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RATIONAL GENERAL SOLUTIONS OF HIGHER ORDER ALGEBRAIC ODES*

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Abstract This paper generalizes the method of Ngô and Winkler (2010, 2011) for finding rational general solutions of a first order non-autonomous algebraic ordinary differential equation (AODE) to the case of a higher order AODE, provided a proper parametrization of its solution hypersurface. The authors reduce the problem of finding the rational general solution of a higher order AODE to finding the rational general solution of an associated system. The rational general solutions of the original AODE and its associated system are in computable 1-1 correspondence. The authors give necessary and sufficient conditions for the associated system to have a rational solution based on proper reparametrization of invariant algebraic space curves. The authors also relate invariant space curves to first integrals and characterize rationally solvable systems by rational first integrals.

Key words Algebraic ODE, associated system, invariant algebraic space curve, rational first integral, rational general solution.

1 Introduction

An algebraic ordinary differential equation (AODE) is of the form $F(x, y, y', \cdots, y^{(n-1)}) = 0$, for F a polynomial over a differential field. By well established methods in differential algebra, cf. e.g., Ritt[1], the solution set of an AODE can be decomposed into the components defined by their prime differential ideals. The general component is the one on which the separant of F does not vanish. In [2], Hubert gave a method for finding the general solution of a first order non-autonomous AODE by computing a Gröbner basis of the prime differential ideal defining the

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general component. Feng and Gao[3,4] presented a method for explicitly computing the rational general solution of a first order autonomous AODE based on a rational parametrization of its corresponding algebraic curve. The degree bounds for curve parametrization derived by Sendra and Winkler[5] make this method completely algorithmic. Subsequently, the rational general solution of a first order non-autonomous AODE was studied by Ngô and Winkler[6,7] based on a proper parametrization of its corresponding algebraic surface. Their method is algorithmic in the generic (non-dicritical) case by using the degree bound in [8].

In this paper, the results of Ngô and Winkler are generalized to the case of a higher order AODE with a rational solution hypersurface. In fact, based on a proper parametrization of the solution hypersurface, we can obtain an associated first order rational system, which has a very special structure making it amenable to rational solution. Moreover, the rational general solutions of the original AODE and its associated system are in computable 1-1 correspondence. We present a method for finding a rational general solution of the associated system based on proper parametrizations of its invariant algebraic space curves. We also relate invariant space curves to first integrals and characterize rationally solvable systems by rational first integrals.

Note that we only consider AODEs with rational solution hypersurfaces, i.e., having a proper parametrization. Whereas for curves and surfaces unirationality coincides with rationality, it is well known that not every unirational hypersurface in higher dimensional space can be properly parametrized[9−11] Also, we don't have a degree bound for invariant algebraic space curves for the case of $n > 2$. A similar problem arises in generalizing Hubert's[2] method to AODEs of higher order; namely, to determine a bound on the number of derivations to be considered. Nevertheless, we present a general method for finding rational general solutions of higher order AODEs, relative to the problems mentioned above.

The paper is organized as follows. In Section 2, we introduce some known concepts and results related to differential polynomials and rational general solutions. In Section 3, we explain how to derive the associated first order rational system, and we prove the correspondence of rational general solutions of the original AODE and its associated system. Section 4 is devoted to presenting our method for finding rational general solutions of associated systems. In Section 5, based on the form of the generators of the ideal defined by a rational general solution and the relation between invariant algebraic space curves and rational first integrals, we give a criterion for deciding when the associated system has a rational general solution. Finally, we concluded with some open problems in Section 6.

2 Preliminaries

In this section, we recall some well known concepts and results in differential algebra. More details can be found in [1, 12]. Most notions in the case of two differential indeterminates have been introduced in [6]. For completeness, we give some similar notions for n differential indeterminates here.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. $\mathbb{K}(x)$ is the differential field of rational functions in x with the usual derivation $\frac{d}{dx}$ (which we also denote by $'$) and $s_1, s_2, \cdots, s_n$ are n differential indeterminates over $\mathbb{K}(x)$. The j-th derivative of s_i is denoted by s_{ij}; for s_i we also write s_{i0}. The differential polynomial ring $\mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ is the ring consisting of all polynomials in s_i $(1 \leq i \leq n)$ and all their derivatives. In the following, we consider two differential rings $\mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ and $\mathbb{K}(x)\{y\}$, where y is another differential indeterminate. Note that y_j denotes the j-th derivative of y. Most notions defined for n differential indeterminates can also be applied to the case of a single indeterminate.

Let $F \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ be a differential polynomial. The i-th derivative of F is denoted by $F^{(i)}$. For $F^{(1)}$ we also write F'. The order of F with respect to s_i is the greatest j such that s_{ij} occurs in F, denoted by $\mathrm{ord}_{s_i}(F)$. We let $\mathrm{ord}_{s_i}(F) = -1$ if F does not involve any s_{ij}.

Definition 1 Let $F, G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$. Suppose that the indeterminate s_i appears effectively in both of them, where $1 \le i \le n$. F is said to be of higher rank than G (or G of lower rank than F) in s_i if one of the following conditions holds:

a) $\mathrm{ord}_{s_i}(F) > \mathrm{ord}_{s_i}(G)$;

b) $\mathrm{ord}_{s_i}(F) = \mathrm{ord}_{s_i}(G) = j$ and $\deg_{s_{ij}}(F) > \deg_{s_{ij}}(G)$.

Definition 2 Let $A = \{s_{ik} \mid i = 1, 2, \cdots, n, k \in \mathbb{N}\}$. The ord-lex ranking on A is the total order defined as follows: $s_{ik} < s_{jl}$ iff $k < l$ or $k = l$ and $i < j$.

The ord-lex ranking defined in Definition 2 is considered as the ranking in the differential ring $\mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ throughout this paper. Note that the ord-lex ranking is an orderly ranking. For any $F \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} \backslash \mathbb{K}(x)$, the greatest derivative occurring in F with respect to the ord-lex ranking is called the leader of F. The leading coefficient with respect to the leader of F is called the initial of F, the partial derivative with respect to the leader of F is called the separant of F. For a usual multivariate polynomial, its initial is defined as its leading coefficient with respect to a chosen variable. If all s_{ik} in ord-lex ranking are regarded as new algebraic variables and the chosen variable corresponds to the leader of the given differential polynomial, then the two initials in differential and algebraic cases are the same.

Definition 3 Let F and G be two differential polynomials in $\mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$. Then G is reduced with respect to F iff G is of lower rank than F in the indeterminate defining the leader of F.

Let $\mathbb{A} \subset \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$. The differential polynomial set $\mathbb{A}$ is called autoreduced iff no elements of $\mathbb{A}$ belongs to $\mathbb{K}(x)$ and each element of $\mathbb{A}$ is reduced with respect to all the others.

Definition 4 Let $F \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$. By Ritt's reduction, for any $P \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$, there exists a representation

$$S^k I^l P = \sum_{i \ge 0} Q_i F^{(i)} + R,$$

where S is the separant of F, I is the initial of F, $Q_i \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$, $F^{(i)}$ is the i-th derivatives of F, $k, l \in \mathbb{N}$, the leader of P is of at least as high rank as the leader of $F^{(i)}$, and $R \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ is reduced with respect to F. Here, R is called the differential pseudo remainder of P with respect to F, denoted by $\mathrm{prem}(P, F)$.

Let us recall some notations about ideals, following [1]. We will denote by $\langle \mathbb{P} \rangle$ the algebraic ideal generated by an algebraic polynomial set $\mathbb{P}$, by $[\mathbb{P}]$ the differential ideal generated by a differential polynomial set $\mathbb{P}$, and by $\{\mathbb{P}\}$ the radical ideal of $[\mathbb{P}]$. This notation clashes with the usual notation of sets; if the meaning is not clear from the context, we will explicitly clarify it. Now, let $F \in \mathbb{K}(x)\{y\}$ be an irreducible differential polynomial and S the separant of F. It has been proved in [1, chap II, sect. 14] that

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where $\{F\} : S = \{H \in \mathbb{K}(x)\{y\} \mid SH \in \{F\}\}$. The differential ideals $\{F\} : S$ and $\{F, S\}$ define the general component and the singular component of $F(y) = 0$ respectively. Moreover, $\{F\} : S$ is a prime differential ideal in the differential ring $\mathbb{K}(x)\{y\}$. By Ritt ([1, chap II, sect. 6]), any prime differential ideal different from the unit ideal has a generic zero.

Definition 5 Let $F \in \mathbb{K}(x)\{y\}$ be an irreducible differential polynomial of order $n-1$ in y. The equation $F(y) = F(x, y, y', \cdots, y^{(n-1)}) = 0$ is called an algebraic ordinary differential equation (AODE). A generic zero of the prime differential ideal $\{F\} : S$ is called a general solution of the AODE $F(y) = 0$.

In [1, chap II, sect. 13], Ritt also proved that for any $P \in \mathbb{K}(x)\{y\}$, $P \in \{F\} : S$ if and only if $\mathrm{prem}(P, F) = 0$. So this gives us an algorithmic way to check whether a solution of $F(y) = 0$ is a general solution via pseudo division in the differential ring $\mathbb{K}(x)\{y\}$.

Theorem 6 *A zero η of F is a general solution of $F(y) = 0$ if and only if for any $P \in \mathbb{K}(x)\{y\}$, if $P(\eta) = 0$, then we have* $\mathrm{prem}(P, F) = 0$, *and vice versa.*

In the next section, Definition 5 and the statement in Theorem 6 will be generalized from a single AODE to the case of a system.

3 Associated System of Higher Order AODEs

Let $\mathcal{K}$ be the universal extension field over $\mathbb{K}$. In this paper, we are interested in looking for rational general solutions $y = f(x) \in \mathcal{K}(x)$ of the higher order AODE

$$F(x, y, y', \cdots, y^{(n-1)}) = 0, \tag{1}$$

where $F \in \mathbb{K}[x, y, y_1, \cdots, y_{n-1}]$ is an irreducible polynomial over $\mathbb{K}$ and $n > 2$. By regarding $x, y, y', \cdots, y^{(n-1)}$ as independent algebraic variables, the algebraic equation $F(x, y, y_1, \cdots, y_{n-1}) = 0$ defines an algebraic hypersurface $\mathcal{S}$ in the affine space $\mathcal{A}^{n+1}(\mathcal{K})$. We call $\mathcal{S}$ the solution hypersurface of the differential equation (1).

Assume that the solution hypersurface $\mathcal{S}$ is rational, i.e., it has a proper parametrization

$$\mathcal{P}(s_1, s_2, \cdots, s_n) = (\chi_1(s_1, s_2, \cdots, s_n), \cdots, \chi_{n+1}(s_1, s_2, \cdots, s_n)),$$

where $\chi_i \in \mathbb{K}(s_1, s_2, \cdots, s_n) \subseteq \mathcal{K}(s_1, s_2, \cdots, s_n)$. Observe that this is a strong assumption. We know that in low dimension (2 and 3), every unirational variety is rational. But not for higher dimension[9]. Since $\mathcal{P}$ is a birational map $\mathcal{A}^n(\mathcal{K}) \to \mathcal{S} \subset \mathcal{A}^{n+1}(\mathcal{K})$, there exists a birational inverse map $\mathcal{P}^{-1}$ defined on $\mathcal{S}$.

A rational solution $y = f(x)$ of the differential equation (1) is an element of $\mathcal{K}(x)$ such that

$$F(x, f(x), f'(x), \cdots, f^{(n-1)}(x)) = 0.$$

It follows that the parametric space curve $\mathcal{C} = (x, f(x), f'(x), \cdots, f^{(n-1)}(x))$ lies on the solution hypersurface $\mathcal{S}$. $\mathcal{C}$ is called the solution curve of $y = f(x)$.

Definition 7 Let $y = f(x) \in \mathcal{K}(x)$ be a rational solution of the AODE $F(x, y, y', \cdots, y^{(n-1)}) = 0$ and $\mathcal{C}$ its solution curve. $\mathcal{C}$ is parametrizable by $\mathcal{P}$ iff $\mathcal{C}$ is almost contained in $\mathrm{image}(\mathcal{P}) \cap \mathrm{domain}(\mathcal{P}^{-1})$, i.e., except for finitely many points on $\mathcal{C}$, where $\mathcal{P}$ is a proper parametrization of the solution hypersurface $\mathcal{S}$.

Lemma 8 *Let the solution hypersurface $\mathcal{S}$ of $F(x, y, y', \cdots, y^{(n-1)}) = 0$ be rational via the proper parametrization*

$$\mathcal{P}(s_1, s_2, \cdots, s_n) = (\chi_1(s_1, s_2, \cdots, s_n), \chi_2(s_1, s_2, \cdots, s_n), \cdots, \chi_{n+1}(s_1, s_2, \cdots, s_n)).$$

Then $F(x, y, y', \cdots, y^{(n-1)}) = 0$ has a rational solution, which is parametrizable by $\mathcal{P}$, if and only if the system

$$\begin{aligned} &\chi_1(s_1(x), s_2(x), \cdots, s_n(x)) = x, \\ &\chi_i(s_1(x), s_2(x), \cdots, s_n(x))' = \chi_{i+1}(s_1(x), s_2(x), \cdots, s_n(x)), \quad i = 2, 3, \cdots, n \end{aligned} \tag{2}$$

has a rational solution $(s_1(x), s_2(x), \cdots, s_n(x))$.

Proof ($\Longrightarrow$) Assume that $y = f(x)$ is a rational solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$, which is parametrizable by $\mathcal{P}$. From

$$(s_1(x), s_2(x), \cdots, s_n(x)) := \mathcal{P}^{-1}(x, f(x), f'(x), \cdots, f^{(n-1)}(x)),$$

we get

$$\begin{aligned}\mathcal{P}(s_1(x), s_2(x), \cdots, s_n(x)) &= \mathcal{P}(\mathcal{P}^{-1}(x, f(x), f'(x), \cdots, f^{(n-1)}(x))) \\ &= (x, f(x), f'(x), \cdots, f^{(n-1)}(x)),\end{aligned}$$

which means

$$\chi_1(s_1(x), s_2(x), \cdots, s_n(x)) = x, \quad \chi_i(s_1(x), s_2(x), \cdots, s_n(x)) = f^{(i-2)}(x), \quad i = 2, 3, \cdots, n+1,$$

i.e., $(s_1(x), s_2(x), \cdots, s_n(x))$ is a solution of System (2). Moreover, $(s_1(x), s_2(x), \cdots, s_n(x))$ is a rational solution because $\mathcal{P}^{-1}$ is a birational map and $f(x)$ is a rational function.

($\Longleftarrow$) If rational functions $s_1 = s_1(x), s_2 = s_2(x), \cdots, s_n = s_n(x)$ satisfy System (2), then it follows from $F(\mathcal{P}(s_1(x), s_2(x), \cdots, s_n(x))) = 0$ that $y = \chi_2(s_1(x), s_2(x), \cdots, s_n(x))$ is a rational solution of the differential equation $F(x, y, y', \cdots, y^{(n-1)}) = 0$. ∎

System (2) can be transformed by differentiating the first equation and expanding the remaining equations. We obtain a linear system of equations in $s_i'(x)$ of the form

$$\boldsymbol{A} \cdot (s_1'(x), s_2'(x), \cdots, s_n'(x))^{\mathrm{T}} = \boldsymbol{B},$$

where

$$\boldsymbol{A} = \begin{pmatrix} \frac{\partial \chi_1(s_1, s_2, \cdots, s_n)}{\partial s_1} & \cdots & \frac{\partial \chi_1(s_1, s_2, \cdots, s_n)}{\partial s_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_n(s_1, s_2, \cdots, s_n)}{\partial s_1} & \cdots & \frac{\partial \chi_n(s_1, s_2, \cdots, s_n)}{\partial s_n} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 \\ \chi_3(s_1, s_2, \cdots, s_n) \\ \vdots \\ \chi_{n+1}(s_1, s_2, \cdots, s_n) \end{pmatrix}.$$

Solving this linear system in $s_i'(x)$ according to Cramer's rule, assuming that $\det(\boldsymbol{A}) \neq 0$, we see that $(s_1(x), s_2(x), \cdots, s_n(x))$ is a solution of the system of rational differential equations

$$s_i' = \frac{U_i(s_1, s_2, \cdots, s_n)}{V_i(s_1, s_2, \cdots, s_n)}, \quad i = 1, 2, \cdots, n, \tag{3}$$

where $U_i, V_i \in \mathbb{K}[s_1, s_2, \cdots, s_n]$ can be computed in terms of $\chi_1, \chi_2, \cdots, \chi_{n+1}$. In fact, U_i and V_i are the numerator and denominator of $\frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}$, respectively, where $\boldsymbol{A}_i$ is the matrix formed by replacing the i-th column of $\boldsymbol{A}$ by $\boldsymbol{B}$.

If $\det(\boldsymbol{A}) = 0$, then $(s_1(x), s_2(x), \cdots, s_n(x))$ is a solution of the system

$$\overline{A}_i(s_1, s_2, \cdots, s_n) = 0, \quad i = 1, 2, \cdots, n-1, \qquad \overline{A}(s_1, s_2, \cdots, s_n) = 0,$$

where $\overline{A}_i(s_1, s_2, \cdots, s_n)$ and $\overline{A}(s_1, s_2, \cdots, s_n)$ are numerators of $\det(\boldsymbol{A}_i)$ $(1 \leq i \leq n-1)$ and $\det(\boldsymbol{A})$, respectively.

Definition 9 System (3) is called the associated system of the AODE (1) with respect to $\mathcal{P}(s_1, s_2, \cdots, s_n)$.

Remark 10 The associated system in the new indeterminates $s_1, s_2, \cdots, s_n$ is of a very special form; it is

1) autonomous (coefficients do not depend on the variable x),

2) of order 1, and

3) of degree 1 with respect to $s'_1, s'_2, \cdots, s'_n$.

We transform (3) into polynomial relations by setting

$$A_i := V_i s'_i - U_i \quad \text{for} \quad 1 \le i \le n.$$

Then the differential polynomial set $\mathbb{A} = \{A_1, A_2, \cdots, A_n\} \subset \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ is an autoreduced set relative to the ord-lex ranking. Note that the initial and separant of A_i are the same. According to the definition of autoreduced set and Proposition 1 in [12, chap I, sect. 9], we have the following proposition.

Proposition 11 *Let $\mathbb{A} = \{A_1, A_2, \cdots, A_n\}$, where $A_i = V_i s'_i - U_i$ for $1 \le i \le n$. For any differential polynomial $G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$, there exists the following representation by consecutive reductions with respect to the autoreduced set $\mathbb{A}$*

$$V_1^{l_1} V_2^{l_2} \cdots V_n^{l_n} G = \sum_{j \ge 0} Q_{1j} A_1^{(j)} + \sum_{j \ge 0} Q_{2j} A_2^{(j)} + \cdots + \sum_{j \ge 0} Q_{nj} A_n^{(j)} + R,$$

where $l_i \in \mathbb{N}$, $A_i^{(j)}$ is the j-th derivatives of A_i, the leader of G is of at least as high rank as the leader of $A_i^{(j)}$, $Q_{ij} \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$, $i = 1, 2, \cdots, n$, and R is reduced with respect to $\mathbb{A}$. Here, R is called the differential pseudo remainder of G with respect to $\mathbb{A}$, denoted by $\mathrm{prem}(G, \mathbb{A})$.

Remark 12 As the order of A_i is 1 and the degree of A_i with respect to s'_i $(1 \le i \le n)$ is 1, it follows that $\mathrm{prem}(G, \mathbb{A})$ is always a polynomial in $\mathbb{K}(x)[s_1, s_2, \cdots, s_n]$ for any differential polynomial $G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$.

By replacing s'_i by z_i and s_i by t_i in $\mathbb{A}$, we get the polynomial set

$$\mathbb{B} = \{z_i V_i(t_1, t_2, \cdots, t_n) - U_i(t_1, t_2, \cdots, t_n) \mid i = 1, 2, \cdots, n\}.$$

Since $z_i V_i - U_i$ is linear with respect to z_i, $\mathbb{B}$ is an irreducible triangular set ([13, sect. 4.1]) with respect to the order $t_1 < t_2 < \cdots < t_n < z_1 < z_2 < \cdots < z_n$ of the algebraic indeterminates. It follows that $\mathbb{B}$ is a characteristic set of the prime polynomial ideal $\mathrm{sat}(\mathbb{B}) = \langle \mathbb{B} \rangle : S_{\mathbb{B}}^{\infty} = \{G \in \mathbb{K}(x)[t_1, t_2, \cdots, t_n, z_1, z_2, \cdots, z_n] \mid (\prod_{i=1}^n V_i)^q G \in \langle \mathbb{B} \rangle$ for some $q \in \mathbb{N}\}$ ([13, Theorems 6.2.4 and 6.2.14]) in Ritt's definition ([1, chap I, sect. 5]). Therefore, $[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} \mid (\prod_{i=1}^n V_i)^q G \in [\mathbb{A}]$ for some $q \in \mathbb{N}\}$ is a prime differential ideal (see [1, chap V, sect. 3]). By Proposition 2.1 in [14] and the fact that $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ is prime (which implies $[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{\mathbb{A}\} : S_{\mathbb{A}}^{\infty}$), we have the following decomposition

$$\{\mathbb{A}\} = ([\mathbb{A}] : S_{\mathbb{A}}^{\infty}) \bigcap \left(\bigcap_{i=1}^{n} \{\mathbb{A}, V_i\} \right),$$

where $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ and $\bigcap_{i=1}^n \{\mathbb{A}, V_i\}$ define the general component and the singular component of $\mathbb{A}$. Observe that every solution of System (3) is a zero of $\mathbb{A}$ for which none of the V_i's vanish, i.e., it is a zero of $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$. Therefore, a general solution of System (3) can be defined as follows.

Definition 13 A generic zero of the prime differential ideal $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ is said to be a general solution of System (3). A general solution $(s_1(x), s_2(x), \cdots, s_n(x))$ of System (3) is rational iff all $s_i(x)$ are rational functions in x.

Theorem 6 gives a method for deciding whether a solution of a single AODE is in fact a general solution. Now, we generalize this to the case of a system of AODEs of the form (3).

Theorem 14 *A rational solution $\bar{s}(x) = (\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x))$ of System (3) is a rational general solution, if and only if*

$$\forall\ G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} : \big(G(\bar{s}(x)) = 0 \iff \mathrm{prem}(G, \mathbb{A}) = 0\big), \tag{4}$$

where $\mathbb{A} = \{A_1, A_2, \cdots, A_n\}$ and $A_i = V_i s_i' - U_i$ for $1 \le i \le n$.

Proof In order to prove that $\bar{s}$ is a general solution of (3) if and only if (4) holds, we have to show

$$\forall\ G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} :\ \big(G \in [\mathbb{A}] : S_{\mathbb{A}}^{\infty} \Longleftrightarrow \mathrm{prem}(G, \mathbb{A}) = 0\big),$$

i.e.,

$$[\mathbb{A}] : S_{\mathbb{A}}^{\infty} \ =\ \{G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} \mid \mathrm{prem}(G, \mathbb{A}) = 0\}.$$

But $\mathbb{A}$ is a characteristic set of the prime differential ideal $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$, which means

$$\{G \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\} \mid \mathrm{prem}(G, \mathbb{A}) = 0\} \ =\ [\mathbb{A}] : S_{\mathbb{A}}^{\infty}.$$

So the theorem is proved. ∎

In fact, Theorem 14 provides an algorithmic way for checking whether a solution of System (3) is a general solution via pseudo division. Next, we turn this into a criterion involving only algebraic (not differential) polynomials in the indeterminates $s_1, s_2, \cdots, s_n$.

Lemma 15 *A rational solution $(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x))$ of System (3) is a rational general solution if and only if for any $G \in \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$, $G(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0$ implies $G = 0$ in $\mathbb{K}(x)[s_1, s_2, \cdots, s_n]$.*

Proof ($\Longrightarrow$) Since $G \in \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$, we have $\mathrm{prem}(G, \mathbb{A}) = G$. It follows from Theorem 14 that $G(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0$ implies $G = 0$.

($\Longleftarrow$) Let $H \in \mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$, and $R = \mathrm{prem}(H, \mathbb{A})$, where $\mathbb{A} = \{A_1, A_2, \cdots, A_n\}$ and $A_i = V_i s_i' - U_i$ for $1 \le i \le n$. We need to prove

$$H(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0 \iff R = 0.$$

Obviously, $R \in \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$. If $H(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0$, then $R(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0$. So, by the assumption, $R = 0$. Conversely, let $R = 0$. From Proposition 11, we get

$$V_1^{l_1} V_2^{l_2} \cdots V_n^{l_n} H = \sum_{j \ge 0} Q_{1j} A_1^{(j)} + \sum_{j \ge 0} Q_{2j} A_2^{(j)} + \cdots + \sum_{j \ge 0} Q_{nj} A_n^{(j)},$$

and consequently

$$\prod_{i=1}^{n} V_i^{l_i}(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) \cdot H(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0.$$

Since $(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x))$ is a solution of System (3), we know $V_i(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) \neq 0$ for any $1 \le i \le n$. Therefore, $H(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x)) = 0$. According to Theorem 14, the rational solution $(\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_n(x))$ is a rational general solution of System (3). ∎

Based on the previous preparation, we can prove the following conclusion, which reduces our problem of finding a rational general solution of a higher order AODE to finding a rational general solution of its associated system.

Theorem 16 *Let $y = f(x)$ be a rational general solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$. If $y = f(x)$ is parametrizable by $\mathcal{P}(s_1, s_2, \cdots, s_n) = (\chi_1(s_1, s_2, \cdots, s_n), \cdots, \chi_{n+1}(s_1, s_2, \cdots, s_n))$, then*

$$(s_1(x), s_2(x), \cdots, s_n(x)) = \mathcal{P}^{-1}(x, f(x), f'(x), \cdots, f^{(n-1)}(x))$$

is a rational general solution of System (3) *when* $\det(\boldsymbol{A}) \neq 0$. *Conversely, let* $(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))$ *be a rational general solution of System* (3), *then*

$$\widehat{y} = \chi_2(\widehat{s}_1(x-c), \widehat{s}_2(x-c), \cdots, \widehat{s}_n(x-c))$$

is a rational general solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$, *where* $c = \chi_1(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x)) - x$.

Proof According to the process for obtaining System (3) after Lemma 8, it is clear to see that $(s_1(x), s_2(x), \cdots, s_n(x))$ is a rational solution of this system when $\det(\boldsymbol{A}) \neq 0$. Suppose that G is a differential polynomial in $\mathbb{K}(x)\{s_1, s_2, \cdots, s_n\}$ such that $G(s_1(x), s_2(x), \cdots, s_n(x)) = 0$. Let $R = \text{prem}(G, \mathbb{A})$, where $\mathbb{A} = \{A_1, A_2, \cdots, A_n\}$, $A_i = V_i s_i' - U_i$ for $1 \leq i \leq n$, then $R \in \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$. Moreover, we have

$$R(s_1(x), s_2(x), \cdots, s_n(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x), \cdots, f^{(n-1)}(x))) = 0.$$

Let

$$R(\mathcal{P}^{-1}(x, y, y_1, \cdots, y_{n-1})) = \frac{M(x, y, y_1, \cdots, y_{n-1})}{N(x, y, y_1, \cdots, y_{n-1})},$$

then $M(x, y, y', \cdots, y^{(n-1)})$ is a differential polynomial satisfying the condition

$$M(x, f(x), f'(x), \cdots, f^{(n-1)}(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$ and both F and M are the $(n-1)$-th order differential polynomials, we have

$$I^l M(x, y, y', \cdots, y^{(n-1)}) = M_0 F,$$

where I is the initial of F, $l \in \mathbb{N}$ and M_0 is a differential polynomial of order $n-1$ in $\mathbb{K}(x)\{y\}$. Therefore,

$$\begin{aligned} R(s_1, s_2, \cdots, s_n) &= R(\mathcal{P}^{-1}(\mathcal{P}(s_1, s_2, \cdots, s_n))) \\ &= \frac{I^l(\mathcal{P}(s_1, s_2, \cdots, s_n))M(\mathcal{P}(s_1, s_2, \cdots, s_n))}{I^l(\mathcal{P}(s_1, s_2, \cdots, s_n))N(\mathcal{P}(s_1, s_2, \cdots, s_n))} \\ &= \frac{M_0(\mathcal{P}(s_1, s_2, \cdots, s_n))F(\mathcal{P}(s_1, s_2, \cdots, s_n))}{I^l(\mathcal{P}(s_1, s_2, \cdots, s_n))N(\mathcal{P}(s_1, s_2, \cdots, s_n))} \\ &= 0. \end{aligned}$$

According to Theorem 14, we know that $(s_1(x), s_2(x), \cdots, s_n(x))$ is a rational general solution of (3).

Next, we need to construct a rational general solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$ from a rational general solution of System (3). Assume that $(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))$ is a rational general solution of (3). We have

$$\chi_1(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x)) = x + c \tag{5}$$

by substituting $\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x)$ into $\chi_1(s_1, s_2, \cdots, s_n)$, where c is a constant. It follows that $\chi_1(\widehat{s}_1(x-c), \widehat{s}_2(x-c), \cdots, \widehat{s}_n(x-c)) = x$. Therefore, $\widehat{y} = \chi_2(\widehat{s}_1(x-c), \widehat{s}_2(x-c), \cdots, \widehat{s}_n(x-c))$ is a rational solution of $F(x, y, y', \cdots, y^{(n-1)}) = 0$. Moreover, it is necessary to prove that $\widehat{y}$ is a general solution. Let $G \in \mathbb{K}(x)\{y\}$ such that $G(\widehat{y}) = 0$ and $R = \text{prem}(G, F)$ the differential pseudo remainder of G with respect to F. Obviously, $R(\widehat{y}) = 0$. We only need to prove that $R = 0$. Assume that $R \neq 0$, then

$$\begin{aligned}&R(\chi_1(s_1, s_2, \cdots, s_n), \chi_2(s_1, s_2, \cdots, s_n), \cdots, \chi_{n+1}(s_1, s_2, \cdots, s_n))\\&= \frac{W(s_1, s_2, \cdots, s_n)}{Z(s_1, s_2, \cdots, s_n)} \in \mathbb{K}(s_1, s_2, \cdots, s_n).\end{aligned}$$

As $R(\chi_1(\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_n), \chi_2(\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_n), \cdots, \chi_{n+1}(\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_n)) = 0$, it follows that $W(\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_n) = 0$. By Lemma 15, we have $W(s_1, s_2, \cdots, s_n) = 0$. Hence,

$$R(\chi_1(s_1, s_2, \cdots, s_n), \chi_2(s_1, s_2, \cdots, s_n), \cdots, \chi_{n+1}(s_1, s_2, \cdots, s_n)) = 0.$$

Since F is irreducible and $\deg_{y^{(n-1)}}(R) < \deg_{y^{(n-1)}}(F)$, it follows that $R = 0$ in $\mathbb{K}[x, y, y_1, \cdots, y_{n-1}]$. Therefore, $\widehat{y}$ is a rational general solution of differential equation $F(x, y, y', \cdots, y^{(n-1)}) = 0$. ∎

Remark 17 Note that the constant c in Theorem 16 is necessary. Indeed, if $(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))$ is a general solution of System (3), according to the autonomousness of this system, $(\widehat{s}_1(x + c_1), \widehat{s}_2(x + c_1), \cdots, \widehat{s}_n(x + c_1))$ is also a general solution of (3), where c_1 is a constant. Assume that $\chi_1(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x)) = x$. Then

$$\chi_1(\widehat{s}_1(x + c_1), \widehat{s}_2(x + c_1), \cdots, \widehat{s}_n(x + c_1)) = x + c_1.$$

Therefore, $y = \chi_2(\widehat{s}_1(x + c_1), \widehat{s}_2(x + c_1), \cdots, \widehat{s}_n(x + c_1))$ is not even a solution of Equation (1). This means that whenever we get one general solution of System (3), we have to reparametrize the solution by translating a constant. Later, in Example 25, the constant $-c_2$ will play the role of the constant c in Theorem 16.

By Theorem 16, we know that the rational general solutions of the original higher order AODE and its associated system are corresponding with respect to the given proper parametrization. In the following, we will present our methods for finding a rational general solution of the associated system.

4 Invariant Algebraic Space Curves and Rational General Solutions of Associated Systems

A rational solution of System (3) is an n-tuple of rational functions $(s_1(x), s_2(x), \cdots, s_n(x))$ which satisfies the given system. A solution $(s_1(x), s_2(x), \cdots, s_n(x))$ is trivial if all the $s_i(x)$ are constant. Each non-trivial rational solution of (3) represents a rational algebraic space curve for which the rational solution itself is a rational parametrization. Geometrically, such a rational algebraic space curve can be implicitly defined by the intersection of at least $n - 1$ algebraic hypersurfaces. Therefore, it is possible to compute a non-trivial rational solution of (3) by finding the implicit rational algebraic space curve (i.e., the implicit defining equations of those algebraic hypersurfaces such that this space curve is determined by their intersection) of the possible non-trivial rational solutions first, and then choosing suitable parametrizations of the space curve which satisfy System (3). Because trivial rational solutions are easy to compute, we restrict our attention to the computation of non-trivial rational solutions. From now on, we simply write rational solution for a non-trivial rational solution, if no confusion can arise.

We consider the differential operator

$$\mathcal{D} = \sum_{j=1}^{n} U_j W_j \frac{\partial}{\partial s_j},$$

where $W_j = \frac{\mathrm{lcm}(V_1,V_2,\cdots,V_n)}{V_j}$, U_j and V_j are the numerator and denominator of the right hand sides of System (3), respectively. Hence, for any $H \in \mathbb{K}[s_1, s_2, \cdots, s_n]$, $\mathcal{D}(H) = \sum_{j=1}^{n} U_j W_j H_{s_j}$, where H_{s_j} is the partial derivative of H with respect to s_j.

Definition 18 Let $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \in \mathbb{K}[s_1, s_2, \cdots, s_n]$. If $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m$ satisfy

$$\mathcal{D}(\widetilde{H}_i) \in \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle, \quad i = 1, 2, \cdots, m, \tag{6}$$

and $\langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle \subseteq \mathbb{K}[s_1, s_2, \cdots, s_n]$ is a 1-dimensional ideal, then $\boldsymbol{Z}(\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m)$, the intersection of the respective hypersurfaces defined by $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m$, is an invariant algebraic space curve of System (3).

Observe that if $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m$ satisfy (6), then for any $P \in \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle$, we have

$$\mathcal{D}(P) \in \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle. \tag{7}$$

Indeed, assume that $P = \sum_{i=1}^{m} C_i \widetilde{H}_i$, where $C_i \in \mathbb{K}[s_1, s_2, \cdots, s_n]$, then

$$\mathcal{D}(P) = \sum_{i=1}^{m} \mathcal{D}(C_i)\widetilde{H}_i + \sum_{i=1}^{m} C_i\, \mathcal{D}(\widetilde{H}_i) \in \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle.$$

Therefore, Definition 18 does not depend on the choice of a set of generators of the ideal. Note that only irreducible space curves can be parametrizable. From now on, we only consider irreducible invariant algebraic space curves.

Let $(s_1(x), s_2(x), \cdots, s_n(x))$ be a rational solution of System (3). This solution defines a parametric space curve $\mathcal{C}$. Let

$$\mathcal{I}_{\mathcal{C}} = \{G \in \mathbb{K}[s_1, s_2, \cdots, s_n] \mid G(s_1(x), s_2(x), \cdots, s_n(x)) = 0\} \subseteq \mathbb{K}[s_1, s_2, \cdots, s_n]$$

be the implicit ideal determined by $\mathcal{C}$. As $\mathcal{I}_{\mathcal{C}}$ has a generic zero $(s_1(x), s_2(x), \cdots, s_n(x))$ which only depends on one parameter, it is a 1-dimensional prime ideal. For every prime ideal $\mathcal{I}$ in $\mathbb{K}[s_1, s_2, \cdots, s_n]$ there exists an irreducible regular chain $\mathbb{T}$, such that $\mathcal{I} = \mathrm{sat}(\mathbb{T})$, and $\dim(\mathcal{I}) = n - |\mathbb{T}|$, where

$$\mathrm{sat}(\mathbb{T}) = \{G \in \mathbb{K}[s_1, s_2, \cdots, s_n] \mid H^q G \in \langle \mathbb{T} \rangle \text{ for some } q \in \mathbb{N}\},$$

H is the product of initials of all polynomials in $\mathbb{T}$, and $|\mathbb{T}|$ is the number of polynomials in $\mathbb{T}$. These facts can be found in [15, Proposition 3.4], [16, Theorem 3.1] and [17, Theorem 3.3]. Therefore, there exists an irreducible regular chain $\{H_1, H_2, \cdots, H_{n-1}\}$, such that $\mathcal{I}_{\mathcal{C}} = \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. Here and subsequently, we assume w.l.o.g that $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$.

Lemma 19 *Let $\widetilde{H}_i \in \mathbb{K}[s_1, s_2, \cdots, s_n]$ such that $\boldsymbol{Z}(\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m)$ is an irreducible invariant algebraic space curve of System* (3). *Then there exists an irreducible regular chain $\{H_1, H_2, \cdots, H_{n-1}\}$, $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$, such that*

$$\langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle = \mathrm{sat}(H_1, H_2, \cdots, H_{n-1}). \tag{8}$$

Furthermore,

$$\mathcal{D}(H_i) \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1}), \quad \textit{for } i = 1, 2, \cdots, n-1. \tag{9}$$

Proof By assumption, we know that $\langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \rangle \subseteq \mathbb{K}[s_1, s_2, \cdots, s_n]$ is a 1-dimensional prime ideal. It follows from the previous remark that there exists an irreducible regular chain

$\{H_1, H_2, \cdots, H_{n-1}\}$ such that (8) holds and $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$. We have the ideal inclusion

$$\langle H_1, H_2, \cdots, H_{n-1}\rangle \subseteq \mathrm{sat}(H_1, H_2, \cdots, H_{n-1}) = \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m\rangle.$$

So, from (7), we get $\mathcal{D}(H_i) \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. ▮

Lemma 20 *Let* $\{H_1, H_2, \cdots, H_{n-1}\}$ *be an irreducible regular chain, where* $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$ *satisfy System* (9)*. Then there exist* $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m \in \mathbb{K}[s_1, s_2, \cdots, s_n]$ *such that*

$$\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}) = \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m\rangle.$$

Furthermore, $\boldsymbol{Z}(\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m)$ *is an irreducible invariant algebraic space curve of System* (3).

Proof According to Hilbert's basis theorem $\mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$ has a finite basis, say $\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m$. So $\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}) = \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m\rangle$. Since $\widetilde{H}_i \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$, there exists $q_i \in \mathbb{N}$ such that

$$H^{q_i}\widetilde{H}_i \in \langle H_1, H_2, \cdots, H_{n-1}\rangle, \quad i = 1, 2, \cdots, m, \tag{10}$$

where H is the product of initials of $H_1, H_2, \cdots, H_{n-1}$. Assume that $H^{q_i}\widetilde{H}_i = \sum_{j=1}^{n-1} P_{ij}H_j$, where $P_{ij} \in \mathbb{K}[s_1, s_2, \cdots, s_n]$. Then

$$\begin{aligned} H^{q_i}\,\mathcal{D}(\widetilde{H}_i) = \mathcal{D}(H^{q_i}\widetilde{H}_i) - \mathcal{D}(H^{q_i})\widetilde{H}_i &= \sum_{j=1}^{n-1}\mathcal{D}(P_{ij})H_j + \sum_{j=1}^{n-1}P_{ij}\,\mathcal{D}(H_j) - \mathcal{D}(H^{q_i})\widetilde{H}_i \\ &\in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1}). \end{aligned}$$

According to the irreducibility of $\langle H_1, H_2, \cdots, H_{n-1}\rangle$, we know that $\mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$ is a prime ideal. As $H \notin \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$, we have

$$\mathcal{D}(\widetilde{H}_i) \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1}) = \langle \widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m\rangle.$$

The H_i form a regular chain, so we have $\dim(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1})) = 1$. Therefore, $\boldsymbol{Z}(\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_m)$ is an invariant algebraic space curve. ▮

According to Lemmas 19 and 20, we know that the condition in System (9) gives an alternative definition for irreducible invariant algebraic space curve, i.e., $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$ is an irreducible invariant algebraic space curve of System (3) if and only if $\{H_1, H_2, \cdots, H_{n-1}\}$ is an irreducible regular chain, where $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$ satisfy System (9). For determining invariant algebraic space curves we choose an upper bound for the degrees of $H_1, H_2, \cdots, H_{n-1}$, make an ansatz for the undetermined coefficients, and compute H_i by solving the corresponding algebraic equations in their coefficients (In fact, one can compute the normal form of $\mathcal{D}(H_i)$ modulo the Gröbner bases $\mathbb{G}$ of the ideal $\mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$ first, denoted by $\mathrm{nform}(\mathcal{D}(H_i), \mathbb{G})$, then the system of equations on the coefficients of H_i can be obtained by setting the coefficients of $\mathrm{nform}(\mathcal{D}(H_i), \mathbb{G})$ with respect to s_i equal to 0). Then we decide whether $\{H_1, H_2, \cdots, H_{n-1}\}$ is irreducible regular chain, in the affirmative case, we can get the implicit representation of the invariant algebraic space curve by computing $\mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. If Definition 18 is used for computing invariant algebraic space curves, more undetermined coefficients are involved. This increases the complexity of computation. Moreover, the number of generators of the ideal of the invariant space curve can be arbitrary.

Lemma 21 *Let* $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$ *be a rational invariant algebraic space curve of System* (3)*, where* $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$*, and* $(s_1(x), s_2(x), \cdots, s_n(x))$ *is a rational parametrization of* $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$*. If* $V_j(s_1(x), s_2(x), \cdots, s_n(x)) \neq 0$ *for* $j = 1, 2, \cdots, n$*, then*

$$s_1'(x)\cdot\frac{U_k(s_1(x), s_2(x), \cdots, s_n(x))}{V_k(s_1(x), s_2(x), \cdots, s_n(x))} = s_k'(x)\cdot\frac{U_1(s_1(x), s_2(x), \cdots, s_n(x))}{V_1(s_1(x), s_2(x), \cdots, s_n(x))}, \quad k = 2, 3, \cdots, n. \tag{11}$$

Proof By assumption, we have $\mathcal{D}(H_i) \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. Observe that $H_i \in \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. Since $(s_1(x), s_2(x), \cdots, s_n(x))$ is a parametrization of $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$, we have

$$\mathcal{D}(H_i)(s_1(x), s_2(x), \cdots, s_n(x)) = 0, \quad H_i(s_1(x), s_2(x), \cdots, s_{i+1}(x)) = 0. \tag{12}$$

By expanding the first equation and differentiating the second one with respect to x of (12), we have

$$\begin{aligned} A_i &:= \sum_{j=1}^{i+1} H_{is_j}(s_1(x), s_2(x), \cdots, s_{i+1}(x)) \cdot U_j(s_1(x), s_2(x), \cdots, s_n(x)) \\ &\quad \cdot W_j(s_1(x), s_2(x), \cdots, s_n(x)) = 0, \\ B_i &:= \sum_{j=1}^{i+1} H_{is_j}(s_1(x), s_2(x), \cdots, s_{i+1}(x)) \cdot s_j'(x) = 0, \end{aligned}$$

for $i = 1, 2, \cdots, n-1$. According to the irreducibility of $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$, we know that $\{H_1, H_2, \cdots, H_{n-1}\}$ is an irreducible regular chain, which means that H_i as a univariate polynomial with respect to s_{i+1} is irreducible over $\mathbb{K}(s_1(x), s_2(x), \cdots, s_i(x))$. Therefore,

$$H_{is_{i+1}}(s_1(x), s_2(x), \cdots, s_{i+1}(x)) \neq 0, \quad \text{for } i = 1, 2, \cdots, n-1. \tag{13}$$

In fact, assume that $H_{is_{i+1}}(s_1(x), s_2(x), \cdots, s_{i+1}(x)) = 0$. By combining this with the second equation of (12), we see that H_i is not squarefree with respect to s_{i+1}. This is a contradiction to the irreducibility of H_i. In particularly, $H_{1s_2}(s_1(x), s_2(x) \neq 0$. Therefore, the system determined by $A_1 = 0$ and $B_1 = 0$ has the non-zero solution $(H_{1s_1}(s_1(x), s_2(x), H_{1s_2}(s_1(x), s_2(x))$. So that the determinant of coefficients matrix is equal to 0, i.e., $\det(\boldsymbol{M}_2) = 0$, where

$$\boldsymbol{M}_2 = \begin{pmatrix} U_1(s_1(x), s_2(x), \cdots, s_n(x)) & U_2(s_1(x), s_2(x), \cdots, s_n(x)) \\ \cdot W_1(s_1(x), s_2(x), \cdots, s_n(x)) & \cdot W_2(s_1(x), s_2(x), \cdots, s_n(x)) \\ s_1'(x) & s_2'(x) \end{pmatrix}.$$

In the following, we will prove by induction $\det(\boldsymbol{M}_k) = 0$, for $k = 3, 4, \cdots, n$, where

$$\boldsymbol{M}_k = \begin{pmatrix} U_1(s_1(x), s_2(x), \cdots, s_n(x)) & U_k(s_1(x), s_2(x), \cdots, s_n(x)) \\ \cdot W_1(s_1(x), s_2(x), \cdots, s_n(x)) & \cdot W_k(s_1(x), s_2(x), \cdots, s_n(x)) \\ s_1'(x) & s_k'(x) \end{pmatrix}.$$

Assuming $\det(\boldsymbol{M}_k) = 0$ to hold for all $k < l$, we will prove it for $k = l$. In fact, the system determined by $A_{l-1} = 0$ and $B_{l-1} = 0$ can be rewritten as

$$\left\{ \begin{aligned} & H_{(l-1)s_1}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot U_1 W_1|_{(s_1(x), s_2(x), \cdots, s_n(x))} \\ & + H_{(l-1)s_l}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot U_l W_l|_{(s_1(x), s_2(x), \cdots, s_n(x))} \\ = & -\sum_{j=2}^{l-1} H_{(l-1)s_j}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot U_j W_j|_{(s_1(x), s_2(x), \cdots, s_n(x))} \\ & H_{(l-1)s_1}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot s_1'(x) + H_{(l-1)s_l}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot s_l'(x) \\ = & -\sum_{j=2}^{l-1} H_{(l-1)s_j}|_{(s_1(x), s_2(x), \cdots, s_l(x))} \cdot s_j'(x). \end{aligned} \right.$$

If $\det(\boldsymbol{M}_l) \neq 0$, then by Cramer's rule

$$H_{(l-1)s_l}(s_1(x), s_2(x), \cdots, s_l(x))) = -\frac{\sum_{j=2}^{l-1} H_{(l-1)s_j}(s_1(x), s_2(x), \cdots, s_l(x)) \det(\boldsymbol{M}_j)}{\det(\boldsymbol{M}_l)}.$$

Since $\det(\boldsymbol{M}_j) = 0$ for $j = 2, 3, \cdots, l-1$, $H_{(l-1)s_l}(s_1(x), s_2(x), \cdots, s_l(x)) = 0$. This is a contradiction to (13). Therefore, $\det(\boldsymbol{M}_k) = 0$ for $k = 2, 3, \cdots, n$. Taking into account that $V_j(s_1(x), s_2(x), \cdots, s_n(x)) \neq 0$ for $j = 1, 2, \cdots, n$, we see that System (11) holds. ∎

The condition $V_j(s_1(x), s_2(x), \cdots, s_n(x)) \neq 0$ in the above lemma means $V_j \notin \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$. The parametrization problem for algebraic plane curves has been studied intensively (e.g., in [18–19]). Therefore, the key point for computing a rational parametrization of the invariant algebraic space curve mentioned in Lemma 21 is to birationally project it to a plane curve (see [20]). This lemma tells us that not every rational parametrization of a rational invariant algebraic space curve can provide a rational solution of System (3). They are the candidates of rational solutions. In the following, we will give a theorem which provides a necessary and sufficient condition for the associated system to have a rational solution. Before that, we need to introduce an important property of proper parametrizations of rational space curves.

Lemma 22 *Let $\mathcal{P}_1(t)$ be a proper parametrization of an affine rational space curve $\mathcal{C}$, and let $\mathcal{P}_2(t)$ be any other rational parametrization of $\mathcal{C}$.*

a) *There exists a non-constant rational function $R(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(R(t))$.*

b) *$\mathcal{P}_2(t)$ is proper if and only if there exists a linear rational function $L(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(L(t))$.*

Proof a) Assume that $\varphi : \mathcal{C} \to \widehat{\mathcal{C}}$ is a birational mapping from the space curve $\mathcal{C}$ onto the plane curve $\widehat{\mathcal{C}}$. $\widehat{\mathcal{C}}$ is also rational. Indeed, since the space curve $\mathcal{C}$ is properly parametrized by $\mathcal{P}_1(t)$, the plane curve $\widehat{\mathcal{C}}$ is properly parametrized by $\widehat{\mathcal{P}}_1(t) = \varphi(\mathcal{P}_1(t))$. Similarly, $\widehat{\mathcal{P}}_2(t) = \varphi(\mathcal{P}_2(t))$ is a parametrization of the rational plane curve $\widehat{\mathcal{C}}$. According to Lemma 4.17 in [19], we have $\widehat{\mathcal{P}}_2(t) = \widehat{\mathcal{P}}_1(R(t))$, where $R(t)$ is a non-constant rational function. It follows that

$$\mathcal{P}_2(t) = \varphi^{-1}(\widehat{\mathcal{P}}_2(t)) = \varphi^{-1}(\widehat{\mathcal{P}}_1(R(t))) = \mathcal{P}_1(R(t)).$$

b) Since φ is birational, $\mathcal{P}_2(t)$ is proper if and only if $\widehat{\mathcal{P}}_2(t)$ is proper. So the statement follows from the appropriate statement for plane curves. ∎

Note that not all components in the computed proper parametrization of a rational space curve are constant. In other words, if $(s_1(x), s_2(x), \cdots, s_n(x))$ is a proper parametrization of the given rational space curve, then at least one of $s_i(x)$ is non-constant. In the sequel we present a method for finding a rational solution of System (3) based on the proper parametrization of its invariant algebraic space curve.

Theorem 23 *Let $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$ be a rational invariant algebraic space curve of System (3), such that $V_j \notin \mathrm{sat}(H_1, H_2, \cdots, H_{n-1})$ for $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, n$, where $H_i \in \mathbb{K}[s_1, s_2, \cdots, s_{i+1}]$, and let $(s_1(x), s_2(x), \cdots, s_n(x))$ be an arbitrary proper rational parametrization of $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$. Then System (3) has a rational solution*

$$(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x)) = (s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x))$$

corresponding to $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$ if and only if there exists a linear rational transformation $T(x) = \frac{ax+b}{cx+d}$ which is a rational solution of one of the following autonomous differential equations

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}, \quad \text{if } s_i'(x) \neq 0, \quad i = 1, 2, \cdots, n.$$

Proof Suppose that $(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))$ is a rational solution of System (3) corresponding to the invariant algebraic space curve $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$. Then

$$\widehat{s}_i'(x) = \frac{U_i(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))}{V_i(\widehat{s}_1(x), \widehat{s}_2(x), \cdots, \widehat{s}_n(x))}, \quad i = 1, 2, \cdots, n. \tag{14}$$

Since $(s_1(x), s_2(x), \cdots, s_n(x))$ is a proper parametrization of the rational space curve $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$, according to Lemma 22 a), there exists a non-constant rational function $T(x)$, such that the two rational parametrizations are related like

$$\widehat{s}_i(x) = s_i(T(x)), \quad i = 1, 2, \cdots, n. \tag{15}$$

By (14) and (15), we have

$$s_i'(T(x)) \cdot T'(x) = \frac{U_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}, \quad i = 1, 2, \cdots, n.$$

Note that at least one of $s_i(x)$ is non-constant. Therefore, $T(x)$ satisfies at least one of the following autonomous differential equations

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}, \quad \text{if } s_i'(x) \neq 0, \quad i = 1, 2, \cdots, n.$$

As the above autonomous differential equations are of degree 1 with respect to $T'(x)$, it follows from Theorem 2.7 and Corollary 3.11 in [3] that their rational solution $T(x)$ is a linear rational function.

Conversely, assume w.l.o.g. that $s_1(x)$ is non-constant and $T(x)$ is a rational solution of the differential equation

$$T'(x) = \frac{1}{s_1'(T(x))} \cdot \frac{U_1(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}.$$

By Lemma 21, we have

$$s_1'(x) \cdot \frac{U_k(s_1(x), s_2(x), \cdots, s_n(x))}{V_k(s_1(x), s_2(x), \cdots, s_n(x))} = s_k'(x) \cdot \frac{U_1(s_1(x), s_2(x), \cdots, s_n(x))}{V_1(s_1(x), s_2(x), \cdots, s_n(x))}, \quad k = 2, 3, \cdots, n.$$

If m of $s_2'(x), s_3'(x), \cdots, s_n'(x)$ are not equal to 0, where $0 \leq m \leq n-1$, assume w.l.o.g. that $s_i'(x) \neq 0$ for $i = 2, 3, \cdots, m+1$, and $s_{m+2}'(x) = s_{m+3}'(x) = \cdots = s_n'(x) = 0$. Then $s_j(x) = c_j$ for $m+2 \leq j \leq n$ and

$$\frac{1}{s_i'(x)} \cdot \frac{U_i(s_1(x), s_2(x), \cdots, s_n(x))}{V_i(s_1(x), s_2(x), \cdots, s_n(x))} = \frac{1}{s_1'(x)} \cdot \frac{U_1(s_1(x), s_2(x), \cdots, s_n(x))}{V_1(s_1(x), s_2(x), \cdots, s_n(x))}, \quad i = 2, 3, \cdots, m+1.$$

Therefore, $T(x)$ is also a rational solution of the following differential equations

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))}, \quad i = 2, 3, \cdots, m+1.$$

It follows that

$$(s_1(T(x)), s_2(T(x)), \cdots, s_{m+1}(T(x)), c_{m+2}, c_{m+3}, \cdots, c_n)$$

is a rational solution corresponding to $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$. A rational solution corresponding to $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2, \cdots, H_{n-1}))$ is $(s_1(T(x)), s_2(T(x)), \cdots, s_n(T(x)))$. ▮

In the following, we give a sufficient criterion for a rational solution to be a rational general solution.

Theorem 24 *Suppose that* $(s_1(x), s_2(x), \cdots, s_n(x))$ *is a rational solution of System* (3). *Assume that the defining ideal of* $(s_1(x), s_2(x), \cdots, s_n(x))$ *is of the form*

$$\langle A_1 - c_1B_1, A_2 - c_2B_2, \cdots, A_m - c_mB_m\rangle, \tag{16}$$

where $A_i, B_i \in \mathbb{K}[s_1, s_2, \cdots, s_n]$, *and* $\{c_1, c_2, \cdots, c_m\}$ *is an algebraically independent set of constants over* $\mathbb{K}$. *Then* $(s_1(x), s_2(x), \cdots, s_n(x))$ *is a rational general solution of System* (3).

Proof Let $\mathcal{I} := \langle G_1, G_2, \cdots, G_m\rangle \cap \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$. Because of the algebraic independence of the constants c_i, $\mathcal{I} = \{0\}$. Therefore, $G(s_1(x), s_2(x), \cdots, s_n(x)) = 0$ implies $G = 0$ for any $G \in \mathbb{K}(x)[s_1, s_2, \cdots, s_n]$. Because of Lemma 15, this implies that $(s_1(x), s_2(x), \cdots, s_n(x))$ is a rational general solution. ▮

Example 25 Consider the differential equation

$$F = 3xy'' - 3yy'' + 2y'^2 - 6y' = 0.$$

The solution hypersurface $F(x, y, z, w) = 3xw - 3yw + 2z^2 - 6z = 0$ of this differential equation is properly parametrized by

$$\mathcal{P}(s_1, s_2, s_3) = (s_1 + s_2 - s_3^2, s_1^2 s_3 + s_2 - s_3^2, 3s_1s_3, 6s_3).$$

The inverse map is

$$\mathcal{P}^{-1}(x, y, z, w) = \left(\frac{3xw - 3yw + 2z^2}{3w}, \frac{w^3 + 36yw - 24z^2}{36w}, \frac{w}{6}\right).$$

Therefore, the associated system of the original differential equation with respect to $\mathcal{P}(s_1, s_2, s_3)$ is

$$\begin{cases} s_1' = 1, \\ s_2' = \dfrac{2s_3^2}{s_1}, \\ s_3' = \dfrac{s_3}{s_1}. \end{cases} \tag{17}$$

Assume that we want to compute the invariant algebraic space curves $\boldsymbol{Z}(\mathrm{sat}(H_1, H_2))$ such that $\deg(H_1) \le 2$ and $\deg(H_2) \le 1$. First, we look for an invariant algebraic space curve $\boldsymbol{Z}(\mathrm{sat}(\widetilde{H}_1, \widetilde{H}_2))$ satisfying $\deg(\widetilde{H}_1) = \deg(\widetilde{H}_2) = 1$. We w.l.o.g. assume that

$$\widetilde{H}_1 = s_2 + b_1s_1 + b_2, \quad \widetilde{H}_2 = s_3 + b_3s_2 + b_4s_1 + b_5.$$

Then the Gröbner basis of $\mathrm{sat}(\widetilde{H}_1, \widetilde{H}_2)$ with respect to the lexicographic order determined by $s_1 < s_2 < s_3$ are

$$\mathbb{G} = [s_2 + b_1s_1 + b_2, s_3 + (b_4 - b_3b_1)s_1 + b_5 - b_3b_2]$$

and

$$\mathcal{D}(\widetilde{H}_1) = 2s_3^2 + b_1s_1, \quad \mathcal{D}(\widetilde{H}_2) = 2b_3s_3^2 + s_3 + b_4s_1.$$

It follows that

$$\begin{aligned}\operatorname{nform}(\mathcal{D}(\widetilde{H}_1),\mathbb{G}) =&(2b_4^2-4b_1b_3b_4+2b_1^2b_3^2)s_1^2+(4b_4b_5-4b_1b_3b_5-4b_2b_3b_4+4b_1b_2b_3^2+b_1)s_1\\&+2b_5^2-4b_2b_3b_5+2b_2^2b_3^2,\\ \operatorname{nform}(\mathcal{D}(\widetilde{H}_2),\mathbb{G}) =&(2b_1^2b_3^3-4b_1b_3^2b_4+2b_3b_4^2)s_1^2+(b_1b_3+4b_3b_4b_5-4b_2b_3^2b_4-4b_1b_3^2b_5\\&+4b_1b_2b_3^3)s_1+2b_2^2b_3^3-4b_2b_3^2b_5-b_5+b_2b_3+2b_3b_5^2.\end{aligned}$$

Therefore, the algebraic system of equations on the coefficients of $\widetilde{H}_1$ and $\widetilde{H}_2$ is

$$\begin{cases} b_4^2-2b_1b_3b_4+b_1^2b_3^2=0,\\ 4b_4b_5-4b_1b_3b_5-4b_2b_3b_4+4b_1b_2b_3^2+b_1=0,\\ b_5^2-2b_2b_3b_5+b_2^2b_3^2=0,\\ b_1^2b_3^3-2b_1b_3^2b_4+b_3b_4^2=0,\\ b_1b_3+4b_3b_4b_5-4b_2b_3^2b_4-4b_1b_3^2b_5+4b_1b_2b_3^3=0,\\ 2b_2^2b_3^3-4b_2b_3^2b_5-b_5+b_2b_3+2b_3b_5^2=0.\end{cases}$$

By solving this system, we obtain the following solution

$$\{b_1=0,b_2=b_2,b_3=b_3,b_4=0,b_5=b_2b_3\},$$

i.e.,

$$\widetilde{H}_1=s_2+b_2,\quad \widetilde{H}_2=s_3+b_3s_2+b_2b_3. \tag{18}$$

Now, we ask for an invariant algebraic space curves $\boldsymbol{Z}(\operatorname{sat}(H_1(s_1,s_2),H_2(s_1,s_2,s_3)))$ such that $\deg(H_1)=2$, $\deg(H_2)=1$. Let

$$H_1=s_2+b_1s_1^2+b_2s_1+b_3,\quad H_2=s_3+b_4s_2+b_5s_1+b_6,$$

then the following solutions are computed by using the same procedure as above

$$\{b_1=0,b_2=0,b_3=b_3,b_4=b_4,b_5=0,b_6=b_3b_4\},$$
$$\{b_1=-b_5^2,b_2=0,b_3=b_3,b_4=0,b_5=b_5,b_6=0\}.$$

The first solution corresponds to the above invariant algebraic space curve $\boldsymbol{Z}(\operatorname{sat}(\widetilde{H}_1,\widetilde{H}_2))$, where $\deg(\widetilde{H}_1)=\deg(\widetilde{H}_2)=1$. For the second solution, it determines another invariant algebraic space curve $\boldsymbol{Z}(\operatorname{sat}(H_1,H_2))$, where

$$H_1=s_2-b_5^2s_1^2+b_3,\quad H_2=s_3+b_5s_1. \tag{19}$$

Let $b_5=0$ in (19), then $\boldsymbol{Z}(\operatorname{sat}(H_1,H_2))=\boldsymbol{Z}(\operatorname{sat}(\widetilde{H}_1,\widetilde{H}_2))$. Therefore, $\boldsymbol{Z}(\operatorname{sat}(\widetilde{H}_1,\widetilde{H}_2))$ is a special case of $\boldsymbol{Z}(\operatorname{sat}(H_1,H_2))$. Actually, $\boldsymbol{Z}(\operatorname{sat}(H_1,H_2))$ is the only invariant algebraic space curves satisfying $\deg(H_1)=2$ and $\deg(H_2)=1$, because the other possible cases of H_1 of degree 2 lead to the computed invariant algebraic space curves reducible. Let $c_1=b_5,c_2=b_3$, as described as above, $\boldsymbol{Z}(\operatorname{sat}(H_1,H_2))$ is also the only invariant algebraic space curve satisfying the above required degree bounds, where

$$H_1=s_2-c_1^2s_1^2+c_2,\quad H_2=s_3+c_1s_1.$$

Since H_1 and H_2 are monic polynomials, $\{H_1, H_2\}$ is irreducible regular chain. Furthermore, $\mathrm{sat}(H_1, H_2) = \langle H_1, H_2 \rangle$. It follows that

$$\boldsymbol{Z}(\mathrm{sat}(H_1, H_2)) = \boldsymbol{Z}(H_1, H_2).$$

In addition,

$$(s_1(x), s_2(x), s_3(x)) = \left(\frac{1}{x}, \frac{c_1^2}{x^2} - c_2, -\frac{c_1}{x}\right)$$

is a proper parametrization of $\boldsymbol{Z}(H_1, H_2)$. Since $s_1'(x) \neq 0$,

$$T'(x) = \frac{1}{s_1'(T(x))} = -T^2(x).$$

Note that both $s_2'(x)$ and $s_3'(x)$ are not equal to 0, and

$$T'(x) = \frac{1}{s_2'(T(x))} \cdot \frac{2s_3(T(x))^2}{s_1(T(x))} = \frac{1}{s_3'(T(x))} \cdot \frac{s_3(T(x))}{s_1(T(x))} = -T^2(x).$$

By solving the above differential equation for $T(x)$, we get

$$T(x) = \frac{1}{x}.$$

It follows from Theorem 23 that

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = (s_1(T(x)), s_2(T(x)), s_3(T(x))) = (x, c_1^2 x^2 - c_2, -c_1 x)$$

is a rational solution of the associated system corresponding to $\boldsymbol{Z}(H_1, H_2)$. Since the Gröbner basis of $\langle H_1, H_2 \rangle$ with respect to the lexicographic order determined by $s_1 < s_2 < s_3 < c_1 < c_2$ is

$$\mathbb{G} = \{s_3 + c_1 s_1, -s_3^2 + s_2 + c_2\},$$

the ideal defined by $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ has generators in the form of (16), where

$$A_1 = s_3, \quad B_1 = -s_1, \quad A_2 = -s_3^2 + s_2, \quad B_2 = -1.$$

According to Theorem 24, $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ is a rational general solution. As

$$\widehat{s}_1(x) + \widehat{s}_2(x) - \widehat{s}_3(x)^2 = x - c_2,$$

it follows from Theorem 16 that a rational general solution of the original differential equation is

$$y = \widehat{s}_1(x + c_2)^2 \widehat{s}_3(x + c_2) + \widehat{s}_2(x + c_2) - \widehat{s}_3(x + c_2)^2 = -c_1(x + c_2)^3 - c_2,$$

where c_1 and c_2 are algebraically independent constants.

Remark 26 Note that $\mathrm{sat}(H_1, H_2) = \langle H_1, H_2 \rangle$ in Example 25 is not a rare special case. As shown in [21], $\mathrm{sat}(\mathbb{T}) = \langle \mathbb{T} \rangle$ if and only if $\mathbb{T}$ is primitive for the given regular chain $\mathbb{T}$. Primitivity can be decided algorithmically, and experimental results show that it occurs quite frequently in practice.

5 Rational First Integrals and Rational General Solutions of Associated Systems

In this section, we study rational general solutions of System (3) via rational first integrals. Relations between irreducible invariant algebraic hypersurfaces and rational first integrals are investigated in [22–23]. However, our notion of invariant algebraic space curves as defined in Section 4 is different from the notion of invariant algebraic hypersurfaces. Therefore, we need to establish the relationship between invariant algebraic space curves and rational first integrals. Based on this relation, we present a sufficient criterion for System (3) to have a rational general solution.

Definition 27 A first integral of System (3) is a non-constant function $R(s_1, s_2, \cdots, s_n)$ such that $\mathcal{D}(R) = 0$, where $\mathcal{D}$ is the differential operator defined in Section 4. A first integral $R(s_1, s_2, \cdots, s_n)$ of System (3) is rational if $R(s_1, s_2, \cdots, s_n) \in \mathbb{K}(s_1, s_2, \cdots, s_n)$.

Note that if $R(s_1, s_2, \cdots, s_n)$ is a first integral of System (3) and $(s_1(x), s_2(x), \cdots, s_n(x))$ is a solution of this system, then there exists a constant c such that

$$R(s_1(x), s_2(x), \cdots, s_n(x)) = c.$$

Definition 28 A rational invariant algebraic space curve of System (3) is called a rational solution space curve if it possesses a rational parametrization which is a solution of the system.

Theorem 29 *Let $\frac{A_i}{B_i} \in \mathbb{K}(s_1, s_2, \cdots, s_n)$, $i = 1, 2, \cdots, m$, be rational first integrals of System (3), such that A_i, B_i are polynomials and $\gcd(A_i, B_i) = 1$.*

a) *If $\mathcal{I}_{1k} = \langle A_1, A_2, \cdots, A_k, B_{k+1}, B_{k+2}, \cdots, B_m\rangle$ and $\mathcal{I}_{2l} = \langle B_1, B_2, \cdots, B_l, A_{l+1}, A_{l+2}, \cdots, A_m\rangle$ are 1-dimensional ideals in $\mathbb{K}[s_1, s_2, \cdots, s_n]$, then $\boldsymbol{Z}(\mathcal{I}_{1k})$ and $\boldsymbol{Z}(\mathcal{I}_{2l})$ are invariant algebraic space curves of System (3), where $1 \le k, l \le m$.*

b) *If $\mathcal{I} = \langle A_1 - c_1B_1, A_2 - c_2B_2, \cdots, A_m - c_mB_m\rangle \subseteq \mathbb{K}(c_1, c_2, \cdots, c_m)[s_1, s_2, \cdots, s_n]$ is a 1-dimensional ideal, then $\boldsymbol{Z}(\mathcal{I})$ is an invariant algebraic space curve of System (3), where c_i are some constants over $\mathbb{K}$. Furthermore, if $\boldsymbol{Z}(\mathcal{I})$ is a rational solution space curve and $\{c_1, c_2, \cdots, c_m\}$ is an algebraically independent set of constants over $\mathbb{K}$, then System (3) has a rational general solution.*

Proof a) According to Definition 27, we have $\mathcal{D}(\frac{A_i}{B_i}) = 0$ for $i = 1, 2, \cdots, m$. This implies

$$\mathcal{D}(A_i) \cdot B_i = A_i \cdot \mathcal{D}(B_i).$$

As $\gcd(A_i, B_i) = 1$, we have $A_i | \mathcal{D}(A_i)$ and $B_i | \mathcal{D}(B_i)$, where $i = 1, 2, \cdots, m$. Therefore,

$$\mathcal{D}(A_1), \mathcal{D}(A_2), \cdots, \mathcal{D}(A_k), \mathcal{D}(B_{k+1}), \mathcal{D}(B_{k+2}), \cdots, \mathcal{D}(B_m) \in \mathcal{I}_{1k},$$
$$\mathcal{D}(B_1), \mathcal{D}(A_2), \cdots, \mathcal{D}(B_l), \mathcal{D}(A_{l+1}), \mathcal{D}(A_{l+2}), \cdots, \mathcal{D}(A_m) \in \mathcal{I}_{2l},$$

where $1 \le k, l \le m$. Note that $\mathcal{I}_{1k}$ and $\mathcal{I}_{2l}$ are 1-dimensional ideals. So, from Definition 18, we see that $\boldsymbol{Z}(\mathcal{I}_{1k})$ and $\boldsymbol{Z}(\mathcal{I}_{2l})$ are invariant algebraic space curves of System (3).

b) Let $P_i = \frac{\mathcal{D}(B_i)}{B_i}$. Since $B_i | \mathcal{D}(B_i)$, P_i is a polynomial in $\mathbb{K}[s_1, s_2, \cdots, s_n]$. For $i = 1, 2, \cdots, m$, we have

$$\mathcal{D}(A_i - c_iB_i) = \mathcal{D}(A_i) - c_i\,\mathcal{D}(B_i) = \frac{A_i}{B_i}\,\mathcal{D}(B_i) - c_i\,\mathcal{D}(B_i) = \frac{\mathcal{D}(B_i)}{B_i}(A_i - c_iB_i) = P_i(A_i - c_iB_i).$$

Therefore,

$$\mathcal{D}(A_i - c_iB_i) \in \langle A_i - c_iB_i\rangle \subset \mathcal{I}.$$

$\mathcal{I}$ is 1-dimensional. So, from Definition 18, we see that $\boldsymbol{Z}(\mathcal{I})$ is an invariant algebraic space curve of System (3).

If $\boldsymbol{Z}(\mathcal{I})$ is a rational solution space curve, then a rational parametrization of $\boldsymbol{Z}(\mathcal{I})$ is a rational solution of System (3). Since all the elements c_i of the algebraically independent set $\{c_1, c_2, \cdots, c_m\}$ are transcendental over $\mathbb{K}$, it follows from Theorem 24 that this rational solution is a rational general solution of System (3). ∎

Theorem 29 establishes the relation between invariant algebraic space curves and rational first integrals of System (3). It also gives us a sufficient criterion based on rational first integrals, for deciding whether System (3) has a rational general solution. In the following, we present an example for illustrating our result.

Example 30 (Example 25, cont.) In fact, the associated system (17) in Example 25 has two rational first integrals:

$$R_1 = \frac{s_3}{s_1}, \quad R_2 = -s_3^2 + s_2.$$

Since $\langle s_3 - c_1 s_1, -s_3^2 + s_2 - c_2\rangle$ is a 1-dimensional ideal, we know that $\boldsymbol{Z}(s_3 - c_1 s_1, -s_3^2 + s_2 - c_2)$ is an invariant algebraic space curve of System (17) by Theorem 29 b). A proper parametrization of this curve is $(x, c_1^2 x^2 + c_2, c_1 x)$. Since $x' = 1 \neq 0$, it follows that

$$T'(x) = \frac{1}{s'(T(x))} = 1.$$

Hence, $T(x) = x$. From Theorem 23, we conclude that $(x, c_1^2 x^2 + c_2, c_1 x)$ is a rational solution of (17). Therefore, $\boldsymbol{Z}(s_3 - c_1 s_1, -s_3^2 + s_2 - c_2)$ is a rational solution space curve. It follows from Theorem 29 that $(x, c_1^2 x^2 + c_2, c_1 x)$ is a rational general solution of System (17) when c_1 and c_2 are algebraically independent constants. By an argument similar to the one in Example 25, we see that

$$y = c_1(x - c_2)^3 + c_2$$

is a rational general solution of the original AODE in Example 25.

6 Conclusions and Problems

In this paper, we have studied rational general solutions of higher order AODEs. We have presented a solution method based on a proper parametrization of the corresponding solution hypersurface. Experimental results show that if we only have an improper parametrization of the solution hypersurface, a general solution of a higher order AODE can still be obtained in many examples. But this solution might not be rational. Further study is needed on this problem. Another open problem concerns the degree bound for the defining polynomials of an irreducible invariant algebraic space curve.

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