Solving Linear Ordinary Differential Systems in Hyperexponential Extensions

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ABSTRACT

Let F be a differential field generated from the rational functions over some constant field by one hyperexponential extension. We present an algorithm to compute the solutions in F^n of systems of n first-order linear ODEs. Solutions in F of a scalar ODE of higher order can be determined by an algorithm of Bronstein and Fredet. Our approach avoids reduction to the scalar case. We also give examples to show how this can be applied to integration.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms

General Terms

Algorithms, Experimentation, Theory

Keywords

Linear Differential Equations over hyperexponential extension, Closed form solutions, Computer Algebra

1. INTRODUCTION

In the literature many results on solving scalar linear ordinary differential equations in various differential fields can be found, e.g., [15, 5, 7, 10]. In principle these can also be used for solving linear ordinary differential systems as they can always be reduced to scalar equations by the cyclic vector method or other uncoupling strategies. However, this reduction in general results in equations with much more complicated coefficients, which can take long to compute and to solve. This suggests that algorithms avoiding uncoupling might be preferable. In the present paper we extend the algorithm for computing closed form solutions in $C(x)$, where C is a field of constants, of systems with coefficients from $C(x)$ without uncoupling presented in [1] to accept more general equations living in $C(x,t)$, with t being transcendental and hyperexponential over $C(x)$, and compute its solutions in $C(x,t)$. This is related to computing exponential solutions as considered in [11], but there the systems are homogeneous only and more general solutions are considered. We show how to deal with the inhomogeneous part from $C(x,t)$, but we need to impose certain conditions on the system matrix. First we start with matrices still having entries from $C(x)$ and later we show how this generalizes to matrices with entries from $C(x,t)$ subject to some conditions. Related considerations for systems in $C(x)[t,t^{-1}]$ were also made in [9, Chap. 7] even for several hyperexponential generators, but no direct algorithm to compute the solutions in $C(x,t)$ nor even in $C(x)[t,t^{-1}]$ was provided there. We also illustrate what makes it so difficult to obtain a complete algorithm for any system with coefficients from $C(x,t)$. As an application of our algorithm we show that integrals involving certain Special Functions can be reduced to differential systems with coefficients of the type we consider.

Most algorithms for computing solutions of linear ordinary differential equations proceed by investigating the singularities of the coefficients to obtain information on the singularities of the solutions. For systems this is more difficult than for scalar equations and a main tool for computing indicial equations in this case is super-reduction [2, 1, 11]. From super-reduced systems we can compute all the integer slopes of the Newton-polygon and determine the corresponding characteristic polynomials [12]. If we are interested in one particular polynomial corresponding to a given integer slope k then k-simple systems as introduced in [12] are just what we need and the condition of being super-reduced is too strong. Recently an algorithm for directly computing k-simple forms of first-order differential systems at $x=0$ with coefficients from $C((x))$ was developed in [8, Chap. 4], eliminating the need to compute a super-reduced form first. We present a rational version of this algorithm at any place in $C(x)$ for systems with coefficients in $C(x)$ providing an alternative to the rational algorithm for super-reduction presented in [2]. In particular, it can also be utilized in the algorithm given in [1] replacing the super-reduction there. In addition, we observe that the algorithm for directly computing k-simple forms can also be applied in more general differential fields, which we exploit in our algorithm as well. Generalizations of concepts such as Newton polygon, characteristic polynomials, normal forms, etc. for linear differential systems with coefficients from exponential extensions of

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$\mathbb{C}((x))$ can be found in [3, 4].

We set up the formal setting in Section 2 and state our problem. All fields are implicitly understood to be of characteristic zero. After Section 3, which describes the algorithm for solving the problem, the next sections present the subalgorithms in more detail. In particular Section 4 presents a rational algorithm to compute indicial and characteristic polynomials and Section 5.2 shows how this can be utilized in $C(x)$ to compute the bounds on solutions in $C(x)[t, t^{-1}]$. Then in Section 6 we illustrate our algorithm and show how it can be applied to integration. We observe in Section 7 that our algorithm can also deal with more general situations and we present more non-trivial integrals. In Section 8 we conclude with discussing the limitations of our algorithm and directions for future work.

2. PROBLEM SETTING

Let C be a field and x transcendental over C, we consider the differential field $(K, \delta) = (C(x), \frac{d}{dx})$, i.e., $\delta x = 1$. Now, let $a \in K$ such that there are no $k \in \mathbb{Z} \setminus \{0\}$ and $g \in K^*$ with $a = \frac{\delta g}{kg}$. Let $t \neq 0$ in some extension of K with $\delta t = at$, i.e., t is hyperexponential over K, and consider the differential field extension $(F, \delta) := (K(t), \delta)$. According to [6, Thm. 5.1.2] the condition on a implies that t is transcendental over K and the constant field of F is C. We keep this setting throughout the paper and consider the following problem.

PROBLEM 1. *Given $A \in K^{n\times n}$ and $\mathbf{f}_0, \dots, \mathbf{f}_m \in F^n$, compute a C-vector space basis of all solutions $(\mathbf{y}, \mathbf{c}) \in F^n \times C^{m+1}$ such that*

$$\delta \mathbf{y} + A\mathbf{y} = \sum_{i=0}^{m} c_i \mathbf{f}_i. \tag{1}$$

Often one needs to consider the affine case where $c_0 = 1$ is fixed, which is what we will do in the examples later. The necessary modifications are trivial.

For the presentation of our work we need to define a few notions for polynomials $p \in K[t]$, see, e.g., [6] for more details. First, note that $\delta p \in K[t]$ as well. If $\gcd(p, \delta p) = 1$ then p is called *normal*, if $p \mid \delta p$ then p is called *special*. Irreducible polynomials p are either normal or special. Any $p \in K[t]$ can be factored into a normal part and a special part, which only have normal or special irreducible factors respectively. The above condition on a also implies that the only special irreducible polynomial $p \in K[t]$ is $p = t$.

Regarding valuations we will rely on the following definitions. Any irreducible $p \in C[x]$ (or $p = \frac{1}{x}$) induces a valuation $\nu_p : C(x) \to \mathbb{Z} \cup \{\infty\}$, where $\nu_p(f)$ is given by the maximal ν such that $p \nmid \operatorname{den}(fp^{-\nu})$ (or by $\nu_p(f) := \deg(\operatorname{den}(f)) - \deg(\operatorname{num}(f))$ for $p = \frac{1}{x}$). It can be thought of as the multiplicity of the zero of f at the roots of p with negative values corresponding to poles. We define the degree of the derivation at p as $\omega_p := \inf_{f\in C(x)^*} \nu_p(\frac{\delta f}{f})$. Any valuation ν_p has a canonical projection π_p from the valuation ring $\{f \in C(x) \mid \nu_p(f) \geq 0\}$ to the residue field $C[x]/(p)$ (or C for $p = \frac{1}{x}$) associated to it, which can be thought of as the evaluation of $f \in C(x)$ at the roots of p. The leading term of the p-adic expansion of an $f \in C(x)^*$ is $\pi_p(fp^{-\nu_p(f)})p^{\nu_p(f)}$. If $\nu_p(f) \geq \nu_p(\delta p) - 1$ then we define the *residue of f at p* by $\operatorname{res}_p(f) := \pi_p(f\frac{p}{\delta p})$. Analogously, for $p \in K[t]$ (or $p = \frac{1}{t}$) we define ν_p, π_p, and $\omega_p := \inf_{f\in K(t)^*} \nu_p(\frac{\delta f}{f})$ on $K(t)$. For vectors $\mathbf{f}$ we define the valuation $\nu_p(\mathbf{f})$ as the minimum of the valuations of the entries, likewise we define $\nu_p(A)$ for matrices.

3. ALGORITHM

Our algorithm follows the same three major steps that already have been used for scalar ODEs in [7]. The important ingredient for computing the solutions is to obtain lower bounds on the possible values of $\nu_p(\mathbf{y})$ for all p.

1. Compute the normal part of the universal denominator, i.e., some $b \in K[t] \setminus \{0\}$ such that any solution $\mathbf{y}$ of (1) satisfies $b\mathbf{y} \in K[t, t^{-1}]^n$.
2. Determine degree bounds λ_0, λ_1 for the Laurent polynomials $\tilde{\mathbf{y}} \in K[t, t^{-1}]^n$ such that $\tilde{\mathbf{y}}/b$ satisfies (1).
3. For the ansatz $\tilde{\mathbf{y}} = \sum_{i=\lambda_0}^{\lambda_1} \mathbf{y}_i t^i$ compute the rational solutions $\mathbf{y}_i \in C(x)^n$ and the corresponding $c_i \in C$.

Essentially, in the first two steps the result is determined by the inhomogeneous part of the system. However there is also the possibility of some cancellation in the homogeneous part, which will be our main concern in Sections 5 and 7. A lower bound for $\nu_p(\delta\mathbf{y} + A\mathbf{y})$ is easily obtained from the inhomogeneous part: $\nu_p(\delta\mathbf{y} + A\mathbf{y}) \geq \min_i \nu_p(\mathbf{f}_i)$. Disregarding cancellation cases we then can bound $\nu_p(\mathbf{y})$ from below based on ω_p and $\nu_p(A)$, since in most cases we have

$$\nu_p(\mathbf{y}) = \nu_p(\delta\mathbf{y} + A\mathbf{y}) - \min(\omega_p, \nu_p(A)). \tag{2}$$

This bound on $\nu_p(\mathbf{y})$ has then to be modified in order to include all values of $\nu_p(\mathbf{y})$ where the standard relation (2) is not satisfied.

The considerations above need to be executed for finitely many p only since we have the a-priori bound $\nu_p(\mathbf{y}) \geq 0$ for almost all p as will be detailed later. In Step 1 the basic fact to consider is that for any normal irreducible $p \in K[t]$ we have that $\nu_p(f) \neq 0$ implies $\nu_p(\delta f) = \nu_p(f) - 1$ and $\nu_p(f) = 0$ implies $\nu_p(\delta f) \geq 0$, hence $\omega_p = -1$. For Step 2, however, with $t \mid \delta t$ we have that $\nu_t(f) \neq 0$ implies $\nu_t(\delta f) = \nu_t(f)$ and $\nu_t(f) = 0$ implies $\nu_t(\delta f) \geq 0$. Hence $\omega_t = 0$, and similarly $\omega_{1/t} = 0$. For more details we refer to [6, Chapter 4].

Once we have determined b we substitute $\mathbf{y} = \tilde{\mathbf{y}}/b$ in (1) and multiply by b to obtain the new system

$$\delta\tilde{\mathbf{y}} + \left(A - \frac{\delta b}{b} I_n\right)\tilde{\mathbf{y}} = \sum_{i=0}^{m} c_i \mathbf{f}_i b. \tag{3}$$

Then we do the remaining computations in Steps 2 and 3 based on this system. After computing λ_0 and λ_1 we either focus on the place $p = t$ or $p = \frac{1}{t}$ in the third step. Starting from t^{λ_0} or t^{λ_1} respectively we successively proceed through the powers of t. For each power t^i by comparing its coefficients, or, more precisely, multiplying (3) by t^{-i} and applying π_p, we obtain a differential system of the form

$$\delta\mathbf{y}_i + (A + (ia - \tilde{b})I_n)\mathbf{y}_i = \sum_{j=0}^{m_i} \tilde{c}_{i,j}\tilde{\mathbf{f}}_{i,j} \tag{4}$$

with coefficients from $C(x)$, where $\tilde{b} = \pi_p(\frac{\delta b}{b})$. In order to compute all solutions $\mathbf{y}_i \in C(x)^n$ and $\tilde{c}_{i,j} \in C$ of these systems we apply the algorithm described in [1] modified to obtain indicial equations based on the algorithm described in Section 4.1 instead of super-reduction. Plugging in the solutions in the ansatz made for $\tilde{\mathbf{y}}$ in (3) generates a new inhomogeneous part with higher ν_t or lower $\nu_{1/t}$ respectively and possibly with a different m. Then we proceed with the next power of t until we eventually consider t^{λ_1} or t^{λ_0}. After that the remaining inhomogeneous part has to vanish, which provides conditions on the remaining free constants.

4. INDICIAL AND CHARACTERISTIC POLYNOMIALS

For computing indicial and characteristic polynomials we consider the following setting.

Let $k \in \mathbb{N}$ and $p \in C[x]$ irreducible or $p = \frac{1}{x}$. Furthermore, let $\alpha \in \mathbb{N}^n$, $\tilde{D}, N \in K^{n\times n}$ such that $\nu_p(\tilde{D}), \nu_p(N) \geq 0$, then for $D := p^{\operatorname{diag}(\alpha)}(I_n + p\tilde{D})$ we consider the operator

$$L(\mathbf{y}) = Dp^{k-\omega_p}\delta\mathbf{y} + N\mathbf{y}. \tag{5}$$

Note that the definition of D implies $\nu_p(D) \geq 0$. Operators of the the form $\delta\mathbf{y} + A\mathbf{y}$ have to be multiplied from the left by an appropriate factor to match this form.

Let $\mathbf{y} = \mathbf{g}e^{\int w}$ with $\mathbf{g} \in K^n$ and $w \in K$ such that $\nu_p(\mathbf{g}) = 0$ and $\nu_p(w) \geq \omega_p - k$, then the equation $L(\mathbf{y}) = 0$ reads

$$L(\mathbf{y}) = (Dp^{k-\omega_p}\delta\mathbf{g} + wDp^{k-\omega_p}\mathbf{g} + N\mathbf{g})e^{\int w} = 0.$$

With $D_0 := \pi_p(p^{\operatorname{diag}(\alpha)})$, $N_0 := \pi_p(N)$, $\mathbf{g}_0 := \pi_p(\mathbf{g})$, and $w_0 := \pi_p(wp^{k-\omega_p})$ we have

$$\pi_p(Dp^{k-\omega_p}\delta\mathbf{g} + wDp^{k-\omega_p}\mathbf{g} + N\mathbf{g}) = D_0\pi_p(p^{k-\omega_p}\delta\mathbf{g}) + (w_0D_0 + N_0)\mathbf{g}_0.$$

Hence from $\nu_p(\delta\mathbf{g}) > \omega_p - k$ and $\mathbf{g}_0 \neq 0$ we deduce that the polynomial

$$P_k(\mu) := \det(\mu D_0 + N_0)$$

has a root at $\mu = w_0$. If $P_k \not\equiv 0$, then L is called k-simple at p and P_k is called the characteristic polynomial ($k > 0$) or the indicial polynomial ($k = 0$) at p. In other words, if L is k-simple, then by computing P_k we get finitely many candidates for w_0. For $k = 0$ this is equivalent to the notion of simple systems used in [1].

As not every L is k-simple at p we need to compute an equivalent operator

$$\tilde{L}(\mathbf{z}) = SL(T\mathbf{z}) = SDp^{k-\omega_p}T\delta\mathbf{z} + S(NT + Dp^{k-\omega_p}(\delta T))\mathbf{z}$$

with $S, T \in K^{n\times n}$ invertible such that $\tilde{L}$ obeys the same structure above and is k-simple at p.

Then for general $\mathbf{g} \in K^n$ and $w \in K$ with $\nu_p(w) \geq \omega_p - k$ and $L(\mathbf{g}e^{\int w}) = 0$ we normalize to $\tilde{\mathbf{g}} := T^{-1}\mathbf{g}p^{-\nu_p(T^{-1}\mathbf{g})}$ and $\tilde{w} := w + \nu_p(T^{-1}\mathbf{g})\frac{\delta p}{p}$. So we have $\nu_p(\tilde{\mathbf{g}}) = 0$ and $\tilde{L}(\tilde{\mathbf{g}}e^{\int \tilde{w}}) = 0$. Hence we have $P_k(\tilde{w}_0) = 0$ where $\tilde{w}_0 = \pi_p(wp^{k-\omega_p}) + \delta_{k,0}\nu_p(T^{-1}\mathbf{g})\pi_p((\delta p)p^{-1-\omega_p})$. Moreover, observe that $\nu_p(T) \leq \nu_p(\mathbf{g}) - \nu_p(T^{-1}\mathbf{g}) \leq -\nu_p(T^{-1})$.

4.1 Computing k-simple forms directly

Using the notation from above we will show in this section, given any such operator $L(\mathbf{y}) = Dp^{k-\omega_p}\delta\mathbf{y} + N\mathbf{y}$, how to compute $S, T \in K^{n\times n}$ invertible such that the operator $SL(T\mathbf{z})$ is k-simple at p, i.e., the corresponding $P_k(\mu)$ is not the zero polynomial. The rational algorithm given below is an immediate generalization of the algorithm developed for $p = x$ by Carole El Bacha in [8, Chap. 4]. The main difference is in verifying that the term $SDp^{k-\omega_p}(\delta T)$ in the equivalence transformation does not interfere.

The algorithm repeats the same step again and again, at each step applying some equivalence transformation determined according to the three cases shown below. At the beginning and after each step of the algorithm we perform a normalizing transformation: we multiply each row of the current operator L by $p^{-\min(\alpha_i,\nu_p(\mathbf{n}_i))}$ from the left, where $\mathbf{n}_i$ is the i-th row of N, which can be summarized as some $S = p^{\operatorname{diag}(\beta)}$, and we apply a permutation matrix P such that the operator $PSL(P^{-1}\mathbf{y})$ has $\alpha \in \mathbb{N}^n$ with $\alpha_1 \leq \dots \leq \alpha_n$. If the resulting operator is k-simple at p, then we collect all the transformations done so far into the overall transformation matrices S and T and stop, otherwise we proceed with the next step. If the input is of the form $\delta\mathbf{y} + A\mathbf{y}$, then the initial normalization multiplies each row by $p^{-\min(\omega_p-k,\nu_p(\mathbf{a}_i))}$ from the left instead in order to obtain the form (5), where $\mathbf{a}_i$ is the i-th row of A.

By the normalization in between the steps the sum $|\alpha| := \alpha_1 + \dots + \alpha_n$ is either decreased or at least stays the same. As long as the operator is not k-simple, which happens for $\alpha = 0$ at latest, the steps ensure that $|\alpha|$ will be decreased eventually. The idea of the transformations applied below is to make one row of N_0 zero for which the corresponding α_i is greater than zero, since then at the first part of the normalizing transformation α_i will be decreased in this situation.

For the rest of the section we denote the residue field $C[x]/(p)$ or C of the valuation ν_p uniformly by K_p. When constructing transformation matrices by elements from K_p we actually refer to canonical representatives from K (w.r.t. π_p). We also use r to denote the rank of the matrix D_0 and observe that

$$D_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

We subdivide the matrix N_0 into the same block sizes as D_0 above. Below we assume that the operator is not k-simple yet, in other words the rows of the matrix

$$\mu D_0 + N_0 = \begin{pmatrix} \mu I_r + N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \in K_p(\mu)^{n\times n}$$

are linearly dependent.

Case 1.

We check whether the rows of the submatrix $(N_{21}\ N_{22})$ are linearly dependent. If they are not, we proceed with Case 2 below. If they are, then for some $i > r$ we can determine a vector $\mathbf{u} \in K_p^{n-i}$ such that $(0, \dots, 0, 1, \mathbf{u})N_0 = 0$. Let $j \in \{i+1, \dots, n\}$ be maximal such that $\alpha_i = \alpha_j$ and define $\tilde{\mathbf{u}} = (-u_1, \dots, -u_{j-i}, 0, \dots, 0) \in K_p^{n-i}$. Then we apply the transformation

$$S = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & \mathbf{u} \\ 0 & 0 & I_{n-i} \end{pmatrix}, \quad T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & \tilde{\mathbf{u}} \\ 0 & 0 & I_{n-i} \end{pmatrix}.$$

We have that $\nu_p(\delta T) \geq \omega_p$ and only its i-th row may be non-zero, which implies $\nu_p(Sp^{\operatorname{diag}(\alpha)}(I_n + p\tilde{D})p^{k-\omega_p}\delta T) \geq \alpha_i + k$, in particular $\nu_p(SDp^{k-\omega_p}\delta T) > 0$. So the new N_0 has all zeros in its i-th row and, since $\alpha_i > 0$, the subsequent normalizing transformation will decrease $|\alpha|$.

Case 2.

We refine the subdivision of N_0 from above by splitting off the first q rows and columns for the maximal $q \in \{0, \dots, r\}$

such that

$$N_0 = \begin{pmatrix} N_{11} & 0 & 0 \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix}$$

and check whether the rows of the submatrix $(N_{32}\ N_{33})$ are linearly dependent. If they are not, we proceed with Case 3 below. If they are, then we apply the transformation

$$S = \begin{pmatrix} p^{-1}I_q & 0 \\ 0 & I_{n-q} \end{pmatrix}, \quad T = \begin{pmatrix} pI_q & -pD_{12} \\ 0 & I_{n-q} \end{pmatrix},$$

where D_{12} is the corresponding submatrix obtained from $\pi_p(\tilde{D})$ by deleting the first q columns of the first q rows

$$\pi_p(\tilde{D}) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

We have that $\nu_p(\delta T) \geq 1+\omega_p$ and only its first q rows may be non-zero, which implies $\nu_p(Sp^{\mathrm{diag}(\alpha)}(I_n + p\tilde{D})p^{k-\omega_p}\delta T) \geq k$ where only the first q rows can have valuation ν_p exactly k. So at worst $SDp^{k-\omega_p}\delta T$ contributes to the first q rows of N_0, for $k = 0$, but does not interfere with the last $n-r$ rows in any case. This transformation does not change α, but the new N_0 has $(0\ N_{32}\ N_{33})$ as its last $n - r$ rows, which are linearly dependent. So $|\alpha|$ will be decreased either now by normalizing or at latest after the next step, which will be Case 1 then.

Case 3.

We apply a $n \times n$ permutation matrix P acting on the rows $\{q+1, \ldots, r\}$ only in order to ensure that for the operator $PL(P^{-1}\mathbf{y})$ with the same subdivision of N_0 from above we can determine a vector $\mathbf{u} \in K_p^{n-q-1}$ such that $(0, \ldots, 0, 1, \mathbf{u})N_0 = 0$. Analogous to Case 1 we define $\tilde{\mathbf{u}} = (-u_1, \ldots, -u_{r-q-1}, 0, \ldots, 0) \in K_p^{n-q-1}$ and then we apply the transformation

$$S = \begin{pmatrix} I_q & 0 & 0 \\ 0 & 1 & \mathbf{u} \\ 0 & 0 & I_{n-q-1} \end{pmatrix}, \quad T = \begin{pmatrix} I_q & 0 & 0 \\ 0 & 1 & \tilde{\mathbf{u}} \\ 0 & 0 & I_{n-q-1} \end{pmatrix}.$$

If $p \in C[x]$ then we have $\omega_p = -1$ and $\nu_p(\delta T) \geq 0$, if $p = \frac{1}{x}$ then we have $K_p = C$ and $\delta T = 0$. In any case $\nu_p(\delta T) > \omega_p$, which implies $\nu_p(Sp^{\mathrm{diag}(\alpha)}(I_n + p\tilde{D})p^{k-\omega_p}\delta T) > k$, in particular $\nu_p(SDp^{k-\omega_p}\delta T) > 0$. So the effect of this transformation is that the new N_0 has an increased value of q. The normalization will not change the operator and can be skipped immediately after this step. This will just result in Case 3 being applied until we are in one of the other two cases, which happens for $q = r$ at latest.

To see that we actually can construct the transformation described in Case 3 consider the following. The rows of

$$\mu D_0 + N_0 = \begin{pmatrix} \mu I_q + N_{11} & 0 & 0 \\ N_{21} & \mu I_{r-q} + N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix} \in K_p(\mu)^{n \times n}$$

are linearly dependent. As $\mu I_q + N_{11}$ is invertible we see that the rows of the submatrix

$$\begin{pmatrix} \mu I_{r-q} + N_{22} & N_{23} \\ N_{32} & N_{33} \end{pmatrix}$$

are linearly dependent, which remains true after specializing $\mu = 0$. But the rows of $(N_{32}\ N_{33})$ are linearly independent since we would not have reached Case 3 otherwise. Therefore, $q < r$ and in particular we can find a transformation of the type described above after a suitable permutation.

5. COMPUTING THE BOUNDS

5.1 Normal part of the denominator

This step is rather trivial in our present setting, but it will be more involved in the generalization considered later in Section 7. For now at any irreducible $p \in K[t]$ with $t \nmid p$ we have (2) as long as $\nu_p(\mathbf{y}) \neq 0$. This is owed to the fact that $\nu_p(A) = 0 > -1 = \omega_p$ implies $\nu_p(\delta\mathbf{y} + A\mathbf{y}) = \nu_p(\delta\mathbf{y}) = \nu_p(\mathbf{y}) - 1$ in this case. So with $\lambda_p := 1 + \min_i \nu_p(\mathbf{f}_i)$ we simply can take

$$b = \prod_p p^{-\lambda_p} \in K[t] \setminus \{0\},$$

where the product runs over the finitely many p such that $\lambda_p < 0$. Then any $\mathbf{y} \in K(t)^n$ that is a solution of (1) for some $c_0, \ldots, c_m \in C$ satisfies

$$b\mathbf{y} \in K[t, t^{-1}]^n.$$

5.2 Degree bounds

We now have to find bounds on the possible values of λ_0 and λ_1 in the solutions $\tilde{\mathbf{y}} = \sum_{i=\lambda_0}^{\lambda_1} \mathbf{y}_i t^i$ of (3). The system matrix $\tilde{A}$, however, has entries from $K(t)^{n \times n}$ now, but it is the sum of $A \in K^{n \times n}$ and $\frac{\delta b}{b} I_n$ for some $b \in K(t)^*$. So in particular we still have $\nu_t(\tilde{A}) = 0$ and $\nu_{1/t}(\tilde{A}) = 0$, which is what we need in the following.

Both for $p = t$ and $p = \frac{1}{t}$ we have $\omega_p = 0$ as well as $\nu_p(\tilde{A}) = 0$, hence (2) reads $\nu_p(\mathbf{y}) = \nu_p(\delta\mathbf{y} + \tilde{A}\mathbf{y})$. In general we have $\nu_p(\mathbf{y}) \leq \nu_p(\delta\mathbf{y} + \tilde{A}\mathbf{y})$ only, so we need to determine the cases where we have strict inequality. Consequently, we are reduced to the problem of finding solutions $\mathbf{g}t^\lambda$ with $\mathbf{g} \in K^n$ and $\lambda \in \mathbb{Z}$ of the homogeneous system

$$\delta\mathbf{y} + \pi_p(\tilde{A})\mathbf{y} = 0.$$

We proceed by an algorithm computing a finite set of candidates for λ, which we present in the proof of the following theorem.

THEOREM 2. *Let $A \in C(x)^{n \times n}$ and $\delta t = at$ where $a \in C(x)$ such that there are no $k \in \mathbb{Z} \setminus \{0\}$ and $g \in C(x)^*$ with $a = \frac{\delta g}{kg}$. Then we can compute a finite set $\Lambda \subset \mathbb{Z}$ such that for any $\mathbf{y} = \mathbf{g}t^\lambda$ with $\mathbf{g} \in C(x)^n$, $\lambda \in \mathbb{Z}$, and $\delta\mathbf{y} + A\mathbf{y} = 0$ we have $\lambda \in \Lambda$.*

PROOF. Choose $p \in C[x]$ irreducible (or $p = \frac{1}{x}$) such that $\nu_p(a) < \omega_p$ or $\mathrm{res}_p(a) \notin \mathbb{Q}$. We can do this because if for each irreducible $p \in C[x]$ we have $\nu_p(a) \geq -1$ and $\mathrm{res}_p(a) \in \mathbb{Q}$ then $a = \tilde{a} + \sum_{i=1}^N r_i \frac{\delta p_i}{p_i}$ for some $\tilde{a} \in C[x]$, $r_i \in \mathbb{Q}$, and $p_i \in C[x]$ and by assumption on a it follows that $\tilde{a} \neq 0$, which implies $\nu_{1/x}(a) < 1$. Let $a_0 := \pi_p(ap^{-\nu_p(a)})$, $p_0 := \pi_p((\delta p)p^{-1-\omega_p})$, and $\beta := \omega_p - \nu_p(a)$. For constructing Λ we distinguish two cases.

Case 1: $\beta > 0$

Compute a β-simple form of $\delta\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding characteristic polynomial $P_\beta(\mu) \in C[x]/(p)[\mu]$ (resp. $\in C[\mu]$). Determine the set

$$\tilde{\Lambda} := \{\lambda \in \mathbb{Z} \setminus \{0\} \mid P_\beta(\lambda a_0) = 0\}.$$

Next, compute a 0-simple form of $\delta\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding transformation matrix $T \in C(x)^{n\times n}$ and indicial polynomial $P_0(\mu) \in C[x]/(p)[\mu]$ (resp. $\in C[\mu]$). If $P_0(\nu p_0) = 0$ has a solution $\nu \in \mathbb{Z}$, then set $\Lambda := \tilde{\Lambda} \cup \{0\}$, otherwise set $\Lambda := \tilde{\Lambda}$.

Case 2: $\beta = 0$

Compute a 0-simple form of $\delta\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding transformation matrix $T \in C(x)^{n\times n}$ and indicial polynomial $P_0(\mu) \in C[x]/(p)[\mu]$ (resp. $\in C[\mu]$). Determine the set

$$\Lambda := \{\lambda \in \mathbb{Z} \mid \exists \nu \in \mathbb{Z} : P_0(\nu p_0 + \lambda a_0) = 0\},$$

which is finite since p_0 and a_0 are $\mathbb{Q}$-linearly independent because of $\frac{a_0}{p_0} = \text{res}_p(a) \notin \mathbb{Q}$.

Now, we verify the desired property of Λ. For $\mathbf{y} = \mathbf{g}t^\lambda$ as above we have $0 = \delta\mathbf{y} + A\mathbf{y} = (\delta\mathbf{g} + \lambda a\mathbf{g} + A\mathbf{g})t^\lambda$, hence $\delta\mathbf{g} + \lambda a\mathbf{g} + A\mathbf{g} = 0$. Again we treat the two cases separately.

Case 1: $\beta > 0$

If $\lambda \neq 0$, then the term $\lambda a\mathbf{g}$ dominates and λa_0 is a root of the characteristic polynomial P_β, hence $\lambda \in \Lambda$. If $\lambda = 0$, then $\nu_p(T^{-1}\mathbf{g})p_0$ is a root of the indicial polynomial P_0 and we deduce $\lambda \in \Lambda$.

Case 2: $\beta = 0$

In this case $\nu_p(T^{-1}\mathbf{g})p_0 + \lambda a_0$ is a root of the indicial polynomial P_0 and we have $\lambda \in \Lambda$. $\square$

6. APPLICATION

We briefly illustrate our algorithm by applying it to a differential system arising in the computation of a closed form of an integral. There are several approaches to indefinite integrals of similar type which lead to related differential systems, for example see [13, 14] and references therein.

Example.

Consider the following indefinite integral involving the Legendre function for generic parameter n:

$$\int P_n(x) - x^{n-1}P_{n+1}(x)\,dx.$$

Assuming an antiderivative can be found in the differential field $\mathbb{Q}(n)(x, x^n, P_n(x), P_{n+1}(x))$, it can be proven to have the form $y_0P_n(x) + y_1P_{n+1}(x)$ with $y_0, y_1 \in \mathbb{Q}(n)(x, x^n)$, see [14]. Differentiating this and comparing coefficients of P_n and P_{n+1} yields the following differential system for y_0, y_1.

$$\delta\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \begin{pmatrix} \frac{(n+1)x}{1-x^2} & \frac{n+1}{1-x^2} \\ -\frac{n+1}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -x^{n-1} \end{pmatrix}$$

Let us write $\mathbf{y} = (y_0, y_1)^T$ and $C(x, t)$ with $C = \mathbb{Q}(n)$ and $\delta t = \frac{n}{x}t$ in the following. The inhomogeneous part of the system becomes $(1, -t/x)^T$ and we immediately obtain

$$b = 1$$

for the normal part of the denominator. Applying Theorem 2 to bound ν_t we choose $p = x$, for which $\omega_p = -1$, $a_0 = n$, $p_0 = 1$, and $\beta = 0$, and computing a 0-simple form of the above operator at p we see that already after normalizing it to have the form (5)

$$p^{-\omega_p}\delta\mathbf{y} + \begin{pmatrix} \frac{(n+1)x^2}{1-x^2} & \frac{(n+1)x}{1-x^2} \\ -\frac{(n+1)x}{1-x^2} & -\frac{(n+1)x^2}{1-x^2} \end{pmatrix}\mathbf{y}$$

is 0-simple with indicial polynomial $P_0(\mu) = \mu^2$. This yields $\Lambda = \{\lambda \in \mathbb{Z} \mid \exists \nu \in \mathbb{Z} : P_0(\nu + \lambda n) = 0\} = \{0\}$. From (2) we obtain $\nu_t(\mathbf{y}) \geq 0$, so the general bound is

$$\lambda_0 = \min(0, \Lambda) = 0.$$

Similarly we obtain a bound for $\nu_{1/t}$ by Theorem 2 again. In this example we could just reuse the Λ from before since the matrix is the same, but we want to vary and see the computation for $p = \frac{1}{x}$ this time. We have $\omega_p = 1$, $a_0 = n$, $p_0 = -1$, and $\beta = 0$ and after normalizing we obtain again the operator

$$p^{-\omega_p}\delta\mathbf{y} + \begin{pmatrix} \frac{(n+1)x^2}{1-x^2} & \frac{(n+1)x}{1-x^2} \\ -\frac{(n+1)x}{1-x^2} & -\frac{(n+1)x^2}{1-x^2} \end{pmatrix}\mathbf{y},$$

which is 0-simple at p and has indicial polynomial $P_0(\mu) = \mu^2 - (n+1)^2$, giving $\Lambda = \{-1, 1\}$. We obtain $\nu_{1/t}(\mathbf{y}) \geq -1$ from (2). This yields the general bound

$$\lambda_1 = \max(1, \Lambda) = 1.$$

Now we need to determine the coefficients in $\mathbf{y} = \mathbf{y}_0 + \mathbf{y}_1 t$. Let us start with $\mathbf{y}_1 \in C(x)^2$ satisfying

$$\delta\mathbf{y}_1 + \begin{pmatrix} \frac{n}{x} + \frac{(n+1)x}{1-x^2} & \frac{n+1}{1-x^2} \\ -\frac{n+1}{1-x^2} & \frac{n}{x} - \frac{(n+1)x}{1-x^2} \end{pmatrix}\mathbf{y}_1 = \begin{pmatrix} 0 \\ -1/x \end{pmatrix},$$

for which we, refraining from detailing the computation, get

$$\mathbf{y}_1 = (x/n, -1/n)^T$$

using the algorithm presented in [1] and likewise we compute

$$\mathbf{y}_0 = (-x/n, 1/n)^T.$$

Altogether, the original system above has $y_0 = \frac{x}{n}(x^n - 1)$ and $y_1 = -\frac{1}{n}(x^n - 1)$ as solution. Hence we computed a closed form of the integral

$$\int P_n(x) - x^{n-1}P_{n+1}(x)\,dx = \frac{x^n - 1}{n}(xP_n(x) - P_{n+1}(x)).$$

7. GENERALIZATION

In certain cases we can also successfully apply our algorithm to systems with coefficients $A \in K(t)^{n\times n}$. More specifically, the generalization presented in this section can deal with system matrices $A \in K(t)^{n\times n}$ subject to the following conditions.

1. If $\mu_0 = \nu_t(A) < 0$, then we require that the matrix $\pi_t(At^{-\mu_0}) \in K^{n\times n}$ is invertible.
2. If $\mu_1 = \nu_{1/t}(A) < 0$, then we require that the matrix $\pi_{1/t}(At^{\mu_1}) \in K^{n\times n}$ is invertible.

These conditions ensure that we can compute all solutions in $K(t)^n$. In particular, they ensure that the bounds λ_0 and λ_1 on the solutions in $K[t, t^{-1}]$ of the intermediate system (3) can be computed as explained in Section 7.2. We will discuss why these conditions are important in Section 8. If the matrix A violates these conditions, then we still could compute all solutions by reducing the problem to the scalar case and applying the results from [7].

7.1 Normal part of the denominator

The algorithm described in Section 4.1 can also be applied successfully to operators with coefficients from $K(t)$ at any irreducible $p \in K[t]$, provided $\omega_p < 0$, i.e., $t \nmid p$. Based on this we can prove the following theorem, which not only relies on (2), but also on the fact that for 0-simple operators (5) at such a p we have

$$\nu_p(Dp^{-1}\delta\mathbf{y}+N\mathbf{y})=\nu_p(\mathbf{y})$$

as long as $\nu_p(\mathbf{y})\pi_p(\delta p)$ is not a root of the indicial polynomial. By also allowing positive values of the bound λ_p for $\nu_p(\mathbf{y})$ in the computation below also parts of the numerator can be determined in some cases.

THEOREM 3. *Let $A\in K(t)^{n\times n}$, $\mathbf{f}_0,\dots,\mathbf{f}_m\in K(t)^n$ and $\delta t=at$ where $a\in K$ such that there are no $k\in\mathbb{Z}\setminus\{0\}$ and $g\in K^*$ with $a=\frac{\delta g}{kg}$. Then we can compute $b\in K(t)^*$ such that for all $\mathbf{y}\in K(t)^n$ with $L(\mathbf{y}):=\delta\mathbf{y}+A\mathbf{y}\in\operatorname{span}_C\{\mathbf{f}_0,\dots,\mathbf{f}_m\}$ we have*

$$b\mathbf{y}\in K[t,t^{-1}]^n.$$

PROOF. We consider all irreducible $p\in K[t]$ with $\omega_p<0$, i.e., $t\nmid p$ and $\omega_p=-1$, such that $\nu_p(A)<0$ or $\min_i\nu_p(\mathbf{f}_i)<-1$. These are finitely many and for each of them we compute transformation matrices $S_p,T_p\in K(t)^{n\times n}$ such that the operator $S_pL(T_p\mathbf{z})$ is 0-simple at p as well as the corresponding indicial polynomial $P_{p,0}(\mu)\in K[t]/(p)[\mu]$. From this we determine

$$\begin{aligned}n_p&:=\min(\nu_p(S_p\mathbf{f}_0),\dots,\nu_p(S_p\mathbf{f}_m)),\\ \mu_p&:=\min\{\mu\in\mathbb{Z}\mid P_{p,0}(\mu\pi_p(\delta p))=0\},\\ \lambda_p&:=\min(n_p,\mu_p)+\nu_p(T_p).\end{aligned}$$

Then we compute the following product over all p considered

$$b:=\prod_p p^{-\lambda_p}\in K(t)^*.$$

Now we fix $\mathbf{y}\in K(t)^n$ with $L(\mathbf{y})\in\operatorname{span}_C\{\mathbf{f}_0,\dots,\mathbf{f}_m\}$. For any irreducible $p\in K[t]$ with $t\nmid p$ we verify that $-\nu_p(b)\le\nu_p(\mathbf{y})$. First, assume p was considered in the computation above. So $\nu_p(S_pL(T_p\mathbf{z}))\ge n_p$ for $\mathbf{z}=T_p^{-1}\mathbf{y}$, which implies that either $P_{p,0}(\nu_p(\mathbf{z})\pi_p(\delta p))\neq0$ and $\nu_p(\mathbf{z})=\nu_p(S_pL(T_p\mathbf{z}))\ge n_p$ or $P_{p,0}(\nu_p(\mathbf{z})\pi_p(\delta p))=0$. Therefore we have $-\nu_p(b)=\lambda_p\le\nu_p(\mathbf{z})+\nu_p(T_p)\le\nu_p(\mathbf{y})$. Assuming p was not considered in the computation above instead, then in particular $\nu_p(A)\ge0$ and $\min_i\nu_p(\mathbf{f}_i)\ge-1$. If $\nu_p(\mathbf{y})\neq0$ then $\min_i\nu_p(\mathbf{f}_i)\le\nu_p(L(\mathbf{y}))=\nu_p(\delta\mathbf{y})=\nu_p(\mathbf{y})-1$ as in (2). Hence $-\nu_p(b)=0\le\nu_p(\mathbf{y})$. Altogether we obtain $b\mathbf{y}\in K[t,t^{-1}]^n$. □

7.2 Degree bounds

Now we need to compute degree bounds for the Laurent polynomial solutions $\tilde{\mathbf{y}}\in K[t,t^{-1}]^n$ of (3). The system matrix $\tilde{A}=A-\frac{\delta b}{b}I_n$ has coefficients from $K(t)$, but the contribution from $\frac{\delta b}{b}I_n$ has the property that $\nu_t(\frac{\delta b}{b})\ge0$ and $\nu_{1/t}(\frac{\delta b}{b})\ge0$. So $\tilde{A}$ also satisfies both conditions stated at the beginning of this section.

For computing the bounds on $\nu_p(\tilde{\mathbf{y}})$ for $p=t$ and $p=\frac{1}{t}$ we distinguish two cases each. If $\mu_p:=\nu_p(\tilde{A})<0$, then we have that $\pi_p(Ap^{-\mu_p})\in K^{n\times n}$ is invertible hence (2) is always true and reads $\nu_p(\tilde{\mathbf{y}})=\nu_p(\delta\tilde{\mathbf{y}}+\tilde{A}\tilde{\mathbf{y}})-\mu_p$. If $\mu_p\ge0$, then we have $\nu_p(\tilde{\mathbf{y}})\le\nu_p(\delta\tilde{\mathbf{y}}+\tilde{A}\tilde{\mathbf{y}})$ in general and strict inequality occurs iff $\pi_p((\delta\tilde{\mathbf{y}}+\tilde{A}\tilde{\mathbf{y}})p^{-\lambda})=0$ or equivalently

$$\delta(\mathbf{g}p^\lambda)+\pi_p(\tilde{A})\mathbf{g}p^\lambda=0$$

for $\lambda=\nu_p(\tilde{\mathbf{y}})$ and $\mathbf{g}=\pi_p(\tilde{\mathbf{y}}p^{-\lambda})$. To decide this we can rely on Theorem 2 again.

7.3 Computing the coefficients

After we determined the bounds λ_0 and λ_1 we need to compute the coefficients $\mathbf{y}_i\in C(x)^n$ and $c_i\in C$ such that $\tilde{\mathbf{y}}=\sum_{i=\lambda_0}^{\lambda_1}\mathbf{y}_it^i$ solves (3) as explained in Section 3. The conditions imposed on A above have the nice property that the systems arising from coefficient comparison of the powers of t are either of the shape (4) or pure algebraic equations with invertible matrix.

7.4 Examples

Again we illustrate the algorithm in this generalized setting along the computation of integrals involving Special Functions of similar type as before. The first example gives rise to a system for which $\nu_{1/t}(A)<0$ and illustrates how the coefficients of $\tilde{\mathbf{y}}\in K[t,t^{-1}]^n$ can be determined in this situation. The second example deals with matrices A and $\tilde{A}$ having entries from $K(t)$, which not all are from $K[t,t^{-1}]$, and we also have a closer look at what can happen to the inhomogeneous part during the solution process.

Example.

Consider the following indefinite integral involving the Bessel function for generic parameter n:

$$\int x(e^{2x}-n^2)J_n(e^x)\,dx.$$

Assuming an antiderivative can be found in the differential field $\mathbb{Q}(n)(x,e^x,J_n(e^x),J_{n+1}(e^x))$, it can be proven to have the form $y_0J_n(e^x)+y_1J_{n+1}(e^x)$ with $y_0,y_1\in\mathbb{Q}(n)(x,e^x)$. Differentiating this and comparing coefficients of J_n and J_{n+1} yields the following differential system for y_0,y_1.

$$\delta\begin{pmatrix}y_0\\y_1\end{pmatrix}+\begin{pmatrix}n&e^x\\-e^x&-(n+1)\end{pmatrix}\begin{pmatrix}y_0\\y_1\end{pmatrix}=\begin{pmatrix}x(e^{2x}-n^2)\\0\end{pmatrix}$$

Let us write $\mathbf{y}=(y_0,y_1)^T$ and $C(x,t)$ with $C=\mathbb{Q}(n)$ and $\delta t=t$ in the following. The inhomogeneous part of the system becomes $(x(t^2-n^2),0)^T$ and we obtain

$$b=1$$

for the normal part of the denominator as there is no polynomial to consider in Theorem 3. In order to bound $\nu_t(\mathbf{y})$ we are facing the standard situation $\nu_t(A)\ge0$ and we apply Theorem 2 to the matrix

$$\pi_t(A)=\begin{pmatrix}n&0\\0&-(n+1)\end{pmatrix}$$

with $p=\frac{1}{x}$, for which $\omega_p=1$, $a_0=1$, $p_0=-1$, and $\beta=1$. The normalized operator

$$p^{1-\omega_p}\delta\mathbf{y}+\begin{pmatrix}n&0\\0&-(n+1)\end{pmatrix}\mathbf{y}$$

is already 1-simple with characteristic polynomial $P_1(\mu)=(\mu+n)(\mu-(n+1))$. Hence $\tilde{\Lambda}=\emptyset$. The normalized operator

$$\begin{pmatrix}p&0\\0&p\end{pmatrix}p^{-\omega_p}\delta\mathbf{y}+\begin{pmatrix}n&0\\0&-(n+1)\end{pmatrix}\mathbf{y}$$

is already 0-simple with the indicial polynomial $P_0(\mu)=-n(n+1)$, hence $\Lambda=\emptyset$. So from (2) we obtain $\lambda_0=0$. For bounding $\nu_{1/t}(\mathbf{y})$ we realize that $\nu_{1/t}(A)=-1$ and

$$\pi_{1/t}(At^{-1})=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

is invertible, so (2) yields the general bound $\nu_{1/t}(\mathbf{y}) \geq -1$, i.e., $\lambda_1 = 1$. Now we show how to compute the coefficients $\mathbf{y}_i \in C(x)^2$ of $\mathbf{y} = \mathbf{y}_1 t + \mathbf{y}_0$ when $\nu_{1/t}(A) < 0$. Since $\pi_{1/t}(At^{-1})$ is invertible, for each coefficient $\mathbf{y}_i$ it suffices to solve a linear system with this matrix. These are obtained by comparing coefficients of $t^{\lambda_1-\nu_{1/t}(A)}$ down to $t^{\lambda_0-\nu_{1/t}(A)}$. More explicitly, after we insert the ansatz for $\mathbf{y}$ into the differential system multiplying by t^{-2} and applying $\pi_{1/t}$ yields

$$\pi_{1/t}(At^{-1})\mathbf{y}_1 = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

hence $\mathbf{y}_1 = (0, x)^T$. After we plug this in we compare coefficients of t^1 in the system to obtain

$$\pi_{1/t}(At^{-1})\mathbf{y}_0 = \begin{pmatrix} 0 \\ nx-1 \end{pmatrix}.$$

After plugging the solution $\mathbf{y}_0 = (-nx+1, 0)^T$ in the differential system the inhomogeneous part vanishes, so the $\mathbf{y} = \mathbf{y}_1 t + \mathbf{y}_0$ we computed really is a solution. Summarizing, the original system above has the only solution $y_0 = -nx+1$ and $y_1 = xe^x$ in $\mathbb{Q}(n)(x, e^x)$. Thus we computed the following closed form of the integral

$$\int x(e^{2x} - n^2)J_n(e^x)\,dx = (-nx+1)J_n(e^x) + xe^x J_{n+1}(e^x).$$

Example.

Consider the following indefinite integral involving the Legendre function for generic parameter n:

$$\int \frac{x}{\cosh(x)^2} P_n(\tanh(x))\,dx.$$

Following the same structural principles as in the previous examples we are looking for an antiderivative of the form $y_0 P_n(\tanh(x)) + y_1 P_{n+1}(\tanh(x))$ with $y_0, y_1 \in \mathbb{Q}(n)(x, e^x)$. We obtain the following differential system for y_0, y_1.

$$\delta \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \begin{pmatrix} (n+1)\tanh(x) & n+1 \\ -(n+1) & -(n+1)\tanh(x) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{x}{\cosh(x)^2} \\ 0 \end{pmatrix}$$

Again let $C(x, t)$ with $C = \mathbb{Q}(n)$ and $\delta t = t$, then we can represent $\tanh(x)$ by $\frac{t^2-1}{t^2+1}$ and $\cosh(x)$ by $\frac{t^2+1}{2t}$. The matrix has $\nu_t(A) = \nu_{1/t}(A) = 0$, so the conditions are trivially satisfied. For determining the normal part of the denominator following the proof of Theorem 3 we need to consider the polynomial $p = t^2 + 1$ only. Bringing the system into the form (5) it becomes 0-simple at p with indicial polynomial $P_{p,0}(\mu) = \mu^2 - 4(n+1)^2$ and right hand side $(\frac{4xt^2}{t^2+1}, 0)^T$. From this we obtain $n_p = -1$ and $\mu_p = \infty$ since there is no $\mu \in \mathbb{Z}$ such that $P_{p,0}(-2\mu) = 0$. So we get $\lambda_p = -1$ and

$$b = t^2 + 1.$$

Then the system (3) satisfied by $\tilde{\mathbf{y}} = (t^2+1)(y_0, y_1)^T \in K[t, t^{-1}]^2$ has the matrix

$$\tilde{A} = \begin{pmatrix} n-1-\frac{2n}{t^2+1} & n+1 \\ -(n+1) & -(n+3)+\frac{2n+4}{t^2+1} \end{pmatrix}.$$

Relying on (2) only for computing the degree bounds we obtain $\lambda_0 = 2$ and $\lambda_1 = 0$. Normally $\lambda_0 > \lambda_1$ implies that there is no non-zero solution of the system. But we still have to update these values by applying Theorem 2 to $\nu_t(\tilde{A})$ as well as $\nu_{1/t}(\tilde{A})$, which gives the corrected bounds $\lambda_0 = 0$ and $\lambda_1 = 2$. So we have $\tilde{\mathbf{y}} = \mathbf{y}_2 t^2 + \mathbf{y}_1 t + \mathbf{y}_0$ and proceed with comparing coefficients of t^2 then t^1 and t^0 in the system. First we multiply by t^{-2} and apply $\pi_{1/t}$ to obtain

$$\delta \mathbf{y}_2 + \begin{pmatrix} n+1 & n+1 \\ -(n+1) & -(n+1) \end{pmatrix} \mathbf{y}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solutions $\mathbf{y}_2 = c_1(1, -1)^T + c_2(x, -x - \frac{1}{n+1})^T \in C(x)^2$ for arbitrary $c_1, c_2 \in C$ as determined by the algorithm given in [1]. Plugging this in we obtain the new right hand side of the system with increased m:

$$\begin{pmatrix} \frac{4xt^2}{t^2+1} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} \frac{2nt^2}{t^2+1} \\ \frac{(2n+4)t^2}{t^2+1} \end{pmatrix} + c_2 \begin{pmatrix} \frac{2nxt^2}{t^2+1} \\ (x + \frac{1}{n+1})\frac{(2n+4)t^2}{t^2+1} \end{pmatrix}.$$

We proceed to comparing coefficients of t^1 yielding

$$\delta \mathbf{y}_1 + \begin{pmatrix} n & n+1 \\ -(n+1) & -(n+2) \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which only has the solution $\mathbf{y}_1 = (0, 0)^T \in C(x)^2$ for any $c_1, c_2 \in C$. Comparing coefficients of t^0 now yields

$$\delta \mathbf{y}_0 + \begin{pmatrix} n-1 & n+1 \\ -(n+1) & -(n+3) \end{pmatrix} \mathbf{y}_0 = \begin{pmatrix} 4x \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 2n \\ 2n+4 \end{pmatrix} + c_2 \begin{pmatrix} 2nx \\ (2n+4)(x + \frac{1}{n+1}) \end{pmatrix}.$$

In this case, for all $c_1, c_2 \in C$ there is exactly one solution $\mathbf{y}_0 \in C(x)^2$, depending linearly on c_1, c_2:

$$\mathbf{y}_0 = \begin{pmatrix} -(n+3)x - n - 2 \\ (n+1)(x+1) \end{pmatrix} + c_1 \begin{pmatrix} -n^2 - 3n - 1 \\ n^2 + n - 1 \end{pmatrix} + c_2 \begin{pmatrix} -(n^2+3n+1)x - (n+1)(n+2) \\ (n^2+n-1)x + \frac{n^3+3n^2+2n-1}{n+1} \end{pmatrix}.$$

After plugging this back in some inhomogeneous part remains, which has to be zero:

$$\begin{pmatrix} \frac{2(n+2)((n+1)x+n)}{t^2+1} \\ \frac{2(n+2)(n+1)(x+1)}{t^2+1} \end{pmatrix} + c_1 \begin{pmatrix} \frac{2n(n+1)(n+2)}{t^2+1} \\ \frac{2n(n+1)(n+2)}{t^2+1} \end{pmatrix} + c_2 \begin{pmatrix} \frac{2n(n+1)(n+2)(x+1)}{t^2+1} \\ \frac{2n(n+2)((n+1)x+n+2)}{t^2+1} \end{pmatrix}.$$

Thereby we obtain $c_1 = \frac{1}{n(n+1)}$ and $c_2 = -\frac{1}{n}$. Altogether, rewriting the result in terms of hyperbolic functions the original system has $y_0 = \frac{1}{n}(\frac{1}{n+1} - x\tanh(x))$ and $y_1 = \frac{x}{n}$ as solution, which yields the following closed form of the integral

$$\int \frac{x}{\cosh(x)^2} P_n(\tanh(x))\,dx = \frac{1}{n}(\frac{1}{n+1} - x\tanh(x))P_n(\tanh(x)) + \frac{x}{n}P_{n+1}(\tanh(x)).$$

8. DISCUSSION AND FUTURE WORK

While for scalar ODEs with coefficients in $C(x, t)$ there are known algorithms to compute the bounds on solutions in $C(x)[t, t^{-1}]$, the corresponding situation for differential systems is not so clear if one wishes to avoid any reduction to the scalar case. We presented an algorithm that works in many cases but appropriate transformations that enable the computation of degree bounds in all cases have not been found yet. The standard approach to compute a 0-simple form at $p = t$ fails in several ways, which we briefly discuss.

On the one hand the algorithm presented in Section 4.1 does not terminate for all inputs for $p = t$. This is due to the interference of the term $SD(\delta T)$ in the transformation done in Case 3, which in general prevents the necessary zeros from being introduced in the $(q+1)$-th row. On the other hand, even if we always were able to compute a 0-simple form at $p = t$, then the indicial polynomial still would not provide the information needed since in general the derivative of the coefficients in $C(x)$ of the solutions interfere. We illustrate this with the following differential system for $t = e^x$

$$\delta \mathbf{y} + \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1/x - x & 0 & -1 \\ -1 & 0 & 1/x - x & 1 \\ t^{-1} & xt^{-1} & xt^{-1} & 0 \end{pmatrix} \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (x-2)t^2 \end{pmatrix},$$

which has the solutions

$$\mathbf{y} = \begin{pmatrix} 0 \\ (1/x - 2)t^2/4 \\ (2 - 1/x)t^2/4 \\ (2x - 5)t^2/4 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1/x \\ -1/x \\ -1 \end{pmatrix}$$

in $C(x,t)^4$. After multiplying the last row of the operator by t we could think of applying the algorithm from Section 4.1, but this would merely result in constantly increasing $m \in \mathbb{N}$ in the operator $D\delta\mathbf{y} + N\mathbf{y}$ given by

$$D = \begin{pmatrix} 1 & 0 & 0 & mxt \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1/x - x & 0 & -1 \\ -1 & mx & 1/x + (m-1)x & 1 \\ 1 & (1-m)x & (1-m)x & 0 \end{pmatrix},$$

which is not 0-simple at $p = t$ for any m. Alternatively we could think of applying the transformation given by

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & (x - 1/x)t & 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & -t^{-1} & 1 & 0 \\ 0 & (1/x - x)t^{-1} & 0 & t^{-1} \end{pmatrix}$$

instead, which gives the following 0-simple operator at $p = t$

$$\delta \mathbf{z} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1/x - x & 0 \\ 1 & -(1 + 1/x^2) & x & 1/x - 1 - x \end{pmatrix} \mathbf{z}.$$

Interpreting the corresponding indicial polynomial

$$P_0(\mu) = (\mu - x)(\mu + \tfrac{1}{x})(\mu^2 - (x + 2 - \tfrac{1}{x})\mu + x - \tfrac{x+1}{x^2})$$

in the usual way wrongly suggests that the homogeneous equation does not have a solution of the form $\mathbf{g}t^\lambda$. But the contrary is true as

$$\mathbf{z} = (0, t/x, 0, -t/x^2)^T$$

shows. This operator happens to be nice enough so that we can apply Theorem 2, since the matrix D multiplied to $\delta\mathbf{z}$ does not involve t. But in general this cannot be expected.

So it will be necessary to come up with new ways of constructing transformations of the operators in order to obtain the information needed. We already have first ideas for progress towards a general algorithm for determining these bounds, which works on both generators x and t at the same time. But the interplay between these two is rather tricky and subtle for the design of transformation matrices, so it will be a challenging task. Apart from generalizing this work we still need to implement it in software as well.

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