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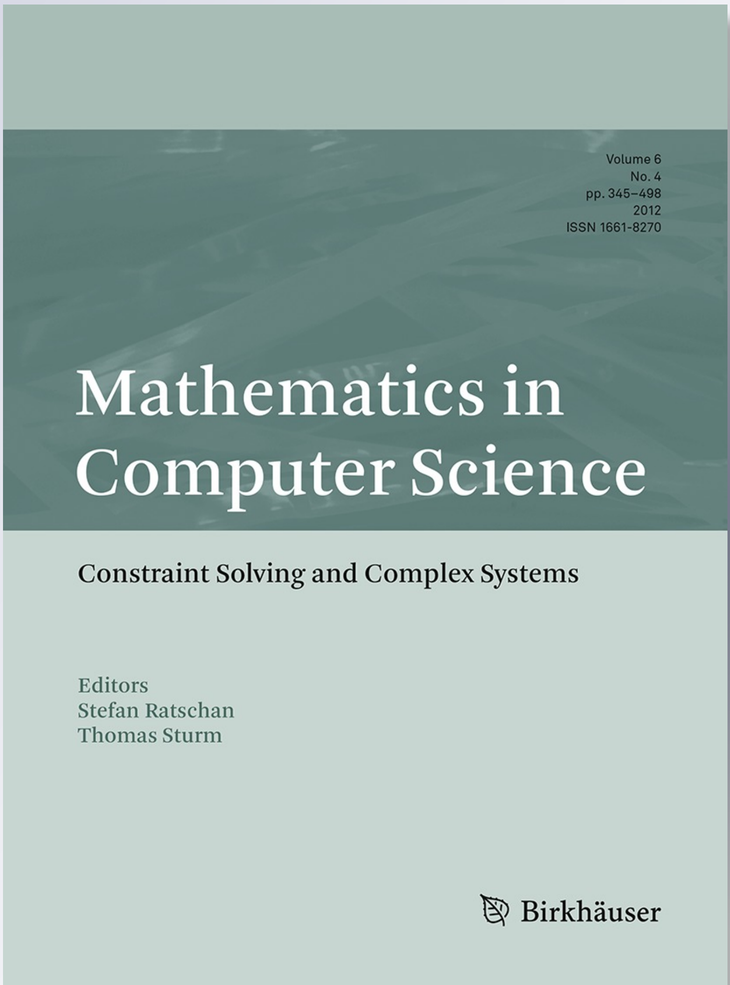

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Rational General Solutions of Trivariate Rational Differential Systems

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Abstract We generalize the method of Ngô and Winkler (J Symbolic Comput 46:1173–1186, 2011) for finding rational general solutions of a plane rational differential system to the case of a trivariate rational differential system. We give necessary and sufficient conditions for the trivariate rational differential system to have a rational solution based on proper reparametrization of invariant algebraic space curves. In fact, the problem for computing a rational solution of the trivariate rational differential system can be reduced to finding a linear rational solution of an autonomous differential equation. We prove that the linear rational solvability of this autonomous differential equation does not depend on the choice of proper parametrizations of invariant algebraic space curves. In addition, two different rational solutions corresponding to the same invariant algebraic space curve are related by a shifting of the variable. We also present a criterion for a rational solution to be a rational general solution.

Keywords Rational general solution · Invariant algebraic space curve · Proper parametrization

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1 Introduction

In this paper, we are interested in looking for rational general solutions of trivariate rational systems of autonomous ordinary differential equations (ODEs) of the form

$$s_i' = \frac{U_i(s_1, s_2, s_3)}{V_i(s_1, s_2, s_3)}, \quad i = 1, 2, 3, \tag{1.1}$$

where $U_i, V_i \in \mathbb{K}[s_1, s_2, s_3]$, $\mathbb{K}$ an algebraically closed field of characteristic zero. For simplicity, the system (1.1) is also called trivariate rational differential system.

Hubert [7] gave a method for finding the general solution of a first order non-autonomous algebraic ODE (AODE) by computing a Gröbner basis of the prime differential ideal defining the general component. Feng and Gao [4,5] presented a method for explicitly computing the rational general solution of a first order autonomous AODE based on a rational parametrization of its corresponding algebraic curve. The degree bounds for curve parametrization derived by Sendra and Winkler [16] make this method completely algorithmic. Subsequently, the rational general solution of a first order non-autonomous AODE was studied by Ngô and Winkler [12,13] based on a proper parametrization of its correponding algebraic surface. In fact, Ngô and Winkler reduced the problem for computing rational general solution of the first order non-autonomous AODE to finding rational general solution of its associated differential system (i.e., a planar rational system of autonomous ODEs). The technique of reparametrization of invariant algebraic curve is used for solving the planar rational differential system. Their method is algorithmic in the non-dicritical case by using the degree bound in [3]. In this paper, we generalize Ngô and Winkler's method in [13] for solving planar rational differential system to the case of trivariate rational differential systems.

A rational solution of the system (1.1) is a 3-tuple of rational functions which satisfies the given system. A solution $(s_1(x), s_2(x), s_3(x))$ is *trivial* if all the $s_i(x)$ are constant. Each non-trivial rational solution of (1.1) represents a rational algebraic space curve for which the rational solution itself is a rational parametrization. Geometrically, such a rational algebraic space curve can be implicitly defined by the intersection of at least 2 algebraic surfaces. Therefore, it is possible to compute a non-trivial rational solution of (1.1) by finding the implicit rational algebraic space curve (i.e. the implicit defining equations of those algebraic surfaces such that this space curve is determined by their intersection) of the possible non-trivial rational solutions first, and then choosing suitable parametrizations of the space curve which satisfy the system (1.1). Because trivial rational solutions are easy to be computed, we restrict our attention to the computation of non-trivial rational solution. From now on, we simply write rational solution for it if no confusion can arise.

Our main idea is to generalize the notion of invariant algebraic curve in [13] for planar rational systems to trivariate rational systems. In fact, we define an invariant algebraic space curve in the trivariate case by using the saturation ideal of a regular chain. Rational solutions of the system (1.1) can be found by the technique of reparametrization of the invariant algebraic space curve. In addition, we give a criterion for deciding when a rational solution would be a rational general solution. Note that only irreducible space curves can be parametrizable. So we only consider irreducible invariant algebraic space curves.

The rest of this paper is organized as follows. In the next section, we introduce some known concepts and results related to differential polynomials and rational general solutions. In Sect. 3, we give the definition of invariant algebraic space curves and explain how to compute them by using undetermined coefficient methods. Section 4 is devoting to presenting our method for computing rational solution of the trivariate rational system of autonomous ODEs. In Sect. 5, the linear rational solvability of an autonomous differential equation and the relationship between two different rational solutions corresponding to the same invariant algebraic space curve are studied. In Sect. 6, we first give some properties of rational general solution of the trivariate differential system, and then present a criterion for a rational solution to be a rational general solution. The paper is concluded with some open problems in Sect. 7.

2 Preliminaries

In this section, we recall some well known concepts and results in differential algebra. More details can be found in [8,12,14].

Let $\mathcal{R}$ be a commutative ring. A map $\delta : \mathcal{R} \to \mathcal{R}$ is said to be a *derivation* on $\mathcal{R}$ if

$$\delta(a+b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b) \quad \text{for all } a, b \in \mathcal{R}.$$

Then $(\mathcal{R}, \delta)$ is called a *differential ring*. Moreover, if $\mathcal{R}$ is a field, $(\mathcal{R}, \delta)$ is called a *differential field*. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The *differential polynomial ring* over $\mathbb{K}$ is the ring consisting of all polynomials in differential indeterminates s_i $(1 \le i \le n)$ and all their derivatives, denoted by $\mathbb{K}\{s_1, \ldots, s_n\}$. The j-th derivative of s_i is denoted by s_{ij}; for s_i we also write s_{i0}. Note that $\mathbb{K}[s_1, \ldots, s_n]$ is the usual polynomial ring.

Here we will consider polynomial rings and differential polynomial rings in 3 indeterminates s_1, s_2, s_3. Let $F \in \mathbb{K}\{s_1, s_2, s_3\}$ be a differential polynomial. The ith derivative of F is denoted by $F^{(i)}$. For $F^{(1)}$ we also write F'. The *order* of F with respect to s_i is the greatest j such that s_{ij} occurs in F, denoted by $\mathrm{ord}_{s_i}(F)$. For convention we define $\mathrm{ord}_{s_i}(F) = -1$ if F does not involve any s_{ij}.

Definition 2.1 Let $F, G \in \mathbb{K}\{s_1, s_2, s_3\}$. Suppose that the indeterminate s_i appears effectively in both of them, where $1 \le i \le 3$. F is said to be *of higher rank than* G (or G *of lower rank than* F) in s_i if one of the following conditions holds:

(a) $\mathrm{ord}_{s_i}(F) > \mathrm{ord}_{s_i}(G)$;
(b) $\mathrm{ord}_{s_i}(F) = \mathrm{ord}_{s_i}(G) = j$ and $\deg_{s_{ij}}(F) > \deg_{s_{ij}}(G)$.

Definition 2.2 Let $A = \{s_{ik} \mid i = 1, \ldots, 3, k \in \mathbb{N}\}$. The *ord-lex ranking* on A is the total order defined as follows: $s_{ik} < s_{jl}$ iff $k < l$ or $k = l$ and $i < j$.

The ord-lex ranking defined in Definition 2.2 is considered as the ranking in the differential ring $\mathbb{K}\{s_1, s_2, s_3\}$ throughout this paper. Note that the ord-lex ranking is an orderly ranking. For any $F \in \mathbb{K}\{s_1, s_2, s_3\} \backslash \mathbb{K}$, the greatest derivative occurring in F with respect to the ord-lex ranking is called the *leader* of F. The leading coefficient with respect to the leader of F is called the *initial* of F, the partial derivative with respect to the leader of F is called the *separant* of F.

Definition 2.3 Let F and G be two differential polynomials in $\mathbb{K}\{s_1, s_2, s_3\}$. Then G is *reduced with respect to* F iff G is of lower rank than F in the indeterminate defining the leader of F.

Let $\mathbb{A} \subset \mathbb{K}\{s_1, s_2, s_3\}$. The differential polynomial set $\mathbb{A}$ is called *autoreduced* iff no elements of $\mathbb{A}$ belongs to $\mathbb{K}$ and each element of $\mathbb{A}$ is reduced with respect to all the others.

Consider the differential polynomial set $\mathbb{A} = \{A_1, A_2, A_3\} \subset \mathbb{K}\{s_1, s_2, s_3\}$, where $A_i = V_i s_i' - U_i$, and U_i, V_i come from the system (1.1). It is easy to see that the initial and separant of A_i are the same. Observe that $\mathbb{A}$ is an autoreduced set relative to the ord-lex ranking

$$s_1 < s_2 < s_3 < \cdots < s_{1k} < s_{2k} < s_{3k} < \cdots, \quad k \in \mathbb{N},$$

which is given by Definition 2.2. According to Proposition 1 in [8, chap I, sect. 9], we have the following conclusion.

Proposition 2.4 *Let* $\mathbb{A} = \{A_1, A_2, A_3\}$*, where* $A_i = V_i s_i' - U_i$ *for* $1 \le i \le 3$*. For any differential polynomial* $G \in \mathbb{K}\{s_1, s_2, s_3\}$*, there exists the following representation by consecutive reductions with respect to the autoreduced set* $\mathbb{A}$

$$V_1^{l_1} \cdots V_3^{l_3} G = \sum_{j \ge 0} Q_{1j} A_1^{(j)} + \sum_{j \ge 0} Q_{2j} A_2^{(j)} + \sum_{j \ge 0} Q_{3j} A_3^{(j)} + R,$$

where $l_i \in \mathbb{N}$, $A_i^{(j)}$ *is the* jth *derivatives of* A_i*, and the leader of* G *is of at least as high rank as the leader of* $A_i^{(j)}$, $Q_{ij} \in \mathbb{K}\{s_1, s_2, s_3\}$, $i = 1, 2, 3$, *and* R *is reduced with respect to* $\mathbb{A}$*. Here,* R *is called the differential pseudo remainder of* G *with respect to* $\mathbb{A}$*, denoted by* $\mathrm{prem}(G, \mathbb{A})$.

Remark 2.5 As the order of A_i is 1 and the degree of A_i with respect to s_i' is 1, it follows that $\mathrm{prem}(G, \mathbb{A})$ is always a polynomial in $\mathbb{K}[s_1, s_2, s_3]$ for any differential polynomial $G \in \mathbb{K}\{s_1, s_2, s_3\}$.

Note that

$$[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{G \in \mathbb{K}\{s_1, s_2, s_3\} \mid \left(\prod_{i=1}^{3} V_i\right)^q G \in [\mathbb{A}] \text{ for some } q \in \mathbb{N}\}$$

is a prime differential ideal. From this, it can be easily prove that

$$[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{\mathbb{A}\} : S_{\mathbb{A}}^{\infty}.$$

According to Proposition 2.1 in [6], we have the following decomposition

$$\{\mathbb{A}\} = ([\mathbb{A}] : S_{\mathbb{A}}^{\infty}) \bigcap \left(\bigcap_{i=1}^{3} \{\mathbb{A}, V_i\}\right),$$

where $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ and $\bigcap_{i=1}^{3}\{\mathbb{A}, V_i\}$ define the *general component* and the *singular component* of $\mathbb{A}$, respectively. Observe that every solution of the system (1.1) is a zero of $\mathbb{A}$ for which none of the V_i's vanish, i.e, it is a zero of $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$. Therefore, a general solution of the system (1.1) can be defined as follows.

Definition 2.6 A generic zero of the prime differential ideal $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ is said to be a *general solution* of the system (1.1). A general solution $(s_1(x), s_2(x), s_3(x))$ of the system (1.1) is *rational* iff all $s_i(x)$ are rational functions in x.

As well known, the notion of general solution of a single ODE is defined by Ritt [14] as a generic zero of a prime differential ideal. Definition 2.6 generalizes it to the case of differential systems. Some properties will be given in Sect. 6.

3 Invariant Algebraic Space Curves

In this section, we define invariant algebraic space curves and explain how to compute them by using undetermined coefficients method.

For convenience, we define the differential operator

$$\mathcal{D} = \sum_{j=1}^{3} U_j W_j \frac{\partial}{\partial s_j},$$

where $W_j = \frac{\operatorname{lcm}(V_1, V_2, V_3)}{V_j}$, U_j and V_j are the numerator and denominator of the right side of the system (1.1), respectively. Hence, for any $H \in \mathbb{K}[s_1, s_2, s_3]$, $\mathcal{D}(H) = \sum_{j=1}^{3} U_j W_j H_{s_j}$, where H_{s_j} is the partial derivative of H w.r.t. s_j. Let $\mathbb{T}$ be any triangular set in $\mathbb{K}[s_1, s_2, s_3]$, the *saturation* of $\mathbb{T}$ is the ideal

$$\operatorname{sat}(\mathbb{T}) := \{G \in \mathbb{K}[s_1, s_2, s_3] \mid H^q G \in \langle \mathbb{T} \rangle \text{ for some } q \in \mathbb{N}\},$$

where H is the product of initials of all polynomials in $\mathbb{T}$. Note that we use the algebraic variable ordering $s_1 < s_2 < s_3$ in the polynomial ring $\mathbb{K}[s_1, s_2, s_3]$.

Let $(s_1(x), s_2(x), s_3(x))$ be a rational solution of the system (1.1). This solution defines a parametric space curve $\mathcal{C}$. Let

$$\mathcal{I}_{\mathcal{C}} = \{G \in \mathbb{K}[s_1, s_2, s_3] \mid G(s_1(x), s_2(x), s_3(x)) = 0\}$$

the implicit ideal determined by $\mathcal{C}$. As $\mathcal{I}_{\mathcal{C}}$ has a generic zero $(s_1(x), s_2(x), s_3(x))$ which only depends on one parameter, it is a 1-dimensional prime ideal. For every prime ideal $\mathcal{I} \in \mathbb{K}[s_1, s_2, s_3]$, there exists an irreducible regular chain $\mathbb{T}$, such that $\mathcal{I} = \operatorname{sat}(\mathbb{T})$, and $\dim(\mathcal{I}) = n - |\mathbb{T}|$, where $|\mathbb{T}|$ is the number of polynomials in $\mathbb{T}$. These facts can be found in [2, Theorem 3.3], [9, Theorem 3.1] and [10, Proposition 3.4]. Therefore, there exists an irreducible regular chain $\{H_1, H_2\}$, such that $\mathcal{I}_{\mathcal{C}} = \operatorname{sat}(H_1, H_2)$. Here and subsequently, we assume w.l.o.g that $H_1 \in \mathbb{K}[s_1, s_2]$, $H_2 \in \mathbb{K}[s_1, s_2, s_3]$.

Lemma 3.1 *Let $(s_1(x), s_2(x), s_3(x))$ be a rational solution of the system (1.1) and $\{H_1(s_1, s_2), H_2(s_1, s_2, s_3)\}$ the irreducible regular chain such that $\mathcal{I}_\mathcal{C} = \mathrm{sat}(H_1, H_2)$, where $\mathcal{I}_\mathcal{C}$ is the implicit ideal determined by the parametric space curve $\mathcal{C} = (s_1(x), s_2(x), s_3(x))$. Then*

$$\mathcal{D}(H_1) \in \mathrm{sat}(H_1, H_2), \quad \mathcal{D}(H_2) \in \mathrm{sat}(H_1, H_2).$$

Proof By the assumption, we have $H_i \in \langle H_1, H_2\rangle \subseteq \mathrm{sat}(H_1, H_2) = \mathcal{I}_\mathcal{C}$. Hence,

$$H_i(s_1(x), s_2(x), s_3(x)) = 0, \quad i = 1, 2.$$

By differentiating H_i w.r.t. x, we have

$$\sum_{j=1}^{3} H_{is_j}(s_1(x), s_2(x), s_3(x)) \cdot s_j'(x) = 0, \quad i = 1, 2.$$

It follows that

$$\sum_{j=1}^{3} H_{is_j}(s_1(x), s_2(x), s_3(x)) \cdot \frac{U_j(s_1(x), s_2(x), s_3(x))}{V_j(s_1(x), s_2(x), s_3(x))} = 0, \quad i = 1, 2.$$

By clearing the common denominator, we have

$$\mathcal{D}(H_i)(s_1(x), s_2(x), s_3(x)) = 0, \quad i = 1, 2,$$

i.e. $\mathcal{D}(H_i) \in \mathrm{sat}(H_1, H_2)$ for $i = 1, 2$. □

Definition 3.2 Let $H_1 \in \mathbb{K}[s_1, s_2]$, $H_2 \in \mathbb{K}[s_1, s_2, s_3]$. If $\mathcal{D}(H_1), \mathcal{D}(H_2) \in \mathrm{sat}(H_1, H_2)$, and $\{H_1, H_2\}$ is an irreducible regular chain, then $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ is an *invariant algebraic space curve* of the system (1.1).

If we choose an upper bound for the degrees of H_1 and H_2, we can make an ansatz for the undetermined coefficients, and determine H_i by solving the corresponding algebraic equations in these coefficients. In fact, one of the computing ways is to use Gröbner bases theory. Assume that the degrees of H_1 and H_2 are given, the system of equations on the coefficients of H_i can be obtained by equating the normal form of $\mathcal{D}(H_i)$ modulo $\mathbb{G}$ (denoted by $\mathrm{nform}(\mathcal{D}(H_i), \mathbb{G})$) to zero, where $\mathbb{G}$ is the Gröbner basis of the saturation ideal $\mathrm{sat}(H_1, H_2)$. By solving the obtained system, we can get the defining polynomials of two surfaces $H_1 = 0$ and $H_2 = 0$. Then we decide whether $\{H_1, H_2\}$ is irreducible regular chain, in the affirmative case, we can get the implicit representation of the invariant algebraic space curve by computing $\mathrm{sat}(H_1, H_2)$. Note that $\mathrm{sat}(\mathbb{T}) = \langle\mathbb{T}\rangle$ is not a rare special case. As shown in [11], $\mathrm{sat}(\mathbb{T}) = \langle\mathbb{T}\rangle$ if and only if $\mathbb{T}$ is primitive for the given regular chain $\mathbb{T}$. Primitivity can be decided algorithmically, and experimental results show that it occurs quite frequently in practice.

Example Consider the trivariate polynomial system of autonomous ODEs

$$s_1' = s_1 s_3 - s_2, \quad s_2' = 2s_1^2 - s_1 s_2, \quad s_3' = s_1^2. \tag{3.1}$$

First, we look for an invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1(s_1, s_2), H_2(s_1, s_2, s_3)))$ satisfying $\deg(H_1) = \deg(H_2) = 1$. We w.l.o.g. assume that the regular chain $\{H_1, H_2\} = \{s_2 + c_1 s_1 + c_2, s_3 + c_3 s_2 + c_4 s_1 + c_5\}$. Then the Gröbner basis of $\mathrm{sat}(H_1, H_2)$ w.r.t. the lexicographic order determined by $s_1 < s_2 < s_3$ is

$$\mathbb{G} = \{s_2 + c_1 s_1 + c_2, s_3 + (c_4 - c_3 c_1)s_1 + c_5 - c_3 c_2\},$$

and

$$\mathcal{D}(H_1) = c_1 s_3 s_1 - s_2 s_1 - c_1 s_2 + 2s_1^2, \quad \mathcal{D}(H_2) = c_4 s_3 s_1 - c_3 s_2 s_1 - c_4 s_2 + (2c_3 + 1)s_1^2.$$

It follows that

$$\begin{aligned}
\mathrm{nform}(\mathcal{D}(H_1), \mathbb{G}) &= (c_1 - c_1 c_4 + c_1^2 c_3 + 2)s_1^2 + (c_1^2 + c_2 - c_1 c_5 + c_1 c_2 c_3)s_1 + c_1 c_2,\\
\mathrm{nform}(\mathcal{D}(H_2), \mathbb{G}) &= (c_1 c_3 - c_4^2 + c_1 c_3 c_4 + 2c_3 + 1)s_1^2 + (c_1 c_4 + c_2 c_3 - c_4 c_5 + c_2 c_3 c_4)s_1 + c_2 c_4.
\end{aligned}$$

Note that $\mathrm{nform}(\mathcal{D}(H_i), \mathbb{G})$ is the normal form of $\mathcal{D}(H_i)$ modulo $\mathbb{G}$ as indicated before this example. Therefore, the algebraic system of equations on the coefficients of H_1 and H_2 is

$$\begin{cases} c_1 - c_1c_4 + c_1^2c_3 + 2 = 0 \\ c_1^2 + c_2 - c_1c_5 + c_1c_2c_3 = 0 \\ c_1c_2 = 0 \\ c_1c_3 - c_4^2 + c_1c_3c_4 + 2c_3 + 1 = 0 \\ c_1c_4 + c_2c_3 - c_4c_5 + c_2c_3c_4 = 0 \\ c_2c_4 = 0. \end{cases}$$

By solving this system, we obtain the solution

$$\{c_1 = -1, c_2 = 0, c_3 = -1 - c_4, c_4 = c_4, c_5 = -1\}.$$

This gives an invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$, where

$$H_1 = s_2 - s_1, \quad H_2 = s_3 - (1 + c_4)s_2 + c_4s_1 - 1.$$

Now we ask for an invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(\tilde{H}_1(s_1, s_2), \tilde{H}_2(s_1, s_2, s_3)))$ such that $\deg(\tilde{H}_1) = 2$, $\deg(\tilde{H}_2) = 1$. Assume that

$$\tilde{H}_1 = s_2 + c_1s_1^2 + c_2s_1 + c_3, \quad \tilde{H}_2 = s_3 + c_4s_2 + c_5s_1 + c_6,$$

then the following solutions are computed by using the same procedure as above

$$\{c_1 = 0, c_2 = -1, c_3 = 0, c_4 = -1 - c_5, c_5 = c_5, c_6 = -1\},$$
$$\{c_1 = 3/2, c_2 = -4, c_3 = 0, c_4 = 0, c_5 = 2, c_6 = -4\}.$$

The first solution corresponds to the computed invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$, where $\deg(H_1) = \deg(H_2) = 1$. For the second solution, it determines another invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(\tilde{H}_1, \tilde{H}_2))$, where

$$\tilde{H}_1 = s_2 + \frac{3}{2}s_1^2 - 4s_1, \quad \tilde{H}_2 = s_3 + 2s_1 - 4.$$

In fact, $\mathbf{Z}(\mathrm{sat}(\tilde{H}_1, \tilde{H}_2))$ is the only invariant algebraic space curve satisfying $\deg(\tilde{H}_1) = 2$ and $\deg(\tilde{H}_2) = 1$ in the above example, because the other possible cases of $\tilde{H}_1$ of degree 2 lead to algebraic systems of equations having no solution. Observe that H_i and $\tilde{H}_i$ in the above example are monic polynomials, $\{H_1, H_2\}$ and $\{\tilde{H}_1, \tilde{H}_2\}$ are irreducible regular chain. Furthermore, $\mathrm{sat}(H_1, H_2) = \langle H_1, H_2\rangle$ and $\mathrm{sat}(\tilde{H}_1, \tilde{H}_2) = \langle \tilde{H}_1, \tilde{H}_2\rangle$. It follows that $\mathbf{Z}(\mathrm{sat}(H_1, H_2)) = \mathbf{Z}(H_1, H_2)$ and $\mathbf{Z}(\mathrm{sat}(\tilde{H}_1, \tilde{H}_2)) = \mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$. Therefore, one of the computed invariant algebraic space curves is a space line determined by the intersection of surfaces $H_1 = 0$ and $H_2 = 0$, the other one is a space conic determined by the intersection of $\tilde{H}_1 = 0$ and $\tilde{H}_2 = 0$.

4 Rational Solutions

In this section, we present a method for finding a rational solution of the system (1.1) based on proper parametrizations of its invariant algebraic space curve. First, we introduce an important property of proper parametrizations of a rational space curve.

Lemma 4.1 *Let $\mathcal{P}_1(t)$ be a proper, i.e. rationally invertible, parametrization of an affine rational space curve $\mathcal{C}$, and let $\mathcal{P}_2(t)$ be any other rational parametrization of $\mathcal{C}$.*

(a) There exists a non-constant rational function $R(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(R(t))$.
(b) $\mathcal{P}_2(t)$ is proper if and only if there exists a linear rational function $L(t)$ such that $\mathcal{P}_2(t) = \mathcal{P}_1(L(t))$.

Proof Assume that $\varphi : \mathcal{C} \to \widehat{\mathcal{C}}$ is a birational mapping from the space curve $\mathcal{C}$ onto the plane curve $\widehat{\mathcal{C}}$. $\widehat{\mathcal{C}}$ is also rational. Indeed, since the space curve $\mathcal{C}$ is properly parametrized by $\mathcal{P}_1(t)$, the plane curve $\widehat{\mathcal{C}}$ is properly parametrized by $\widehat{\mathcal{P}}_1(t) = \varphi(\mathcal{P}_1(t))$. Similarly, $\widehat{\mathcal{P}}_2(t) = \varphi(\mathcal{P}_2(t))$ is a parametrization of rational plane curve $\widehat{\mathcal{C}}$. According to Lemma 4.17 in [17], we have $\widehat{\mathcal{P}}_2(t) = \widehat{\mathcal{P}}_1(R(t))$, where $R(t)$ is a non-constant rational function. It follows that

$$\mathcal{P}_2(t) = \varphi^{-1}(\widehat{\mathcal{P}}_2(t)) = \varphi^{-1}(\widehat{\mathcal{P}}_1(R(t))) = \mathcal{P}_1(R(t)).$$

This proves (a). Since φ is birational, $\mathcal{P}_2(t)$ is proper if and only if $\widehat{\mathcal{P}}_2(t)$ is proper. Therefore, the statement (b) follows from the appropriate statement for plane curves. □

Remark 4.2 Note that not all components in the computed proper parametrization of a rational space curve are constant. In other words, if $(s_1(x), s_2(x), s_3(x))$ is a proper parametrization of the given rational space curve, then at least one of $s_i(x)$ is non-constant.

In the following, we give our method for finding rational solutions of the system (1.1).

Lemma 4.3 *Let* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ *be a rational invariant algebraic space curve of the system* (1.1)*, where* $H_1 \in \mathbb{K}[s_1, s_2]$, $H_2 \in \mathbb{K}[s_1, s_2, s_3]$*, and* $(s_1(x), s_2(x), s_3(x))$ *is a rational parametrization of* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$*. If* $V_j(s_1(x), s_2(x), s_3(x)) \neq 0$ *for* $j = 1, 2, 3$*, then*

$$s_1'(x) \cdot \frac{U_k(s_1(x), s_2(x), s_3(x))}{V_k(s_1(x), s_2(x), s_3(x))} = s_k'(x) \cdot \frac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))}, \quad k = 2, 3. \tag{4.1}$$

Proof According to Definition 3.2, we have $\mathcal{D}(H_i) \in \mathrm{sat}(H_1, H_2)$. Observe that $H_i \in \mathrm{sat}(H_1, H_2)$. Since $(s_1(x), s_2(x), s_3(x))$ is a parametrization of $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$, we have

$$\mathcal{D}(H_i)(s_1(x), s_2(x), s_3(x)) = 0, \quad H_i(s_1(x), \ldots, s_{i+1}(x)) = 0. \tag{4.2}$$

By expanding the first equation and differentiating the second one w.r.t. x in (4.2), we have

$$A_i := \sum_{j=1}^{i+1} H_{is_j}(s_1(x), \ldots, s_{i+1}(x)) \cdot U_j(s_1(x), s_2(x), s_3(x)) \cdot W_j(s_1(x), s_2(x), s_3(x)) = 0,$$

$$B_i := \sum_{j=1}^{i+1} H_{is_j}(s_1(x), \ldots, s_{i+1}(x)) \cdot s_j'(x) = 0,$$

for $i = 1, 2$. According to the irreducibility of $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$, we know that $\{H_1, H_2\}$ is an irreducible regular chain, which means that H_i as an univariate polynomial with respect to s_{i+1} is irreducible over $\mathbb{K}(s_1(x), \cdots, s_i(x))$. It follows that

$$H_{1s_2}(s_1(x), s_2(x)) \neq 0, \quad H_{2s_3}(s_1(x), s_2(x), s_3(x)) \neq 0. \tag{4.3}$$

In fact, assume that $H_{is_{i+1}}(s_1(x), \ldots, s_{i+1}(x)) = 0$. By combining it with the second equation of (4.2), we know that H_i is not squarefree with respect to s_{i+1}. This is a contradiction with the irreducibility of H_i. In particularly, $H_{1s_2}(s_1(x), s_2(x)) \neq 0$. Therefore, the system determined by $A_1 = 0$ and $B_1 = 0$ has the non-zero solution

$$(H_{1s_1}(s_1(x), s_2(x)), H_{1s_2}(s_1(x), s_2(x))).$$

So that the determinant of coefficients matrix is equal to 0, i.e. $\det(\mathbf{M}_2) = 0$, where

$$\mathbf{M}_2 = \begin{pmatrix} U_1 W_1(s_1(x), s_2(x), s_3(x)) & U_2 W_2(s_1(x), s_2(x), s_3(x)) \\ s_1'(x) & s_2'(x) \end{pmatrix}.$$

In the following, we need to prove $\det(\mathbf{M}_3) = 0$, where

$$\mathbf{M}_3 = \begin{pmatrix} U_1W_1(s_1(x), s_2(x), s_3(x)) & U_3W_3(s_1(x), s_2(x), s_3(x)) \\ s_1'(x) & s_3'(x) \end{pmatrix}.$$

In fact, the system determined by $A_2 = 0$ and $B_2 = 0$ can be rewritten as

$$\begin{cases} H_{2s_1}|_{(s_1(x),s_2(x),s_3(x))} \cdot U_1W_1|_{(s_1(x),s_2(x),s_3(x))} + H_{2s_3}|_{(s_1(x),s_2(x),s_3(x))} \cdot U_3W_3|_{(s_1(x),s_2(x),s_3(x))} \\ \quad = -H_{2s_2}|_{(s_1(x),s_2(x),s_3(x))} \cdot U_2W_2|_{(s_1(x),s_2(x),s_3(x))} \\ H_{2s_1}|_{(s_1(x),s_2(x),s_3(x))} \cdot s_1'(x) + H_{2s_3}|_{(s_1(x),s_2(x),s_3(x))} \cdot s_3'(x) = -H_{2s_2}|_{(s_1(x),s_2(x),s_3(x))} \cdot s_2'(x). \end{cases}$$

If $\det(\mathbf{M}_3) \neq 0$, then by Cramer's rule,

$$H_{2s_3}(s_1(x), s_2(x), s_3(x))) = -\frac{H_{2s_2}(s_1(x), s_2(x), s_3(x)) \det(\mathbf{M}_2)}{\det(\mathbf{M}_3)}.$$

Since $\det(\mathbf{M}_2) = 0$, $H_{2s_3}(s_1(x), s_2(x), s_3(x))) = 0$. This is a contradiction to the second inequation in (4.3). Taking into account that $V_j(s_1(x), \cdots, s_n(x)) \neq 0$ for $j = 1, 2, 3$, we see that the system (4.1) holds. □

The condition $V_j(s_1(x), s_2(x), s_3(x)) \neq 0$ in the above lemma means $V_j \notin \mathrm{sat}(H_1, H_2)$. The parametrization problem for algebraic plane curves has been studied intensively, e.g. in [15–17]. Therefore the key point for computing a rational parametrization of the invariant algebraic space curve mentioned in Lemma 4.3 is to birationally project it to a plane curve (see [1]). Lemma 4.3 tells us that not every rational parametrization of a rational invariant algebraic space curve can provide a rational solution of the system (1.1). They are the candidates of rational solutions. In what follows, we give a theorem which provides necessary and sufficient conditions for the rational system to have a rational solution.

Theorem 4.4 *Let $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ be a rational invariant algebraic space curve of the system (1.1), such that $V_j \notin \mathrm{sat}(H_1, H_2)$ for $1 \leq j \leq 3$, and let $(s_1(x), s_2(x), s_3(x))$ be an arbitrary proper rational parametrization of the space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$. Then the system (1.1) has a rational solution*

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = (s_1(T(x)), s_2(T(x)), s_3(T(x)))$$

corresponding to $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ if and only if there exists a linear rational transformation $T(x) = \frac{ax+b}{cx+d}$ which is a rational solution of one of the following autonomous differential equations

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}, \quad \text{if } s_i'(x) \neq 0,\ i = 1, 2, 3. \tag{4.4}$$

Proof Suppose that $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ is a rational solution of the system (1.1) corresponding to the invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$. Then

$$\widehat{s}_i'(x) = \frac{U_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))}{V_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))}, \quad i = 1, 2, 3. \tag{4.5}$$

Since $(s_1(x), s_2(x), s_3(x))$ is a proper parametrization of rational space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$. According to Lemma 4.1(a), there exists a non-constant rational function $T(x)$, such that the two rational parametrizations of the same algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ are related like

$$\widehat{s}_i(x) = s_i(T(x)), \quad i = 1, 2, 3. \tag{4.6}$$

By (4.5) and (4.6), we have

$$s_i'(T(x)) \cdot T'(x) = \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}, \quad i = 1, 2, 3.$$

According to Remark 4.2, we know that at least one of $s_i(x)$ is non-constant. Therefore, $T(x)$ satisfies at least one of the following autonomous differential equations

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}, \quad \text{if } s_i'(x) \neq 0, \quad i = 1, 2, 3.$$

As the above autonomous differential equations are of degree 1 with respect to $T'(x)$, it follows from Theorem 2.7 and Corollary 3.11 in [4] that their rational solution $T(x)$ is a linear rational function.

Conversely, assume w.l.o.g that $s_1(x)$ is non-constant and $T(x)$ is a rational solution of the differential equation

$$T'(x) = \frac{1}{s_1'(T(x))} \cdot \frac{U_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}.$$

By Lemma 4.3 we have

$$s_1'(x) \cdot \frac{U_k(s_1(x), s_2(x), s_3(x))}{V_k(s_1(x), s_2(x), s_3(x))} = s_k'(x) \cdot \frac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))}, \quad k = 2, 3.$$

There are three cases in the following.

1. if $s_2'(x) = s_3'(x) = 0$, then $\frac{U_2(s_1(x),s_2(x),s_3(x))}{V_2(s_1(x),s_2(x),s_3(x))} = \frac{U_3(s_1(x),s_2(x),s_3(x))}{V_3(s_1(x),s_2(x),s_3(x))} = 0$ and $s_2(x) = c_1$, $s_3(x) = c_2$. In this case, $(s_1(T(x)), c_1, c_2)$ is a rational solution corresponding to the invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$.
2. Else if one of $s_2(x)$ and $s_3(x)$ is not equal to 0, assume w.l.o.g that $s_2'(x) \neq 0$ and $s_3'(x) = 0$, then $s_3(x) = c_3$ and
$$\frac{1}{s_2'(x)} \cdot \frac{U_2(s_1(x), s_2(x), s_3(x))}{V_2(s_1(x), s_2(x), s_3(x))} = \frac{1}{s_1'(x)} \cdot \frac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))}.$$
Therefore, $T(x)$ is also a rational solution of the following differential equation
$$T'(x) = \frac{1}{s_2'(T(x))} \cdot \frac{U_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_2(s_1(T(x)), s_2(T(x)), s_3(T(x)))}.$$
It follows that $(s_1(T(x)), s_2(T(x)), c_3)$ is a rational solution corresponding to $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$.
3. Otherwise, $s_2'(x) \neq 0$ and $s_3'(x) \neq 0$. By an similar argument with above, a rational solution corresponding to $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ is $(s_1(T(x)), s_2(T(x)), s_3(T(x)))$. □

Example Continue considering the system (3.1) in the example given in Sect. 3. In fact, it is easy to see that

$$(s_1(x), s_2(x), s_3(x)) = (x, x, x + 1)$$

is a proper parametrization of the invariant algebraic space curve $\mathbf{Z}(H_1, H_2)$. Since $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = T^2(x),$$

we have $T(x) = -\frac{1}{x}$. According to Theorem 4.4,

$$(s_1(T(x)), s_2(T(x)), s_3(T(x))) = \left(-\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x} + 1\right)$$

is a rational solution of the system (3.1) corresponding to $\mathbf{Z}(H_1, H_2)$. Similarly,

$$(s_1(x), s_2(x), s_3(x)) = \left(x, -\frac{3}{2}x^2 + 4x, -2x + 4\right)$$

is a proper parametrization of the invariant algebraic space curve $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$. Note that $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = -\frac{1}{2}T^2(x),$$

we have $T(x) = \frac{2}{x}$. According to Theorem 4.4,

$$(s_1(T(x)), s_2(T(x)), s_3(T(x))) = \left(\frac{2}{x}, -\frac{6}{x^2} + \frac{8}{x}, -\frac{4}{x} + 4\right)$$

is a rational solution of the system (3.1) corresponding to $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$.

5 Linear Rational Solvability and Relations Between Two Rational Solutions

It can be seen from Theorem 4.4 that the problem for computing a rational solution of the system (1.1) is reduced to finding a linear rational solution of the autonomous differential equation

$$T'(x) = \frac{1}{s_1'(T(x))} \cdot \frac{U_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}$$

if $s_1'(x) \neq 0$. In this section, we consider the linear rational solvability of the above autonomous differential equation and analyze the relationship between two different rational solutions corresponding to the same invariant algebraic space curve. From now on, we w.l.o.g. assume that $s_1'(x) \neq 0$.

Theorem 5.1 *Let* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ *be a rational invariant algebraic space curve of the system* (1.1)*, such that* $V_j \notin \mathrm{sat}(H_1, H_2)$ *for* $j = 1, 2, 3$*. If* $\mathcal{P} = (s_1(x), s_2(x), s_3(x))$ *and* $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ *are two different proper parametrizations of* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$*, then*

$$T'(x) = \frac{U_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), s_3(T(x)) \cdot s_1'(T(x))} \tag{5.1}$$

has a linear rational solution $T(x)$ *if and only if*

$$\widehat{T}'(x) = \frac{U_1(\widehat{s}_1(\widehat{T}(x)), \widehat{s}_2(\widehat{T}(x)), \widehat{s}_3(\widehat{T}(x)))}{V_1(\widehat{s}_1(\widehat{T}(x)), \widehat{s}_2(\widehat{T}(x)), \widehat{s}_3(\widehat{T}(x))) \cdot \widehat{s}_1'(\widehat{T}(x))} \tag{5.2}$$

has a linear rational solution $\widehat{T}(x)$*. Moreover,* $\mathcal{P}(T(x)) = \widehat{\mathcal{P}}(\widehat{T}(x))$*.*

Proof Assume that $T(x) = \frac{ax+b}{cx+d}$ is a linear rational solution of (5.1). Then the rational solution of the system (1.1) corresponding to $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ is $(s_1(T(x)), s_2(T(x)), s_3(T(x)))$. As $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ is a proper parametrization of the same space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$, there exists $\widehat{T}(x) = \frac{\widehat{a}x+\widehat{b}}{\widehat{c}x+\widehat{d}}$, such that

$$s_i(T(x)) = \widehat{s}_i(\widehat{T}(x)), \quad i = 1, 2, 3. \tag{5.3}$$

In particular, $s_1(T(x)) = \widehat{s}_1(\widehat{T}(x))$. It follows that

$$\begin{aligned}\widehat{s}_1'(\widehat{T}(x))\widehat{T}'(x) = s_1'(T(x))T'(x) &= \frac{U_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_1(s_1(T(x)), s_2(T(x)), s_3(T(x)))} \\ &= \frac{U_1(\widehat{s}_1(\widehat{T}(x)), \widehat{s}_2(\widehat{T}(x)), \widehat{s}_3(\widehat{T}(x)))}{V_1(\widehat{s}_1(\widehat{T}(x)), \widehat{s}_2(\widehat{T}(x)), \widehat{s}_3(\widehat{T}(x)))}.\end{aligned}$$

Therefore, $\widehat{T}(x)$ is a linear rational solution of (5.2), and vice versa. Furthermore, since $\mathcal{P} = (s_1(x), s_2(x), s_3(x))$ and $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$, we get from (5.3) that $\mathcal{P}(T(x)) = \widehat{\mathcal{P}}(\widehat{T}(x))$. □

It can be seen from Theorem 5.1 that the solvability of the differential equation in (4.4) does not depend on the choice of the proper parametrization of the invariant algebraic space curve. According to Theorem 5.1, we know that it is possible to get two different rational solutions from two different proper parametrizations of the same invariant algebraic space curve. In fact, they are related to each other by a shifting of the variable.

Theorem 5.2 *Let* $(s_1(x), s_2(x), s_3(x))$ *and* $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ *be two rational solutions of the system* (1.1) *corresponding to the same invariant algebraic space curve. Then there exists a constant c such that*

$$(s_1(x+c), s_2(x+c), s_3(x+c)) = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)).$$

Proof Since $(s_1(x), s_2(x), s_3(x))$ and $(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$ are two proper parametrizations of the same invariant algebraic space curve. Therefore, there exists a linear rational function $T(x)$ such that

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = (s_1(T(x)), s_2(T(x)), s_3(T(x))).$$

It follows that

$$\begin{aligned} s_i'(T(x))T'(x) &= \widehat{s}_i'(x) \\ &= \frac{U_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))}{V_i(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))} \\ &= \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))} \\ &= s_i'(T(x)). \end{aligned}$$

Assume w.l.o.g. that $s_1'(x) \neq 0$, then $T'(x) = 1$. By solving this differential equation, we have $T(x) = x + c$ for some constant c. □

Remark 5.3 In fact, the transformation from one solution into another one in Theorem 5.2 can be computed. Let $\mathcal{P} = (s_1(x), s_2(x), s_3(x))$ and $\widehat{\mathcal{P}} = (\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x))$. By considering these two rational solutions as the proper parametrizations, then $\mathcal{P}^{-1} \circ \widehat{\mathcal{P}}$ is the transformation from the solution $\mathcal{P}$ to the solution $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}^{-1} \circ \mathcal{P}$ is the transformation from the solution $\widehat{\mathcal{P}}$ to the solution $\mathcal{P}$, where $*^{-1}$ represents the inverse of the parametrization.

Example For the computed invariant algebraic space curve $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$ in the example given in Sect. 3, it is easy to see that

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = \left(-\frac{x}{2} + 2, -\frac{3}{8}x^2 + x + 2, x\right)$$

is its another proper parametrization. Note that $s_1'(x) \neq 0$, by solving the differential equation

$$\widehat{T}'(x) = \frac{\widehat{s}_1(\widehat{T}(x))\widehat{s}_3(\widehat{T}(x)) - \widehat{s}_2(\widehat{T}(x))}{\widehat{s}_1'(\widehat{T}(x))} = \left(-\frac{1}{\widehat{T}(x)} + 2\right)^2,$$

we have $\widehat{T}(x) = \frac{4x}{x+1}$. It follows that

$$(\widehat{s}_1(\widehat{T}(x)), \widehat{s}_2(\widehat{T}(x)), \widehat{s}_3(\widehat{T}(x))) = \left(\frac{2}{x+1}, \frac{8x+2}{(x+1)^2}, \frac{4x}{x+1}\right)$$

is another rational solution corresponding to $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$. Let

$$\mathcal{P} = \left(\frac{2}{x}, -\frac{6}{x^2} + \frac{8}{x}, -\frac{4}{x} + 4\right), \quad \widehat{\mathcal{P}} = \left(\frac{2}{x+1}, \frac{8x+2}{(x+1)^2}, \frac{4x}{x+1}\right),$$

which is two rational solutions corresponding to the same invariant algebraic space curve $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$ and $\mathcal{P}$ is computed in the example given in Sect. 4. Observe that the transformation from $\mathcal{P}$ to $\widehat{\mathcal{P}}$ is $T(x) = x + 1$. In fact, since $\mathcal{P}^{-1}(s_1, s_2, s_3) = \frac{2}{s_1}$,

$$T(x) = \mathcal{P}^{-1} \circ \widehat{\mathcal{P}} = \frac{2}{\frac{2}{x+1}} = x + 1.$$

According to Remark 5.3, we have $\mathcal{P}(T(x)) = \widehat{\mathcal{P}}$.

6 Rational General Solutions

Observe that our previous method can compute rational solutions of the system (1.1). A natural question is how to check whether the computed solution is a general solution. In the section, we present a method for deciding it by characterizing the form for the generators of the ideal defined by its rational solution.

It is known that for a single ODE, the pseudo division in differential ring can be used to check whether its a solution is a general solution. The following lemma generalizes it to the case of a system.

Lemma 6.1 *A rational solution* $(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x))$ *of the system* (1.1) *is a rational general solution, if and only if*

$$\forall G \in \mathbb{K}\{s_1, s_2, s_3\} : G(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0 \iff \text{prem}(G, \mathbb{A}) = 0,$$

where $\mathbb{A} = \{A_1, A_2, A_3\}$ *and* $A_i = V_i s_i' - U_i$ *for* $1 \le i \le 3$.

Proof Let $\mathcal{I} = \{G \in \mathbb{K}\{s_1, s_2, s_3\} \mid \text{prem}(G, \mathbb{A}) = 0\}$. It is proved that $\mathbb{A}$ is a characteristic set of the prime differential ideal $[\mathbb{A}] : S^\infty$ in [14, chap V, sect. 3], which means

$$\mathcal{I} = [\mathbb{A}] : S_{\mathbb{A}}^\infty. \tag{6.1}$$

Therefore, for any $G \in \mathbb{K}\{s_1, s_2, s_3\}$, $G \in [\mathbb{A}] : S_{\mathbb{A}}^\infty$ if and only if $\text{prem}(G, \mathbb{A}) = 0$. Combining Definition 2.6, the theorem is proved. □

In fact, Lemma 6.1 provides an algorithmic way for checking whether a solution of the system (1.1) is a general solution via pseudo division. Next we turn this into a criterion involving only algebraic (not differential) polynomials in the indeterminates s_1, s_2 and s_3.

Lemma 6.2 *A rational solution* $(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x))$ *of the system* (1.1) *is a rational general solution if and only if for any* $G \in \mathbb{K}[s_1, s_2, s_3]$, $G(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0$ *implies* $G = 0$ *in* $\mathbb{K}[s_1, s_2, s_3]$.

Proof ($\Longrightarrow$) Since $G \in \mathbb{K}[s_1, s_2, s_3]$, we have $\text{prem}(G, \mathbb{A}) = G$. It follows from Lemma 6.1 that $G(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0$ implies $G = 0$.
($\Longleftarrow$) Let $H \in \mathbb{K}\{s_1, s_2, s_3\}$ and $R = \text{prem}(H, \mathbb{A})$, where $\mathbb{A} = \{A_1, A_2, A_3\}$ and $A_i = V_i s_i' - U_i$ for $1 \le i \le 3$. We need to prove

$$H(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0 \iff R = 0.$$

Obviously, $R \in \mathbb{K}[s_1, s_2, s_3]$. If $H(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0$, then

$$R(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0.$$

So, by the assumption, $R = 0$. Conversely, let $R = 0$, from Proposition 2.4, we get

$$V_1^{l_1} \cdots V_3^{l_3} H = \sum_{j \ge 0} Q_{1j} A_1^{(j)} + \sum_{j \ge 0} Q_{2j} A_2^{(j)} + \sum_{j \ge 0} Q_{3j} A_3^{(j)}.$$

It follows that

$$\prod_{i=1}^{3} V_i^{l_i}(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) \cdot H(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0.$$

Since $(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x))$ is a solution of the system (1.1), we know

$$V_i(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) \ne 0$$

for any $1 \le i \le 3$. Therefore, $H(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x)) = 0$. According to Lemma 6.1, the rational solution $(\bar{s}_1(x), \bar{s}_2(x), \bar{s}_3(x))$ is a rational general solution of the system (1.1). □

The following theorem gives us a criterion for a rational solution to be a rational general solution.

Theorem 6.3 *Suppose that* $(s_1(x), s_2(x), s_3(x))$ *is a rational solution of the system* (1.1)*, which is implicitly represented by the ideal generated by* G_i *with the following form*

$$G_i = A_i - c_i B_i, \quad i = 1, \ldots, m, \tag{6.2}$$

where $A_i, B_i \in \mathbb{K}[s_1, s_2, s_3]$*, and* c_i *are independent transcendental constants over* $\mathbb{K}$*. Then* $(s_1(x), s_2(x), s_3(x))$ *is a rational general solution of the system* (1.1).

Proof Let $\mathcal{I} = \langle G_1, \cdots, G_m \rangle \cap \mathbb{K}[s_1, s_2, s_3]$. Because of the independence of the constants c_i, $\mathcal{I} = \{0\}$. Therefore, $G(s_1(x), s_2(x), s_3(x)) = 0$ implies $G = 0$ for any $G \in \mathbb{K}[s_1, s_2, s_3]$. Because of Lemma 6.2 this implies that $(s_1(x), s_2(x), s_3(x))$ is a rational general solution. □

Example Consider the following trivariate rational system of autonomous ODEs

$$s_1'(x) = 1, \quad s_2'(x) = -\frac{s_3}{s_1^2}, \quad s_3'(x) = \frac{2s_3}{s_1}.$$

By using undetermined coefficients method introduced in Sect. 3, an invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ can be computed under $\deg(H_1) \leq 1$ and $\deg(H_2) \leq 2$, where

$$\begin{aligned} H_1 &= s_2 + c_1 s_1 + c_2, \\ H_2 &= s_3 + c_3 s_2^2 + c_4 s_2 s_1 + c_6 s_2 + (-c_1^2 c_3 + c_1 c_4 - c_1) s_1^2 + (-2c_1 c_2 c_3 + c_1 c_6 + c_2 c_4) s_1 - c_2^2 c_3 + c_2 c_6. \end{aligned}$$

Obviously, $\mathbf{Z}(\mathrm{sat}(H_1, H_2)) = \mathbf{Z}(H_1, H_2)$. A proper parametrization of $\mathbf{Z}(H_1, H_2)$ is

$$(s_1(x), s_2(x), s_3(x)) = \left(\frac{1}{x}, -\frac{c_1}{x} - c_2, \frac{c_1}{x^2}\right).$$

Since $s_1'(x) \neq 0$,

$$T'(x) = \frac{1}{s_1'(T(x))} = -T^2(x).$$

Note that both $s_2'(x)$ and $s_3'(x)$ are not equal to 0, and

$$T'(x) = -\frac{1}{s_2'(T(x))} \cdot \frac{s_3(T(x))}{s_1^2(T(x))} = \frac{1}{s_3'(T(x))} \cdot \frac{2s_3(T(x))}{s_1(T(x))} = -T^2(x).$$

By solving the above differential equation for $T(x)$, we get

$$T(x) = \frac{1}{x}.$$

It follows from Theorem 4.4 that

$$(s_1(T(x)), s_2(T(x)), s_3(T(x))) = (x, -c_1 x - c_2, c_1 x^2)$$

is a rational solution of the given system corresponding to $\mathbf{Z}(H_1, H_2)$. Note that this solution can be implicitly represented by the ideal $\langle -s_3 + c_1 s_1^2, s_3 + s_2 s_1 + c_2 s_1 \rangle$, which has generators of the form in (6.2), where

$$A_1 = -s_3, \quad B_1 = -s_1^2, \quad A_2 = s_3 + s_2 s_1, \quad B_2 = -s_1.$$

According to Theorem 6.3, $(x, -c_1 x - c_2, c_1 x^2)$ is a rational general solution of the given system.

7 Conclusions

In this paper, we have presented a method for finding rational general solutions of trivariate rational differential systems. Our method can be extended to multivariate rational differential systems. In addition, there is a connection between invariant algebraic space curves and rational first integrals. This relationship helps us to study rational general solutions of rational differential systems via rational first integrals and vice versa. Observe that the undetermined coefficients method for computing invariant algebraic space curve is time consuming, especially when the polynomial degree is increased. In fact, the time complexity of this method is as high as of computing a Gröbner basis. Moreover, we don't have a degree bound for irreducible invariant algebraic space curves. This is similar to the problem arising in a generalization of Hubert's method [7] to higher order ODEs; namely to determine a bound on the number of derivations to be considered. Therefore, it would be interesting to replace the undetermined coefficients method by other possibly more efficient methods or even develop some methods for finding invariant algebraic space curves in other representations without involving the degree bound problem, e.g. looking for parametric representations of invariant algebraic space curves directly.

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