathcal{Q}_2(T) = -T^2.$$

Hence, $T(x) = \frac{1}{x}$. So the rational general solution of the associated system is

$$s(x) = \mathcal{Q}_1(T(x)) = -\frac{2c}{1+cx^2}, \quad t(x) = \mathcal{Q}_2(T(x)) = -\frac{2cx}{1+cx^2}.$$

We observe that

$$\chi_1(s(x),t(x)) = x - \frac{1}{c}.$$

Therefore, the rational general solution of (9) is

$$y = \chi_2\left(s\left(x+\frac{1}{c}\right), t\left(x+\frac{1}{c}\right)\right) = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}((x+c)^2+3c).$$

3. A group of affine linear transformations

Up to now, we have considered parametrizable ODEs of order 1 independently. We have mentioned in the introduction that some equations (or their associated systems) are easier to solve than others. So, the natural question is whether a given equation can be transformed into an easier one. As a first step in this direction, we develop in this section a family of birational transformations preserving certain characteristics of the rational solutions of the corresponding equations.

Precisely, we define a group of affine linear transformations on $\mathbb{K}(x)^3$ mapping an integral curve of the space to another one. By an integral curve of the space we mean a parametric curve of the form $\mathcal{C}(x) = (x, f(x), f'(x))$. So this group can act on the set of all AODEs of order 1 and it is compatible with the solution curves of the corresponding differential equations. Therefore, the group orbits partition the set of all AODEs of order 1. Most of the observations in this section are elementary but we prove them for the sake of completeness.

Let $L : \mathbb{K}(x)^3 \longrightarrow \mathbb{K}(x)^3$ be an affine linear transformation defined by

$$L(v) = Av + B,$$

where A is an invertible 3×3 matrix over $\mathbb{K}$, B is a column vector over $\mathbb{K}$ and v is a column vector over $\mathbb{K}(x)$. We want to determine A and B such that for any $f \in \mathbb{K}(x)$, there exists $g \in \mathbb{K}(x)$ with

$$L\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} = A\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + B = \begin{pmatrix} x \\ g(x) \\ g'(x) \end{pmatrix},$$

i.e., L maps an integral curve to an integral curve. By choosing some special rational functions for $f(x)$, we see that A and B must be of the forms

$$A := \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ c \\ b \end{pmatrix},$$

where a, b and c are in $\mathbb{K}$ and $a \neq 0$. Let $\mathcal{G}$ be the set of all such affine linear transformations. We represent the elements of $\mathcal{G}$ by a pair of matrices $[A, B]$. Let

$$L_i := \left[\begin{pmatrix} 1 & 0 & 0 \\ b_i & a_i & 0 \\ 0 & 0 & a_i \end{pmatrix}, \begin{pmatrix} 0 \\ c_i \\ b_i \end{pmatrix}\right], \quad i = 1, 2,$$

be two elements in $\mathcal{G}$. The usual composition of maps defines a multiplication on $\mathcal{G}$ as

$$L_1 \circ L_2 = \left[\begin{pmatrix} 1 & 0 & 0 \\ b_1 + a_1 b_2 & a_1 a_2 & 0 \\ 0 & 0 & a_1 a_2 \end{pmatrix}, \begin{pmatrix} 0 \\ c_1 + a_1 c_2 \\ b_1 + a_1 b_2 \end{pmatrix}\right]$$

and an inverse operation as

$$L_1^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 \\ -\frac{b_1}{a_1} & \frac{1}{a_1} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{c_1}{a_1} \\ -\frac{b_1}{a_1} \end{pmatrix}\right].$$

Hence $\mathcal{G}$ is a group with the unit element (the identity map)

$$I = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right].$$

This group can be naturally generalized to higher dimensional spaces; i.e., to the case of higher order AODEs.

LEMMA 3.1. *The group $\mathcal{G}$ defines a group action on $\mathcal{AODE}$ by*

$$\begin{array}{lcl} \mathcal{G} \times \mathcal{AODE} & \to & \mathcal{AODE} \\ (L, F) & \mapsto & L \cdot F = (F \circ L^{-1})(x, y, y') = F\left(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}, -\frac{b}{a} + \frac{1}{a}y'\right), \end{array}$$

where

$$L := \left[\begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ b \end{pmatrix} \right].$$

PROOF. We have

$$\begin{aligned} (L_1 \circ L_2) \cdot F = F \circ (L_1 \circ L_2)^{-1} &= F \circ (L_2^{-1} \circ L_1^{-1}) \\ &= (F \circ L_2^{-1}) \circ L_1^{-1} \\ &= L_1 \cdot (L_2 \cdot F), \end{aligned}$$

and $I \cdot F = F$. Therefore, this is an action of the group $\mathcal{G}$ on the set $\mathcal{AODE}$. $\square$

REMARK 3.1. Let $F \in \mathcal{PODE}$ and $\mathcal{P}(s,t)$ be a proper parametrization of the solution surface, then $(L \circ \mathcal{P})(s,t)$ is a proper parametrization of the solution surface of $(L \cdot F)$, because

$$(L \cdot F)((L \circ \mathcal{P})(s,t)) = F(L^{-1}((L \circ \mathcal{P})(s,t))) = F(\mathcal{P}(s,t)) = 0.$$

Therefore, $(L \cdot F) \in \mathcal{PODE}$. Moreover, the group $\mathcal{G}$ also defines a group action on $\mathcal{PODE}$.

The action of $\mathcal{G}$ on $\mathcal{PODE}$ induces a partition of $\mathcal{PODE}$ into classes for which the solvability, and in particular the rational solvability, is an invariant property. In the next theorem we state that the associated system is also invariant.

THEOREM 3.1. *Let $F \in \mathcal{PODE}$, and $L \in \mathcal{G}$. For every proper rational parametrization $\mathcal{P}$ of the surface $F(x,y,z) = 0$, the associated system of $F(x,y,y') = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x,y,y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal.*

PROOF. Let $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ be a proper rational parametrization of $F(x,y,z) = 0$. Then $L \cdot F$ can be parametrized by $(L \circ \mathcal{P})(s,t)$. The associated system of $F(x,y,y') = 0$ w.r.t. $\mathcal{P}(s,t)$ is $\left\{ s' = \frac{f_1}{g}, t' = \frac{f_2}{g} \right\}$ where

$$f_1 = \begin{vmatrix} 1 & \chi_{1t} \\ \chi_3 & \chi_{2t} \end{vmatrix}, \quad f_2 = \begin{vmatrix} \chi_{1s} & 1 \\ \chi_{2s} & \chi_3 \end{vmatrix}, \quad \text{and} \quad g = \begin{vmatrix} \chi_{1s} & \chi_{1t} \\ \chi_{2s} & \chi_{2t} \end{vmatrix}.$$

We have

$$(L \circ \mathcal{P})(s,t) = (\chi_1, b\chi_1 + a\chi_2 + c, b + a\chi_3),$$

where a, b and c are constants and $a \neq 0$. So the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $(L \circ \mathcal{P})$ is $\left\{ s' = \frac{\tilde{f}_1}{\tilde{g}}, t' = \frac{\tilde{f}_2}{\tilde{g}} \right\}$ where

$$\tilde{f}_1 = \begin{vmatrix} 1 & \chi_{1t} \\ b + a\chi_3 & b\chi_{1t} + a\chi_{2t} \end{vmatrix} = af_1, \quad \tilde{f}_2 = \begin{vmatrix} \chi_{1s} & 1 \\ b\chi_{1s} + a\chi_{2s} & b + a\chi_3 \end{vmatrix} = af_2$$

and

$$\tilde{g} = \begin{vmatrix} \chi_{1s} & \chi_{1t} \\ b\chi_{1s} + a\chi_{2s} & b\chi_{1t} + a\chi_{2t} \end{vmatrix} = ag.$$

Therefore, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal. □

Clearly, specially interesting classes of $\mathcal{PODE}$ are those containing autonomous parametrizable ODEs. Algorithmically, if we are given an equation in $\mathcal{PODE}$ and we want to check whether it is in the autonomous class, we may apply to the equation a generic element in $\mathcal{G}$ (i.e., introducing undetermined elements in the description of $L \in \mathcal{G}$) and afterwards require the coefficients of the resulting equation not to depend on x. In the next corollary we describe the type of associated system we get for these equations.

Corollary 3.1. *Let $F \in \mathcal{PODE}$ and $L \in \mathcal{G}$ such that*

$$(L \cdot F)(x, y, y') = 0$$

is an autonomous AODE. There exists a proper rational parametrization $\mathcal{P}(s, t)$ of $F(x, y, z) = 0$ such that its associated system is of the form

$$\left\{ s' = 1, \ t' = \frac{M(t)}{N(t)} \right\}. \tag{10}$$

Proof. Since $(L \cdot F)(x, y, y') = 0$ is an autonomous parametrizable ODE, the plane algebraic curve $(L \cdot F)(y, z) = 0$ is rational, and for every proper rational parametrization $(f(t), g(t))$ of $(L \cdot F)(y, z) = 0$ the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $\mathcal{P}(s, t) = (s, f(t), g(t))$ is of the form $\{s' = 1, t' = g(t)/f'(t)\}$. □

Remark 3.2. The converse of Corollary 3.1 is not true. Indeed, we consider the equation

$$F(x, y, y') = y - y'^2 - y' - y'x = 0.$$

It belongs to $\mathcal{PODE}$ and it can be properly parametrized as

$$\mathcal{P}_1(s, t) = (s, t^2 + t + ts, t).$$

The associated system w.r.t. $\mathcal{P}_1(s, t)$ is $\{s' = 1, t' = 0\}$ that is of the form (10). Let us see that the class of $F(x, y, y') = 0$ does not contain any autonomous equation. A generic transformation yields

$$(L \cdot F)(x, y, y') = -\frac{1}{a^2}y'^2 - \frac{1}{a}y' + 2\frac{b}{a^2}y' - \frac{1}{a}xy' + \frac{1}{a}y + \frac{b}{a} - \frac{b^2}{a^2} - \frac{c}{a},$$

and from here the conclusion is clear.

EXAMPLE 3.1. As in Example 2.1 we consider the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0.$$

We first check whether in the class of F there exists an autonomous AODE. For this, we apply a generic L to F to get

$$(L\cdot F)(x,y,y') = \frac{1}{a^2}y'^2 + \frac{3}{a}y' - \frac{2b}{a^2}y' - \frac{2}{a}y + \frac{2b}{a}x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a}.$$

Therefore, for every $a \neq 0$ and b such that $2b - 3a = 0$, we get an autonomous AODE. In particular, for $a = 1$, $b = 3/2$ and $c = 0$ we get

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix}\right],$$

i.e., we obtain

$$F(L^{-1}(x,y,y')) \equiv y'^2 - 2y - \frac{9}{4} = 0.$$

The last equation can be parametrized by $\mathcal{P}_2(s,t) = \left(s, \frac{t^2}{2} - \frac{9}{8}, t\right)$. Its associated system is $\{s' = 1, t' = 1\}$. Therefore, this is also the associated system of the given differential equation w.r.t. the parametrization

$$(L \circ \mathcal{P}_2)(s,t) = \left(s, \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8}, t - \frac{3}{2}\right).$$

The general invariant algebraic curve of this associated system is $s - t + \tilde{c} = 0$, where $\tilde{c}$ is an arbitrary constant. Again using the algorithm RATSOLVE in [**NW11**] we obtain a rational general solution of this associated system, namely: $s(x) = x,\ \ t(x) = x + \tilde{c}$. Therefore, we see that the rational general solution of the given differential equation is

$$y = \frac{t(x)^2}{2} - \frac{3}{2}s(x) - \frac{9}{8} = \frac{1}{2}\left(\left(x + \tilde{c} - \frac{3}{2}\right)^2 + 3\left(\tilde{c} - \frac{3}{2}\right)\right).$$

Now, it is clear that this rational general solution is equivalent to the rational general solution computed in Example 2.1 up to a change of the arbitrary constant. In fact, we have (see $\mathcal{P}_0(s,t)$ in Example 2.1)

$$((L \circ \mathcal{P}_2)^{-1} \circ \mathcal{P}_0)(s,t) = \left(\frac{t}{s} + \frac{2s + t^2}{s^2}, \frac{t}{s} + \frac{3}{2}\right).$$

This birational mapping transforms the planar curve $\left(-\frac{2c}{1 + cx^2}, -\frac{2cx}{1 + cx^2}\right)$ in Example 2.1 into the planar curve $\left(x - \frac{1}{c}, x + \frac{3}{2}\right)$, whose defining equation is $s - t + \frac{1}{c} + \frac{3}{2} = 0$.

4. Solvable AODEs and their associated systems

Based on these observations, the study of parametrizable ODEs can be reduced to the study of their normal forms with respect to, for instance, an affine linear transformation in $\mathcal{G}$. In this section, we describe some of the special parametrizable AODEs that are good candidates for normal forms. They are classified in [**Pia33**], Chapter V; and in [**Mur60**], Chapter A2, Part I as *those solvable for* y', *those solvable for* y and *those solvable for* x. One can derive from these special AODEs new differential equations of order 1 and of degree 1, which are of the same complexity in terms of rational solvability. In fact, the three special types are, under minimal requirements, in $\mathcal{PODE}$ and they have an obvious proper parametrization. So we can interpret the results in the light of our algebraic geometric approach.

4.1. Equations solvable for y'. We consider a differential equation solvable for y', i.e., $y' = G(x,y)$, where $G(x,y)$ is a rational function. Then we need not change the variable because it is already in the desired form for applying Darboux's theory (see equation (7)).

Since $G(x,y)$ is rational, $(s,t,G(s,t))$ is a parametrization of the solution surface, and hence the equation belongs to $\mathcal{PODE}$; moreover, it is proper because $\mathbb{K}(s,t,G(s,t)) = \mathbb{K}(s,t)$.

If we apply an affine linear transformation $L \in \mathcal{G}$ to $F = y' - G(x,y)$, then

$$(L \cdot F)(x,y,y') = -\frac{b}{a} + \frac{1}{a}y' - G\left(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}\right).$$

Therefore, the new differential equation is of the same form. In other words, the property of being solvable for y' is invariant in the class, and we do not enlarge this class by applying the transformations in $\mathcal{G}$.

The associated system, via the parametrization $(s,t,G(s,t))$, is

$$\{s' = 1, t' = G(s,t)\}$$

and the single rational ODE derived from the system (see equation 7) is the original equation

$$\frac{dt}{ds} = G(s,t).$$

4.2. Equations solvable for y. Let the differential equation be of the form $y = G(x,y')$, where $G(x,y)$ is a rational function. A typical example is Clairaut's equation in Example 4.1. Let us assume that G is a rational function. Clearly this type of equations belongs to $\mathcal{PODE}$ since $(s,G(s,t),t)$ is a proper parametrization of the solution surface $y = G(x,z)$.

In this class, if we apply an affine linear transformation $L \in \mathcal{G}$ to $F = y - G(x,y')$, then

$$(L \cdot F)(x,y,y') = -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a} - G\left(x, -\frac{b}{a} + \frac{1}{a}y'\right).$$

Therefore, this class is also closed under the group action of $\mathcal{G}$, i.e., we do not enlarge this class by applying the transformations in $\mathcal{G}$.

The associated system, via the parametrization $(s,G(s,t),t)$, is

$$\left\{s' = 1, t' = \frac{t - G_s(s,t)}{G_t(s,t)}\right\},$$

where G_s and G_t are the partial derivatives of $G(s,t)$ w.r.t. s and t, respectively. Moreover, the single rational ODE derived from the system (see equation (7)) is

$$\frac{dt}{ds} = \frac{t - G_s(s,t)}{G_t(s,t)},$$

which is of the desired form.

Let us see that one gets the same equation using the classical reasoning. One can differentiate the equation w.r.t. x to obtain

$$y' = G_x(x,y') + G_{y'}(x,y') \cdot y'',$$

where G_x and $G_{y'}$ are the partial derivatives of $G(x,y')$ w.r.t. x and y', respectively. Denoting y' by $\tilde{y}$, one can rewrite the above differential equation in the form

$$\tilde{y} = G_x(x,\tilde{y}) + G_{\tilde{y}}(x,\tilde{y}) \cdot \frac{d\tilde{y}}{dx},$$

or equivalently,

$$\frac{d\tilde{y}}{dx} = \frac{\tilde{y} - G_x(x,\tilde{y})}{G_{\tilde{y}}(x,\tilde{y})}.$$

Example 4.1. [*Clairaut's equation*] Let f be a smooth function of one variable. We consider Clairaut's differential equation

$$y = y'x + f(y').$$

This is a differential equation solvable for y and it can be parametrized by

$$\mathcal{P}_3(s,t) = (s, st + f(t), t).$$

If f is rational, then $\mathcal{P}_3(s,t)$ is a proper rational parametrization of the differential equation. The associated system w.r.t. $\mathcal{P}_3(s,t)$ is $\{s' = 1, t' = 0\}$. The set of irreducible invariant algebraic curves is

$$\{t - c = 0 \mid c \text{ is an arbitrary constant}\}.$$

Using the algorithm RATSOLVE in [**NW11**], we obtain $(s(x), t(x)) = (x, c)$ as a rational general solution of the associated system. So we get the rational general solution of Clairaut's differential equation, namely:

$$y = cx + f(c).$$

4.3. Equations solvable for x. We consider a differential equation of the form $x = G(y,y')$. Assuming that G is rational, this AODE belongs to $\mathcal{PODE}$, because $(G(s,t), s, t)$ is a proper parametrization of the solution surface.

If we apply an affine linear transformation $L \in \mathcal{G}$ to $F = x - G(y,y')$, then

$$(L \cdot F)(x,y,y') = x - G\left(-\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}, -\frac{b}{a} + \frac{1}{a}y'\right).$$

The degree of x in this equation is no longer linear. So this class is not closed under the action of the group $\mathcal{G}$.

The associated system, via the parametrization $(G(s,t), s, t)$, is

$$\left\{s' = t,\ t' = \frac{1 - tG_s(s,t)}{G_t(s,t)}\right\}$$

where G_s and G_t are the partial derivatives of G w.r.t. s and t, respectively. Moreover, the single rational ODE derived from the system (see equation (7)) is

$$\frac{dt}{ds} = \frac{1 - tG_s(s,t)}{tG_t(s,t)}.$$

Let us see that one gets the same equation using the classical reasoning. One can differentiate the equation w.r.t. y to obtain

$$\frac{dx}{dy} = G_y(y,y') + G_{y'}(y,y') \cdot \frac{dy'}{dy}.$$

Let $\tilde{y} = y'$, then we have

$$\frac{1}{\tilde{y}} = G_y(y,\tilde{y}) + G_{\tilde{y}}(y,\tilde{y}) \cdot \frac{d\tilde{y}}{dy}.$$

So we have transformed the differential equation $x = G(y,y')$ to a new differential equation of order 1 and of degree 1 in the desired form, namely

$$\frac{d\tilde{y}}{dy} = \frac{1 - \tilde{y}G_y(y,\tilde{y})}{\tilde{y}G_{\tilde{y}}(y,\tilde{y})}.$$

We summarize the three classes, and their geometric interpretation, in the following table:

	Solvable for y'	Solvable for y	Solvable for x
AODE	$y' = G(x,y)$	$y = G(x,y')$	$x = G(y,y')$
Proper Parametrization	$(s,t,G(s,t))$	$(s,G(s,t),t)$	$(G(s,t),s,t)$
Associated System	$\begin{cases} s' = 1 \\ t' = G(s,t) \end{cases}$	$\begin{cases} s' = 1 \\ t' = \dfrac{t - G_s(s,t)}{G_t(s,t)} \end{cases}$	$\begin{cases} s' = t \\ t' = \dfrac{1 - tG_s(s,t)}{G_t(s,t)} \end{cases}$
Equation (7)	$\dfrac{dt}{ds} = G(s,t)$	$\dfrac{dt}{ds} = \dfrac{t - G_s(s,t)}{G_t(s,t)}$	$\dfrac{dt}{ds} = \dfrac{1 - tG_s(s,t)}{tG_t(s,t)}$

Example 4.2. As we have already mentioned, if $F \in \mathcal{PODE}$ is solvable for y', then all elements in the class are solvable for y'; similarly if $F \in \mathcal{PODE}$ is solvable for y. However, this is not the case for equations solvable in x. So, if we are given $F \in \mathcal{PODE}$ we may try to check whether there exists $L \in \mathcal{G}$ such that $(L \cdot F)$ is solvable for x. For this purpose, we apply a generic transformation in $\mathcal{G}$, and afterwards require that $(L \cdot F)$ be linear in x. For instance, let us consider the equation

$$F(x,y,y') \equiv -3x - 4x^2 + 4xy - y^2 + 2xy' + 2y - yy' + 8 - 8y' + 2y'^2 = 0,$$

which belongs to $\mathcal{PODE}$. Note that

$$(s^2 + st - 2t^2,\ 2s^2 + 2st - 4t^2 + s,\ 2 + t)$$

is a proper parametrization of $F(x,y,z) = 0$. Applying a generic transformation in $\mathcal{G}$ one gets a quadratic polynomial in x, and the coefficient of x^2 is

$$\frac{-(2a+b)^2}{a^2}.$$

So if we take, for instance, $a = 1, b = -2$ and $c = 0$ we get an equation in the class solvable for x; indeed, we get $x - y^2 - y\,y' + 2y'^2 = 0$.

5. Parametrizable ODEs with special geometric shapes

In [**FG04, FG06**], an autonomous AODE is associated to a plane algebraic curve. Accordingly, an autonomous AODE possessing a rational general solution is associated to a rational plane curve. In fact, these are special AODEs in $\mathcal{PODE}$, whose solution surfaces are cylindrical surfaces over a rational plane curve. Observe that the action of an element in $\mathcal{G}$ on an autonomous AODE typically results in a non-autonomous one. Hence, the resulting AODE has the same associated system and the same rational solvability. Therefore, autonomy is not an intrinsic property of an AODE with respect to rational solvability. In this section, we consider some classes in $\mathcal{PODE}$ having special geometric shapes and one of the classes is a generalization of autonomous AODEs.

5.1. Differential equations of pencil type. We first consider parametrizable ODEs whose solution surface is a pencil of rational curves. More precisely, we assume that $F(x, y, z) = 0$ is the defining equation of an algebraic curve over the algebraic closure $\overline{\mathbb{K}(x)}$ of $\mathbb{K}(x)$ and that it is $\mathbb{K}(x)$-parametrizable; i.e., $\mathbb{K}(x)$ is the optimal field of the parametrization of the curve. The latter assumption is always fulfilled if the degree of the curve is odd (cf. [**SWPD08**], Chapter 5). With these assumptions, the surface $F(x, y, z) = 0$ has a proper parametrization of the form

$$\mathcal{P}_4(s, t) = (s, f(s, t), g(s, t)), \tag{11}$$

where f and g are rational functions in s and t. Indeed, letting $(f(s, t), g(s, t)) \in \mathbb{K}(s)(t)^2$ be a proper parametrization of the curve (recall that Lüroth's theorem is valid over every field), then $\mathbb{K}(s)(f(s, t), g(s, t)) = \mathbb{K}(s)(t)$. $\mathcal{P}_4$ parametrizes the surface $F(x, y, z) = 0$ and $\mathbb{K}(s, f(s, t), g(s, t)) = \mathbb{K}(s, t)$; hence it is proper.

The surface parametrized by (11) is called a *pencil of rational curves*. In this case, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}_4(s, t)$ is

$$\left\{ s' = 1,\ t' = \frac{-f_s(s, t) + g(s, t)}{f_t(s, t)} \right\}, \tag{12}$$

where f_s and f_t are the partial derivatives of f w.r.t. s and t, respectively. The derived differential equation from the associated system (see equation (7)) is

$$\frac{dt}{ds} = \frac{-f_s(s, t) + g(s, t)}{f_t(s, t)}. \tag{13}$$

In fact, there are several cases, in which the associated system (12) and the derived ODE (13) are simple: it can be separable or homogeneous. For instance, if $f(s, t)$ and $g(s, t)$ are homogeneous polynomials of degree $m + 1$ and m, respectively, then the derived differential equation (13) is homogeneous. In this case, we can write

$$f(s, t) = s^{m+1} f\left(1, \frac{t}{s}\right),\ g(s, t) = s^m g\left(1, \frac{t}{s}\right).$$

So the birational change of parameters $s^* = s,\ t^* = \dfrac{t}{s}$ transforms $(s, f(s, t), g(s, t))$ into the parametrization

$$(s, s^{m+1} f_1(t), s^m f_2(t)).$$

We consider, in the next subsections, the following two cases:

- [Cylindrical type] $f(s,t) = \lambda s + f_1(t)$ and $g(s,t) = f_2(t)$;
- [Quasi-cylindrical type] $f(s,t) = s^{m+1} f_1(t)$ and $g(s,t) = s^m f_2(t)$;

where f_1, f_2 are non-constant rational functions such that $(f_1(t), f_2(t))$ is proper; i.e., $\mathbb{K}(f_1(t), f_2(t)) = \mathbb{K}(t)$.

5.1.1. *Differential equations of cylindrical type.*

DEFINITION 5.1. Let $F \in \mathcal{PODE}$. F is of *cylindrical type* iff $F(x,y,z) = 0$ has a proper rational parametrization of the form

$$\mathcal{P}_5(s,t) = (0, f_1(t), f_2(t)) + s(1, \lambda, 0) = (s, \lambda s + f_1(t), f_2(t)), \tag{14}$$

where λ is a constant and $f_1(t)$ is non-constant; i.e., F can be written as

$$G(y - \lambda x, y') = 0, \tag{15}$$

where $G(u,v) = 0$ is a rational curve.

It is clear that an autonomous AODE with rational solutions is a special case of cylindrical type, corresponding to $\lambda = 0$.

Note that the properness of $\mathcal{P}_5(s,t)$ is equivalent to the properness of $(f_1(t), f_2(t))$ because

$$\mathbb{K}(s, \lambda s + f_1(t), f_2(t)) = \mathbb{K}(s)(f_1(t), f_2(t)) = \mathbb{K}(s)(t).$$

If an AODE can be parametrized by a proper rational parametrization of the form

$$\mathcal{P}_6(s,t) = (f_1(t), f_2(t), f_3(t)) + s(1, \lambda, 0), \tag{16}$$

where λ is a constant and $f_2'(t) - \lambda f_1'(t) \neq 0$, then by a change of parameters we can bring it to the standard cylindrical type. Indeed, one can apply the birational transformation $\{s^* = f_1(t) + s,\ t^* = t\}$.

THEOREM 5.1. *Every parametrizable ODE of cylindrical type is transformable into an autonomous AODE by the transformation*

$$L := \left[\begin{pmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\lambda \end{pmatrix} \right].$$

As a consequence (see Theorem 3.1 *and Corollary* 3.1*), every parametrizable ODE of cylindrical type has a parametrization w.r.t. which its associated system is of the form* $\left\{ s' = 1,\ t' = \dfrac{M(t)}{N(t)} \right\}$*, where M, N are polynomials in one variable over $\mathbb{K}$.*

PROOF. We have

$$L \cdot F = F(L^{-1}(x, y, y')) = G(y, y' + \lambda),$$

which is an autonomous AODE. □

The associated system w.r.t. the parametrization in (14) is

$$\left\{ s' = 1,\ t' = \frac{f_2(t) - \lambda}{f_1'(t)} \right\}. \tag{17}$$

A rational general solution of this system, if it exists, is of the form

$$(s(x), t(x)) = \left(x + c, \frac{\alpha x + \beta}{\gamma x + \delta} \right),$$

where $\alpha, \beta, \gamma, \delta$ are constants and c is an arbitrary constant. Here we use the fact that the second differential equation in the associated system is autonomous. So from [**FG04, FG06**] we know the exact degree of a possible rational solution, which in this case is 1. This exact degree bound is derived from an exact degree bound for curve parametrizations in [**SW01**].

In this case, a rational general solution of $G(y - \lambda x, y') = 0$ is

$$y(x) = f_1(t(x-c)) + \lambda x = f_1\left(\frac{\alpha(x-c)+\beta}{\gamma(x-c)+\delta}\right) + \lambda x,$$

where c is an arbitrary constant.

REMARK 5.1. If the integral

$$\varphi(t) = \int \frac{f_1'(t)}{f_2(t) - \lambda} dt = \frac{P_1(t)}{P_2(t)}$$

is a rational function, then the general irreducible invariant algebraic curve of the system (17) is

$$P_1(t) - sP_2(t) - cP_2(t) = 0,$$

where c is an arbitrary constant. Hence, the system (17) has a general solution of the form $(x, t(x))$, where $t(x)$ is an algebraic function satisfying the equation

$$P_1(t(x)) - xP_2(t(x)) - cP_2(t(x)) = 0.$$

So a general solution of $G(y - \lambda x, y') = 0$ is an algebraic solution given by

$$y = f_1(t(x)) + \lambda x.$$

By Theorem 5.1, the autonomous AODEs are the representatives of parametrizable ODEs of cylindrical type. In order to check whether an $F(x, y, y') = 0$ in $\mathcal{PODE}$ is equivalent to a parametric ODE of cylindrical type, we proceed as follows. First we apply a generic transformation $L \in \mathcal{G}$, say $G(a, b, x, y, y') = (L \cdot F)(x, y, y')$. Then we consider the differential equation

$$G(a, b, c, x, y, y') = 0 \tag{18}$$

and determine a, b, c such that the new differential equation is an autonomous AODE; i.e., the coefficients of $G(a, b, c, x, y, y')$ w.r.t. x must be all zero except for the coefficient of degree 0.

5.1.2. *Differential equations of quasi-cylindrical type.*

DEFINITION 5.2. Let $G(u, v) = 0$ be a rational plane curve. A differential equation of the form

$$G\left(\frac{y}{x^{m+1}}, \frac{y'}{x^m}\right) = 0, \tag{19}$$

is called of *quasi-cylindrical type*.

Of course, there are other differential equations which are transformable into this type via linear affine transformations. These are of the form

$$G\left(\frac{ay + bx + c}{x^{m+1}}, \frac{ay' + b}{x^m}\right) = 0, \tag{20}$$

where a, b and c are constants and $a \neq 0$.

Suppose that $(f_1(t), f_2(t))$ is a proper rational parametrization of $G(u,v) = 0$. Then the solution surface of (19) can be properly parametrized by

$$\mathcal{P}_7(s,t) = (s, s^{m+1} f_1(t), s^m f_2(t)). \tag{21}$$

Note that $\mathcal{P}_7(s,t)$ is proper, because it is a special case of the parametrization considered in (11). With respect to $\mathcal{P}_7(s,t)$ the associated system is separable

$$\left\{ s' = 1,\ t' = \frac{-(m+1)f_1(t) + f_2(t)}{s f_1'(t)} \right\}. \tag{22}$$

Therefore, we can always decide whether the differential equation (19) has a rational general solution or not.

5.2. Differential equations of cone type. A rational conical surface (say with vertex at the origin) can be parametrized as

$$s\,\mathcal{E}(t)$$

where $\mathcal{E}(t)$ is a space curve parametrization; if the curve $\mathcal{E}(t)$ is contained in a plane passing through the origin, the surface is that plane. This motivates the following definition.

DEFINITION 5.3. A parametrizable ODE is of *cone type* if its solution surface has a parametrization of the form

$$\mathcal{P}_8(s,t) = (s^{m_1} f_1(t), s^{m_2} f_2(t), s^{m_3} f_3(t)), \tag{23}$$

where f_1, f_2, f_3 are rational functions and m_1, m_2, m_3 are integers.

- *When $m_1 = 1, f_1(t) = 1$ and $m_2 = m_3 + 1$, we obtain a quasi-cylindrical surface.*

The associated system w.r.t. (23) is

$$\begin{cases} s' = \dfrac{s^{1-m_1} f_2'(t) - s^{1-m_2+m_3} f_3(t) f_1'(t)}{m_1 f_1(t) f_2'(t) - m_2 f_1'(t) f_2(t)}, \\[2ex] t' = \dfrac{-m_2 s^{-m_1} f_2(t) + m_1 s^{-m_2+m_3} f_3(t) f_1(t)}{m_1 f_1(t) f_2'(t) - m_2 f_1'(t) f_2(t)}. \end{cases} \tag{24}$$

- *In fact, we consider the case $m_2 = m_1 + m_3$, i.e., the parametrization is*

$$\mathcal{P}_9(s,t) = (s^{m_1} f_1(t), s^{m_1+m_3} f_2(t), s^{m_3} f_3(t)), \tag{25}$$

  in which $\mathcal{P}_7(s,t)$ is a special case.

Then the derived differential equation is separable, namely:

$$\frac{dt}{ds} = \frac{-m_2 f_2(t) + m_1 f_1(t) f_3(t)}{(f_2'(t) - f_3(t) f_1'(t))s}. \tag{26}$$

By integration we can decide if the associated system has a general invariant algebraic curve and proceed as in the algorithm RATSOLVE ([**NW11**]) to check the existence of a rational general solution of the system (24) with $m_2 = m_1 + m_3$.

The differential equation corresponding to the parametrization (25) is of the form

$$G\left(\frac{y^{m_1}}{x^{m_1+m_3}}, \frac{y'^{m_1}}{x^{m_3}}\right) = 0, \tag{27}$$

where $G(u,v) = 0$ is a rational planar curve.

In general, from the form (27) we do not know whether the surface is rational or not. However, in some special cases, we can decide this property. For instance, if the rational curve $G(u, v) = 0$ has a rational parametrization of the form

$$(g(t)^{m_1}, h(t)^{m_1}), \tag{28}$$

then the surface defined by (27) can be parametrized by

$$\mathcal{P}_{10}(s, t) = (s^{m_1}, s^{m_1+m_3} g(t), s^{m_3} h(t)). \tag{29}$$

This parametrization is proper if $(g(t)^{m_1}, h(t)^{m_1})$ is proper and $\gcd(m_1, m_3) = 1$. Then we can continue applying our method for deciding the existence of a rational general solution and computing it in the affirmative case.

6. Conclusion

We have described an algebraic geometric approach to classify parametrizable ODEs of order 1 w.r.t. their rational solvability. These classes are the orbits generated by a group of affine linear transformations acting on AODEs. AODEs in the same equivalence class share important characteristics, such as the associated system, and the complexity of determining general rational solutions. We have pointed out some interesting classes in this equivalence relation. This is the first step towards classifying AODEs w.r.t. a more general group of birational transformations preserving certain characteristics of the rational solutions of AODEs. Finally, we have analyzed some classes of AODEs having general rational solutions. It turns out that being autonomous is not a characteristic property of such a class. Some geometric properties of differential equations carry over to representatives of their corresponding classes, which can obviously be solved rationally.

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DK Computational Mathematics, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria
E-mail address: Ngo.Chau@risc.jku.at

Dpto. de Matemáticas, Universidad de Alcalá, Alcalá de Henares/Madrid, Spain. Member of the Research Group ASYNACS (Ref. CCEE2011/R34)
E-mail address: Rafael.Sendra@uah.es

Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria
E-mail address: Franz.Winkler@risc.jku.at