INFINITELY MANY CONGRUENCES FOR BROKEN 2–DIAMOND PARTITIONS MODULO 3

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Abstract. In 2007, Andrews and Paule introduced the family of functions $\Delta_k(n)$ which enumerate the number of broken k–diamond partitions for a fixed positive integer k. Since then, numerous mathematicians have considered partitions congruences satisfied by $\Delta_k(n)$ for small values of k. In this work, we focus our attention on the function $\Delta_2(n)$ and explicitly identify infinitely many Ramanujan–like congruences modulo 3 which are satisfied by this function.

1. Introduction

Broken k-diamond partitions were introduced in 2007 by Andrews and Paule [2]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms. Namely,

$$\sum_{n=0}^{\infty}\Delta_k(n)q^n = \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}$$
$$= q^{(k+1)/12}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \quad k\geq 1,$$

where we recall the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \quad (q=e^{2\pi i\tau}).$$

In their original work, Andrews and Paule proved that, for all $n\geq 0$,

$$\Delta_1(2n+1)\equiv 0 \pmod 3. \tag{1.1}$$

They also conjectured a few other congruences modulo 2 satisfied by certain families of broken k-diamond partitions.

Since then, a number of authors [3, 4, 5, 7, 8, 10] have provided proofs of additional congruences satisfied by broken k-diamond partitions. However, most of these works have focused on congruences modulo primes other than 3; the exceptions to this are Hirschhorn

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and Sellers [4] who re-proved (1.1) by finding an explicit representation of the generating function for $\Delta_1(2n+1)$ which implied (1.1) and Mortenson [5] who developed a statistic on the partitions enumerated by $\Delta_1(2n+1)$ which naturally breaks these partitions into three subsets of equal size (thus proving (1.1) combinatorially).

Our overarching goal in this work is to focus attention on the function $\Delta_2(n)$ which enumerates broken 2–diamond partitions and to prove infinitely many Ramanujan–like congruences modulo 3 which are satisfied by this function. In order to do so, we will prove the following congruence result related to the generating function for $\Delta_2(3n+1)$:

Theorem 1.1.

$$\sum_{n=0}^{\infty} \Delta_2(3n+1)q^n \equiv 2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2} \pmod 3$$

We will prove this theorem in the next section. Before doing so, we make a number of important remarks.

First, note that the product

$$\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2}$$

which appears in the result in Theorem 1.1 is a function of q^5. Thus, when written as a power series in q,

$$2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2}$$

will only contain powers of q raised to powers of the form $5n+1$. Therefore, when we replace n on the left–hand side of the equation in Theorem 1.1 by $5n, 5n+2, 5n+3$ and $5n+4$, we immediately have the following congruences:

Corollary 1.2. *For all* $n \geq 0$,

$$\begin{aligned} \Delta_2(15n+1) &\equiv 0 \pmod 3, \\ \Delta_2(15n+7) &\equiv 0 \pmod 3, \\ \Delta_2(15n+10) &\equiv 0 \pmod 3, \quad \textit{and} \\ \Delta_2(15n+13) &\equiv 0 \pmod 3 \end{aligned}$$

This is a gratifying result, but much more can be said. Note that Theorem 1.1 implies that

$$\sum_{n=0}^{\infty} \Delta_2(3n+1)q^n \equiv 2q\left(\sum_{n=0}^{\infty} q^{5n(n+1)/2}\right)^2 \pmod 3 \tag{1.2}$$

thanks to a well–known result of Gauss; see Andrews [1, Corollary 2.10] for more information. Thus, (1.2) implies that

$$\Delta_2(3n+1) \equiv 2r(n) \pmod 3$$

where $r(n)$ is the number of representations of n as

$$n = 1 + \frac{5j(j+1)}{2} + \frac{5k(k+1)}{2}$$

for nonnegative integers j and k. By completing the square, we see that this is the same as counting the number of representations of $40n+10$ as

$$40n + 10 = (5(2j+1))^2 + (5(2k+1))^2. \tag{1.3}$$

We are then reminded of the following theorem attributed to Fermat:

Theorem 1.3. *A positive integer N is representable as a sum of two squares if and only if in the prime factorization of N, each prime $p \equiv 3 \pmod 4$ appears an even number of times.*

With these comments in mind, we can now prove the following corollary to Theorem 1.1.

Corollary 1.4. *Let $p \equiv 3 \pmod 4$ be prime and let $r = \frac{3}{4}(p(4k+3)-1)+1$ where $0 \le k \le p-1$ and $p \ne 4k+3$. Then, for all $n \ge 0$, $\Delta_2(3p^2n + r) \equiv 0 \pmod 3$.*

Remark 1.5. Two remarks are in order here. First, note that exactly one of the values of k, $0 \le k \le p-1$, is the value such that $p = 4k+3$. This is the value of k which is excluded in the statement of Corollary 1.4. Moreover, for every **other** value of k, $0 \le k \le p-1$, we know $p \nmid 4k+3$. Secondly, notice that Corollary 1.4 explicitly provides $p-1$ different Ramanujan–like congruences modulo 3 which are satisfied by Δ_2 for each prime $p \equiv 3 \pmod 4$. Thus, we can explicitly state infinitely many different congruences satisfied by Δ_2 modulo 3.

Proof. (of Corollary 1.4) Note that

$$\Delta_2\left(3p^2n + \frac{3}{4}(p(4k+3)-1)+1\right) = \Delta_2\left(3(p^2n + \frac{1}{4}(p(4k+3)-1))+1\right).$$

Thanks to (1.3), we see that we want to consider the number of ways to represent

$$40\left(p^2n + \frac{1}{4}(p(4k+3)-1)\right) + 10$$

as a sum of two squares. However, notice that

$$\begin{aligned} 40\left(p^2n + \frac{1}{4}(p(4k+3)-1)\right) + 10 &= 40p^2n + 10p(4k+3) \\ &= p(40pn + 10(4k+3)). \end{aligned}$$

From the way in which we chose the values of k, we know that $p \nmid 4k+3$. And, of course, $p \nmid 10$ since $p \equiv 3 \pmod 4$. Thus, since $p \mid 40pn$, we see that $p \nmid 40pn + 10(4k+3)$. Therefore, the prime p appears in the factorization of $40(p^2n + \frac{1}{4}(p(4k+3)-1)) + 10$ exactly one time (which is odd). By Theorem 1.3, we know that the number of ways to represent $40\left(p^2n + \frac{1}{4}(p(4k+3)-1)\right) + 10$ as a sum of two squares is zero. Therefore, for all $n \ge 0$,

$$\Delta_2\left(3p^2n + \frac{3}{4}(p(4k+3)-1)+1\right) \equiv 0 \pmod 3.$$

□

Hence, for example, we see that, for all $n \geq 0$,

$$\begin{aligned} \Delta_2(27n+16) &\equiv 0 \pmod 3, \\ \Delta_2(27n+25) &\equiv 0 \pmod 3, \end{aligned}$$

$$\begin{aligned} \Delta_2(147n+16) &\equiv 0 \pmod 3, \\ \Delta_2(147n+58) &\equiv 0 \pmod 3, \\ \Delta_2(147n+79) &\equiv 0 \pmod 3, \\ \Delta_2(147n+100) &\equiv 0 \pmod 3, \\ \Delta_2(147n+121) &\equiv 0 \pmod 3, \\ \Delta_2(147n+142) &\equiv 0 \pmod 3, \end{aligned}$$

and, just for fun,

$$\Delta_2(3 \cdot (100000000003)^2 n + 225000000007) \equiv 0 \pmod 3$$

along with 100000000001 other such congruences.

2. Proof of Theorem 1.1

We define

$$\sum_{n=0}^{\infty} a(n) q^n := \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{5n})(1-q^{10n})^2}{(1-q^{3n})(1-q^{30n})}.$$

Note that

$$\prod_{n=1}^{\infty} \frac{1-q^{30n}}{(1-q^{10n})^3} \times \prod_{n=1}^{\infty} \frac{1-q^{3n}}{(1-q^n)^3} \times \sum_{n=0}^{\infty} a(n) q^n = \sum_{n=0}^{\infty} \Delta_2(n) q^n.$$

This implies that $\Delta_2(n) \equiv a(n) \pmod 3$ for all $n \geq 0$. We note that

$$\begin{aligned} & q^5 \prod_{n=1}^{\infty} (1-q^{30n})^2 (1-q^{15n})^2 (1-q^{3n})^{12} \sum_{n=1}^{\infty} a(n) q^n \\ =& q^5 \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{3n})^{11}(1-q^{5n})(1-q^{10n})^2(1-q^{15n})^2(1-q^{30n}) \\ =& \eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau). \end{aligned}$$

We also note that

$$\begin{aligned}&U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))\\=&U_3\Big(q^5\prod_{n=1}^{\infty}(1-q^{30n})^2(1-q^{15n})^2(1-q^{3n})^{12}\sum_{n=1}^{\infty}a(n)q^n\Big)\\=&\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}U_3\Big(q^5\sum_{n=0}^{\infty}a(n)q^n\Big)\\=&\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}\sum_{n=0}^{\infty}a(3n-5)q^n\\=&q^7\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}\sum_{n=0}^{\infty}a(3n+1)q^n\end{aligned}$$

where U_3 is the usual operator on power series defined by

$$U_3\left(\sum_{n\geq 0}b(n)q^n\right)=\sum_{n\geq 0}b(3n)q^n.$$

Our goal now is to prove that

$$\sum_{n=0}^{\infty}a(3n+1)q^n\equiv 2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2}\pmod 3 \tag{2.1}$$

or, equivalently,

$$\begin{aligned}&U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))\\=&q^2\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}\sum_{n=0}^{\infty}a(3n+1)q^n\\\equiv&q^2\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}\times 2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2}\pmod 3.\end{aligned}$$

Note that

$$q^2\prod_{n=1}^{\infty}(1-q^{10n})^2(1-q^{5n})^2(1-q^{n})^{12}\times 2q\prod_{n=1}^{\infty}\frac{(1-q^{10n})^4}{(1-q^{5n})^2}=2\eta(10\tau)^6\eta^{12}(\tau).$$

Thus, we want to prove

$$U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))\equiv 2\eta(10\tau)^6\eta^{12}(\tau)\pmod 3.$$

As in [10], we use [6, Theorem 1.64] to find that $\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau)$ is a modular form of weight 9 for the group $\Gamma_0(360)$ with character $\chi(d):=\left(\frac{-3}{d}\right)$. This implies $U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))$ is also a modular form of weight 9 for the group $\Gamma_0(360)$ with character $\chi(d):=\left(\frac{-3}{d}\right)$. Again using [6, Theorem 1.64] we find that $\eta(10\tau)^6\eta^{12}(\tau)$ is a modular form of weight 9 for the group $\Gamma_0(360)$ with character $\left(\frac{-1}{d}\right)$. Therefore,

$$\big\{U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))\big\}^2-\big\{\eta(10\tau)^6\eta^{12}(\tau)\big\}^2 \tag{2.2}$$

is a modular form of weight 18 for the group $\Gamma_0(360)$. Using Sturm's criterion [9] we see that we need to check that $\frac{18}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(360)]+1 = \frac{18}{12}\cdot 360(1+1/2)(1+1/3)(1+1/5)+1 = 1297$ coefficients are congruent to 0 modulo 3 in (2.2) in order to prove that the modular form in (2.2) is congruent to 0 modulo 3. We have checked that this is the case using `MAPLE`.

Note that (2.2) is of the form $X^2 - Y^2 = (X-Y)(X+Y)$, where

$$X := U_3(\eta(2\tau)\eta^{11}(3\tau)\eta(5\tau)\eta^2(10\tau)\eta^2(15\tau)\eta(30\tau))$$

and

$$Y := \eta(10\tau)^6\eta^{12}(\tau).$$

By coefficient comparison we easily find that $X - Y \not\equiv 0 \pmod 3$. Hence $X + Y \equiv 0 \pmod 3$ and this implies $X \equiv 2Y \pmod 3$. This proves (2.1), and because $a(n) \equiv \Delta_2(n) \pmod 3$ for all $n \geq 0$, this proves Theorem 1.1. □

3. Concluding Remarks

We close with two comments. First, we note that the following congruence properties (which are similar to the result given in Theorem 1.1) can also be proven using our techniques:

$$\sum_{n=0}^{\infty} \Delta_2(3n)q^n \equiv \prod_{n=1}^{\infty} \frac{(1-q^{2n})^6(1-q^{5n})^2}{(1-q^n)^4(1-q^{10n})^2} \pmod 3$$

and

$$\sum_{n=0}^{\infty} \Delta_2(3n+2)q^n \equiv 2\prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{5n})(1-q^{10n})}{(1-q^{2n})} \pmod 3$$

Unfortunately, these do not appear to readily imply any particular Ramanujan–like congruences modulo 3 for broken 2–diamond partitions.

Secondly, it is a straightforward matter to prove that, for all $n \geq 0$,

$$\Delta_2(27n+7) \equiv \Delta_2(3n+1) \pmod 3,$$

a truly satisfying "internal" congruence because it gives us the ability to generate infinitely many non–nested arithmetic progressions which yield congruences modulo 3 for broken 2–diamond partitions.

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